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Some sums involving Farey fractions I

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1. Introduction.

It is our aim in this paper to give some refinements of theorems proved by Hall [6] and the first author [9] on some sums involving Farey fractions.

Let F_n ($n \in \boldsymbol{N}$) be the Farey series of order n, that is, the set of all irreducible fractions between 0 and 1 with denominators $\leqq n$ and arranged in the increasing order of magnitude: $F_n = \{h/k \mid 0 \leqq h \leqq k \leqq n, (h, k)=1\}$; for any term h/k (<1) of F_n we denote by h'/k' its successor in F_n and by Q_n the set of all pairs (k, k') of the denominators of such consecutive fractions in F_n. For any function $f: \boldsymbol{N} \times \boldsymbol{N} \rightarrow \boldsymbol{C}$, writing

$$S_n = \sum_{(k, k') \in Q_n} f(k, k'),$$

Lehner and Newman [11] proved the sum formula (see also Mitsui [14], pp. 106-109)

$$S_n = f(1, 1) + \sum_{r=2}^{n} \sum_{\substack{k=1 \\ (k, r)=1}}^{r} \{f(k, r) + f(r, k) - f(k, r-k)\}. \tag{1}$$

The interest in this formula arises due to the fact that a sum involving Farey fractions is transformed into one which does not. Lehner-Newman [11] and the first author [9] discussed, among other things, the applications of the sum formula (1) to the evaluation of certain infinite series. Recently, the second and third named authors [18] made use of an extension of the sum formula (1) (to be found in Apostol [1], p. 111) to proving several identities involving Riemann's zeta-function and, in particular, those of Briggs, Chowla, Kempner and Mientka [2], Gupta [5], Hans and Dumir [7] and Williams [21]. In section 2 of this paper we state refinements of the first author's sharpenings of Hall's results [6] and establish some preliminary results for the proofs of these results. The preliminary results obtained in section 2 also enable us to sharpen various

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results due to Lehner-Newman [11] and Mikolás [13], which will be carried out in our forthcoming paper with the same title. In section 3 proofs of theorems are given.

2. Statement of theorems and prerequisites.

Hall considered sums defined by

$$S_n(m)=\sum_{(k,\,k')\in Q_n}(kk')^{-m}$$

for integral $m\geqq 2$ and established the following

THEOREM (Hall). *As $n\to\infty$, we have*

$$S_n(2)=\frac{12}{\pi^2 n^2}\left\{\log n+C+\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}\right\}+O\left(\frac{\log^2 n}{n^3}\right), \tag{2}$$

$$S_n(m)=2\frac{\zeta(m-1)}{\zeta(m)\,n^m}+O\left(\frac{\log^\theta n}{n^{m+1}}\right), \quad \text{for } m\geqq 3, \tag{3}$$

where θ is 0 or 1 according as $m\geqq 4$ or $m=3$, C is Euler's constant, ζ is the Riemann zeta-function and ζ' its derivative.

Recently, the first author [9] refined Hall's result (2) above by proving that the O-estimate for the error term could be reduced to $O_\varepsilon(n^{-3}t^{1+\gamma}(n)u^{1+\varepsilon}(n))$ for each $\varepsilon>0$, where t and u are abbreviations for log and log log of the variables considered, and the meaning of γ will be clarified in the proof of Lemma 9, its best known value being 2/3. Also he noted that Hall's result is best possible in the sense that the error estimate is given by $O(n^{-m-1}t^\theta)$ by proving that

$$S_n(m)=2\frac{\zeta(m-1)}{\zeta(m)\,n^m}+\frac{3\theta t}{\zeta(2)\,n^4}+O\left(\frac{1}{n^{m+1}}\right), \quad \text{for } m\geqq 3.$$

In section 3 we establish the following sharper results:

THEOREM 1. *For each $\varepsilon>0$, we have, as $n\to\infty$*

$$S_n(2)=\frac{12}{\pi^2 n^2}\left\{t+C+\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}\right\}+\frac{4U(n)t}{n^3}+O_\varepsilon\left(\frac{tu^{1+\varepsilon}}{n^3}\right),$$

where $t=t(n)=\log n$, $u=u(n)=\log\log n$ and $U(n)$ is given by (7).

THEOREM 2. *For each $\varepsilon>0$, we have, as $n\to\infty$*

$$S_n(3)=\frac{2\zeta(2)\zeta^{-1}(3)}{n^3}+\frac{3\zeta^{-1}(2)}{n^4}\left\{t+C-\frac{1}{4}-\frac{\zeta'(2)}{\zeta(2)}+2\zeta^2(2)c_2(n)\right\}$$
$$+\frac{12U(n)t}{n^5}+O_\varepsilon\left(\frac{tu^{1+\varepsilon}}{n^5}\right),$$

where for $\lambda>1$

(4) $$c_{\lambda}(n)=\sum_{m=1}^{\infty}\frac{\mu(m)}{m^{\lambda}}P_1\left(\frac{n}{m}\right),$$

and t, u have the same meanings as in Theorem 1.

Now we fix some notation, recall some known results and establish necessary lemmas. Let $\mu(n)$ denote the Möbius function, $\phi(n)$ the Euler totient function and for $n\in\boldsymbol{N}$, $P_n(v)$ the periodic Bernoulli polynomial of degree n, so that, in particular, $P_1(v)=\{v\}-1/2$ and $P_2(v)=\{v\}^2-\{v\}+1/6$ ($\{v\}$ being the fractional part of v). Further, we write

(5) $$E(x)=\sum_{n\leq x}\phi(n)-\frac{x^2}{2\zeta(2)},$$

(6) $$H(x)=\sum_{n\leq x}\frac{\phi(n)}{n}-\frac{x}{\zeta(2)},$$

(7) $$U(x)=\sum_{n\leq x}\frac{\mu(n)}{n}P_1\left(\frac{x}{n}\right).$$

In order to make the error estimates as good as possible appearing in the lemmas below we shall use the following two best known estimates concerning $U(x)$ and the average of the Möbius function due, respectively, to Saltykov [**17**] and Walfisz [**20**], although we do not need such deeper results for the proofs of our theorems: For each $\varepsilon>0$, as $x\to\infty$

(8) $$U(x)=O_{\varepsilon}(\lambda_{\varepsilon}(x)),$$

where

(9) $$\lambda(x)=\lambda_{\varepsilon}(x)\equiv t^{\gamma}u^{1+\varepsilon},$$

and

(10) $$M(x)=\sum_{n\leq x}\mu(n)=O(x\delta_A(x)),$$

as $x\to\infty$, where

(11) $$\delta_A(x)=\exp(-At^{3/5}u^{-1/5}),$$

A being a positive constant, not necessarily the same at each occurrence (Walfisz [**20**], p. 191 and also p. 181).

