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Asymptotic wave functions and energy distributions for long-range perturbations of the d'Alambert equation

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Introduction.

In this paper, we study the asymptotic behavior for t (time)$\rightarrow\infty$ of acoustic waves which propagate in some inhomogeneous fluids filled in an exterior domain Ω of $\boldsymbol{R}^n$, requiring that the perturbation by inhomogeneous fluids is *long-range*. For each wave, the corresponding asymptotic wave function will be constructed from the initial data (Theorem 3.1), and the asymptotic distribution of the wave energy will be calculated by use of the asymptotic wave function (Theorems 5.1$\sim$5.3).

In case of homogeneous fluids, these problems have been studied by Wilcox [8] (cf., also Kitahara [1] and Wilcox [9], where are developed the wave propagation phenomena in anisotropic homogeneous media of strongly propagative class). The principal result there states that each solution $w_0(x, t)$ of the d'Alambert equation

$$\partial_t^2 w_0(x, t)=c^2\Delta w_0(x, t) \quad \text{in } \Omega \tag{0.1}$$

($\partial_t=\partial/\partial t$, Δ is the Laplacian in $\boldsymbol{R}^n$ and $c>0$) is asymptotically equal for $t\rightarrow\infty$ to a diverging spherical wave of the form

$$w_0^\infty(x, t)=\frac{1}{\sqrt{2}}\sqrt{c}\,r^{-(n-1)/2}F_0(c^{-1}r-t, \tilde{x}) \quad (r=|x|,\ \tilde{x}=x/|x|), \tag{0.2}$$

where the wave profile $F_0(s, \tilde{x})$, $s\in\boldsymbol{R}$, is calculated from the initial data. Moreover, this result is used to the calculation of the asymptotic energy distributions. In [8] the function (0.2) is called the asymptotic wave function. If $\Omega=\boldsymbol{R}^n$, the profile $F_0(s, \tilde{x})$ is the Radon transform of the initial data, and in general case, it is modified by use of the Møller wave operators in the scattering theory. So, the asymptotic wave function (0.2) is closely related to the translation representation of Lax-Phillips [2].

Now, in inhomogeneous fluids, the propagation of acoustic waves is governed by a perturbed d'Alambert equation of the form

(0.3) $$\partial_t^2 w(x, t)=c(x)^2 p(x)\nabla\cdot\left\{\frac{1}{p(x)}\nabla w(x, t)\right\},$$

where ∇ is the gradient in $\boldsymbol{R}^n$, and $c(x)$ and $p(x)$ represent the local speed of sound and the equilibrium density of fluid, respectively. The inhomogeneity of fluid is called *short-range* if $c(x)$ and $p(x)$ satisfy for some $c>0$, $p>0$ and $\delta>0$,

(0.4) $$\nabla^l\{c(x)-c\}=O(r^{-1-\delta}),\quad \nabla^l\{p(x)-p\}=O(r^{-1-\delta})$$

($|l|=0, 1, 2$) near infinity, where $l=(l_1, \cdots, l_n)$ are multi-indices with $|l|=l_1+\cdots+l_n$. In this case the influence of the perturbation on solutions at large t is negligible, and every solution is asymptotically equal for $t\to\infty$ to a solution of the free d'Alambert equation in $\boldsymbol{R}^n$ (see, e.g., Mochizuki [4] or Reed-Simon [7]). Thus, in the short-range problem, the asymptotic wave function can be constructed by the same method of Wilcox [8] and forms also a diverging spherical wave. On the other hand, *long-range* problems admit a weaker rate of convergence of $c(x)$ and $p(x)$ themselves (in this paper we adopt a more general class where $c(x)$ and $p(x)$ need not converge at infinity; see, Assumptions and Examples in § 1). So, their influence remains on solutions at large t, and the asymptotic wave function is no longer a diverging spherical wave. The main purpose of this paper is to determine it as a modified diverging spherical wave. The modification will be done by use of an approximate phase for the steady state wave propagation problem. As a result, we can see that the ordinary Møller wave operators do not in general exist for our long-range problem.

Our result is based on two spectral representations for the operator $L=-c(x)^2p(x)\nabla\cdot\left\{\frac{1}{p(x)}\nabla\right\}$ acting on the Hilbert space $\mathfrak{H}=L^2(\Omega;\, c(x)^{-2}p(x)^{-1})$ (a precise formulation of L is given in § 1). As will be seen, our long-range class corresponds to the "oscillating" long-range potentials in quantum mechanical system introduced in Mochizuki-Uchiyama [6]. Namely, we are able to obtain the approximate phase as an approximate solution of a Riccati equation. In the present case, it is given for each $\sigma\in\boldsymbol{R}-\{0\}$ by

(0.5) $$\rho(x, \sigma)=-i\sigma\xi(x)+\frac{n-1}{2}\log r-\frac{1}{2}\log c(x);$$

$$\xi(x)=\int^r c(r'\tilde{x})^{-1}dr'.$$

With this phase function, following the argument of [6], we construct in § 2 unitary operators $\mathcal{F}_\pm:\mathfrak{H}\to L^2(\boldsymbol{R}_\pm\times S^{n-1})$ ($\boldsymbol{R}_+=(0, \infty)$, $\boldsymbol{R}_-=(-\infty, 0)$ and S^{n-1} is the unit sphere in $\boldsymbol{R}^n$) which attain spectral representations for L (Theorem 2.2):

(0.6) $$[\mathscr{F}_{\pm}Lf](\sigma, \tilde{x})=\sigma^2[\mathscr{F}_{\pm}f](\sigma, \tilde{x}), \quad (\sigma, \tilde{x})\in \boldsymbol{R}_{\pm}\times S^{n-1}.$$

In §§3 and 4, we consider L^2-solutions $w(\cdot, t)$ of (0.3). For each initial data $\{w(x, 0), \partial_t w(x, 0)\}=\{f_1(x), f_2(x)\}$, the corresponding asymptotic wave function is defined by

(0.7) $$w^{\infty}(x, t)=\frac{1}{\sqrt{2}}\sqrt{c(x)p(x)}\, r^{-(n-1)/2}F(\xi(x)-t, \tilde{x}),$$

where $\xi(x)$ is as given in (0.5) and the wave profile $F(s, \tilde{x})$ is a generalized Radon transform of the initial data:

(0.8) $$F(s, \tilde{x})=\frac{-i}{2\sqrt{\pi}}\int_{\boldsymbol{R}_+}\exp\{i\sigma s\}[\mathscr{F}_+(f_1+iL^{-1/2}f_2)](\sigma, \tilde{x})d\sigma$$
$$+\frac{-i}{2\sqrt{\pi}}\int_{\boldsymbol{R}_-}\exp\{i\sigma s\}[\mathscr{F}_-(f_1-iL^{-1/2}f_2)](\sigma, \tilde{x})d\sigma .$$

Comparing (0.7) with the spectral inversion formula of $w(x, t)$, we obtain the desired property

(0.9) $$\|w(\cdot, t)-w^{\infty}(\cdot, t)\|^2\equiv\int_{\Omega}c(x)^{-2}p(x)^{-1}|w(x, t)-w^{\infty}(x, t)|^2dx\to 0 \quad \text{as } t\to\infty.$$

The stationary phase method will play an important role for this purpose.

Finally, in §5 we consider solutions of (0.3) with finite energy. In this case, the asymptotic wave functions are similarly constructed for the fluid velocity $\nabla w(x, t)$ and pressure $\partial_t w(x, t)$, and they are applied to the calculation of the asymptotic energy distributions. As will be seen, the corresponding results of Wilcox [8; Lecture 8] are extended to our long-range problem.

§1. Assumptions and a Hilbert space formulation of the problem.

Let Ω be an infinite domain in $\boldsymbol{R}^n$ with smooth compact boundary $\partial\Omega$ lying inside some sphere $\{x\in\boldsymbol{R}^n; |x|=R_0\}$ $(R_0>0)$. We consider in Ω the following initial-boundary value problem for (0.3):

(1.1) $$\partial_t^2 w(x, t)=c(x)^2p(x)\nabla\cdot\left\{\frac{1}{p(x)}\nabla w(x, t)\right\}, \qquad (x, t)\in\Omega\times\boldsymbol{R},$$

(1.2) $$Bw(x, t)=\begin{bmatrix} w(x, t) \quad \text{or} \\ \{\nu(x)\cdot\nabla+d(x)\}\, w(x, t)\end{bmatrix}=0, \qquad (x, t)\in\partial\Omega\times\boldsymbol{R},$$

(1.3) $$w(x, 0)=f_1(x) \quad \text{and} \quad \partial_t w(x, 0)=f_2(x), \qquad x\in\Omega.$$

Here $\nu(x)=(\nu_1(x), \cdots, \nu_n(x))$ is the outer unit normal to the boundary and $d(x)$

is a smooth non-negative function on $\partial\Omega$, and thus, B represents the usual Dirichlet or Robin boundary condition.

Throughout this paper, we require the following conditions on $c(x)$ and $p(x)$.

ASSUMPTIONS. $c(x)$ and $p(x)$ are twice continuously differentiable and are uniformly positive in $\bar{\Omega}=\Omega\cup\partial\Omega$:

(c. 1) $$0<c_1\leqq c(x)\leqq c_2<\infty,$$

(p. 1) $$0<p_1\leqq p(x)\leqq p_2<\infty.$$

Moreover, for some $0\leqq\gamma<1$, $\gamma/2<\delta_0\leqq 1$, $1/2<\delta_1\leqq 1$ and $(2+\gamma)/4<\delta_2\leqq 1$ (the requirement $\delta_j\leqq 1$ ($j=0, 1, 2$) does not restrict the generality).

(c. 2) $$\liminf_{r\to\infty}\{\gamma c(x)-r\partial_r c(x)\}\geqq 0,$$

(c. 3) $$\partial_r c(x)=O(r^{-1}),\quad \nabla\partial_r c(x)=O(r^{-1-\delta_1}),$$

(c. 4) $$(\nabla-\tilde{x}\partial_r)c(x)=O(r^{-1-\delta_2}),$$

(c. 5) $$-r^{-2}\Lambda c(x)\equiv(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)c(x)=O(r^{-1-2\delta_2}),$$

(p. 2) $$V_p(x)\equiv-\frac{\Delta p(x)}{2p(x)}+3\Big(\frac{\nabla p(x)}{2p(x)}\Big)^2=O(r^{-1-\delta_0})$$

as $r\to\infty$, where $r=|x|$, $\tilde{x}=x/|x|$ and $\partial_r=\partial/\partial r$.

In the above assumptions we do not require that the functions $c(x)$ and $p(x)$ themselves converge at infinity. As we see in the following examples, our theory includes some inhomogeneous fluids which are inhomogeneous also at infinity.

EXAMPLE 1. Let $c_0(x)$ be a uniformly positive function such that

$$c_0(x)\to c_0,\quad \nabla^l c_0(x)=O(r^{-1-|l|\rho})\ (|l|=1, 2)\quad \text{as}\quad r\to\infty$$

for some $1/2<\rho\leqq 1$. Then the function

$$c(x)=c_0(x)+a\sin(\log r)\ (|a|<c_0,\ \text{small})$$

satisfies (c. 2)~(c. 5) with $\gamma=|a|(c_0^2-a^2)^{-1/2}$ ($<\min\{1, 4\rho-2\}$), $\delta_1=1$ and $\delta_2=\rho$.

EXAMPLE 2. Let $c_0(x)$ be as given above. Then the function

$$c(x)=c_0(x)+a\sin r^{\varepsilon}/r^{\varepsilon}\ (0\leqq\varepsilon<1/2;\ |a|<c_0,\ \text{small})$$

satisfies (c. 2)~(c. 5) with $\gamma=\varepsilon|a|/c_0$ ($<\min\{1, 4\rho-2\}$), $\delta_1=1-\varepsilon$ and $\delta_2=\rho$. Here r^{ε} with $\varepsilon=0$ is regarded as $\log r$.

EXAMPLE 3. Let $p_0(x)$ be a uniformly positive function such that

$$p_0(x)=O(r^{-|l|\rho})\ (|l|=0, 1, 2)\quad \text{as}\quad r\to\infty$$

for some $(2+\gamma)/4<\rho\leqq 1$. Then the function

$$p(x)=p_0(x)+b\sin r^{\varepsilon}\quad (0\leqq\varepsilon<(2-\gamma)/4\,;\ |b|\ \text{small})$$

satisfies (p. 2) with $\delta_0=1-2\varepsilon$.

