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Some examples of pseudofree S^1-actions on homotopy $(4m+1)$-spheres

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1. Introduction.

Let S^1 be the circle group. In [**15**] and [**16**], Montgomery and Yang introduced the notion of pseudofree S^1-action (see § 2) and classified all pseudofree S^1-actions on homotopy seven-spheres. Recently, Petrie constructed many pseudofree S^1-actions on homotopy $(4m+3)$-spheres with different isotropy groups and slice representations ([**18**], [**19**]).

In this paper, we construct infinitely many pseudofree S^1-actions on homotopy $(4m+1)$-spheres with the following properties: (i) they are S^1-homotopy equivalent to some fixed linear pseudofree S^1-action φ on S^{4m+1}, (ii) their isotropy groups and slice representations coincide with those of (S^{4m+1}, φ), (iii) their equivariant Pontrjagin classes of the tangent bundles are different from one another.

The method of our construction is due to Petrie [**18**], [**19**] and Hsiang [**7**].

The paper is organized as follows:

In § 2, we state our main theorem precisely. In § 3, we prove a preliminary lemma. In §§ 4 and 5, we consider a quasi-equivalence and S^1-transversality respectively. In § 6, we construct an S^1-normal map. In § 7, we consider a signature of an orbit manifold, which is an obstruction to performing equivariant surgery. In § 8, we prove the main theorem.

2. Notations and the main theorem.

In [**15**] and [**16**], a differentiable action of the circle group S^1 on a compact smooth manifold is said to be *pseudofree* if it is an effective action for which every isotropy group is finite and the set of exceptional orbits is finite but not void. Let M be a compact pseudofree S^1-manifold. Let $S^1/\boldsymbol{Z}_n$ be a singular orbit of M and let V_x be the slice representation space of the isotropy group $\boldsymbol{Z}_n$ at $x \in S^1/\boldsymbol{Z}_n$, where $\boldsymbol{Z}_n$ denotes the cyclic group of order n. We remark that the equivalent class $\{V_x\}$ is independent of the choice of $x \in S^1/\boldsymbol{Z}_n$. So we can define an invariant $I(M)$ of M by

$$I(M)=\{(S^1/\boldsymbol{Z}_n, \{V_x\}) \mid S^1/\boldsymbol{Z}_n: \text{a singular orbit of } M, x\in S^1/\boldsymbol{Z}_n\}.$$

Let S^1 be the circle group consisting of complex numbers of absolute value one. For a sequence $p=(p_1, p_2, \cdots, p_m)$ of positive integers, we define the S^1-action φ_p on the complex m-dimensional vector space $\boldsymbol{C}^m$ by

$$\varphi_p(s, (z_1, z_2, \cdots, z_m))=(s^{p_1}z_1, s^{p_2}z_2, \cdots, s^{p_m}z_m).$$

Denote by $S^{2m-1}(p_1, p_2, \cdots, p_m)$ the unit sphere in $\boldsymbol{C}^m$ with this action φ_p. Here we remark that the S^1-action on $S^{2m-1}(p_1, p_2, \cdots, p_m)$ is pseudofree (resp. free) if $(p_i, p_j)=1$ for $i\neq j$ and $p_i>1$ for some $1\leqq i\leqq m$ (resp. $p_1=p_2=\cdots=p_m=1$).

Denote by G_i ($i\geqq 1$) the stable homotopy group $\pi_{n+i}(S^n)$ ($n\geqq i+2$). We put $s(k)=\prod_{i=1}^{k}|G_i|$ where $|G_i|$ denotes the order of the group G_i.

Let ES^1 be a universal S^1-space.

In this paper, we shall show the following theorem:

MAIN THEOREM. *Let $m\geqq 3$ be an integer and let $p_1, p_2, \cdots, p_{2m+1}$ be positive odd integers such that $(p_i, p_j)=1$ for $i\neq j$, $p_i>1$ for some $1\leqq i\leqq 2m+1$ and $(p_i, s(4m-1))=1$ for $1\leqq i\leqq 2m+1$. Then there are infinitely many closed pseudo-free S^1-manifolds Σ with the following properties:*

(i) *Σ is S^1-homotopy equivalent to $S^{4m+1}(p_1, p_2, \cdots, p_{2m+1})$,*

(ii) *$I(\Sigma)=I(S^{4m+1}(p_1, p_2, \cdots, p_{2m+1}))$,*

(iii) *the total Pontrjagin classes*

$$p(ES^1\times_{S^1}T\Sigma)\in H^*(ES^1\times_{S^1}\Sigma;\boldsymbol{Z})$$
$$\cong H^*(ES^1\times_{S^1}S^{4m+1}(p_1, p_2, \cdots, p_{2m+1});\boldsymbol{Z})$$

are different from one another, where $T\Sigma$ denotes the tangent bundle of Σ.