What we actually need is, as far as the order estimate is concerned, that

$$\lambda(x)=O(x^{\alpha})\qquad\text{for some }\alpha\in[0,1],$$

and

$$\delta(x)=O(t^{-2}),\ x\delta(x)\uparrow\uparrow.$$

We make good use of the following

LEMMA 1 (Euler-Maclaurin's summation formula, cf. Rademacher [**16**], p. 14).

Let $a<b$ be integers, n a positive integer and $f\in C^n([a, b])$. Then

$$\sum_{j=a+1}^{b} f(j)=\int_a^b f(x)dx+\sum_{k=1}^{n}\frac{(-1)^k B_k}{k!}(f^{(k-1)}(b)-f^{(k-1)}(a))$$
$$+\frac{(-1)^{n-1}}{n!}\int_a^b P_n(x)f^{(n)}(x)dx\,,$$

where B_k is the k-th Bernoulli number determined by $z/(e^z-1)=\sum_{k=0}^{\infty}(B_k/k!)z^k$.

The following result should be compared with that of Pillai and Chowla [**15**]:

LEMMA 2. *Let $\{a_n\}_{n=1}^{\infty}$ be a complex sequence such that $|a_n|<K<\infty$ and $\eta:[3, \infty)\to\boldsymbol{R}^+$ be a function decreasing to zero, with $x\eta(x)$ increasing for large x. For $x\geqq 3$, $1\leqq Q\leqq x$, write*

$$\theta(x, Q)=\sum_{n\leq Q}a_n\left\{\frac{x}{n}\right\}^2.$$

If $\sum_{n\leq x}a_n=O(x\eta(x))$ as $x\to\infty$, then $\theta(x, Q)=O(x\eta_1(x))$ uniformly in x and Q, where $\eta_1(x)=(\eta(x))^{1/3}+(\eta(x))^{-2/3}\eta(x(\eta(x))^{1/3})$.

PROOF. Let $i=[(\eta(x))^{-1/3}]$ and suppose that $Q\geqq x\eta_1(x)$ since otherwise the lemma is trivial. We have

$$\theta(x, Q)=\Big(\sum_{n\leq x/i}+\sum_{j+1\leq k\leq i-1}\ \sum_{x/(k+1)<n\leq x/k}+\sum_{x/(j+1)<n\leq Q}\Big)a_n\left\{\frac{x}{n}\right\}^2,$$

where $j=[x/Q]$. Noting that $\{x/n\}=(x/n)-k$ for n satisfying $x/(k+1)<n\leqq x/k$, we have

$$\theta(x, Q)\ll\frac{x}{i}+x^2\sum_{j\leq k\leq i-1}|S_1(x, k)|+2x\sum_{j\leq k\leq i-1}k|S_2(x, k)|+\sum_{j\leq k\leq i-1}k^2|S_3(x, k)|\,, \tag{12}$$

where

$$S_1(x, k)=\sum_{n\in I_k}\frac{a_n}{n^2}\,,\qquad S_2(x, k)=\sum_{n\in I_k}\frac{a_n}{n}\,,\qquad S_3(x, k)=\sum_{n\in I_k}a_n\,,$$

with $I_k=(x/(k+1), x/k]$ for $j<k\leqq i-1$ and $I_j=(x/(j+1), Q]$.

Since $x\eta(x)$ is increasing and $\eta(x)$ is decreasing for large x, clearly for $j\leqq k\leqq i-1$

$$S_3(x, k)\ll\frac{x}{k}\eta\Big(\frac{x}{i}\Big). \tag{13}$$

Since $\sum_{n\leq x}a_n=O(x\eta(x))$ as $x\to\infty$, we deduce, by partial summation, that for $j<k\leqq i-1$

$$S_2(x, k)\ll\eta\Big(\frac{x}{i}\Big),$$

while for $j=k$

$$S_2(x, k)=\sum_{x/(j+1)<n\leq Q}\frac{a_n}{n}\ll\eta\left(\frac{x}{i}\right),$$

so that for $j\leqq k\leqq i-1$

$$S_2(x, k)\ll\eta\left(\frac{x}{i}\right). \tag{14}$$

Again, by partial summation, we have for $j<k\leqq i-1$

$$S_1(x, k)\ll\frac{\eta\left(\frac{x}{i}\right)}{x/i},$$

while for $j=k$

$$S_1(x, k)\ll\frac{\eta\left(\frac{x}{i}\right)}{x/i},$$

so that for $j\leqq k\leqq i-1$

$$S_1(x, k)\ll\frac{\eta\left(\frac{x}{i}\right)}{x/i}. \tag{15}$$

Collecting (12) through (15), we obtain

$$\theta(x, Q)\ll\frac{x}{i}+xi^2\eta\left(\frac{x}{i}\right). \tag{16}$$

Since $i=[(\eta(x))^{-1/3}]$, the lemma follows from (16).

LEMMA 3. *As* $x\to\infty$

$$E(x)=-xU(x)+O(x\delta_A(x)) \tag{17}$$

and

$$H(x)=-U(x)+O(\delta_A(x)), \tag{18}$$

where $E(x)$, $H(x)$, $U(x)$ *and* $\delta_A(x)$ *are as given in* (5), (6), (7) *and* (11), *respectively.*

PROOF. Using the well-known identity $\phi(n)=\sum_{d\delta=n}\mu(d)\delta$, we have

$$\begin{aligned}\Phi(x)&\equiv\sum_{n\leq x}\phi(n)=\sum_{d\leq x}\mu(d)\Big(\sum_{\delta\leq x/d}\delta\Big)\\&=\frac{1}{2}\sum_{d\leq x}\mu(d)\left(\frac{x^2}{d^2}-2\frac{x}{d}P_1\left(\frac{x}{d}\right)+\left\{\frac{x}{d}\right\}^2-\frac{x}{d}+\left[\frac{x}{d}\right]\right)\\&=\frac{x^2}{2}\left(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}-\sum_{d>x}\frac{\mu(d)}{d^2}\right)-xU(x)\\&\quad+\frac{1}{2}\sum_{d\leq x}\mu(d)\left\{\frac{x}{d}\right\}^2-x\sum_{d\leq x}\frac{\mu(d)}{d}+\frac{1}{2}\end{aligned} \tag{19}$$

by (7) and the well-known $\sum_{d\leqq x}\mu(d)[x/d]=1$ for $x\geqq 1$.