The steady state wave propagation problem for (1.1)~(1.3) is:

$$\text{(1.4)}\qquad \begin{cases} -c(x)^2 p(x)\nabla\cdot\left\{\dfrac{1}{p(x)}\nabla u\right\}-\kappa^2 u=f(x) & \text{in } \Omega \\ Bu=0 \quad \text{on } \partial\Omega, & \end{cases}$$

where $\kappa=\sigma+i\tau$ is a complex number such that $\sigma\in\boldsymbol{R}_{\pm}$ and $\tau\geqq 0$. We put

$$\text{(1.5)}\qquad u_p=p(x)^{-1/2}u \quad \text{and} \quad f_p(x)=p(x)^{-1/2}f(x)$$

in (1.4). Then we have

$$\text{(1.6)}\qquad \begin{cases} -c(x)^2\{\Delta-V_p(x)\}u_p-\kappa^2 u_p=f_p(x) & \text{in } \Omega \\ B_p u_p=\begin{bmatrix} u_p \quad \text{or} \\ \{\nu(x)\cdot\nabla+d_p(x)\}u_p \end{bmatrix}=0 & \text{on } \partial\Omega, \end{cases}$$

where $V_p(x)$ is the function given in (p. 2) and

$$\text{(1.7)}\qquad d_p(x)=d(x)+\frac{\nu(x)\cdot\nabla p(x)}{2p(x)} \quad (x\in\partial\Omega).$$

Let $\mathfrak{H}=L^2(\Omega\,;c(x)^{-2}p(x)^{-1})$ be the Hilbert space with norm

$$\text{(1.8)}\qquad \|f\|=\left\{\int_{\Omega}c(x)^{-2}p(x)^{-1}|f(x)|^2dx\right\}^{1/2},$$

and let $\mathfrak{H}_1=L^2(\Omega\,;c(x)^{-2})$ be the Hilbert space with norm

$$\text{(1.9)}\qquad |||f|||=\left\{\int_{\Omega}c(x)^{-2}|f(x)|^2dx\right\}^{1/2}.$$

The norm of the usual L^2-space $L^2(\Omega)$ is given by

$$\text{(1.10)}\qquad \|f\|_0=\left\{\int_{\Omega}|f(x)|^2dx\right\}^{1/2}.$$

Under (c. 1) and (p. 1) the above three norms are equivalent:

$$\text{(1.11)}\qquad \sqrt{p_1}\|f\|\leqq|||f|||\leqq\sqrt{p_2}\|f\|,$$

$$\text{(1.12)}\qquad c_1|||f|||\leqq\|f\|_0\leqq c_2|||f|||.$$

We define the operator L acting in $\mathfrak{H}$ and L_p acting in $\mathfrak{H}_1$ as follows:

(1.13)
$$\begin{cases} \mathcal{D}(L)=\{u\in H^2(\Omega)\,;\, Bu|_{\partial\Omega}=0\} \\ Lu=-c(x)^2p(x)\nabla\cdot\left\{\dfrac{1}{p(x)}\nabla u\right\} \quad \text{for} \quad u\in\mathcal{D}(L), \end{cases}$$

(1.14)
$$\begin{cases} \mathcal{D}(L_p)=\{u_p\in H^2(\Omega)\,;\, B_pu_p|_{\partial\Omega}=0\} \\ L_pu_p=-c(x)^2\{\Delta-V_p(x)\}u_p \quad \text{for} \quad u_p\in\mathcal{D}(L_p), \end{cases}$$

where $H^k(\Omega)$ $(k=1, 2, \cdots)$ denotes the usual Sobolev space with norm

(1.15)
$$\|f\|_{H^k}=\{\sum_{|l|\leq k}\|\nabla^l f\|_0^2\}^{1/2}.$$

Then as is easily seen (cf., [4], [7]), we have the

LEMMA 1.1. (a) *Both L and L_p are positive selfadjoint operators. Moreover, they are unitary equivalent to each other:*

(1.16)
$$L=J_pL_pJ_p^{-1},$$

where $J_p:\mathfrak{H}_1\to\mathfrak{H}$ is the multiplication operator defined by

(1.17)
$$J_pf_p=\sqrt{p(x)}f_p(x) \qquad \textit{for} \quad f_p\in\mathfrak{H}_1.$$

(b) *Let V be the multiplication operator given by $c(x)^2V_p(x)$. Then we have*

(1.18)
$$L_p=L_1+V,$$

where L_1 is the operator L_p with $V_p(x)\equiv 0$.

The relations (1.16) and (1.18) will be used in the next section to obtain spectral representations for L. In the remainder of this section we shall use the positivity of L to obtain solutions of the initial-boundary value problem (1.1)~(1.3). Our concern will be restricted to the solution in $\mathfrak{H}$ or the solution with finite energy.

Let H be the positive square root of $L: H=\sqrt{L}>0$, and let $\mathcal{D}(H)$ [or $\mathcal{D}(H^{-1})$] denote the domain of H [or H^{-1}]. Then for given initial data $\{f_1, f_2\}\in\mathcal{D}(L)\times\{\mathcal{D}(H)\cap\mathcal{D}(H^{-1})\}$, (1.1)~(1.3) is reduced to the evolution equation

(1.19)
$$\partial_t^2 w(t)+H^2w(t)=0, \quad w(0)=f_1, \quad \partial_t w(0)=f_2$$

in $\mathfrak{H}$. Thus, we can write down the solution of (1.1)~(1.3) as follows:

(1.20)
$$w(\cdot, t)=\cos(Ht)f_1+H^{-1}\sin(Ht)f_2.$$

Note that $t\to w(\cdot, t)\in H^k(\Omega)$ $(k=0, 1, 2)$ is $(2-k)$-times continuously differentiable in $t\in\boldsymbol{R}$. We put

(1.21)
$$w_1(\cdot, t)=Hw(\cdot, t) \quad \text{and} \quad w_2(\cdot, t)=\partial_t w(\cdot, t).$$

Then the wave energy at time t is given by

$$E(w, \Omega, t)=\|w_1(\cdot, t)\|^2+\|w_2(\cdot, t)\|^2. \tag{1.22}$$

We define $\bar{\mathscr{D}}(H)$ [$\bar{\mathscr{D}}(H^{-1})$] to be the closure of $\mathscr{D}(H)$ [$\mathscr{D}(H^{-1})$] in the norm $\|Hf\|$ [$\|H^{-1}f\|$]. Then since $\mathscr{D}(H)$ and $\mathscr{D}(H^{-1})$ are both dense in $\mathfrak{H}$, we have the

LEMMA 1.2. *H [H^{-1}] can be extended to a unitary operator from $\bar{\mathscr{D}}(H)$ [$\bar{\mathscr{D}}(H^{-1})$] onto $\mathfrak{H}$. The extended operator will be denoted by $\bar{H}$ [$\overline{H^{-1}}$].*

By means of this lemma we can easily verify the following

PROPOSITION 1.1. (a) *For $\{f_1, f_2\}\in\mathfrak{H}\times\bar{\mathscr{D}}(H^{-1})$ and $t\in\boldsymbol{R}$ we define the weak solution $w(\cdot, t)\in\mathfrak{H}$ of* (1.1)$\sim$(1.3) *as follows:*

$$w(\cdot, t)=\frac{1}{2}\exp\{-iHt\}(f_1+i\overline{H^{-1}}f_2)+\frac{1}{2}\exp\{iHt\}(f_1-i\overline{H^{-1}}f_2). \tag{1.23}$$

Then it is an extension of (1.20) *and $t\to w(\cdot, t)$ remains continuous for all $t\in\boldsymbol{R}$. Moreover,*

$$\|w(\cdot, t)\|\leqq\|f_1\|+\|\overline{H^{-1}}f_2\| \qquad \text{for any} \quad t\in\boldsymbol{R}. \tag{1.24}$$

(b) *For $\{f_1, f_2\}\in\bar{\mathscr{D}}(H)\times\mathfrak{H}$ and $t\in\boldsymbol{R}$ we define the pair $\{w_1(\cdot, t), w_2(\cdot, t)\}$ as follows:*

$$\begin{cases} w_1(\cdot, t)=\dfrac{1}{2}\exp\{-iHt\}(\bar{H}f_1+if_2)+\dfrac{1}{2}\exp\{iHt\}(\bar{H}f_1-if_2) \\ w_2(\cdot, t)=-\dfrac{i}{2}\exp\{-iHt\}(\bar{H}f_1+if_2)+\dfrac{i}{2}\exp\{iHt\}(\bar{H}f_1-if_2). \end{cases} \tag{1.25}$$

Then it is an extension of (1.21) *and $t\to\{w_1(\cdot, t), w_2(\cdot, t)\}\in\mathfrak{H}\times\mathfrak{H}$ remains continuous for all $t\in\boldsymbol{R}$. Moreover, the wave energy defined by* (1.22) *and* (1.25) *is independent of $t\in\boldsymbol{R}$:*

$$E(w, \Omega, t)=E(w, \Omega, 0)=\|\bar{H}f_1\|^2+\|f_2\|^2. \tag{1.26}$$

REMARK 1.1. $\bar{\mathscr{D}}(H)$ is the so-called Beppo-Levi space.

§ 2. Spectral representations for L.

In this section we shall construct spectral representations for L. A semi-abstract theory of [**6**; I] is applicable (under a slight modification) to show the principle of limiting absorption for (1.6). Then following the same argument of [**6**; II], we can have unitary operators $\mathscr{F}_\pm:\mathfrak{H}\to L^2(\boldsymbol{R}_\pm\times S^{n-1})$ which diagonalize L.

We consider the exterior boundary-value problem

$$\begin{cases} -\Delta u+q(x)u-\kappa^2c(x)^{-2}u=c(x)^{-2}f(x) & \text{in } \Omega \\ B_pu=0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here $q(x)$ stands for the functions $q(x)\equiv 0$ and $q(x)=V_p(x)$. Namely, it is a real-valued continuous function on $\bar{\Omega}$ behaving like

$$q(x)=O(r^{-1-\delta_0}) \quad \text{as} \quad r\to\infty. \tag{2.2}$$

We define the selfadjoint operator M in $\mathfrak{H}_1$ as follows:

$$\begin{cases} \mathcal{D}(M)=\{u\in H^2(\Omega)\,;\, B_p u|_{\partial\Omega}=0\} \\ Mu=-c(x)^2\{\Delta-q(x)\}u \quad \text{for} \quad u\in\mathcal{D}(M). \end{cases} \tag{2.3}$$

Note that $M=L_1$ if $q(x)\equiv 0$ and $M=L_p$ if $q(x)=V_p(x)$. Thus, M is lower semi-bounded in $\mathfrak{H}_1$, and the following ellipticity estimate holds for any $u\in\mathcal{D}(M)$ (cf., e.g., Mizohata [3]).

$$\|u\|_{H^2}\leqq C\{|||Mu|||+|||u|||\} \quad \text{for some} \quad C>0. \tag{2.4}$$

The selfadjointness of M shows that if κ^2 is non-real and $f\in\mathfrak{H}_1$, (2.1) has a unique solution

$$u=u(\cdot,\kappa\,;f)=R_M(\kappa^2)f \quad \text{in } \mathfrak{H}_1, \tag{2.5}$$

where $R_M(\kappa^2)$ is the resolvent of M. We shall show that the limit

$$u(\cdot,\sigma\,;f)=\lim_{\tau\downarrow 0}u(\cdot,\sigma+i\tau\,;f) \quad (\sigma\in\boldsymbol{R}_\pm) \tag{2.6}$$

exists in some locally square integrable class. For this aim, we have only to check two propositions which correspond to Assumptions 1 and 2 of [6; I]. Proposition 2.1 will be a uniqueness result (or growth property) for solutions of

$$-\Delta u+q(x)u-\sigma^2 c(x)^{-2}u=0 \quad \text{in } \Omega. \tag{2.7}$$

Proposition 2.2 will be related to the radiation condition, i.e., required properties of an approximate solution $k(x,\kappa)$ of the Riccati equation

$$\partial_r k+\frac{n-1}{r}k-k^2+q(x)-\kappa^2 c(x)^{-2}=O(r^{-1-\delta}) \tag{2.8}$$

will be summarized there. Here and in the following we put

$$\delta=\min\{\delta_0,\delta_1,\delta_2\} \quad \text{and} \quad \tilde{\delta}=\min\{\delta, 2\delta_2-1\}. \tag{2.9}$$

In [6; I] is studied the equation

$$-\Delta u+\{V_1(x)+V_s(x)\}u-\zeta u=f(x) \quad \text{in } \Omega,$$

where $V_1(x)$ is an "oscillating" long-range potential, $V_s(x)$ is a short-range potential and ζ is a complex number. A radiation condition is introduced based on the function $\sqrt{\zeta-V_1(x)}$. In the present case (2.1), this function should be replaced by $\kappa c(x)^{-1}$. Then a solution of (2.8) is (cf., (8.15) of [**6**; I]):

$$k(x, \kappa)=-i\kappa c(x)^{-1}+\frac{n-1}{2r}-\frac{\partial_r c(x)}{2c(x)}. \tag{2.10}$$

PROPOSITION 2.1. *For each* $\sigma\in\boldsymbol{R}-\{0\}$, *let* $u\in H^2_{\mathrm{loc}}(\bar{\Omega})$ *be a non-trivial solution of* (2.7). *Then we have for any* $\beta>\gamma$,

$$\liminf_{R\to\infty} R^{\beta}\int_{S(R)}\{|\partial_r u|^2+|u|^2\}\,dS=\infty, \tag{2.11}$$

where $S(R)=\{x\,;\,|x|=R\}$.