REMARK 2.1. By Lemma 4.8 of Kakutani [**9**], we have

$$H^*(ES^1\times_{S^1}\Sigma;\boldsymbol{Z})\cong H^*(ES^1\times_{S^1}S^{4m+1}(p_1, p_2, \cdots, p_{2m+1});\boldsymbol{Z})$$
$$=\boldsymbol{Z}[c]/(qc^{2m+1}),$$

where $q=\prod_{i=1}^{2m+1}p_i$ and $\deg c=2$. It follows from the properties (i) and (ii) of the main theorem that

$$p(ES^1\times_{S^1}T\Sigma)\equiv\prod_{i=1}^{2m+1}(1+p_i^2c^2) \qquad \text{mod } q.$$

REMARK 2.2. When $p_1=p_2=\cdots=p_{2m+1}=1$, similar result has been obtained in Hsiang [7].

3. A preliminary lemma.

Let n be a positive integer. Let V and W be orthogonal representation spaces of $\boldsymbol{Z}_n$ with $\dim_{\boldsymbol{R}} V=\dim_{\boldsymbol{R}} W=k$ and let $S(V)$ (resp. $D(V)$) denote the unit sphere (resp. the unit disk) in V. Let

$$\rho_V,\ \rho_W : \boldsymbol{Z}_n \longrightarrow O(k)$$

be the representations of $\boldsymbol{Z}_n$ afforded by V, W respectively. Then a $\boldsymbol{Z}_n$-action on $O(k)$ is given by

$$s\circ A=\rho_W(s)A\rho_V(s)^{-1} \quad \text{for} \quad s\in\boldsymbol{Z}_n,\ A\in O(k).$$

We denote by $O(V, W)$ this $\boldsymbol{Z}_n$-space.

Let $F(S^{k-1}, S^{k-1})$ denote the space of homotopy equivalences of S^{k-1} with the compact-open topology. It is well-known that $F(S^{k-1}, S^{k-1})$ has two connected components $F^+(S^{k-1}, S^{k-1})$ and $F^-(S^{k-1}, S^{k-1})$ representing maps of degree $+1$ and -1 respectively. A $\boldsymbol{Z}_n$-action on $F(S^{k-1}, S^{k-1})$ is given by

$$(s\circ f)(v)=\rho_W(s)f(\rho_V(s)^{-1}v) \quad \text{for} \quad s\in\boldsymbol{Z}_n,\ f\in F(S^{k-1}, S^{k-1}),\ v\in S^{k-1}.$$

Denote by $F(S(V), S(W))$ this $\boldsymbol{Z}_n$-space.

If n is odd, then we have

$$\rho_V(\boldsymbol{Z}_n),\ \rho_W(\boldsymbol{Z}_n)\subset SO(k).$$

Therefore $SO(k)$ (resp. $F^+(S^{k-1}, S^{k-1})$) is a $\boldsymbol{Z}_n$-subspace of $O(V, W)$ (resp. $F(S(V), S(W))$). Denote by $SO(V, W)$ (resp. $F^+(S(V), S(W))$) this $\boldsymbol{Z}_n$-subspace of $O(V, W)$ (resp. $F(S(V), S(W))$). Remark that we have the natural inclusion

$$SO(V, W)\subset F^+(S(V), S(W)).$$

Let $[Y, Z]$ denote the homotopy classes of maps of Y to Z.

LEMMA 3.1. *Let n be a positive odd integer and let U be a complex $\boldsymbol{Z}_n$-representation space such that $\boldsymbol{Z}_n$ acts freely on $S(U)$ and $\dim_{\boldsymbol{R}} U=2m$. Let V and W be orthogonal $\boldsymbol{Z}_n$-representation spaces such that $\dim_{\boldsymbol{R}} V=\dim_{\boldsymbol{R}} W=k\geqq 2m+2$. Assume that*

(i) $(n, s(2m-1))=1$,

(ii) *there is a $\boldsymbol{Z}_n$-map*

$$f: S(U) \longrightarrow F^+(S(V), S(W))$$

such that $[f]=0\in[S^{2m-1}, F^+(S(V), S(W))]$ $(\cong\pi_{2m-1}(F^+(S(V), S(W)))$,

(iii) $SO(V, W)^{\boldsymbol{Z}_n}\neq\emptyset$.