Now by partial summation and (10), we obtain $\sum_{d>x}\mu(d)/d^2=O(x^{-1}\delta_A(x))$ and $\sum_{d\leqq x}\mu(d)/d=-\sum_{d>x}\mu(d)/d=O(\delta_A(x))$. Further by (10) and Lemma 2, we find $\sum_{d\leqq x}\mu(d)\{x/d\}^2=O(x\delta_A(x))$. Using these in (19) and noting (5), we obtain (17).

Further, we have

$$\sum_{n\leqq x}\frac{\phi(n)}{n}=\sum_{d\leqq x}\frac{\mu(d)}{d}\left[\frac{x}{d}\right]$$

$$=\sum_{d\leqq x}\frac{\mu(d)}{d}\left(\frac{x}{d}-P_1\left(\frac{x}{d}\right)-\frac{1}{2}\right)$$

$$=\frac{x}{\zeta(2)}-U(x)+O(\delta_A(x))\,, \tag{20}$$

so that (6) implies (18). This completes the proof of Lemma 3.

REMARK 1. From Lemma 3 follows a refinement over Pillai and Chowla's result [**15**], namely, that $E(x)-xH(x)=o(x)$. Using different arguments, and, in particular, the fact that $E(x)\neq o(x\log u)$ due to Pillai and Chowla [**15**], Suryanarayana [**19**] has recently deduced this refinement.

LEMMA 4.

$$\int_1^\infty\frac{E(t)}{t^2}dt=\frac{3}{\pi^2}.$$

More precisely, as $x\to\infty$

$$\int_1^x\frac{E(t)}{t^2}dt=\frac{3}{\pi^2}+O(\delta_A(x))\,. \tag{21}$$

PROOF. In fact, by partial summation, together with (5)

$$\sum_{n\leqq x}\frac{\phi(n)}{n}=\frac{6x}{\pi^2}+\frac{E(x)}{x}-\frac{3}{\pi^2}+\int_1^x\frac{E(t)}{t^2}dt\,.$$

Comparing this with (20) and using (18), we obtain the lemma.

LEMMA 5. *For positive integral* n *and each* $\varepsilon>0$, *we have as* $x\to\infty$

$$\Phi_n(x)\equiv\sum_{mn\leqq x}\frac{\phi(mn)}{mn}=\frac{6x}{\pi^2\psi(n)}+O_\varepsilon(\lambda_\varepsilon(x))$$

uniformly in x *and* n, *where* $\psi(n)$ *is the Dedekind* ψ*-function defined by* $\psi(n)=\sum_{d\delta=n}\mu^2(d)\delta=n\prod_{p\mid n}(1+1/p)$, *the product ranging over all prime factors of* n *and* $\lambda(x)$ *is given by* (9).

PROOF. Let p be a prime such that $(p,n)=1$ and $\alpha\in\boldsymbol{N}$. Then

$$\bar{\Phi}_{np^\alpha}(x)=\frac{\phi(p^\alpha)}{p^\alpha}\sum_{\substack{mnp^\alpha\leq x\\(p,m)=1}}\frac{\phi(mn)}{mn}+\sum_{\substack{mnp^\alpha\leq x\\p|m}}\frac{\phi(mn)}{mn}$$

$$=\frac{\phi(p^\alpha)}{p^\alpha}\Bigl(\sum_{mnp^\alpha\leq x}\frac{\phi(mn)}{mn}-\sum_{\substack{mnp^\alpha\leq x\\p|m}}\frac{\phi(mn)}{mn}\Bigr)+\sum_{\substack{mnp^\alpha\leq x\\p|m}}\frac{\phi(mn)}{mn}$$

$$=\frac{\phi(p^\alpha)}{p^\alpha}\bar{\Phi}_n\Bigl(\frac{x}{p^\alpha}\Bigr)+\frac{1}{p}\sum_{\substack{mnp^\alpha\leq x\\p|m}}\frac{\phi(mn)}{mn}$$

$$=\Bigl(1-\frac{1}{p}\Bigr)\bar{\Phi}_n\Bigl(\frac{x}{p^\alpha}\Bigr)+\frac{1}{p}\bar{\Phi}_{np}\Bigl(\frac{x}{p^\alpha}\Bigr). \tag{22}$$

Taking $\alpha=1$ in (22), we get inductively

$$\bar{\Phi}_{np}(x)=\Bigl(1-\frac{1}{p}\Bigr)\sum_{r=0}^{c-2}p^{-r}\bar{\Phi}_n\Bigl(\frac{x}{p^{r+1}}\Bigr),$$

where $c=[\log x/\log p]$. Thus, by (22) again

$$\bar{\Phi}_{np^\alpha}(x)=\Bigl(1-\frac{1}{p}\Bigr)\bar{\Phi}_n\Bigl(\frac{x}{p^\alpha}\Bigr)+\Bigl(1-\frac{1}{p}\Bigr)\sum_{r=0}^{c-2}p^{-r-1}\bar{\Phi}_n\Bigl(\frac{x}{p^{\alpha+r+1}}\Bigr)$$

$$=\Bigl(1-\frac{1}{p}\Bigr)\sum_{r=0}^{\infty}p^{-r}\bar{\Phi}_n\Bigl(\frac{x}{p^{r+\alpha}}\Bigr). \tag{23}$$

The lemma is true for $n=1$ by (8) and (19). Suppose the result to be true for $1\leqq N\leqq n-1$ $(n\geqq 2)$. There exists a prime p such that $n=Np^\alpha$, $(p, N)=1$ and $1\leqq N\leqq n-1$. By induction hypothesis, we have

$$\bar{\Phi}_N(x)=\frac{6x}{\pi^2\psi(N)}+O_\varepsilon(\lambda(x))$$

uniformly in x and N. Thus by (23)

$$\bar{\Phi}_n(x)=\bar{\Phi}_{Np^\alpha}(x)=\frac{6x(1-p^{-1})}{\pi^2\psi(N)p^\alpha}\sum_{r=0}^{\infty}p^{-2r}+O_\varepsilon((1-p^{-1})\lambda(x)\sum_{r=0}^{\infty}p^{-r})$$

$$=\frac{6x}{\pi^2\psi(Np^\alpha)}+O_\varepsilon(\lambda(x)),$$

which proves the lemma.