PROOF. Note that

$$2\beta c(x)^{-2}+r\partial_r c(x)^{-2}=\frac{2}{c(x)^2}\{\gamma c(x)-r\partial_r c(x)\}+\frac{2(\beta-\gamma)}{c(x)^2}.$$

Then it follows from (c. 1) and (c. 2) that there exist constants $R_1\geqq R_0$ and $C_1\geqq 0$ depending on β such that

$$2\beta c(x)^{-2}+r\partial_r c(x)^{-2}\geqq C_1 \quad \text{for} \quad |x|\geqq R_1. \tag{2.12}$$

By means of (2.2), (2.12) and the local version of (2.4), we can apply Theorem 1.1 of Mochizuki [**5**] to obtain the above assertion. Q. E. D.

PROPOSITION 2.2. *We put*

$$\Pi_{\pm}=\{\kappa=\sigma+i\tau\,;\ \sigma\in\boldsymbol{R}_{\pm} \text{ and } \tau\geqq 0\}, \tag{2.13}$$

and let $k(x, \kappa)$, $(x, \kappa)\in\Omega\times\Pi_{\pm}$, *be as given in* (2.10). *Let* $K_{\pm}$ *be any compact set of* $\Pi_{\pm}$. *Then there exist* $C_j=C_j(K_{\pm})>0$ $(j=2\sim 6)$ *such that for any* $(x, \kappa)\in\Omega\times K_{\pm}$,

$$\left|\partial_r k+\frac{n-1}{r}k-k^2+q(x)-\kappa^2 c(x)^{-2}\right|\leqq C_2(1+r)^{-1-\delta}, \tag{2.14}$$

$$|k(x, \kappa)|\leqq C_3, \tag{2.15}$$

$$\mp\mathrm{Im}\, k(x, \kappa)=\pm\sigma c(x)^{-1}\geqq C_4, \tag{2.16}$$

$$\mathrm{Re}\, k(x, \kappa)-\frac{n-1-\beta}{2r}\geqq C_5 r^{-1} \quad (\beta>\gamma), \tag{2.17}$$

$$|(\nabla-\tilde{x}\partial_r)k(x, \kappa)|\leqq C_6 r^{-1-\delta}. \tag{2.18}$$

PROOF. (2.15) and (2.16) are obvious from (c. 1) and (c. 3). (2.17) follows from (c. 2) since we have

$$\mathrm{Re}\, k-\frac{n-1-\beta}{2r}=\tau c(x)^{-1}+\frac{1}{2rc(x)}\{\beta c(x)-r\partial_r c(x)\}.$$

(2.18) follows from (c. 3) and (c. 4) since we have

$$(\nabla-\tilde{x}\partial_r)k=c(x)^{-2}\Big\{i\kappa(\nabla-\tilde{x}\partial_r)c(x)+\frac{1}{2}c(x)(\nabla-\tilde{x}\partial_r)\partial_r c(x)+\frac{1}{2}\partial_r c(x)(\nabla-\tilde{x}\partial_r)c(x)\Big\}.$$

Finally, (2.14) follows from (c. 3) and (2.2) since we have

$$\partial_r k+\frac{n-1}{r}k-k^2+q(x)-\kappa^2 c(x)^{-2}$$

$$=q(x)+\frac{(n-1)(n-3)}{4r^2}+\Big(\frac{\partial_r c(x)}{2c(x)}\Big)^2-\frac{\partial_r^2 c(x)}{2c(x)}. \qquad \text{Q. E. D.}$$

We denote by $L^2_\mu(\Omega)$, $\mu\in \boldsymbol{R}$, the space of all functions $f(x)$ such that

$$\|f\|^2_\mu=\int_\Omega (1+r)^{2\mu}|f(x)|^2 dx<\infty. \tag{2.19}$$

DEFINITION 2.1. For solutions $u\in H^2_{\mathrm{loc}}(\bar{\Omega})$ of (2.1) with $\kappa\in\Pi_\pm$, the radiation condition at infinity is defined by

$$u\in L^2_{(-1-\alpha)/2}(\Omega) \quad \text{and} \quad \{\partial_r+k(x,\kappa)\}u\in L^2_{(-1+\beta)/2}(\Omega), \tag{2.20}$$

where α, β is any pair of positive numbers satisfying

$$0<\alpha\leqq\beta\leqq 1,\ \gamma<\beta \quad \text{and} \quad \alpha+\beta\leqq 2\delta. \tag{2.21}$$

u satisfying (2.1) and (2.20) with $\kappa\in\Pi_+$ [or $\in\Pi_-$] is called an outgoing [or incoming] solution.

Now, with the aid of the above two propositions, we can follow the same line of proof of [**6**; I] to obtain the following results (cf., Theorems 1∼5 of [**6**; I]).

THEOREM 2.1. (a) *Let $K_\pm$ and α, β be as given above. Then for any $\kappa\in K_\pm$ and $f\in L^2_{(1+\beta)/2}(\Omega)$ (which is dense in $\mathfrak{H}_1$), (2.1) has a unique outgoing [incoming] solution $u=u(\cdot,\kappa;f)$, which also satisfies*

$$\|u\|_{(-1-\alpha)/2}\leqq C_7\|f\|_{(1+\beta)/2}, \tag{2.22}$$

$$\|\{\nabla+\tilde{x}k(x,\kappa)\}u\|_{(-1+\beta)/2}\leqq C_7\|f\|_{(1+\beta)/2}, \tag{2.23}$$

where $C_7=C_7(K_\pm)>0$ is a domain constant independent of f. Moreover, if κ is non-real, this solution coincides with the L^2-solution (2.5).

(b) *We extend $R_M(\kappa^2)$ to real $\kappa=\sigma+i0$ by*

$$[R_M((\sigma+i0)^2)f](x)=u(x,\sigma+i0;f). \tag{2.24}$$

Then $R_M(\kappa^2)f$ is continuous in $L^2_{(-1-\alpha)/2}(\Omega)$ with respect to $(\kappa,f)\in\Pi_\pm\times L^2_{(1+\beta)/2}(\Omega)$. Moreover, let $R^*_M(\kappa^2)\colon L^2_{(1+\alpha)/2}(\Omega)\to L^2_{(-1-\beta)/2}(\Omega)$ be the adjoint of $R_M(\kappa^2)$. Then we have for any $(\kappa,f)\in\Pi_\pm\times L^2_{(1+\beta)/2}(\Omega)$,

$$R^*_M(\kappa^2)f=R_M((-\bar{\kappa})^2)f \quad (\bar{\kappa}\ \textit{being the complex conjugate of}\ \kappa). \tag{2.25}$$

(c) *Let* $\{\mathcal{E}_M(\lambda);\lambda\in\boldsymbol{R}\}$ *be the spectral measure of* M. *Then for any pre-compact set* $e\Subset R_+$ *and* $f,\ g\in L^2_{(1+\beta)/2}(\Omega)$, *we have*

$$((\mathcal{E}_M(e)f,\ g))=\frac{1}{\pi i}\int_{\pm\sqrt{e}}((\{R_M((\sigma+i0)^2)-R_M((-\sigma+i0)^2)\}f,\ g))\sigma d\sigma\,, \tag{2.26}$$

where $\pm\sqrt{e}=\{\sigma\in\boldsymbol{R}_\pm;\ \sigma^2\in e\}$ *and* $((\ ,\ ))$ *denotes the inner product in* $\mathfrak{H}_1$, *or more generally, a duality between* $L^2_{(-1-\alpha)/2}(\Omega)$ *and* $L^2_{(1+\alpha)/2}(\Omega)$. *Thus, the positive part of* M *is absolutely continuous with respect to the Lebesgue measure on* $\sigma^2\in\boldsymbol{R}_+$.

REMARK 2.1. In case $q(x)\equiv 0$, we can choose $\delta_0=1$ in (2.2). So, the condition (2.21) is represented in this case as follows:

$$0<\alpha\leqq\beta\leqq 1,\quad \gamma<\beta\quad \text{and}\quad \alpha+\beta\leqq\min\{2\delta_1,\ 2\delta_2\}. \tag{2.27}$$

In particular, we can choose $\beta=1$ in (2.27).

Let $R_1(\kappa^2)$ and $R_p(\kappa^2)$ denote the operators $R_M(\kappa^2)$ corresponding to $M=L_1$ and $M=L_p$. Then (1.18) and the 'resolvent' equation imply

$$R_p(\kappa^2)=R_1(\kappa^2)\{1-VR_p(\kappa^2)\}\qquad(\kappa\in\Pi_\pm). \tag{2.28}$$

Here $VR_p(\kappa^2)$ is bounded in $L^2_{(1+\beta)/2}(\Omega)$ by (c. 1) and (p. 2). For $\lambda\in\boldsymbol{R}$ let $\mathcal{E}(\lambda)$ be the spectral measure of the operator L. Then noting (1.16), we have from (2.28) and (b), (c) of the above theorem the

LEMMA 2.1. *For any pre-compact set* $e\Subset\boldsymbol{R}_+$ *and* $f,\ g\in L^2_{(1+\beta)/2}(\Omega)$,

$$(\mathcal{E}(e)f,\ g)=\frac{1}{\pi i}\int_{\pm\sqrt{e}}(\{R_1((\sigma+i0)^2)-R_1((-\sigma+i0)^2)\} \cdot\{1-VR_p((\sigma+i0)^2)\}J_p^{-1}f,\ \{1-VR_p((\sigma+i0)^2)\}J_p^{-1}g)\sigma d\sigma\,, \tag{2.29}$$

where $(\ ,\ )$ *is the inner product in* $\mathfrak{H}$.

We put for any $(x,\ \sigma)\in\Omega\times(\boldsymbol{R}-\{0\})$,

$$\rho(x,\ \sigma)=-i\sigma\xi(x)+\frac{n-1}{2}\log r-\frac{1}{2}\log c(x)\,; \tag{2.30}$$

$$\xi(x)=\begin{cases}\int_{R_0}^{r}c(s\tilde{x})^{-1}ds & \text{if}\quad r=|x|>R_0,\\ 0 & \text{if}\quad r=|x|\leqq R_0.\end{cases} \tag{2.31}$$

Then obviously

$$\partial_r\rho(x,\ \sigma)=k(x,\ \sigma+i0)\qquad\text{for}\quad r>R_0, \tag{2.32}$$

and from (c. 1), (c. 4) and (c. 5) we have the

LEMMA 2.2. *Let* $N>1$. *Then there exists a constant* $C_8>0$ *such that for any* $(x,\ \sigma)\in\Omega\times([-N,\ 0)\cup(0,\ N])$,

$$|(\nabla-\tilde{x}\partial_r)\rho|\leqq C_8(1+r)^{-\delta_2}, \tag{2.33}$$

$$|(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)\rho|\leqq C_8(1+r)^{-2\delta_2}. \tag{2.34}$$

Our spectral representations for L will be based on the above two lemmas (cf., [**6**; II]).

For $\sigma\in\boldsymbol{R}-\{0\}$ and $f\in L^2_{(1+\beta)/2}(\Omega)$, where β satisfies (2.27), we put

$$[\mathcal{F}_1(\sigma, r)f](\tilde{x})=\sqrt{\frac{2}{\pi}}\sigma\exp\{\rho(r\tilde{x}, \sigma)\}[R_1((\sigma+i0)^2)f](r\tilde{x}). \tag{2.35}$$

Then (2.20), (2.14), (2.32), Lemma 2.2 and the Green formula show the following proposition (cf., [**6**; II]; Propositions 1.3, 1.4, 2.1 and Lemma 3.2).

PROPOSITION 2.3. (a) *Let α, β be any pair satisfying* (2.27). *Let* $(f, \sigma)\in L^2_{(1+\beta)/2}(\Omega)\times(\boldsymbol{R}-\{0\})$ *and* $u_1=R_1((\sigma+i0)^2)f$. *Then there exists a sequence* $r_l=r_l(\alpha, \beta, f, \sigma)$ *diverging to* ∞ *such that*

$$\lim_{l\to\infty}\int_{S(r_l)}\{r^{-\alpha}|u_1|^2+r^{\beta}|(\nabla+\tilde{x}\partial_r\rho)u_1|^2\}dS=0, \tag{2.36}$$

from which it follows that

$$\lim_{l\to\infty}\int_{S^{n-1}}|\mathcal{F}_1(\sigma, r_l)f|^2dS_{\tilde{x}}=\frac{\sigma}{\pi i}((\{R_1((\sigma+i0)^2)-R_1((-\sigma+i0)^2)\}f, f)). \tag{2.37}$$

(b) *Let* $f\in L^2_1(\Omega)$ *and* $r_l=r_l(\alpha, 1, f, \sigma)$. *Then the strong limit*

$$\mathcal{F}_1(\sigma)f=\operatorname*{s-lim}_{l\to\infty}\mathcal{F}_1(\sigma, r_l)f \quad in \quad L^2(S^{n-1}) \tag{2.38}$$

exists, and $\mathcal{F}_1(\sigma)$ *is a bounded operator from* $L^2_1(\Omega)$ *to* $L^2(S^{n-1})$ *which depends continuously on* $\sigma\in\boldsymbol{R}-\{0\}$.