Let φ be an arbitrary element of $SO(V, W)^{\boldsymbol{Z}_n}$. Then there is a $\boldsymbol{Z}_n$-map

$$F: D(U) \longrightarrow F^+(S(V), S(W))$$

such that $F|S(U)=f$ *and* $F(B(U))=\{\varphi\}$, *where* $B(U)=\{v\in D(U)\,|\,\|v\|\leqq 1/2\}$.

Proof. Define a constant $\boldsymbol{Z}_n$-map $f' : S(U)\to F^+(S(V), S(W))$ by $f'(v)=\varphi$ for all $v\in S(U)$. Then we have

$$[f]=[f']=0\in[S^{2m-1}, F^+(S(V), S(W))].$$

It follows from Kakutani [**9**; Theorem 2.1, Proposition 3.4] that there exists a $\boldsymbol{Z}_n$-map

$$F' : S(U)\times[0, 1] \longrightarrow F^+(S(V), S(W))$$

such that $F'|S(U)\times\{0\}=f'$ and $F'|S(U)\times\{1\}=f$. We define a $\boldsymbol{Z}_n$-map

$$F : D(U) \longrightarrow F^+(S(V), S(W))$$

by putting

$$F(v)=\begin{cases} F'(v/\|v\|, 2\|v\|-1) & \text{if } 1/2\leqq\|v\|, \\ \varphi \quad \text{if } \|v\|\leqq 1/2. \end{cases}$$

It is clear that this map F is well-defined and satisfies the required properties.

Q. E. D.

4. A nice quasi-equivalence.

Let G be a compact Lie group and let X be a compact G-space. Let ξ and η be real G-vector bundles of the same dimension over X. In [**13**] and [**18**], a G-map $\omega : \xi\to\eta$ which is proper, fiber-preserving and degree one on fibers is called *a quasi-equivalence*. Let $\alpha=\eta-\xi\in KO_G(X)$ and define $\alpha\geqq 0$ to mean there exist a G-vector bundle θ over X and a quasi-equivalence $\omega : \xi\oplus\theta\to\eta\oplus\theta$. Denote by $S(\xi)$ (resp. $S(\eta)$) the sphere bundle associated with ξ (resp. η) with respect to some G-invariant metric. A G-map $\omega : S(\xi)\to S(\eta)$ which is fiber-preserving and degree one on fibers is said to be *a quasi-equivalence of G-sphere bundles.* Let V be a G-representation space. We denote by $\underline{V}$ the G-vector bundle

$$V \longrightarrow X\times V \longrightarrow X.$$

Let M be an oriented closed smooth $(2m+1)$-dimensional manifold with a pseudofree S^1-action. Then there are only finite singular orbits, say

$$S^1/\boldsymbol{Z}_{p_1}, S^1/\boldsymbol{Z}_{p_2}, \cdots, S^1/\boldsymbol{Z}_{p_k}\subset M.$$

Let ν_i be the normal bundle of $S^1/\boldsymbol{Z}_{p_i}$ in M and let N_i be an open invariant tubular neighborhood of $S^1/\boldsymbol{Z}_{p_i}$ in M mutually disjoint for $1\leqq i\leqq k$. By the differentiable slice theorem (see Bredon [**5**; VI. §2]), there is an S^1-diffeomorphism $\phi_i : \nu_i\to N_i$ such that $\phi_i|S^1/\boldsymbol{Z}_{p_i}$ is the inclusion of $S^1/\boldsymbol{Z}_{p_i}$ in M. We often identify ν_i with N_i by this S^1-diffeomorphism ϕ_i. The unit disk bundle $D(\nu_i)$ is also identified with a closed invariant tubular neighborhood of $S^1/\boldsymbol{Z}_{p_i}$

in M. Let $x_i \in S^1/\boldsymbol{Z}_{p_i}$ for $1 \leqq i \leqq k$. Denote by U_i the fiber of ν_i over x_i. Then U_i is regarded as an orthogonal $\boldsymbol{Z}_{p_i}$-representation space. Since M is a $(2m+1)$-dimensional pseudofree S^1-manifold, $\boldsymbol{Z}_{p_i}$ acts freely on $S(U_i)$ and $\dim_{\boldsymbol{R}} U_i = 2m$. Moreover if p_i is odd, then U_i has a canonical complex structure (see Atiyah-Singer [**4**; §3]). The canonical map

$$h_i : S^1 \times_{\boldsymbol{Z}_{p_i}} U_i \longrightarrow \nu_i$$

defined by $h_i([s, v]) = sv$ is an isomorphism of smooth orthogonal S^1-vector bundles.