COROLLARY 1. *For positive integral n and each* $\varepsilon>0$

$$\Phi_n(x)\equiv\sum_{mn\leq x}\phi(mn)=\frac{3x^2}{\pi^2\psi(n)}+O_\varepsilon(x\lambda(x))$$

uniformly in x and n.

PROOF. This follows from the theorem of partial summation and Lemma 5.

LEMMA 6. *For* $s>1$, *we have, as* $x\to\infty$

$$\int_1^x \frac{H(t)\log t}{t^{s+1}}\,dt = \frac{\zeta(s)\zeta'(s+1)}{s\zeta^2(s+1)} - \frac{\zeta'(s)}{s\zeta(s+1)} + \frac{\zeta(s)}{s^2\zeta(s+1)}$$
$$-\frac{1}{(s-1)^2\zeta(2)} + O\Big(\frac{\log x}{x^s}\Big).$$

PROOF. For an integer $n>0$ and real $s>1$, we have, by the Euler-Maclaurin sum formula

$$\sum_{m\leq n} m^{-s} = \zeta(s) - \frac{1}{(s-1)n^{s-1}} + \frac{1}{2n^s} - \frac{s}{12n^{s+1}} + O\Big(\frac{1}{n^{s+3}}\Big),$$

so that for real $x\geqq 1$ and $s>1$

$$\sum_{m\leq x} m^{-s} = \zeta(s) - \frac{1}{(s-1)[x]^{s-1}} + \frac{1}{2[x]^s} - \frac{s}{12[x]^{s+1}} + O\Big(\frac{1}{x^{s+3}}\Big)$$
$$= \zeta(s) - \frac{1}{(s-1)x^{s-1}}\Big(1-\frac{\{x\}}{x}\Big)^{1-s} + \frac{1}{2x^s}\Big(1-\frac{\{x\}}{x}\Big)^{-s}$$
$$-\frac{s}{12x^{s+1}}\Big(1-\frac{\{x\}}{x}\Big)^{-s-1} + O(x^{-s-3})$$

(24) $$= \zeta(s) - \frac{1}{(s-1)x^{s-1}} - \frac{P_1(x)}{x^s} + \frac{-\frac{s}{2}P_2(x)}{x^{s+1}} + O\Big(\frac{1}{x^{s+2}}\Big)$$

on simplification. Similarly we have for real $x\geqq 1$ and real $s>1$

(25) $$\sum_{n\leq x} \frac{\log n}{n^s} = -\zeta'(s) - \frac{P_1(x)\log x}{x^s} - \frac{1}{(s-1)^2 x^{s-1}} - \frac{t}{(s-1)x^{s-1}} + O\Big(\frac{1}{x^s}\Big).$$

Now we have on the one hand, by (6) and partial summation

$$\sum_{n\leq x} \frac{\phi(n)\log n}{n^{s+1}} = \frac{1}{(s-1)^2\zeta(2)} - \int_1^\infty \frac{H(t)}{t^{s+1}}\,dt + s\int_1^\infty \frac{H(t)\log t}{t^{s+1}}\,dt$$

(26) $$+\frac{H(x)\log x}{x^s} - \frac{1}{\zeta(2)(s-1)^2 x^{s-1}} - \frac{\log x}{(s-1)\zeta(2)x^{s-1}}$$
$$-s\int_x^\infty \frac{H(t)\log t}{t^{s+1}}\,dt + O\Big(\frac{\log x}{x^s}\Big)$$

after simplification, where in the above the integrals converge in view of the trivial estimate $H(x)=O(\log x)$.

On the other hand, we have

$$\sum_{n\leq x}\frac{\phi(n)\log n}{n^{s+1}}=\sum_{d\leq x}\frac{\mu(d)\log d}{d^{s+1}}\Big(\sum_{\delta\leq x/d}\delta^{-s}\Big)$$

$$+\sum_{d\leq x}\frac{\mu(d)}{d^{s+1}}\Big(\sum_{\delta\leq x/d}\delta^{-s}\log\delta\Big)$$

$$=S_1+S_2, \tag{27}$$

say.

By using (24), we find

$$S_1=\sum_{d\leq x}\frac{\mu(d)\log d}{d^{s+1}}\Big\{\zeta(s)-\frac{d^{s-1}}{(s-1)x^{s-1}}-\frac{P_1\big(\frac{x}{d}\big)d^s}{x^s}+O\Big(\Big(\frac{d}{x}\Big)^{s+1}\Big)\Big\}$$

$$=\frac{\zeta(s)\zeta'(s+1)}{\zeta^2(s+1)}-\frac{\zeta'(2)}{(s-1)\zeta^2(s)x^{s-1}}-\frac{1}{x^s}\sum_{d\leq x}\frac{\mu(d)\log d}{d}P_1\Big(\frac{x}{d}\Big) \tag{28}$$

$$+O(x^{-s}\log x),$$

where in the above we used $\sum_{d>x}d^{-s-1}\mu(d)\log d=O(x^{-s}\log x)$ and that $\sum_{d=1}^{\infty}d^{-s}\mu(d)\log d=\zeta'(s)\zeta^{-2}(s)$ $(s>1)$.

Similarly, by using (25), we get

$$S_2=-\frac{\zeta'(s)}{\zeta(s+1)}-\frac{\log x}{(s-1)\zeta(2)x^{s-1}}+\frac{1}{(s-1)x^{s-1}}\Big(\frac{\zeta'(2)}{\zeta^2(2)}-\frac{1}{(s-1)\zeta(2)}\Big) \tag{29}$$

$$-\frac{U(x)\log x}{x^s}+\frac{1}{x^s}\sum_{d\leq x}\frac{\mu(d)\log d}{d}P_1\Big(\frac{x}{d}\Big)+O\Big(\frac{\log x}{x^s}\Big),$$

where in the above we used $\sum_{n=1}^{\infty}\mu(n)n^{-s}=1/\zeta(s)$. Thus by (27), (28) and (29), we obtain

$$\sum_{n\leq x}\frac{\phi(n)\log n}{n^{s+1}}=\frac{\zeta(s)\zeta'(s+1)}{\zeta^2(s+1)}-\frac{\zeta'(s)}{\zeta(s+1)}-\frac{\log x}{(s-1)\zeta(2)x^{s-1}} \tag{30}$$

$$-\frac{1}{(s-1)^2\zeta(2)x^{s-1}}-\frac{U(x)\log x}{x^s}+O\Big(\frac{\log x}{x^s}\Big).$$