(c) *We choose* $\tilde{\alpha}$, $\tilde{\beta}$ *as follows:*

$$0<\tilde{\alpha}\leqq\tilde{\beta}\leqq 1,\quad \gamma<\tilde{\beta}\quad and\quad \tilde{\alpha}+\tilde{\beta}\leqq 2\tilde{\delta}=\min\{2\delta, 4\delta_2-2\} \tag{2.39}$$

((2.39) *is stronger than* (2.21), *i.e.*, (2.27)). *Then for any* $\sigma\in\boldsymbol{R}-\{0\}$, $\mathcal{F}_1(\sigma)$ *can be extended by continuity to a bounded operator from* $L^2_{(1+\tilde{\beta})/2}(\Omega)$ *to* $L^2(S^{n-1})$. *Denoting the extended operator by* $\mathcal{F}_1(\sigma)$ *again, we have for any* $f\in L^2_{(1+\tilde{\beta})/2}(\Omega)$ *and* $\phi\in L^2(S^{n-1})$,

$$\int_{S^{n-1}}\mathcal{F}_1(\sigma)f\bar{\phi}dS_{\tilde{x}}=\lim_{l\to\infty}\int_{S^{n-1}}\mathcal{F}_1(\sigma, r_l)f\bar{\phi}dS_{\tilde{x}}, \tag{2.40}$$

where $r_l=r_l(\tilde{\alpha}, \tilde{\beta}, f, \sigma)$.

Now we put

$$\mathcal{F}(\sigma)=\mathcal{F}_1(\sigma)\{1-VR_p((\sigma+i0)^2)\}J_p^{-1}, \qquad \sigma\in\boldsymbol{R}-\{0\}. \tag{2.41}$$

Then by use of Lemma 2.1 and Proposition 2.3 we can prove the following spectral representation theorem for L.

THEOREM 2.2. (a) *Let* $\mathscr{F}_\pm$ *be defined by*

$$[\mathscr{F}_\pm f](\sigma, \tilde{x})=[\mathscr{F}(\sigma)f](\tilde{x}) \quad for \quad (\sigma, \tilde{x})\in \boldsymbol{R}_\pm\times S^{n-1}. \tag{2.42}$$

Then $\mathscr{F}_\pm$ *initially defined on* $L^2_{(1+\tilde{\beta})/2}(\Omega)$ *can be extended to a unitary operator from* $\mathfrak{H}$ *onto* $\tilde{\mathfrak{H}}_\pm=L^2(\boldsymbol{R}_\pm\times S^{n-1})$, *which will be denoted by* $\mathscr{F}_\pm$ *again. Here the norm* $\|\ \|_\pm$ *of* $\tilde{\mathfrak{H}}_\pm$ *is defined by*

$$\|\phi_\pm\|_\pm^2=\iint_{\boldsymbol{R}_\pm\times S^{n-1}}|\phi_\pm(\sigma, \tilde{x})|^2 d\sigma dS_{\tilde{x}} \qquad for \quad \phi_\pm\in\tilde{\mathfrak{H}}_\pm. \tag{2.43}$$

(b) *For any bounded Borel function* $a(t)$ *on* $\boldsymbol{R}$ *and any* $f\in\mathfrak{H}$, *we have*

$$a(L)f=\mathscr{F}_\pm^* a(\sigma^2)\mathscr{F}_\pm f=\underset{N\to\infty}{\text{s-lim}}\int_{e_{\pm N}}\mathscr{F}^*(\sigma)a(\sigma^2)[\mathscr{F}_\pm f](\sigma, \cdot)d\sigma \quad in\ \mathfrak{H}, \tag{2.44}$$

where $\mathscr{F}_\pm^*: \tilde{\mathfrak{H}}_\pm\to\mathfrak{H}$ *is the adjoint of* $\mathscr{F}_\pm$, $\mathscr{F}^*(\sigma): L^2(S^{n-1})\to L^2_{(-1-\tilde{\beta})/2}(\Omega)$ *is the adjoint of* $\mathscr{F}(\sigma)$ *and* $e_{+N}=(1/N, N)$, $e_{-N}=(-N, -1/N)$.

We omit the proof of this theorem, since it can be done by the same argument of [**6**; II] (Theorems 3.1 and 4.1).

§ 3. Asymptotic wave functions for $w(x, t)$.

In this section we return to the solution $w(x, t)$ given by (1.23) of the initial-boundary value problem (1.1)~(1.3).

The following spectral representations of $w(x, t)$ is a result of Theorem 2.2 (b).

LEMMA 3.1. *For given any initial data* $f=\{f_1, f_2\}\in\mathfrak{H}\times\mathscr{D}(H^{-1})$, $w(x, t)$ *is expressed as follows:*

$$w(\cdot, t)=\frac{1}{\sqrt{2}}\underset{N,N'\to\infty}{\text{s-lim}}\int_{e_{-N,N'}}\exp\{\mp i\sigma t\}\mathscr{F}^*(\sigma)\tilde{f}_\pm(\sigma, \cdot)d\sigma, \tag{3.1$_\pm$}$$

where $e_{-N,N'}=(-N, -1/N)\cup(1/N', N')$ *and*

$$\tilde{f}_+(\sigma, \tilde{x})=\begin{cases}\dfrac{1}{\sqrt{2}}[\mathscr{F}_+(f_1+i\overline{H^{-1}}f_2)](\sigma, \tilde{x}), & (\sigma, \tilde{x})\in\boldsymbol{R}_+\times S^{n-1}\\ \dfrac{1}{\sqrt{2}}[\mathscr{F}_-(f_1-i\overline{H^{-1}}f_2)](\sigma, \tilde{x}), & (\sigma, \tilde{x})\in\boldsymbol{R}_-\times S^{n-1},\end{cases} \tag{3.2$_+$}$$

$$\tilde{f}_-(\sigma, \tilde{x})=\begin{cases}\dfrac{1}{\sqrt{2}}[\mathscr{F}_+(f_1-i\overline{H^{-1}}f_2)](\sigma, \tilde{x}), & (\sigma, \tilde{x})\in\boldsymbol{R}_+\times S^{n-1}\\ \dfrac{1}{\sqrt{2}}[\mathscr{F}_-(f_1+i\overline{H^{-1}}f_2)](\sigma, \tilde{x}), & (\sigma, \tilde{x})\in\boldsymbol{R}_-\times S^{n-1}.\end{cases} \tag{3.2$_-$}$$

PROOF. We use (2.44) with $a(\lambda)=\exp\{-i\sqrt{\lambda}t\}$ and $=\exp\{i\sqrt{\lambda}t\}$. Then it follows from (1.23) that

$$w(\cdot, t)=\frac{1}{2}\lim_{N'\to\infty}\int_{e_{\pm N'}}\mathcal{F}^*(\sigma)\exp\{-i|\sigma|t\}[\mathcal{F}_\pm(f_1+i\overline{H^{-1}}f_2)](\sigma, \cdot)d\sigma$$

$$+\frac{1}{2}\lim_{N\to\infty}\int_{e_{\mp N}}\mathcal{F}^*(\sigma)\exp\{i|\sigma|t\}[\mathcal{F}_\mp(f_1-i\overline{H^{-1}}f_2)](\sigma, \cdot)d\sigma .$$

This and $(3.2)_\pm$ imply $(3.1)_\pm$. Q. E. D.

We shall use the expression $(3.1)_+$ to construct an asymptotic wave function as $t\to\infty$ corresponding to each solution $w(x, t)$. For the sake of simplicity, we omit the subscript $+$ of $\tilde{f}_+(\sigma, \tilde{x})$ and write it as $\tilde{f}(\sigma, \tilde{x})$. (Note here that an asymptotic wave function for $t\to-\infty$ can be constructed by the same method if we use $(3.1)_-$ in place of $(3.1)_+$. In the following, however, we do not enter into this problem.)

LEMMA 3.2. *Let $\tilde{\mathfrak{H}}=L^2(\boldsymbol{R}\times S^{n-1})$ be the Hilbert space with norm*

$$\|\phi\|=\left\{\iint_{\boldsymbol{R}\times S^{n-1}}|\phi(\sigma, \tilde{x})|^2 d\sigma dS\right\}^{1/2}. \tag{3.3}$$

Then the map $\sim:\mathfrak{H}\times\mathcal{D}(H^{-1})\ni f=\{f_1, f_2\}\to\tilde{f}(\sigma, \tilde{x})\in\tilde{\mathfrak{H}}$ *is unitary.*

PROOF. Obviously $\tilde{\mathfrak{H}}=\tilde{\mathfrak{H}}_+\oplus\tilde{\mathfrak{H}}_-$. Thus, we have $\tilde{f}\in\tilde{\mathfrak{H}}$ and

$$\|\tilde{f}\|^2=\frac{1}{2}\{\|\mathcal{F}_+(f_1+i\overline{H^{-1}}f_2\|_+^2+\|\mathcal{F}_-(f_1-i\overline{H^{-1}}f_2)\|_-^2\}$$

$$=\frac{1}{2}\{\|f_1+i\overline{H^{-1}}f_2\|^2+\|f_1-i\overline{H^{-1}}f_2\|^2\}=\|f_1\|^2+\|\overline{H^{-1}}f_2\|^2 .$$

Since $\overline{H^{-1}}:\mathcal{D}(H^{-1})\to\mathfrak{H}$ and $\mathcal{F}_\pm:\mathfrak{H}\to\tilde{\mathfrak{H}}_\pm$ are unitary (see Lemma 1.2 and Theorem 2.2 (a)), this implies the assertion of the lemma. Q. E. D.

We introduce the following class of functions which is dense in $\tilde{\mathfrak{H}}$:

$$\tilde{\mathcal{D}}_0=\{\phi(\sigma, \tilde{x})\in C_0^\infty(\boldsymbol{R}\times S^{n-1});\ \phi(\sigma, \tilde{x})=0 \text{ near } \sigma=0\}. \tag{3.4}$$

Note that for each $\phi\in\tilde{\mathcal{D}}_0$ there exists a pre-compact set $e\Subset\boldsymbol{R}-\{0\}$ such that the support in σ of ϕ is contained in e, that is, $\phi(\sigma, \tilde{x})\in C_0^\infty(e\times S^{n-1})$. For any $\phi\in\tilde{\mathcal{D}}_0$ we put

$$v_\phi(x, \sigma)=\frac{1}{\sqrt{2\pi}}\exp\{-\rho(x, \sigma)\}\phi(\sigma, \tilde{x})\psi(\xi(x)), \tag{3.5}$$

$$g_\phi(x, \sigma)=-c(x)^2\{\Delta-V_p(x)\}v_\phi(x, \sigma)-\sigma^2 v_\phi(x, \sigma), \tag{3.6}$$

where $\rho(x, \sigma)$ and $\xi(x)$ are as given in (2.8) and (2.9), and $\psi(s)$ is a smooth non-decreasing function of $s\geqq0$ such that $\psi(0)=0$ and $\psi(s)=1$ for $s>1/c_2$ (v_ϕ

and g_ϕ depends also on ψ. However, we do not specify it here). A straight calculation gives

$$\begin{aligned}(3.7)\qquad g_\phi=&\frac{1}{\sqrt{2\pi}}c(x)^2\exp\{-\rho\}\Big[\Big\{\partial_r^2\rho+\frac{n-1}{r}\partial_r\rho-(\partial_r\rho)^2-\sigma^2c(x)^{-2}+V_p(x)\\ &+(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)\rho-((\nabla-\tilde{x}\partial_r)\rho)^2\Big\}\phi\psi-\{(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)\phi\\ &-2(\nabla-\tilde{x}\partial_r)\rho\cdot\nabla\phi\}\psi+\{-\Delta\psi+2\nabla\rho\cdot\nabla\psi\}\phi-2\nabla\psi\cdot\nabla\phi\Big]\\ \equiv&\exp\{-\rho(x,\sigma)\}\zeta_\phi(x,\sigma).\end{aligned}$$

Here $(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)\phi=O(r^{-2})$, $\nabla\phi=O(r^{-1})$, and $\Delta\{\psi(\xi(x))\}$ and $\nabla\{\psi(\xi(x))\}$ vanish in $r>R_0+1$.

In virtue of (2.14), Lemma 2.2 and the above properties, we can easily prove the following lemma.