Let $r > 0$. We put

$$\begin{cases} D(U_i\,;\, r) = \{v \in U_i \mid \|v\| \leqq r\}, \\ S(U_i\,;\, r) = \{v \in U_i \mid \|v\| = r\}, \end{cases}$$

and

$$\begin{cases} D(\nu_i\,;\, r) = h_i(S^1 \times_{\boldsymbol{Z}_{p_i}} D(U_i\,;\, r)), \\ S(\nu_i\,;\, r) = h_i(S^1 \times_{\boldsymbol{Z}_{p_i}} S(U_i\,;\, r)). \end{cases}$$

Here we remark that $D(\nu_i) = D(\nu_i\,;\, 1)$ and $S(\nu_i) = S(\nu_i\,;\, 1)$.

DEFINITION 4.1. Let ξ and η be oriented smooth S^1-vector bundles over M. A quasi-equivalence $\omega : \xi \to \eta$ is said to be a *nice* quasi-equivalence if ω is an orientation-preserving map and there is $r > 0$ such that the restrictions

$$\omega \mid \{\xi \mid D(\nu_i\,;\, r)\} : \xi \mid D(\nu_i\,;\, r) \longrightarrow \eta \mid D(\nu_i\,;\, r) \qquad \text{for} \quad 1 \leqq i \leqq k$$

are isomorphisms of smooth S^1-vector bundles.

PROPOSITION 4.2. *Assume that p_i $(1 \leqq i \leqq k)$ are positive odd integers with $(p_i, s(2m-1)) = 1$ for $1 \leqq i \leqq k$. Let V and W be complex S^1-representation spaces with the following properties:*

(i) $\dim_{\boldsymbol{R}} V = \dim_{\boldsymbol{R}} W \geqq 2m$,

(ii) *V and W are equivalent as orthogonal $\boldsymbol{Z}_{p_i}$-representation spaces for $1 \leqq i \leqq k$,*

(iii) *there is a quasi-equivalence*

$$\omega : M \times V \longrightarrow M \times W .$$

Then there is a nice quasi-equivalence

$$\tilde{\omega} : M \times V \oplus \boldsymbol{C}^1 \longrightarrow M \times W \oplus \boldsymbol{C}^1 .$$

PROOF. It is easy to see that ω induces an orientation-preserving quasi-equivalence of S^1-sphere bundles

$$\theta : M \times S(V \oplus \boldsymbol{C}^1) \longrightarrow M \times S(W \oplus \boldsymbol{C}^1) .$$

Remark that $D(U_i; 2)$ $(\subset D(\nu_i; 2) \subset M)$ is a $\boldsymbol{Z}_{p_i}$-invariant subspace of M. We consider the restriction

$$\theta_i = \theta \,|\, D(U_i; 2) \times S(V \oplus \boldsymbol{C}^1) : D(U_i; 2) \times S(V \oplus \boldsymbol{C}^1) \longrightarrow D(U_i; 2) \times S(W \oplus \boldsymbol{C}^1).$$

Then θ_i is an orientation-preserving quasi-equivalence of $\boldsymbol{Z}_{p_i}$-sphere bundles over $D(U_i; 2)$. The map θ_i yields the following $\boldsymbol{Z}_{p_i}$-map

$$\bar{\theta}_i : D(U_i; 2) \longrightarrow F^+(S(V \oplus \boldsymbol{C}^1), S(W \oplus \boldsymbol{C}^1))$$

by putting $\bar{\theta}_i(d)(v) = \pi(\theta_i(d, v))$ for $d \in D(U_i; 2)$, $v \in S(V \oplus \boldsymbol{C}^1)$, where $\pi : D(U_i; 2) \times S(W \oplus \boldsymbol{C}^1) \to S(W \oplus \boldsymbol{C}^1)$ denotes the projection on the second factor. On the other hand, by the assumption (ii), we have

$$SO(V \oplus \boldsymbol{C}^1, W \oplus \boldsymbol{C}^1)^{\boldsymbol{Z}_{p_i}} \neq \emptyset.$$

Let $\varphi_i \in SO(V \oplus \boldsymbol{C}^1, W \oplus \boldsymbol{C}^1)^{\boldsymbol{Z}_{p_i}}$. It follows from Lemma 3.1 that there exists a $\boldsymbol{Z}_{p_i}$-map

$$\bar{\omega}_i : D(U_i; 2) \longrightarrow F^+(S(V \oplus \boldsymbol{C}^1), S(W \oplus \boldsymbol{C}^1))$$

such that $\bar{\omega}_i \,|\, S(U_i; 2) = \bar{\theta}_i \,|\, S(U_i; 2)$ and $\bar{\omega}_i(D(U_i; 1)) = \{\varphi_i\}$.