Comparing the right sides of (26) and (30) and letting $x\to\infty$, we find, for $s>1$

$$\frac{1}{(s-1)^2\zeta(2)}-\int_1^{\infty}\frac{H(t)}{t^{s+1}}dt+s\int_1^{\infty}\frac{H(t)\log t}{t^{s+1}}dt$$

$$=\frac{\zeta(s)\zeta'(s+1)}{\zeta^2(s+1)}-\frac{\zeta'(s)}{\zeta(s+1)}. \tag{31}$$

Again comparing the right sides of (26) and (30) and on using (31) together with (18), we deduce

$$\int_x^\infty \frac{H(t)\log t}{t^{s+1}}\,dt = O\Big(\frac{\log x}{x^s}\Big). \tag{32}$$

According to (31) and (32), to prove the lemma it is sufficient to show that for $s>1$

$$\int_1^\infty \frac{H(t)}{t^{s+1}}\,dt = \frac{\zeta(s)}{s\zeta(s+1)} - \frac{1}{(s-1)\zeta(2)}. \tag{33}$$

In fact, on using (6) and partial summation, we get

$$\sum_{n\le x}\frac{\phi(n)}{n^{s+1}} = \frac{s}{(s-1)\zeta(2)} - \frac{1}{(s-1)\zeta(2)x^{s-1}} + s\int_1^x \frac{H(t)}{t^{s+1}}\,dt + O\Big(\frac{\log x}{x^s}\Big).$$

By letting $x\to\infty$ in the above, we get (33) in view of $\sum_{n=1}^{\infty}\phi(n)/n^{s+1}=\zeta(s)/\zeta(s+1)$ $(s>1)$. This completes the proof of Lemma 6.

REMARK 2. Since $H(t)=O(\log t)$, the integrals $\int_1^\infty \frac{H(t)}{t^{s+1}}dt$ and $\int_1^\infty \frac{H(t)\log t}{t^{s+1}}dt$ both converge absolutely and uniformly on every compact subset of the half-plane $\{s\in \boldsymbol{C}\mid \operatorname{Re} s>0\}$ and hence define analytic functions. Thus by Lemma 6, (33) and analytic continuation we obtain: For any $s\in\boldsymbol{C}$ with $\operatorname{Re} s>0$,

$$\int_1^\infty \frac{H(t)\log t}{t^{s+1}}\,dt = \frac{\zeta(s)\zeta'(s+1)}{s\zeta^2(s+1)} - \frac{\zeta'(s)}{s\zeta(s+1)} + \frac{\zeta(s)}{s^2\zeta(s+1)} - \frac{1}{(s-1)^2\zeta(2)},$$

$$\int_1^\infty \frac{H(t)}{t^{s+1}}\,dt = \frac{\zeta(s)}{s\zeta(s+1)} - \frac{1}{(s-1)\zeta(2)},$$

so that, in particular, $\int_1^\infty H(t)t^{-2}dt = \zeta^{-1}(2)(C-1-\zeta^{-1}(2)\zeta'(2))$.

LEMMA 7. *There exist positive constants c and d such that for any* $\alpha\in[0, 1]$ *and any* $r\in\boldsymbol{N}$

$$c\log r \leqq \sum_{p\le r}\frac{\log p}{p}\Big(\log\frac{r}{p}\Big)^{\alpha} \leqq d\log r,$$

where the sum ranges over all primes $\leqq r$.

PROOF. It is well-known that $\theta(x)\equiv\sum_{\substack{p\le x\\ p \text{ prime}}}\log p = O(x)$ (Hardy and Wright [8], Theorem 414). Hence

$$\sum_{p\le r}\log p\log\frac{r}{p} = \int_0^r \theta(t)t^{-1}dt = O(r),$$

which in turn yields by partial summation

$$\sum_{p\le r}\log p\Big(\log\frac{r}{p}\Big)\frac{1}{p} = O(\log r).$$

This together with the known $\sum_{p\leq r}(\log p)/p=\log r+O(1)$ (Hardy and Wright [8], Theorem 425) shows that for $\alpha\in[0, 1]$

$$\sum_{p\leq r}\frac{\log p}{p}\left(\log\frac{r}{p}\right)^{\alpha}\leqq\sum_{p\leq r/e}\frac{\log p}{p}\left(\log\frac{r}{p}\right)+\sum_{r/e<p\leq r}\frac{\log p}{p}\ll\log r\,.$$

Further

$$\sum_{p\leq r}\frac{\log p}{p}\left(\log\frac{r}{p}\right)^{\alpha}\geqq\sum_{p\leq r/e}\frac{\log p}{p}=\log r+O(1)\,.$$

This completes the proof of the lemma.

LEMMA 8. *Let* $k\in N$, $t(k)=\sum_{d|k}(\mu(d)/d)\log(k/d)$ *and* $T(x)=\sum_{k\leq x}t(k)$. *Then for each* $\varepsilon>0$, *we have, as* $x\to\infty$

$$T(x)=\frac{x}{\zeta(x)}\left\{\log x-1-\frac{\zeta'(2)}{\zeta(2)}\right\}+H(x)\log x+O_{\varepsilon}(tu^{1+\varepsilon})\,.$$

PROOF. Firstly, we note that

$$\alpha(k)\equiv-\frac{k}{\phi(k)}\sum_{d|k}\frac{\mu(d)\log d}{d}=\sum_{p|k}\frac{\log p}{p-1}\,, \tag{34}$$

the extreme sum on the right extending over all the distinct prime factors of k. We remark parenthetically that Davenport [**3**] was the first to discuss systematically the function $\alpha(k)$. (34) follows on noting that the functions involved on both sides are additive and coincide with each other at arbitrary prime powers. Secondly, we note that by (17), (18), Lemma 4 and the fact that $\delta_A(t)=O((\log t)^{-2})$ as $t\to\infty$

$$\begin{aligned}\int_1^x\frac{H(t)}{t}dt&=O(1)+\int_1^x\frac{E(t)}{t^2}dt+O\left(\int_3^x\frac{\delta_A(t)}{t}dt\right)\\&=O(1)\,.\end{aligned}\tag{35}$$