LEMMA 3.3. *Let $\phi\in\tilde{\mathscr{D}}_0$ and let $e\Subset\boldsymbol{R}-\{0\}$ be a pre-compact set including the support in σ of $\phi(\sigma,\tilde{x})$. Then there exists a constant $C_9>0$ such that for any $(x,\sigma)\in\Omega\times e$,*

$$(3.8)\qquad |\partial_\sigma^l\zeta_\phi(x,\sigma)|\leqq C_9(1+r)^{-(1+\tilde{\delta})}\qquad(l=0,1,2),$$

$$(3.9)\qquad |g_\phi(x,\sigma)|\leqq C_9(1+r)^{-(n-1)/2}(1+r)^{-(1+\tilde{\delta})},$$

$$(3.10)\qquad |v_\phi(x,\sigma)|\leqq C_9(1+r)^{-(n-1)/2}.$$

Moreover, we have

$$(3.11)\qquad \{\partial_r+\partial_r\rho(x,\sigma)\}v_\phi(x,\sigma)=0\quad\text{in}\quad(x,\sigma)\in\{x\,;\,|x|>R_0+1\}\times e,$$

$$(3.12)\qquad B_pv_\phi(x,\sigma)=0\quad\text{on}\quad(x,\sigma)\in\partial\Omega\times e.$$

Let $\tilde{\alpha}$, $\tilde{\beta}$ be as given in Proposition 2.3 (c). Then (3.9) implies that $g_\phi\in L^2_{(1+\tilde{\beta})/2}(\Omega)$, and it follows from (3.10)~(3.12) that v_ϕ gives an outgoing [or incoming] solution of (2.1) with $q(x)=V_p(x)$, $f=g_\phi$, $\alpha=\tilde{\alpha}$ and $\beta=\tilde{\beta}$. Namely, we have

$$(3.13)\qquad v_\phi(\cdot,\sigma)=R_p((\sigma+i0)^2)g_\phi(\cdot,\sigma),\qquad\sigma\in e.$$

PROPOSITION 3.1. *For any $\phi(\sigma,\tilde{x})\in\tilde{\mathscr{D}}_0$, $\mathscr{F}^*(\sigma)\phi(\sigma,\cdot)\in L^2_{(-1-\tilde{\beta})/2}(\Omega)$ is expressed as follows.*

$$(3.14)\qquad \mathscr{F}^*(\sigma)\phi(\sigma,\cdot)=-iJ_pv_\phi+iJ_pR_p((-\sigma+i0)^2)g_\phi.$$

PROOF. For $f\in L^2_{(1+\tilde{\beta})/2}(\Omega)$ we put

$$(3.15)\qquad u_1=R_p((\sigma+i0)^2)J_p^{-1}f=R_1((\sigma+i0)^2)f_1;$$

(3.16) $$f_1=\{1-VR_p((\sigma+i0)^2)\}J_p^{-1}f.$$

Since $f_1\in L^2_{(1+\tilde{\beta})/2}(\Omega)$, we can choose a sequence $r_l=r_l(\tilde{\alpha}, \tilde{\beta}, f_1, \sigma)$ as in Proposition 2.3 (a). Let $\Omega(r_l)=\{x\in\Omega; |x|<r_l\}$. Then by the Green formula, (3.11) and (3.12) we have

(3.17) $$\begin{aligned}&-i\int_{\Omega(r_l)}c(x)^{-2}\{u_1\overline{g_\phi}-J_p^{-1}f\overline{v_\phi}\}\,dx\\&=-i\int_{S(r_l)}\{\partial_r u_1\overline{v_\phi}-u_1\partial_r\overline{v_\phi}\}\,dS\\&=-i\int_{S(r_l)}(\partial_r+\partial_r\rho)u_1\overline{v_\phi}dS-\int_{S(r_l)}2(\mathrm{Im}\,\partial_r\rho)u_1\overline{v_\phi}dS.\end{aligned}$$

Here by (2.30), (2.31) and (3.5),

$$\begin{aligned}-\int_{S(r_l)}2(\mathrm{Im}\,\partial_r\rho)u_1\overline{v_\phi}dS&=\int_{S(r_l)}2\sigma c(x)^{-1}u_1\overline{v_\phi}dS\\&=\int_{S^{n-1}}\sqrt{\frac{2}{\pi}}\sigma\exp\{\rho(r_l\tilde{x}, \sigma)\}u_1(r_l\tilde{x}, \sigma)\overline{\phi(\sigma, \tilde{x})}dS_{\tilde{x}}.\end{aligned}$$

So, noting (2.35) and letting $l\to\infty$ in (3.17), we have

(3.18) $$\begin{aligned}&-i\int_\Omega c(x)^{-2}\{u_1\overline{g_\phi}-J_p^{-1}f\overline{v_\phi}\}\,dx\\&=\lim_{l\to\infty}\int_{S^{n-1}}[\mathcal{F}_1(\sigma, r_l)f_1](\tilde{x})\overline{\phi(\sigma, \tilde{x})}dS_{\tilde{x}}.\end{aligned}$$

By means of Theorem 2.1 (c), the left side of (3.18) equals

$$-i\{((R_p((\sigma+i0)^2)J_p^{-1}f, g_\phi))-((J_p^{-1}f, v_\phi))\}=(f, iJ_pR_p((-\sigma+i0)^2)g_\phi-iv_\phi).$$

On the other hand, by means of Proposition 2.3 (c), the right side equals

$$\int_{S^{n-1}}[\mathcal{F}_1(\sigma)\{1-VR_p((\sigma+i0)^2)\}J_p^{-1}f](\tilde{x})\overline{\phi(\sigma, \tilde{x})}dS_{\tilde{x}}=(f, \mathcal{F}^*(\sigma)\phi(\sigma, \cdot)).$$

Thus, (3.18) proves (3.14). Q. E. D.

In $(3.1)_+$ we assume that $\tilde{f}(\sigma, \tilde{x})\equiv\tilde{f}_+(\sigma, \tilde{x})$ belongs to the class $\tilde{\mathcal{D}}_0$. Then we can use the above proposition to obtain

(3.19) $$w(x, t)=w^\infty(x, t)+q_\gamma(x, t),$$

where $w^\infty(\cdot, t)$ and $q_\gamma(\cdot, t)\in L^2_{(-1-\alpha)/2}(\Omega)$ are defined by

(3.20) $$w^\infty(\cdot, t)=\frac{-i}{\sqrt{2}}\int_R\exp\{-i\sigma t\}J_pv_\gamma(\cdot, \sigma)d\sigma,$$

$$q\tilde{\jmath}(\cdot, t)=\frac{i}{\sqrt{2}}\int_R \exp\{-i\sigma t\} J_p R_p((-\sigma+i0)^2) g\tilde{\jmath}(\cdot, \sigma)d\sigma . \tag{3.21}$$

The above integrals make sense since $v\tilde{\jmath}(\cdot, \sigma)$ and $g\tilde{\jmath}(\cdot, \sigma)$ both are continuous in σ and have compact support. It will be shown that $q\tilde{\jmath}(\cdot, t)$ tends to zero in $\mathfrak{H}$ when $t\to\infty$. Before doing this we extend the correspondence $f\to w^\infty(\cdot, t)$ to arbitrary $f=\{f_1, f_2\}\in\mathfrak{H}\times\mathfrak{D}(H^{-1})$ and prove some basic properties of $w^\infty(x, t)$, including that $w^\infty(\cdot, t)\in\mathfrak{H}$ (see (3.27)).

We put

$$F(s, \tilde{x})=\frac{-i}{\sqrt{2\pi}}\int_R \exp\{i\sigma s\}\tilde{f}(\sigma, \tilde{x})d\sigma . \tag{3.22}$$

The Parseval equality and Lemma 3.2 show that

$$\iint_{R\times S^{n-1}} |F(s, \tilde{x})|^2 ds dS_{\tilde{x}}=\|\tilde{f}\|^2=\|f_1\|^2+\|\overline{H^{-1}}f_2\|^2 . \tag{3.23}$$

Thus, (3.22) can be defined for all $f=\{f_1, f_2\}\in\mathfrak{H}\times\mathfrak{D}(H^{-1})$ and the map $f\to F\in L^2(\boldsymbol{R}\times S^{n-1})$ becomes unitary. In virtue of (1.17), (2.30), (3.5) and (3.20) we have for $\tilde{f}\in\tilde{\mathfrak{D}}_0$,

$$w^\infty(x, t)=\frac{1}{\sqrt{2}}\psi(\xi(x))\sqrt{c(x)p(x)}\, r^{-(n-1)/2}F(\xi(x)-t, \tilde{x}) . \tag{3.24}$$

As is discussed above the right side is defined for all $\tilde{f}\in\tilde{\mathfrak{H}}$. Since $\tilde{\mathfrak{D}}_0$ is dense in $\tilde{\mathfrak{H}}$, we see that the correspondence $f\to w^\infty(x, t)$ can be extended to arbitrary $f\in\mathfrak{H}\times\mathfrak{D}(H^{-1})$ by (3.24).

DEFINITION 3.1. For each $f=\{f_1, f_2\}\in\mathfrak{H}\times\mathfrak{D}(H^{-1})$ the corresponding *wave profile* $F(s, \tilde{x})\in L^2(\boldsymbol{R}\times S^{n-1})$ is the function defined by (3.22). Moreover, the *asymptotic wave function* $w^\infty(x, t)$ corresponding to the solution (1.23) is the modified diverging spherical wave defined by (3.24).

PROPOSITION 3.2. *For any* $f=\{f_1, f_2\}\in\mathfrak{H}\times\bar{D}(H^{-1})$ *the asymptotic wave function* $w^\infty(x, t)$ *satisfies the following properties:*

(a) $t\to w^\infty(\cdot, t)\in\mathfrak{H}$ *is continuous for all* $t\in\boldsymbol{R}$.

(b) $\|w^\infty(\cdot, t)\|$ *is monotone increasing in* $t\in\boldsymbol{R}$ *and*

$$\lim_{t\to+\infty}\|w^\infty(\cdot, t)\|=\frac{1}{\sqrt{2}}\{\|f_1\|^2+\|H^{-1}f_2\|^2\}^{1/2} , \tag{3.25}$$

$$\lim_{t\to-\infty}\|w^\infty(\cdot, t)\|=0 . \tag{3.26}$$

PROOF. Note that $\xi(x)=\int_{R_0}^r c(r'\tilde{x})^{-1}dr'$ is monotone increasing and goes to ∞ as $r\to\infty$. We put $s=\xi(x)$. Then $ds=c(x)^{-1}dr$ and it follows from (1.8) and (3.24) that

$$(3.27)\qquad \|w^{\infty}(\cdot,\ t)\|^2=\frac{1}{2}\int_{S^{n-1}}dS_{\tilde{x}}\int_{R_0}^{\infty}\phi(\xi(x))^2c(x)^{-1}|F(\xi(x)-t,\ \tilde{x})|^2dr$$

$$=\frac{1}{2}\int_{S^{n-1}}dS_{\tilde{x}}\int_{-t}^{\infty}\phi(s+t)^2|F(s,\ \tilde{x})|^2ds\,.$$

Here $|\phi(s+t)|\leqq 1$ for all $s+t\geqq 0$ and $\phi(s+t)=1$ for $s+t\geqq 1/c_2$. Thus, (3.23) and (3.27) imply the assertions (b). To verify (a) note that the Fourier transform of $F(s-t,\ \tilde{x})$ in $L^2(\boldsymbol{R}\,;\,L^2(S^{n-1}))$ is $\exp\{-i\sigma t\}\tilde{f}(\sigma,\ \tilde{x})$. Then by the Parseval formula we have

$$(3.28)\qquad \|w^{\infty}(\cdot,\ t)-w^{\infty}(\cdot,\ t')\|^2$$

$$=\frac{1}{2}\int_{S^{n-1}}dS_{\tilde{x}}\int_{0}^{\infty}\phi(s)^2|F(s-t,\ \tilde{x})-F(s-t',\ \tilde{x})|^2ds$$

$$\leqq\frac{1}{2}\int_{S^{n-1}}dS_{\tilde{x}}\int_{\boldsymbol{R}}|\exp\{-i\sigma t\}-\exp\{-i\sigma t'\}|^2|\tilde{f}(\sigma,\ \tilde{x})|^2d\sigma$$

for all real t and t'. Here $|\exp\{-i\sigma t\}-\exp\{-i\sigma t'\}|^2\leqq 4$ for all real t, t' and σ, and $|\exp\{-i\sigma t\}-\exp\{-i\sigma t'\}|^2\to 0$ as $t'\to t$ for fixed σ. Since $\tilde{f}\in\tilde{\mathfrak{H}}=L^2(\boldsymbol{R}\times S^{n-1})$, the Lebesgue theorem shows that the last integral of (3.28) tends to zero as $t'\to t$. This proves the assertion (a). Q.E.D.

Now the principal result of this paper is stated in the following

THEOREM 3.1. *For any initial data* $\{f_1,\ f_2\}\in\mathfrak{H}\times\mathfrak{D}(H^{-1})$, *let* $w(\cdot,\ t)$ *be the solution in* $\mathfrak{H}$ *of the initial-boundary value problem* (1.1)$\sim$(1.3), *given by* (1.23), *and let* $w^{\infty}(\cdot,\ t)$ *be the corresponding asymptotic wave function. Then*

$$(3.29)\qquad \lim_{t\to\infty}\|w(\cdot,\ t)-w^{\infty}(\cdot,\ t)\|=0\,.$$

To verify this convergence theorem we require one more proposition, the proof of which will be given in the next section.

PROPOSITION 3.3. *Let* $\zeta(x,\ \sigma)$ *be a function of* $(x,\ \sigma)\in\Omega\times\boldsymbol{R}$ *satisfying the following conditions:*

(i) $\zeta(x,\ \sigma)$ *is continuous in* x *and twice continuously differentiable in* σ, *and has the support contained in* $B(R_0)\times e$, *where* $B(R_0)=\{x\,;\,|x|>R_0\}$ *and* $e=(\sigma_1,\ \sigma_2)\Subset R_+$.