We define

$$\omega_i : D(U_i; 2) \times S(V \oplus \boldsymbol{C}^1) \longrightarrow D(U_i; 2) \times S(W \oplus \boldsymbol{C}^1)$$

by putting $\omega_i(d, v) = (d, \bar{\omega}_i(d)(v))$ for $d \in D(U_i; 2)$, $v \in S(V \oplus \boldsymbol{C}^1)$. Then the map ω_i is an orientation-preserving quasi-equivalence of $\boldsymbol{Z}_{p_i}$-sphere bundles such that

$$\omega_i \,|\, S(U_i; 2) \times S(V \oplus \boldsymbol{C}^1) = \theta_i \,|\, S(U_i; 2) \times S(V \oplus \boldsymbol{C}^1)$$

and

$$\omega_i \,|\, D(U_i; 1) \times S(V \oplus \boldsymbol{C}^1) = \mathrm{id} \times \varphi_i.$$

Furthermore, we define

$$\omega_i' : D(\nu_i; 2) \times S(V \oplus \boldsymbol{C}^1) \longrightarrow D(\nu_i; 2) \times S(W \oplus \boldsymbol{C}^1)$$

by putting $\omega_i'(d, v) = s\omega_i(s^{-1}d, s^{-1}v)$ where $s \in S^1$ is chosen as $s^{-1}d \in D(U_i; 2)$. It is easy to see that ω_i' is a well-defined S^1-map and satisfies the following:

(a) ω_i' is an orientation-preserving quasi-equivalence of S^1-sphere bundles over $D(\nu_i; 2)$,

(b) $\omega_i' \,|\, S(\nu_i; 2) \times S(V \oplus \boldsymbol{C}^1) = \theta \,|\, S(\nu_i; 2) \times S(V \oplus \boldsymbol{C}^1)$,

(c) the restriction $\omega_i' \,|\, D(\nu_i; 1) \times S(V \oplus \boldsymbol{C}^1)$ is an isomorphism of smooth orthogonal S^1-sphere bundles.

Then we can define

$$\omega' : M \times S(V \oplus \boldsymbol{C}^1) \longrightarrow M \times S(W \oplus \boldsymbol{C}^1)$$

by

$$\omega'(x)=\begin{cases} \theta(x) & \text{if} \quad x\notin \coprod_{i=1}^{k} D(\nu_i\,;2)\times S(V\oplus \boldsymbol{C}^1)\,, \\ \omega_i'(x) & \text{if} \quad x\in D(\nu_i\,;2)\times S(V\oplus \boldsymbol{C}^1) \quad \text{for } 1\leqq i\leqq k. \end{cases}$$

Let $p: M\times S(W\oplus \boldsymbol{C}^1)\to S(W\oplus \boldsymbol{C}^1)$ be the projection on the second factor. We define

$$\tilde{\omega}: M\times V\oplus \boldsymbol{C}^1 \longrightarrow M\times W\oplus \boldsymbol{C}^1$$

by

$$\tilde{\omega}(y,v)=\begin{cases} (y,\ \|v\|p(\omega'(y,\ v/\|v\|))) & \text{if } v\neq 0\,, \\ (y,0) & \text{if } v=0\,, \end{cases}$$

for $y\in M$, $v\in V\oplus \boldsymbol{C}^1$. It is clear that $\tilde{\omega}$ is a well-defined nice quasi-equivalence.

Q. E. D.

5. S^1-transversality.

Let M be as in §4. Let V and W be complex S^1-representation spaces. We assume that there is a nice quasi-equivalence

$$\omega_0: M\times V \longrightarrow M\times W\,.$$

By definition, there is $r>0$ such that the restrictions $\omega_0|\{D(\nu_i\,;r)\times V\}$ $(1\leqq i\leqq k)$ are isomorphisms of smooth S^1-vector bundles. Let r_i $(1\leqq i\leqq 3)$ be real numbers such that $0<r_1<r_2<r_3<r$.

LEMMA 5.1. *There is a smooth proper S^1-map*

$$\omega_1: M\times V \longrightarrow M\times W$$

with the following properties:

(i) $\omega_1|\{D(\nu_i\,;r_3)\times V\}=\omega_0|\{D(\nu_i\,;r_3)\times V\}$ *for* $1\leqq i\leqq k$,

(ii) ω_1 *and* ω_0 *are properly S^1-homotopic* rel $\coprod_{i=1}^{k} D(\nu_i\,;r_3)\times V$,

(iii) $\omega_1^{-1}(D(\nu_i\,;r_1)\times W)=D(\nu_i\,;r_1)\times V$ *for* $1\leqq i\leqq k$.