Now by (34) we obtain

$$T(x)=\sum_{k\leq x}t(k)=\sum_{k\leq x}\frac{\phi(k)\log k}{k}+\sum_{k\leq x}\frac{\phi(k)\alpha(k)}{k}=S_3+S_4\,, \tag{36}$$

say. By the theorem of partial summation and (35), we have

$$S_3=\frac{x\log x}{\zeta(2)}-\frac{x}{\zeta(2)}+H(x)\log x+O(1)\,. \tag{37}$$

Further by (34)

$$\begin{aligned}S_4&=\sum_{p\delta\leq x}\frac{\phi(p\delta)}{p\delta}\frac{\log p}{p-1}\\&=\sum_{p\delta\leq x}\frac{\phi(\delta)}{\delta}\frac{\log p}{p}+\sum_{\substack{p\delta\leq x\\ p|\delta}}\frac{\phi(\delta)}{\delta}\frac{\log p}{p(p-1)}\end{aligned}$$

Now using successively (6), Lemma 5, the fact that $H(x)=O_\varepsilon(\lambda_\varepsilon(x))$ for each $\varepsilon>0$ (which is an easy consequence of (8) and (18)) and Lemma 7, we find

$$S_4=\frac{x}{\zeta(2)}\Big(\sum_p\frac{\log p}{p^2}+O\Big(\frac{\log x}{x}\Big)\Big)+O_\varepsilon\Big(\sum_{p\leq x}\frac{\log p}{p}\Big(\log\frac{x}{p}\Big)^{\gamma}\Big(\log\log\frac{x}{p}\Big)^{1+\varepsilon}\Big)$$

$$+\frac{x}{\zeta(2)}\Big(\sum_p\frac{\log p}{p^2(p^2-1)}+O\Big(\frac{\log x}{x^3}\Big)\Big)+O_\varepsilon(\lambda_\varepsilon(x))$$

$$(38)\qquad =-\frac{\zeta'(2)}{\zeta^2(2)}x+O_\varepsilon(tu^{1+\varepsilon}).$$

Collecting (36), (37) and (38), we conclude the assertion.

LEMMA 9. *For* $x\geqq 3$ *and* $1\leqq Q\leqq x$, *write*

$$(39)\qquad U(x,\ Q)=\sum_{n\leq Q}\frac{\mu(n)}{n}P_1\Big(\frac{x}{n}\Big).$$

Then for each $\varepsilon>0$

$$(40)\qquad U(x,\ Q)=O_\varepsilon(\lambda_\varepsilon(x))$$

uniformly in x *and* Q, *where* $\lambda_\varepsilon(x)$ *is given by* (9).

Before establishing Lemma 9 we shall explain the meaning of γ. Consider the polynomial $f(y)=\alpha_1 y+\cdots+\alpha_{n+1}y^{n+1}$ part of whose coefficients are rational, say $\alpha_\nu=a_\nu/q$ ($\nu=s+2,\cdots,3s$; $1\leqq s\leqq(n+1)/3$) and by Δ_s we mean the determinant $\det\Big(\binom{s+i+j}{i}a_{s+i+j}\Big)_{1\leq i,j\leq s}$ of order s. Assume that if δ is a fixed number in the interval $0<\delta\leqq 1/3$; $\delta n\leqq s\leqq(n+1)/3$; $s+1\leqq r\leqq 2s(1-\delta)$; $q=p^r$; $(\Delta_s,\ q)=1$, then the following estimate holds:

$$(41)\qquad \sum_{y=1}^{P}\exp(2\pi i f(y))\ll\exp(C_1 n^{\gamma_1})P^{1-C_2 n^{-\gamma_2}},$$

where $\gamma_1\geqq 0$, $\gamma_2\geqq 1$, $1+\gamma_2>\gamma_1$ and the constants C_1 and C_2 depend possibly on δ, γ_1 and γ_2. With these γ_i's we define $\gamma=(\gamma_1+\gamma_2)/(\gamma_1+\gamma_2+1)$. It is known due to Korobov [**10**] that (41) is valid for the choice $\gamma_1=0$ and $\gamma_2=2$, thus yielding the best known value $\gamma=2/3$.

PROOF OF LEMMA 9. First let x be an integer. If $Q\leqq\exp(B\lambda(x))$, B being a constant depending on δ, γ_1 and γ_2, then (40) follows trivially. If $\exp(B\lambda(x))<Q\leqq x\exp(-\sqrt{t})$, then we write

$$U(x,\ Q)=\Big(\sum_{n\leq\exp(B\lambda(x))}+\sum_{\exp(B\lambda(x))<n\leq Q}\Big)\frac{\mu(n)}{n}P_1\Big(\frac{x}{n}\Big)$$

and divide the range of n of the second sum into $O(t)$ subintervals of the form $(2^{i-1}M,\ 2^iM]$. Now Saltykov's result (eqn (55), p. 49)

$$\sum_{n=M}^{M'} \frac{\mu(n)}{n} P_1\left(\frac{x}{n}\right) = O(t^{-1}), \tag{42}$$

valid for M, M' satisfying the conditions $M<M'\leqq 2M$, $\exp(B\lambda(x))\leqq M<M'\leqq x\exp(-\sqrt{t})$, will prove (40).

Finally, if $x\exp(-\sqrt{t})<Q\leqq x$, then in view of the above it is enough to consider

$$\sum_{x\exp(-\sqrt{t})<n\leqq Q} \frac{\mu(n)}{n} P_1\left(\frac{x}{n}\right) \ll \sqrt{t}\,,$$

which proves the contention.

If x is not an integer, we note that

$$U(x, Q) = U([x], Q) + O(1)\,,$$

which completes the proof of the Lemma.

3. Proofs of theorems.

PROOF OF THEOREM 1. Taking $f(x, y)=(xy)^{-2}$ in the sum formula (1), we obtain as in [**9**],

$$S_n(2) = 1 - 4\sum_{r=2}^{n} r^{-3} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} k^{-1} = 1 - 4\sum_{r=2}^{n} r^{-3} S_r'\,,$$

say. Since, by definition, $S_n(2)=o(1)$ as $n\to\infty$, we have

$$S_n(2) = 4\sum_{r=n+1}^{\infty} r^{-3} S_r'\,. \tag{43}$$

It may be noted that the above argument also proves the identity $\sum_{r=1}^{\infty} r^{-3} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} k^{-1} = 5/4$ to be found in [**9**] (Corollary 1) and [**18**] (eqn. (1.10)).