(ii) *There exists a constant* $C_{10}>0$ *such that for any* $(x,\ \sigma)\in B(R_0)\times e$,

$$(3.30)\qquad |\partial_\sigma^l\zeta(x,\ \sigma)|\leqq C_{10}(1+r)^{-1-\tilde{\delta}}\qquad (l=0,\ 1,\ 2).$$

Let $G_{\pm}(x,\ s,\ t)$ *be the function of* $x\in\Omega$, s, $t\in\boldsymbol{R}_+$ *defined by*

$$(3.31)\qquad G_{\pm}(x,\ s,\ t)=\sqrt{c(x)}\,r^{-(n-1)/2}\int_{e}\exp\{\mp i(\sigma^2s+\sigma t-\sigma\xi(x))\}\zeta(x,\ \sigma)d\sigma\,.$$

Then for any fixed $t>0$, $G_{\pm}(\cdot, s, t)$ *is an* $\mathfrak{H}_1$*-valued integrable function of* $s\in \boldsymbol{R}_+$ *and*

(3.32) $$\lim_{t\to\infty}\int_0^\infty |||G_{\pm}(\cdot, s, t)|||ds=0.$$

As a corollary of this proposition we can have the

LEMMA 3.4. *For any* $\tilde{f}(\sigma, \tilde{x})\in\tilde{\mathfrak{D}}_0$ *let* $q_{\tilde{f}}(x, t)$ *be the function defined by* (3.21). *Then we have* $q_{\tilde{f}}(\cdot, t)\in\mathfrak{H}$ *for any* $t>0$ *and*

(3.33) $$\lim_{t\to\infty}\|q_{\tilde{f}}(\cdot, t)\|=0.$$

PROOF. Note that the support in σ of $\tilde{f}(\sigma, \tilde{x})$ is contained in $e_{-N,N}=e_{-N}\cup e_N =(-N, -1/N)\cup(1/N, N)$ if $N>1$ is chosen sufficiently large. In (3.7) with $\phi=\tilde{f}$ we divide $\zeta_{\tilde{f}}(x, \sigma)$ into two terms:

$$\zeta_{\tilde{f}}(x, \sigma)=\zeta_{\tilde{f},+}(x, \sigma)+\zeta_{\tilde{f},-}(x, \sigma),$$

where

$$\zeta_{\tilde{f},\pm}(x, \sigma)=\begin{cases}\zeta_{\tilde{f}}(x, \sigma) & \text{for } (x, \sigma)\in\Omega\times\boldsymbol{R}_{\pm},\\ 0 & \text{for } (x, \sigma)\in\Omega\times\boldsymbol{R}_{\mp}.\end{cases}$$

Then (3.8) implies that $\zeta_{\tilde{f},\pm}(x, \pm\sigma)$ both satisfy conditions (i), (ii) of the above proposition with $e=e_N$. Thus, if we put

$$G_{\tilde{f},\pm}(x, s, t)=\sqrt{c(x)}\,r^{-(n-1)/2}\int_{e_N}\exp\{\mp i(\sigma^2 s+\sigma t-\sigma\xi(x))\}\zeta_{\tilde{f},\pm}(x, \pm\sigma)d\sigma,$$

then

(3.34) $$\lim_{t\to\infty}\int_0^\infty |||G_{\tilde{f},\pm}(\cdot, s, t)|||ds=0.$$

We put for $\tau>0$,

$$q_{\tilde{f},\tau}(\cdot, t)=\frac{i}{\sqrt{2}}\int_{e_N}\exp\{-i\sigma t\}J_pR_p(\sigma^2-i\tau)g_{\tilde{f}}(\cdot, \sigma)d\sigma$$
$$+\frac{i}{\sqrt{2}}\int_{e_{-N}}\exp\{-i\sigma t\}J_pR_p(\sigma^2+i\tau)g_{\tilde{f}}(\cdot, \sigma)d\sigma.$$

Then in virtue of (3.9) we have

$$q_{\tilde{f},\tau}(\cdot, t)=\frac{1}{\sqrt{2}}\int_{e_N}\exp\{-i\sigma t\}J_p\left[\int_0^\infty\exp\{i(L_p-\sigma^2+i\tau)s\}\,ds\right]g_{\tilde{f}}(\cdot, \sigma)d\sigma$$
$$+\frac{1}{\sqrt{2}}\int_{e_{-N}}\exp\{-i\sigma t\}J_p\left[\int_0^\infty\exp\{-i(L_p-\sigma^2-i\tau)s\}\,ds\right]g_{\tilde{f}}(\cdot, \sigma)d\sigma$$

$$=\frac{1}{\sqrt{2}}J_p\int_0^\infty \exp\{i(L_p+i\tau)s\}G_{\tilde{\gamma},+}(\cdot,\ s,\ t)ds$$

$$+\frac{1}{\sqrt{2}}J_p\int_0^\infty \exp\{-i(L_p-i\tau)s\}G_{\tilde{\gamma},-}(\cdot,\ s,\ t)ds\ .$$

Thus,

$$\|q_{\tilde{\gamma},\tau}(\cdot,\ t)\|\leqq\frac{1}{\sqrt{2}}\int_0^\infty\{|||G_{\tilde{\gamma},+}(\cdot,\ s,\ t)|||+|||G_{\tilde{\gamma},-}(\cdot,\ s,\ t)|||\}ds<\infty \tag{3.35}$$

for any $t>0$. Further, since we have for any $h\in L^2_{(1+\tilde{\beta})/2}(\Omega)$ and $\tau>0$,

$$(q_{\tilde{\gamma},\tau}(\cdot,\ t),\ h)=\frac{i}{\sqrt{2}}\int_{e_N}\exp\{-i\sigma t\}((g_{\tilde{\gamma}}(\cdot,\ \sigma),\ R_p(\sigma^2+i\tau)J_p^{-1}h))d\sigma$$

$$+\frac{i}{\sqrt{2}}\int_{e_{-N}}\exp\{-i\sigma t\}((g_{\tilde{\gamma}}(\cdot,\ \sigma),\ R_p(\sigma^2-i\tau)J_p^{-1}h))d\sigma\ ,$$

it follows from Theorem 2.1 (a), (b) and the Lebesgue theorem that

$$\lim_{\tau\downarrow 0}(q_{\tilde{\gamma},\tau}(\cdot,\ t),\ h)=\frac{i}{\sqrt{2}}\int_{e_{-N,N}}\exp\{-i\sigma t\}$$

$$\times((g_{\tilde{\gamma}}(\cdot,\ \sigma),\ R_p((\sigma+i0)^2)J_p^{-1}h))d\sigma=(q_{\tilde{\gamma}}(\cdot,\ t),\ h)\ .$$

$L^2_{(1+\tilde{\beta})/2}(\Omega)$ being dense in $\mathfrak{H}$, this and (3.35) imply that $q_{\tilde{\gamma}}(\cdot,\ t)$ is the weak limit as $\tau\downarrow 0$ of $q_{\tilde{\gamma},\tau}(\cdot,\ t)$ in $\mathfrak{H}$ for any $t>0$. Hence, $q_{\tilde{\gamma}}(\cdot,\ t)\in\mathfrak{H}$ and we have from (3.34) and (3.35)

$$\|q_{\tilde{\gamma}}(\cdot,\ t)\|\leqq\liminf_{\tau\downarrow 0}\|q_{\tilde{\gamma},\tau}(\cdot,\ t)\|\to 0 \qquad \text{as} \quad t\to\infty,$$

which is to be proved. Q.E.D.

PROOF OF THEOREM 3.1. As we see from Propositions 1.1 (a) and 3.2 (b),

$$\|w(\cdot,\ t)-w^\infty(\cdot,\ t)\|\leqq\|w(\cdot,\ t)\|+\|w^\infty(\cdot,\ t)\| \tag{3.36}$$

$$\leqq\left(1+\frac{1}{\sqrt{2}}\right)\{\|f_1\|^2+\|\overline{H^{-1}}f_2\|^2\}^{1/2}$$

for any $\{f_1,\ f_2\}\in\mathfrak{H}\times\mathscr{D}(H^{-1})$ and $t>0$. Note that the set $\mathscr{D}_0=\{\{f_1,\ f_2\}\ ;\tilde{f}\in\tilde{\mathscr{D}}_0\}$ is dense in $\mathfrak{H}\times\mathscr{D}(H^{-1})$ by Lemma 3.2. Then we can choose a sequence $\{f_{1l},\ f_{2l}\}$ ($l=1,\ 2,\ \cdots$) in $\mathscr{D}_0$ which converges as $l\to\infty$ to $\{f_1,\ f_2\}$ in the topology of $\mathfrak{H}\times\mathscr{D}(H^{-1})$. Let $q_l(x,\ t)$ be the function $q_{\tilde{\gamma}}(x,\ t)$ corresponding to the initial data $\{f_{1l},\ f_{2l}\}$. Then we have from (3.19) and (3.36)

$$\|w(\cdot,\ t)-w^\infty(\cdot,\ t)\|\leqq\|q_l(\cdot,\ t)\|$$

$$+\left(1+\frac{1}{\sqrt{2}}\right)\{\|f_1-f_{1l}\|^2+\|\overline{H^{-1}}(f_2-f_{2l})\|^2\}^{1/2}\ .$$

Here the second term of the right side tends to zero as $l\to\infty$, and for fixed any l the first term tends to zero as $t\to\infty$ by Lemma 3.4. Thus, (3.29) holds.

Q. E. D.

Before closing this section, we note that the asymptotic wave function $w^{\infty}(x, t)$ is essentially a diverging spherical wave if the inhomogeneity of fluid is short-range. This assertion will be proved in the following

PROPOSITION 3.4. *Suppose that for some* $c>0$ *and* $p>0$,

$$c(x)-c=O(r^{-1-\delta}) \quad \text{and} \quad p(x)-p=O(r^{-1-\delta}) \quad \text{as} \quad r\to\infty. \tag{3.37}$$

Put

$$w_0^{\infty}(x, t)=\frac{1}{\sqrt{2}}\psi(r-R_0)\sqrt{cp}\, r^{-(n-1)/2}F_0(c^{-1}r-t, \tilde{x})\,; \tag{3.38}$$

$$F_0(s, \tilde{x})=\frac{-i}{\sqrt{2\pi}}\int_R \exp\{i\sigma s\}J(\sigma, \tilde{x})\tilde{f}(\sigma, \tilde{x})d\sigma\,,$$

$$J(\sigma, \tilde{x})=\exp\{-i\sigma c^{-1}R_0-i\sigma\varepsilon(R_0\tilde{x})\}\,;\ \varepsilon(r\tilde{x})=\int_r^{\infty}\{c^{-1}-c(r'\tilde{x})^{-1}\}\,dr'.$$

Then we have

$$\lim_{t\to\infty}\|w^{\infty}(\cdot, t)-w_0^{\infty}(\cdot, t)\|=0. \tag{3.39}$$

PROOF. We have only to prove (3.39) for $\tilde{f}(\sigma, \tilde{x})\in\tilde{\mathscr{D}}_0$. Note that

$$\begin{aligned} w^{\infty}(x, t)-w_0^{\infty}(x, t)&=\chi_R(x)\{w^{\infty}(x, t)-w_0^{\infty}(x, t)\} \\ &+\frac{1-\chi_R(x)}{\sqrt{2}}\{\psi(\xi(x))\sqrt{c(x)p(x)}-\psi(r-R_0)\sqrt{cp}\}r^{-(n-1)/2}F(\xi(x)-t, \tilde{x}) \\ &+\frac{1-\chi_R(x)}{\sqrt{2}}\psi(r-R_0)\sqrt{cp}\, r^{-(n-1)/2}\{F(\xi(x)-t, \tilde{x})-F_0(c^{-1}r-t, \tilde{x})\} \\ &=I_1(x, t)+I_2(x, t)+I_3(x, t), \end{aligned} \tag{3.40}$$

where $\chi_R(x)$ is the characteristic function on $\{x\,;\,|x|\leqq R\}$ $(R>R_0)$. The support in x of $I_1(x, t)$ being compact, we can apply the Riemann-Lebesgue theorem to obtain

$$\|I_1(\cdot, t)\|\to 0 \quad \text{as} \quad t\to\infty. \tag{3.41}$$

Further, by (3.37) we have for some $C_{11}>0$,

$$\|I_2(\cdot, t)\|\leqq C_{11}R^{-1/2-\delta}. \tag{3.42}$$

On the other hand, $\varepsilon(x)=\varepsilon(r\tilde{x})=O(r^{-\delta})$ as $r\to\infty$, and

$$I_3(x, t)=\frac{-i}{2\sqrt{\pi}}\{1-\chi_R(x)\}\sqrt{cp}\, r^{-(n-1)/2}$$

$$\times\int_R \exp\{i\sigma(\xi(x)-t)\}[1-\exp\{-i\sigma\varepsilon(x)\}]\tilde{f}(\sigma, \tilde{x})d\sigma .$$

Thus, choosing N so large that $N\delta>1/2$, we have

$$|I_3(x, t)|\leqq\frac{1}{2\sqrt{\pi}}\{1-\chi_R(x)\}\sqrt{cp}\, r^{-(n-1)/2}\sum_{k=1}^{p}\frac{|\varepsilon(x)|^k}{k!}|G_k(\xi(x)-t, \tilde{x})|$$

$$+C_{12}\{1-\chi_R(x)\}r^{-(n-1)/2-N\delta-\delta},$$

where

$$G_k(s, \tilde{x})=\int_R \exp\{i\sigma s\}\sigma^k\tilde{f}(\sigma, \tilde{x})d\sigma\in L^2(\boldsymbol{R}\times S^{n-1}).$$

Following the argument of Proposition 3.2, we then have

$$\|I_3(\cdot, t)\|\leqq C_{13}R^{-\delta}\left\{\sum_{k=1}^{p}\|G_k\|_{L^2(\boldsymbol{R}\times S^{n-1})}+1\right\}. \tag{3.43}$$

(3.40)$\sim$(3.43) imply (3.39). Q. E. D.