PROOF. The proof is an easy generalization of Wasserman [**22**; Corollary 1.12]. So we omit it.

PROPOSITION 5.2. *There is a smooth proper S^1-map*

$$\omega_2: M\times V \longrightarrow M\times W$$

with the following properties:

(i) $\omega_2|\{D(\nu_i\,;r_2)\times V\}=\omega_0|\{D(\nu_i\,;r_2)\times V\}$ *for* $1\leqq i\leqq k$,

(ii) ω_2 *and* ω_0 *are properly S^1-homotopic* rel $\coprod_{i=1}^{k} D(\nu_i\,;r_2)\times V$,

(iii) $\omega_2^{-1}(D(\nu_i\,;r_1)\times W)=D(\nu_i\,;r_1)\times V$ *for* $1\leqq i\leqq k$,

(iv) ω_2 *is transverse to the zero-section* $M\times\{0\}\subset M\times W$.

PROOF. Let ω_1 be as in Lemma 5.1. Since M is compact and ω_1 is proper, there exists $n>2$ such that

$$\omega_1^{-1}(M\times\{0\})\subset M\times D(V\,;n-2)\,.$$

We put

$$C=M\times\{V-\mathrm{Int}\,D(V\,;n-1)\}\cup\bigcup_{i=1}^{k}\{D(\nu_i\,;r_2)\times V\}\,.$$

Then C is a closed invariant subset of $M\times V$ and ω_1 is transverse to $M\times\{0\}$ along C. We also put

$$U=M\times\{V-D(V\,;n-2)\}\cup\bigcup_{i=1}^{k}\{\mathrm{Int}\,D(\nu_i\,;r_3)\times V\}\,.$$

Then U is an open invariant subset of $M\times V$ such that $C\subset U$ and ω_1 is transverse to $M\times\{0\}$ along U. Since the S^1-action on $M\times V-C$ is free, it follows from Thom Transversality Theorem (see Milnor [**14**; 1.35]) and the differentiable slice theorem that there exists an S^1-map

$$\omega_2: M\times V\longrightarrow M\times W$$

such that

(a) ω_2 is transverse to the zero-section $M\times\{0\}$,

(b) $\omega_2|C=\omega_1|C$,

(c) ω_2 and ω_1 are S^1-homotopic rel C,

(d) $\omega_2^{-1}(D(\nu_i\,;r_1)\times W)=D(\nu_i\,;r_1)\times V$ for $1\leqq i\leqq k$,

(cf. Petrie [**19**; Chapter II. §1], Lee-Wasserman [**11**; Proposition 2.2]). Since ω_1 satisfies the properties (i), (ii), (iii) of Lemma 5.1, ω_2 has our required properties. Q. E. D.

6. An S^1-normal map.

Let M be as in §4. In this section, we assume that p_i $(1\leqq i\leqq k)$ are positive odd integers such that $(p_i,\,s(2m-1))=1$ for $1\leqq i\leqq k$. Let V and W be complex S^1-representation spaces with the following properties:

(6.1) $\dim_R V=\dim_R W\geqq 2m$,

(6.2) V and W are equivalent as orthogonal $\boldsymbol{Z}_{p_i}$-representation spaces for $1\leqq i\leqq k$,

(6.3) there is a quasi-equivalence

$$\omega: M\times V\longrightarrow M\times W\,.$$

By Propositions 4.3 and 5.2, there are $r>0$ and a smooth proper S^1-map

$$\theta: M\times V\oplus \boldsymbol{C}^1 \longrightarrow M\times W\oplus \boldsymbol{C}^1$$

such that

(6.4) $\theta^{-1}(D(\nu_i; r)\times W\oplus \boldsymbol{C}^1)=D(\nu_i; r)\times V\oplus \boldsymbol{C}^1$ for $1\leqq i\leqq k$,

(6.5) the restrictions $\theta|\{D(\nu_i; r)\times V\oplus \boldsymbol{C}^1\}: D(\nu_i; r)\times V\oplus \boldsymbol{C}^1\to D(\nu_i; r)\times W\oplus \boldsymbol{C}^1$ $(1\leqq i\leqq k)$ are isomorphisms of smooth S^1-vector bundles,

(6.6) θ is properly S^1-homotopic to some nice quasi-equivalence,

(6.7) θ is transverse to the zero-section $M\times\{0\}\subset M\times W\oplus \boldsymbol{C}^1$.