Now using the formula

$$\sum_{k=1}^{r} k^{-1} = \log r + C + \frac{1}{2r} + O\left(\frac{1}{r^2}\right)$$

which readily follows from Lemma 1, we deduce that

$$\begin{aligned} S_r' &= \sum_{k=1}^{r} k^{-1} \Big(\sum_{d\mid(k,r)} \mu(d)\Big) \\ &= \sum_{d\mid r} \mu(d) d^{-1} \Big(\sum_{h=1}^{r/d} h^{-1}\Big) \\ &= t(r) + C\phi(r) r^{-1} + O(r^{-2}\sigma(r))\,, \end{aligned} \tag{44}$$

where $t(r)$ is as given in Lemma 8, $r>1$ and $\sigma(r)$ is the sum of all the divisors of r. Now if we put for $a>1$, $S_5^{(a)}=\sum_{r=n+1}^{\infty} r^{-a}t(r)$ and $S_6^{(a)}=\sum_{r=n+1}^{\infty} r^{-a-1}\phi(r)$, then

$$S^{(a)}\equiv \sum_{r=n+1}^{\infty} S_r' r^{-a}=S_5^{(a)}+CS_6^{(a)}+O(n^{-a}) \tag{45}$$

by virtue of the estimate $\sum_{r\leq n}\sigma(r)=O(n^2)$ and partial summation. Also, by partial summation and Lemma 8, we get

$$S_5^{(a)}=\frac{a}{\zeta(2)}\int_{n+1}^{\infty}\frac{\log t}{t^a}dt-a\Big(1+\frac{\zeta'(2)}{\zeta(2)}\Big)\frac{1}{\zeta(2)}\int_{n+1}^{\infty}t^{-a}dt+a\int_{n+1}^{\infty}\frac{H(t)\log t}{t^{a+1}}dt$$
$$-\frac{n\log n}{\zeta(2)(n+1)^a}+\frac{\Big(1+\frac{\zeta'(2)}{\zeta(2)}\Big)n}{\zeta(2)(n+1)^a}-\frac{H(n)\log n}{(n+1)^a}+O_\varepsilon\Big(\frac{\log n(\log\log n)^{1+\varepsilon}}{n^a}\Big).$$

Since

$$\int_{n+1}^{\infty}\frac{\log t}{t^a}dt=-\frac{1}{1-a}\frac{\log(n+1)}{(n+1)^{a-1}}+\frac{1}{(1-a)^2(n+1)^{a-1}},$$
$$\int_{n+1}^{\infty}\frac{dt}{t^a}=\frac{1}{(a-1)(n+1)^{a-1}},\ \log(n+1)=\log n+O\Big(\frac{1}{n}\Big)$$

and $(n+1)^{-\alpha}=n^{-\alpha}+O_\alpha(n^{-\alpha-1})$ for any $\alpha\in\boldsymbol{R}$, using (32) and (18), we obtain

$$S_5^{(a)}=\frac{1}{(a-1)\zeta(2)n^{a-1}}\Big\{\log n+\frac{1}{a-1}-\frac{\zeta'(2)}{\zeta(2)}\Big\}+\frac{U(n)\log n}{n^a}+O_\varepsilon\Big(\frac{\log n(\log\log n)^{1+\varepsilon}}{n^a}\Big). \tag{46}$$

Also, we have, by (24)

$$\sum_{r\leq n}\frac{\phi(r)}{r^{a+1}}=\sum_{d\leq n}\frac{\mu(d)}{d^{a+1}}\Big(\sum_{\delta\leq n/d}\delta^{-a}\Big)$$
$$=\sum_{d\leq n}\frac{\mu(d)}{d^{a+1}}\Big\{\zeta(a)-\frac{1}{a-1}\Big(\frac{d}{n}\Big)^{a-1}-P_1\Big(\frac{n}{d}\Big)\Big(\frac{d}{n}\Big)^a+O\Big(\Big(\frac{d}{n}\Big)^{a+1}\Big)\Big\}$$
$$=\frac{\zeta(a)}{\zeta(a+1)}-\frac{1}{(a-1)\zeta(2)n^{a-1}}-\frac{U(n)}{n^a}+O\Big(\frac{1}{n^a}\Big),$$

so that

$$S_6^{(a)}=\sum_{r=n+1}^{\infty}\frac{\phi(r)}{r^{a+1}}=\frac{1}{(a-1)\zeta(2)n^{a-1}}+\frac{U(n)}{n^a}+O\Big(\frac{1}{n^a}\Big). \tag{47}$$

Thus from (45), (46) and (47), we get for any $a>1$

(48)
$$S^{(a)}=\frac{1}{(a-1)\zeta(2)n^{a-1}}\left\{\log n+C+\frac{1}{a-1}-\frac{\zeta'(2)}{\zeta(2)}\right\}+\frac{U(n)\log n}{n^a}+O_\varepsilon\left(\frac{\log n(\log\log n)^{1+\varepsilon}}{n^a}\right).$$

By taking $a=3$ in (48), Theorem 2 follows by virtue of (43).

REMARK 3. In the course of proof of Theorem 1, in case $m=2$, in [**9**], eqn (28) should be as:

(49)
$$\sum_{d=1}^{r}\mu(d)d^{-1}P_1\left(\frac{r}{d}\right)\log d=\int_1^r U(r,\ x)x^{-1}dx+U(r)\log r,$$

and as such, eqn (14) in [**9**] could not be used; however, Lemma 9 and eqn (8) (of this paper) show that the right side of (49) is indeed $O_\varepsilon((\log r)^{1+\gamma}(\log\log r)^{1+\varepsilon})$.