§4. Proof of Proposition 3.3.

Noting (c. 1), (2.31) and condition (i) on $\zeta(x, \sigma)$, we put

$$\varepsilon=c_1\min\{1/2, \sigma_1\}>0, \tag{4.1}$$

$$\Omega_1=\{x\in\Omega\,;\,|x|<\varepsilon(s+t)\}\quad\text{and}\quad\Omega_2=\Omega-\Omega_1. \tag{4.2}$$

LEMMA 4.1. *There exists a constant* $C_{14}>0$ *such that for any* $x\in\Omega_1$ *and* $s, t\in\boldsymbol{R}_+$,

$$|G_\pm(x, s, t)|\leqq C_{14}r^{-(n-1)/2}(1+r)^{-1-\tilde{\delta}}(s+t)^{-2}. \tag{4.3}$$

PROOF. Integrating by parts gives

$$G_\pm(x, s, t)=\sqrt{c(x)}\,r^{-(n-1)/2} \tag{4.4}$$

$$\times\int_e\left[\left\{\frac{\pm i}{2\sigma s+t-\xi(x)}\partial_\sigma\right\}^2\exp\{\mp i(\sigma^2 s+\sigma t-\sigma\xi(x))\}\right]\zeta(x, \sigma)d\sigma$$

$$=-\sqrt{c(x)}\,r^{-(n-1)/2}\int_e\exp\{\mp i(\sigma^2 s+\sigma t-\sigma\xi(x))\}(2\sigma s+t-\xi(x))^{-2}$$

$$\times\{\partial_\sigma^2\zeta-6s(2\sigma s+t-\xi(x))^{-1}\partial_\sigma\zeta+12s^2(2\sigma s+t-\xi(x))^{-2}\zeta\}\,d\sigma .$$

Here, since $\xi(x)\leqq c_1^{-1}(r-R_0)$, we have

(4.5) $$2\sigma s+t-\xi(x)\geqq\varepsilon(s+t) \quad \text{in} \quad (x, \sigma)\in\Omega_1\times e\,.$$

Condition (ii) on $\zeta(x, \sigma)$, (4.4) and (4.5) imply (4.3). Q. E. D.

The following lemma is obvious from the definition of $G_{\pm}(x, s, t)$.

LEMMA 4.2. *There exists a constant* $C_{15}>0$ *such that for any* $x\in\Omega_2$ *and* $s, t\in \boldsymbol{R}_+$

(4.6) $$|G_{\pm}(x, s, t)|<C_{15}r^{-(n-1)/2}(1+r)^{-1-\tilde{\delta}}\,.$$

LEMMA 4.3. *There exists a constant* $C_{16}>0$ *such that for any* $x\in\Omega_2$, $s\in(1, \infty)$ *and* $t\in\boldsymbol{R}_+$,

(4.7) $$|G_{\pm}(x, s, t)|\leqq C_{16}r^{-(n-1)/2}(1+r)^{-1-\tilde{\delta}}s^{-1/2}\,.$$

PROOF. Let $\omega(\lambda)$ be a C^{∞}-function of $\lambda\in\boldsymbol{R}$ such that $0\leqq\omega(\lambda)\leqq 1$, $\omega(\lambda)=1$ for $|\lambda|\leqq 1/2$ and $=0$ for $|\lambda|\geqq 1$. We divide $G_{\pm}(x, s, t)$ into two parts:

(4.8) $$G_{\pm}(x, s, t)=\sqrt{c(x)}\,r^{-(n-1)/2}\{H_{\pm,1}(x, s, t)+H_{\pm,2}(x, s, t)\}\ ;$$

$$H_{\pm,1}(x, s, t)=\int_e \exp\{\mp i(\sigma^2 s+\sigma t-\sigma\xi(x))\}\,\omega(\sigma-\sigma_c)\zeta(x, \sigma)d\sigma\,,$$

$$H_{\pm,2}(x, s, t)=\int_e \exp\{\mp i(\sigma^2 s+\sigma t-\sigma\xi(x))\}\,\{1-\omega(\sigma-\sigma_c)\}\zeta(x, \sigma)d\sigma\,,$$

where $\sigma_c=(\xi(x)-t)/2s$ is the critical point of the phase function. For the sake of simplicity we put

(4.9) $$h_1(x, s, t, \sigma)=\omega(\sigma-\sigma_c)\zeta(x, \sigma)\,,$$

(4.10) $$h_2(x, s, t, \sigma)=\{1-\omega(\sigma-\sigma_c)\}\zeta(x, \sigma)\,.$$

It then follows from condition (ii) on $\zeta(x, \sigma)$ that

(4.11) $$|\partial_\sigma^l h_j(x, s, t, \sigma)|\leqq C_{17}(1+r)^{-1-\tilde{\delta}} \qquad (l=0, 1, 2 \text{ and } j=1, 2),$$

where $C_{17}>0$ is independent of $x\in\Omega_2$, $s\in(1, \infty)$, $t\in\boldsymbol{R}_+$ and $\sigma\in e$.

Note that

$$\sigma^2 s+\sigma t-\sigma\xi(x)=(\sigma-\sigma_c)^2 s-\sigma_c^2 s$$

and

$$h_1(x, s, t, \sigma)=h_1(x, s, t, \sigma_c)+(\sigma-\sigma_c)\int_0^1(\partial_\sigma h_1)(x, s, t, \sigma_c+(\sigma-\sigma_c)\tau)d\tau\,.$$

Then we have for any sufficiently large N,

(4.12) $$\exp\{\mp i\sigma_c^2 s\}H_{\pm,1}(x, s, t)=h_1(x, s, t, \sigma_c)\int_{-N}^{N}\exp\{\mp i(\sigma-\sigma_c)^2 s\}\,d\sigma$$

$$+\int_{-N}^{N}\exp\{\mp i(\sigma-\sigma_c)^2 s\}(\sigma-\sigma_c)\int_0^1(\partial_\sigma h_1)(x, s, t, \sigma_c+(\sigma-\sigma_c)\tau)d\tau d\sigma\,.$$

Here applying the Fresnel integral formula, we have

$$\lim_{N\to\infty}\int_{-N}^{N}\exp\{\mp i(\sigma-\sigma_c)^2 s\}\,d\sigma=\sqrt{\pi}\exp\{\mp\pi i/4\}\,s^{-1/2}. \tag{4.13}$$

On the other hand, since

$$\lim_{N\to\infty}\int_0^1(\partial_\sigma h_1)(x,\ s,\ t,\ \sigma_c+(\pm N-\sigma_c)\tau)d\tau=0$$

by the Lebesgue theorem, we have integrating by parts and changing the order of integration,

$$\begin{aligned}(4.14)\qquad &\lim_{N\to\infty}\int_{-N}^{N}\exp\{\mp i(\sigma-\sigma_c)^2 s\}(\sigma-\sigma_c)\int_0^1(\partial_\sigma h_1)(x,\ s,\ t,\ \sigma_c+(\sigma-\sigma_c)\tau)d\tau d\sigma\\ &=\mp i(2s)^{-1}\int_0^1\tau d\tau\int_\Sigma\exp\{\mp i(\sigma-\sigma_c)^2 s\}(\partial_\sigma^2 h_1)(x,\ s,\ t,\ \sigma_c+(\sigma-\sigma_c)\tau)d\sigma\,,\end{aligned}$$

where

$$\Sigma=\{\sigma\,;\ \sigma_1-\sigma_c\leqq(\sigma-\sigma_c)\tau\leqq\sigma_2-\sigma_c \text{ and } |(\sigma-\sigma_c)\tau|<1\}$$

if we note that $\omega(\lambda)=0$ for $|\lambda|\geqq 1$ and $h_1(x, s, t, \sigma)=0$ for $\sigma\notin e=(\sigma_1, \sigma_2)$. Taking account of (4.11) with $j=1$, we now have from (4.12)~(4.14) the following inequality.

$$|H_{\pm,1}(x,\ s,\ t)|\leqq C_{17}(1+r)^{-1-\tilde{\delta}}\{\sqrt{\pi}\,s^{-1/2}+(2s)^{-1}2\}. \tag{4.15}$$

Next note that $h_2(x, s, t, \sigma)=0$ in $\{\sigma\in e\,;\ |\sigma-\sigma_c|\leqq 1/2\}$ and

$$|2\sigma s+t-\xi(x)|\geqq s \quad\text{in}\quad \{\sigma\in e\,;\ |\sigma-\sigma_c|\geqq 1/2\}. \tag{4.16}$$

Then integrating by parts gives

$$\begin{aligned}H_{\pm,2}(x,\ s,\ t)&=\int_e\Big[\frac{\pm i}{2\sigma s+t-\xi(x)}\partial_\sigma\exp\{\mp i(\sigma^2 s+\sigma t-\sigma\xi(x))\}\Big]h_2 d\sigma\\ &=\mp i\int_e\exp\{\mp i(\sigma^2 s+\sigma t-\sigma\xi(x))\}\{(2\sigma s+t-\xi(x))^{-1}\partial_\sigma h_2\\ &\qquad -2s(2\sigma s+t-\xi(x))^{-2}h_2\}\,d\sigma\,.\end{aligned}$$

Thus, by (4.16) and (4.11) with $j=2$ we have

$$|H_{\pm,2}(x,\ s,\ t)|\leqq C_{17}(1+r)^{-1-\tilde{\delta}}s^{-1}3\int_e d\sigma\,. \tag{4.17}$$

(4.8), (4.15) and (4.17) prove (4.7). Q. E. D.

PROOF OF PROPOSITION 3.3. Note the inequality

$$\int_0^\infty |||G_\pm(\cdot,\ s,\ t)|||ds \leqq 2\left\{\int_0^\infty |||G_\pm(\cdot,\ s,\ t)|||_{\Omega_1}ds + \int_0^1 |||G_\pm(\cdot,\ s,\ t)|||_{\Omega_2}ds + \int_1^\infty |||G_\pm(\cdot,\ s,\ t)|||_{\Omega_2}ds\right\}. \tag{4.18}$$

By Lemma 4.1 we have

$$\begin{aligned}|||G_\pm|||_{\Omega_1} &\leqq C_{14}(s+t)^{-2}\left\{\int_\Omega c(x)^{-2}r^{-n+1}(1+r)^{-2-2\tilde{\delta}}dx\right\}^{1/2} \\ &\leqq C_{18}(s+t)^{-2}.\end{aligned}$$

By Lemma 4.2 we have for $s\in(0,\ 1]$,

$$\begin{aligned}|||G_\pm|||_{\Omega_2} &\leqq C_{15}\left\{\int_{\Omega_2} c(x)^{-2}r^{-n+1}(1+r)^{-2-2\tilde{\delta}}dx\right\}^{1/2} \\ &\leqq C_{19}\left\{\int_{S^{n-1}} dS_{\tilde{x}}\int_{\varepsilon(s+t)}^\infty (1+r)^{-2-2\tilde{\delta}}dr\right\}^{1/2} \\ &\leqq C_{20}(\varepsilon t)^{-1/2-\tilde{\delta}}.\end{aligned}$$

Moreover, by Lemma 4.3 we have for $s\in(1,\ \infty)$,

$$\begin{aligned}|||G_+|||_{\Omega_2} &\leqq C_{16}s^{-1/2}\left\{\int_{\Omega_2} c(x)^{-2}r^{-n+1}(1+r)^{-2-2\tilde{\delta}}dx\right\}^{1/2} \\ &\leqq C_{21}s^{-1/2}\{\varepsilon(s+t)\}^{-1/2-\tilde{\delta}}.\end{aligned}$$

Applying these inequalities in (4.18), we see that there exists a constant $C_{22}>0$ such that for any $t\in\boldsymbol{R}_+$,

$$\int_0^\infty |||G_+(\cdot,\ s,\ t)|||ds \leqq C_{22}\left\{\int_0^\infty (s+t)^{-2}ds + t^{-1/2-\tilde{\delta}} + \int_1^\infty s^{-1/2}(s+t)^{-1/2-\tilde{\delta}}ds\right\}.$$

The right side tends to zero when $t\to\infty$. Hence, we have (3.32) and the proof is completed. Q. E. D.