We set

$$\begin{cases} X=\theta^{-1}(M\times\{0\})\subset M\times V\oplus \boldsymbol{C}^1, \\ \tilde{\theta}=\theta|X: X\longrightarrow M, \end{cases}$$

and

$$N=\coprod_{i=1}^{k} D(\nu_i; r)\subset M.$$

Then, by (6.6) and (6.7), X is a compact smooth $(2m+1)$-dimensional S^1-manifold and $\tilde{\theta}$ is a smooth S^1-map. It follows from (6.4) and (6.5) that

$$N\subset X\cap M\times\{0\}\subset M\times V\oplus \boldsymbol{C}^1,$$

and

$$\tilde{\theta}^{-1}(N)=N.$$

This shows that the S^1-action on X is pseudofree and $I(X)=I(M)$ (see §2). Moreover $\tilde{\theta}|N: N\to N\subset M$ is the identity. Let $X=\coprod_{i=1}^{t} X_i$ and each X_i is connected. Then, by (6.6), each X_i has a canonical orientation. Define an S^1-map

$$\theta_i=\tilde{\theta}|X_i: X_i\longrightarrow M.$$

By (6.6), we have

LEMMA 6.8. $\sum_{i=1}^{t} \deg\theta_i=+1$.

If $X_i\not\supset N$, then θ_i is not surjective. Hence $\deg\theta_i=0$. Therefore it follows from Lemma 6.8 that there exists a unique i_0 $(1\leqq i_0\leqq t)$ such that $X_{i_0}\supset N$ and $\deg\theta_{i_0}=+1$. Moreover it follows from Petrie [**19**; Chapter II. §7] that

$$TX_{i_0}=\theta_{i_0}^*(TM-\underline{\underline{W}}+\underline{\underline{V}}) \quad \text{in } KO_{S^1}(X_{i_0}).$$

Thus we have proved

PROPOSITION 6.9. *Let V and W be complex S^1-representation spaces which satisfy the properties* (6.1), (6.2) *and* (6.3). *Then there are an oriented closed* $(2m+1)$-*dimensional psendofree S^1-manifold X_0 and a smooth S^1-map $\theta_0: X_0\to M$ with the following properties:*

(i) $\deg \theta_0 = +1$,

(ii) $I(X_0) = I(M)$,

(iii) $TX_0 = \theta_0^*(TM - \underline{\underline{W}} + \underline{V})$ *in* $KO_{S^1}(X_0)$.

REMARK 6.10. In [**19**], the map $\theta_0 : X_0 \to M$ is called an S^1-normal map.

7. A signature of an orbit manifold.

First we require some notations and definitions. Let Y be an oriented closed $(4m+1)$-dimensional pseudofree S^1-manifold $(m>0)$. Denote by N a closed invariant tubular neighborhood of the singular orbits in Y. We put $Z = Y - \operatorname{Int} N$. Let $\pi : (Z, \partial Z) \to (Z/S^1, \partial Z/S^1)$ be the natural projection. Since the S^1-action on Z is free, π is a projection of a principal bundle with group S^1. Thus we can consider the Gysin homomorphism (see Ozeki-Uchida [**17**; §4]):

$$\pi_! : H^{4m+1}(Z, \partial Z ; \boldsymbol{Z}) \longrightarrow H^{4m}(Z/S^1, \partial Z/S^1 ; \boldsymbol{Z}) .$$

It is easy to see that $\pi_!$ is an isomorphism. Let $i_1 : (Y, \phi) \to (Y, N)$, $i_2 : (Z, \partial Z) \to (Y, N)$, $j_1 : (Z/S^1, \partial Z/S^1) \to (Y/S^1, N/S^1)$ and $j_2 : (Y/S^1, \phi) \to (Y/S^1, N/S^1)$ be the natural inclusions. Then they induce the following isomorphisms:

$$i_1^* : H^{4m+1}(Y, N ; \boldsymbol{Z}) \longrightarrow H^{4m+1}(Y ; \boldsymbol{Z}) ,$$

$$i_2^* : H^{4m+1}(Y, N ; \boldsymbol{Z}) \longrightarrow H^{4m+1}(Z, \partial Z ; \boldsymbol{Z}) ,$$

$$j_1^* : H^{4m}(Y/S^1, N/S^1 ; \boldsymbol{Z}) \longrightarrow H^{4m}(Z/S^1, \partial Z/S^1 ; \boldsymbol{Z}) ,$$

$$j_2^* : H^{4m}(Y/S^1, N/S^1 ; \boldsymbol{Z}) \longrightarrow H^{4m}(Y/S^1 ; \boldsymbol{Z}) .$$