REMARK 4. In a private communication Hall raised the problem of estimating $T_n=\sum_{\nu=1}^{N}(l_\nu+l_{\nu+1})^2$, where $l_\nu=\rho_\nu-\rho_{\nu-1}$, ρ_ν being the ν-th fraction appearing in the Farey series of order n, $N=\sum_{i=1}^{n}\phi(i)$ and expected $T_n\sim T(\log n)/n^2$ with $T<24/\pi^2$. The authors are, as yet, unable to obtain an asymptotic formula for T_n, but note that for all large n

$$\frac{24}{\pi^2}\leqq\left(\frac{\log n}{n^2}\right)^{-1}T_n\leqq\frac{48}{\pi^2}.$$

This readily follows from Theorem 1 above on noting that for large n

$$2\left(S_n(2)-\frac{1}{n^2}\right)\leqq T_n=2\sum_{\nu=1}^{N-1}l_\nu^2+2\sum_{\nu=1}^{N-1}l_\nu l_{\nu+1}\leqq 4S_n(2),$$

where the last inequality is a consequence of the Cauchy-Schwarz inequality.

PROOF OF THEOREM 2. Taking $f(x,\ y)=(xy)^{-3}$ in the sum formula (1), we find easily

$$S_n(3)=1-6\sum_{r=2}^{n}r^{-4}\sum_{\substack{k=1\\(k,r)=1}}^{r}k^{-2}-12\sum_{r=2}^{n}r^{-5}\sum_{\substack{k=1\\(k,r)=1}}^{r}k^{-1}.$$

Since $S_n(3)\leqq S_n(2)\rightarrow 0$ as $n\rightarrow\infty$, we obtain

(50)
$$S_n(3)=6S_7+12S^{(5)},$$

where $S_7=\sum_{r=n+1}^{\infty}r^{-4}\sum_{\substack{k=1\\(k,r)=1}}^{r}k^{-2}$ and $S^{(5)}$ is given by (45).

Writing $\phi_2(r)=\sum_{d\delta=r}\mu(d)\delta^2$ (a special case of Jordan's totient function, cf. [**4**], p. 149) and using (24), we find

$$S_7=\sum_{r=n+1}^{\infty} r^{-4}\Big(\sum_{k=1}^{r} r^{-2} \sum_{d\mid (k,r)} \mu(d)\Big)$$

$$=\sum_{r=n+1}^{\infty} r^{-4} \sum_{d\mid r} \mu(d)d^{-2}\Big(\zeta(2)-\frac{d}{r}+\frac{1}{2}\frac{d^2}{r^2}+O\Big(\frac{d^3}{r^3}\Big)\Big)$$

$$=\zeta(2)\sum_{r=n+1}^{\infty}\frac{\phi_2(r)}{r^6}-\sum_{r=n+1}^{\infty}\frac{\phi(r)}{r^6}+O(n^{-5}). \tag{51}$$

Also by (24) (with $s=4$) and (4) (with $\lambda=2$)

$$\sum_{r\leqq n}\frac{\phi_2(r)}{r^6}=\sum_{d\leqq n}\frac{\mu(d)}{d^6}\Big(\sum_{\delta\leqq n/d}\frac{1}{\delta^4}\Big)$$

$$=\sum_{d\leqq n}\frac{\mu(d)}{d^6}\Big(\zeta(4)-\frac{1}{3}\Big(\frac{d}{n}\Big)^3-P_1\Big(\frac{n}{d}\Big)\Big(\frac{d}{n}\Big)^4+O\Big(\frac{d^5}{n^5}\Big)\Big)$$

$$=\frac{\zeta(4)}{\zeta(6)}-\frac{1}{3\zeta(3)n^3}-\frac{c_2(n)}{n^4}+O\Big(\frac{\log n}{n^5}\Big),$$

so that by (47) (with $a=5$) and (51), we obtain

$$S_7=\zeta(2)\Big\{\frac{1}{3\zeta(3)n^3}+\frac{c_2(n)}{n^4}\Big\}-\frac{1}{4\zeta(2)n^4}+O\Big(\frac{\log n}{n^5}\Big)$$

$$=\frac{\zeta(2)}{3\zeta(3)n^3}+\frac{\zeta(2)c_2(n)}{n^4}-\frac{1}{4\zeta(2)n^4}+O\Big(\frac{\log n}{n^5}\Big). \tag{52}$$

Hence by (48) (with $a=5$), (50) and (52)

$$S_n(3)=6\Big\{\frac{\zeta(2)}{3\zeta(3)n^3}+\frac{\zeta(2)c_2(n)}{n^4}-\frac{1}{4\zeta(2)n^4}+O\Big(\frac{\log n}{n^5}\Big)\Big\}$$

$$+12\Big\{\frac{1}{4\zeta(2)n^4}\Big(\log n+C+\frac{1}{4}-\frac{\zeta'(2)}{\zeta(2)}\Big)+\frac{U(n)\log n}{n^5}+O_\varepsilon\Big(\frac{\log n(\log\log n)^{1+\varepsilon}}{n^5}\Big)\Big\}$$

$$=\frac{2\zeta(2)}{\zeta(3)}n^{-3}+\frac{3n^{-4}}{\zeta(2)}\Big\{\log n+C-\frac{1}{4}-\frac{\zeta'(2)}{\zeta(2)}+2\zeta^2(2)c_2(n)\Big\}$$

$$+\frac{12U(n)\log n}{n^5}+O_\varepsilon\Big(\frac{\log n(\log\log n)^{1+\varepsilon}}{n^5}\Big).$$

This completes the proof of Theorem 2.

REMARK 5. The sequence $\{c_\lambda(n)\}_{n=1}^{\infty}$ appearing in Theorem 2 above clearly satisfies: For fixed $\lambda>1$, $|c_\lambda(n)|\leqq(1/2)\zeta(\lambda)$. However, it is of interest to note that $c_\lambda(n)\neq o(1)$ as $n\to\infty$ which readily follows from

$$\liminf_{n\to\infty} c_\lambda(n)\leqq-\frac{1}{2\zeta(\lambda)}. \tag{53}$$

To see this, let u denote a square-free integer in what follows; p_r the r-th prime and $n_r=p_1\cdots p_r$ for $r\in \boldsymbol{N}$. Then to each $v\in \boldsymbol{N}$, there exists an $r_0=r_0(v)$ such that $n_r\equiv 0 \pmod u$ for all $u\leqq v$, and so we have

$$c_\lambda(n_r)=\sum_{u\leq v}\frac{\mu(u)}{u^\lambda}P_1\left(\frac{n_r}{u}\right)+O(v^{1-\lambda})$$

$$=-\frac{1}{2}\sum_{u\leq v}\frac{\mu(u)}{u^\lambda}+O(v^{1-\lambda})$$

$$\longrightarrow -\frac{1}{2\zeta(\lambda)}$$

as $v\to\infty$. This proves (53).

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