§ 5. Asymptotic energy distributions.

In this section we return to the pair $\{w_1(x,\ t),\ w_2(x,\ t)\}$, given by (1.25), which determines the energy at time t of the initial-boundary value problem (1.1)~(1.3). We shall first follow the argument of § 3 to construct the asymptotic wave functions $w_j^\infty(x,\ t)$ $(j=1,\ 2)$ corresponding to $w_j(x,\ t)$, and then apply them to the calculation of the asymptotic distribution of the wave energy for large t.

For $f=\{f_1, f_2\}\in\mathscr{D}(H)\times\mathfrak{H}$ we put

$$\tag{5.1} \tilde{f}_E(\sigma, \tilde{x})=\begin{cases} \dfrac{1}{\sqrt{2}}[\mathscr{F}_+(\bar{H}f_1+if_2)](\sigma, \tilde{x}), & (\sigma, \tilde{x})\in \boldsymbol{R}_+\times S^{n-1} \\ \dfrac{-1}{\sqrt{2}}[\mathscr{F}_-(\bar{H}f_1-if_2)](\sigma, \tilde{x}), & (\sigma, \tilde{x})\in \boldsymbol{R}_-\times S^{n-1}. \end{cases}$$

Then (1.25) and Theorem 2.2 (b) give the following spectral representations for $w_j(x, t)$ (cf., Lemma 3.1):

$$\tag{5.2} \begin{cases} w_1(x, t)=\dfrac{1}{\sqrt{2}}\ \underset{N,N'\to\infty}{\text{s-lim}}\displaystyle\int_{e_{-N,N'}}\frac{\sigma}{|\sigma|}\exp\{-i\sigma t\}\mathscr{F}^*(\sigma)\tilde{f}_E(\sigma, \cdot)d\sigma, \\ w_2(x, t)=\dfrac{1}{\sqrt{2}}\ \underset{N,N'\to\infty}{\text{s-lim}}\displaystyle\int_{e_{-N,N'}}(-i)\exp\{-i\sigma t\}\mathscr{F}^*(\sigma)\tilde{f}_E(\sigma, \cdot)d\sigma. \end{cases}$$

We put

$$\tag{5.3} \begin{cases} F_1(s, \tilde{x})=\dfrac{-i}{\sqrt{2\pi}}\displaystyle\int_{R}\exp\{i\sigma s\}\frac{\sigma}{|\sigma|}\tilde{f}_E(\sigma, \tilde{x})d\sigma, \\ F_2(s, x)=\dfrac{-i}{\sqrt{2\pi}}\displaystyle\int_{R}\exp\{i\sigma s\}(-i)\tilde{f}_E(\sigma, \tilde{x})d\sigma. \end{cases}$$

Then since the map $\tilde{}_E: \mathscr{D}(H)\times\mathfrak{H}\ni f\to\tilde{f}_E\in\tilde{\mathfrak{H}}$ is unitary (cf., Lemma 3.2), we have for any $f\in\mathscr{D}(H)\times\mathfrak{H}$,

$$\tag{5.4} \iint_{R\times S^{n-1}}|F_j(s, \tilde{x})|^2 ds dS_{\tilde{x}}=\|\tilde{f}_E\|^2=\|\bar{H}f_1\|^2+\|f_2\|^2 \qquad (j=1, 2).$$

The function $F_j(s, \tilde{x})$ characterizes the profile of the asymptotic wave function for $w_j(x, t)$ (see Definition 3.1). Namely, the corresponding asymptotic wave function is defined by

$$\tag{5.5} w_j^\infty(x, t)=\frac{1}{\sqrt{2}}\psi(\xi(x))\sqrt{c(x)p(x)}\,r^{-(n-1)/2}F_j(\xi(x)-t, \tilde{x}).$$

We put

$$\tag{5.6} K(\theta_1(t), \theta_2(t); C_0)=\{x\in\Omega;\ \theta_1(t)<\xi(x)<\theta_2(t),\ \tilde{x}\in C_0\},$$

where $\theta_1(t)$, $\theta_2(t)$ are given for each $t\in\boldsymbol{R}$ such that $0\leqq\theta_1(t)\leqq\theta_2(t)\leqq\infty$ and C_0 is a measurable subset of S^{n-1}. Then (5.3)$\sim$(5.5) prove the following

Proposition 5.1. *For any* $\{f_1, f_2\}\in\mathscr{D}(H)\times\mathfrak{H}$ *we have*

$$\tag{5.7} \|w_j^\infty(\cdot, t)\|^2_{K(\theta_1(t),\theta_2(t);C_0)}=\frac{1}{2}\int_{C_0}dS_{\tilde{x}}\int_{\theta_1(t)-t}^{\theta_2(t)-t}\psi(s+t)^2|F_j(s, \tilde{x})|^2 ds.$$

Proof. Let $\chi_K(x)$ be the characteristic function on $K(\theta_1(t), \theta_2(t); C_0)$. Then we have

$$\|w_j^\infty(\cdot,\ t)\|^2_{K(\theta_1(t),\ \theta_2(t);\ C_0)}$$
$$=\frac{1}{2}\int_{S^{n-1}} dS_{\tilde{x}} \int_{R_0}^\infty \chi_K(x)\psi(\xi(x))^2 c(x)^{-1}|F_j(\xi(x)-t,\ \tilde{x})|^2 dr\,.$$

Thus, putting $s=\xi(x)$, we have (5.7) (cf., Proof of Proposition 3.2). Q.E.D.

COROLLARY 5.1. *$\|w_j^\infty(\cdot,\ t)\|^2$ is monotone increasing in $t\in\boldsymbol{R}$ and*

$$\lim_{t\to\infty}\|w_j^\infty(\cdot,\ t)\|^2=\frac{1}{2}\{\|\bar{H}f_1\|^2+\|f_2\|^2\}\,. \tag{5.8}$$

PROOF. Note that $\|w_j(\cdot,\ t)\|^2$ is the quantity (5.7) with $\theta_1(t)=0$, $\theta_2(t)=\infty$ and $C_0=S^{n-1}$. Then we see that it is monotone increasing in $t\in\boldsymbol{R}$ and (5.8) follows from (5.4). Q.E.D.

By use of (5.2), Corollary 5.1 and Propositions 3.1 and 3.3, and following the argument of the proof of Theorem 3.1, we can prove the

PROPOSITION 5.2. *For any initial data $\{f_1,\ f_2\}\in\mathfrak{D}(H)\times\mathfrak{H}$ we have*

$$\lim_{t\to\infty}\|w_j(\cdot,\ t)-w_j^\infty(\cdot,\ t)\|^2=0\,. \tag{5.9}$$

For any $K\subset\Omega$ the wave energy in the set K at time t is defined by

$$E(w,\ K,\ t)=\|w_1(\cdot,\ t)\|_K^2+\|w_2(\cdot,\ t)\|_K^2\,. \tag{5.10}$$

Clearly, if $\{f_1,\ f_2\}\in\mathfrak{D}(H)\times\mathfrak{H}$, then

$$E(w,\ K,\ t)\leqq E(w,\ \Omega,\ t)=\|\bar{H}f_1\|^2+\|f_2\|^2<\infty \tag{5.11}$$

for every measurable set $K\subset\Omega$ and for all $t\in\boldsymbol{R}$.

Now, the first result on the asymptotic energy distributions is the following theorem which describes the asymptotic equi-partition of energy.

THEOREM 5.1. *For any $\{f_1,\ f_2\}\in\mathfrak{D}(H)\times\mathfrak{H}$ we have*

$$\lim_{t\to\infty}\|w_j(\cdot,\ t)\|^2=\frac{1}{2}E(w,\ \Omega,\ 0)\qquad(j=1,\ 2)\,. \tag{5.12}$$

PROOF. (5.12) follows from (5.8) and (5.9) if we note the inequalities

$$\|w_j(\cdot,\ t)\|_K\leqq\|w_j^\infty(\cdot,\ t)\|_K+\|w_j(\cdot,\ t)-w_j^\infty(\cdot,\ t)\|\,, \tag{5.13}$$

$$\|w_j(\cdot,\ t)\|_K\geqq\|w_j^\infty(\cdot,\ t)\|_K-\|w_j(\cdot,\ t)-w_j^\infty(\cdot,\ t)\| \tag{5.14}$$

which hold for every $K\subset\Omega$ and $t\in\boldsymbol{R}$. Q.E.D.

Our next result is the following theorem which describes that the wave energy is almost concentrated in an expanding *spherical* zone of constant thickness for all t large enough.

THEOREM 5.2. *For any $\varepsilon>0$ and $\{f_1,\ f_2\}\in\mathfrak{D}(H)\times\mathfrak{H}$ there exist some constants $\eta>0$ and $t_0\geqq\max\{1,\ \eta\}$ such that for any $t>t_0$,*

$$(5.15)\qquad E(w,\ \Omega,\ 0)-\varepsilon\leqq E(w,\ K(t-\eta,\ t+\eta\ ;\ S^{n-1}),\ t)\leqq E(w,\ \Omega,\ 0)\ .$$

PROOF. By means of (5.4) and Proposition 5.2, there exist $\eta>0$ and $t_0\geqq\max\{1,\ \eta\}$ such that

$$\int_{S^{n-1}}dS_{\tilde{x}}\left[\int_{-\infty}^{-\eta}+\int_{\eta}^{\infty}\right]|F_j(s,\ \tilde{x})|^2ds\leqq\varepsilon/2\qquad(j=1,\ 2),$$

$$\|w_j(\cdot,\ t)-w_j^\infty(\cdot,\ t)\|\sqrt{E(w,\ \Omega,\ 0)}\leqq\varepsilon/12\quad(j=1,\ 2)$$

for all $t>t_0$. Note further that $\psi(s)=1$ for $s\geqq 1$. Then by (5.10) and (5.13) with $K=K(t-\eta,\ t+\eta\ ;\ S^{n-1})$, Proposition 5.1 and (5.4) we have

$$E(w,\ K(t-\eta,\ t+\eta\ ;\ S^{n-1}),\ t)\geqq\sum_{j=1}^{2}\left\{\frac{1}{2}\int_{S^{n-1}}dS_{\tilde{x}}\int_{-\eta}^{\eta}|F_j(s,\ \tilde{x})|^2ds\right.$$

$$\left.-3\|w_j(\cdot,\ t)-w_j^\infty(\cdot,\ t)\|\sqrt{E(w,\ \Omega,\ 0)}\right\}\geqq E(w,\ \Omega,\ 0)-\varepsilon$$

for all $t>t_0$. This proves the first inequality of (5.15). The second inequality is a result of (5.11). Q. E. D.

Finally, we prove the following theorem which describes the asymptotic distribution of energy in cones.

THEOREM 5.3. *Let C be a cone in $\boldsymbol{R}^n$ with vertex at the origin:*

$$(5.16)\qquad C=\boldsymbol{R}_+\times C_0\quad(C_0\ \textit{being a measurable subset of}\ S^{n-1}).$$

Then we have for any $\{f_1,\ f_2\}\in\mathfrak{D}(H)\times\mathfrak{H}$,

$$(5.17)\qquad \lim_{t\to\infty}E(w,\ \Omega\cap C,\ t)=\iint_{\boldsymbol{R}\times C_0}|\tilde{f}_E(\sigma,\ \tilde{x})|^2d\sigma dS_{\tilde{x}}$$

$$=\frac{1}{2}\left\{\iint_{\boldsymbol{R}_+\times C_0}|[\mathcal{F}_+(\bar{H}f_1+if_2)](\sigma,\ \tilde{x})|^2d\sigma dS_{\tilde{x}}\right.$$

$$\left.+\iint_{\boldsymbol{R}_-\times C_0}|[\mathcal{F}_-(\bar{H}f_1-if_2)](\sigma,\ \tilde{x})|^2d\sigma dS_{\tilde{x}}\right\}.$$

PROOF. Since $\Omega\cap C=K(0,\ \infty\ ;\ C_0)$, it follows from (5.10) and (5.13) with $K=K(0,\ \infty\ ;\ C_0)$ and Propositions 5.1, 5.2 that

$$(5.18)\qquad \lim_{t\to\infty}E(w,\ \Omega\cap C,\ t)=\lim_{t\to\infty}\sum_{j=1}^{2}\|w_j(\cdot,\ t)\|_{K(0,\infty;\ C_0)}^2$$

$$=\frac{1}{2}\sum_{j=1}^{2}\int_{C_0}dS_{\tilde{x}}\int_{-\infty}^{\infty}|F_j(s,\ \tilde{x})|^2ds\ .$$

Here by (5.3) and the Parseval equality, (5.19) $$\int_{-\infty}^{\infty}|F_j(s, \tilde{x})|^2 ds=\int_{-\infty}^{\infty}|\tilde{f}_E(\sigma, \tilde{x})|^2 d\sigma \qquad (j=1, 2).$$

Equalities (5.18), (5.19) and (5.1) imply (5.17). Q. E. D.

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