Now we define an isomorphism

$$\Phi : H^{4m+1}(Y ; \boldsymbol{Z}) \longrightarrow H^{4m}(Y/S^1 ; \boldsymbol{Z})$$

by putting $\Phi = j_2^* \circ (j_1^*)^{-1} \circ \pi_! \circ i_2^* \circ (i_1^*)^{-1}$. It is easy to see that Φ is independent of the choice of N. It is well-known that Y/S^1 is an orientable rational homology manifold. Let $[Y] \in H_{4m+1}(Y ; \boldsymbol{Z})$ be the fundamental homology class. Then we define a fundamental homology class $[Y/S^1] \in H_{4m}(Y/S^1 ; \boldsymbol{Z})$ by $\langle \Phi(\omega), [Y/S^1] \rangle = 1$ where $\omega \in H^{4m+1}(Y ; \boldsymbol{Z})$ is chosen as $\langle \omega, [Y] \rangle = 1$. Then the cap-product $\gamma \mapsto \gamma \cap [Y/S^1]$ gives an isomorphism $H^q(Y/S^1 ; \boldsymbol{Q}) \to H_{4m-q}(Y/S^1 ; \boldsymbol{Q})$. It follows that the cup-product defines a non-degenerate quadratic form on $H^{2m}(Y/S^1 ; \boldsymbol{Q})$; the signature of Y/S^1 is by definition the signature of this quadratic form and is denoted by $\operatorname{Sign}(Y/S^1)$.

Let ξ be an S^1-vector bundle over Y. The natural projection $\varphi : ES^1 \times_{S^1} Y \to Y/S^1$ induces an isomorphism

$$\varphi^* : H^*(Y/S^1 ; \boldsymbol{Q}) \longrightarrow H^*(ES^1 \times_{S^1} Y ; \boldsymbol{Q}) ,$$

(see Bredon [**5**; p. 372]). Then we define

$$p_i(\xi/S^1)\in H^{4i}(Y/S^1;\boldsymbol{Q}) \qquad \text{for} \quad 1\leqq i\leqq[(\dim\xi)/2]$$

by

$$\varphi^*(p_i(\xi/S^1))=p_i(ES^1\underset{S^1}{\times}\xi)\in H^{4i}(ES^1\underset{S^1}{\times}Y;\boldsymbol{Q}),$$

where $p_i(ES^1\underset{S^1}{\times}\xi)$ denotes the i-th Pontrjagin class of the vector bundle $ES^1\underset{S^1}{\times}\xi \to ES^1\underset{S^1}{\times}Y$. We also define

$$p(\xi/S^1)=1+p_1(\xi/S^1)+\cdots+p_k(\xi/S^1)\in H^*(Y/S^1;\boldsymbol{Q}),$$

where $k=[(\dim\xi)/2]$.

REMARK 7.1. Since Y/S^1 is an oriented rational homology manifold, we can define the rational Pontrjagin class in the sense of Thom: $p(Y/S^1)\in H^*(Y/S^1;\boldsymbol{Q})$. But $p(Y/S^1)$ does not coincide with $p(TY/S^1)$ in general.

Let M, X_0, θ_0, V and W be as in Proposition 6.9. In the following, we assume that

$$M=S^{4m+1}(p_1, p_2, \cdots, p_{2m+1}),$$

where p_i $(1\leqq i\leqq 2m+1)$ are positive odd integers such that $p_i>1$ for some $1\leqq i\leqq 2m+1$, $(p_i, p_j)=1$ for $i\neq j$ and $(p_i, s(4m-1))=1$ for all $1\leqq i\leqq 2m+1$. We also assume that $m\geqq 2$.

By Petrie [**19**; Chapter II, Theorem 11.1] and Iberkleid [8; Corollary 3.5], we have

PROPOSITION 7.2. *If* $\mathrm{Sign}(X_0/S^1)=\mathrm{Sign}(M/S^1)$, *then we can perform equivariant surgery on* (X_0, θ_0) *to get* (Σ, θ_1) *such that* Σ *is an oriented closed* $(4m+1)$*-dimensional pseudofree* S^1*-manifold and* $\theta_1:\Sigma\to M$ *is an* S^1*-map with the following properties:*

(i) θ_1 *is an* S^1*-homotopy equivalence and* $\deg\theta_1=+1$,

(ii) $I(\Sigma)=I(M)$,

(iii) $T\Sigma=\theta_1^*(TM-\underline{\underline{W}}+\underline{\underline{V}})$ *in* $KO_{S^1}(\Sigma)$.

We give a simple proof of Proposition 7.2 in § 9, where we use only the Browder-Novikov theory [**6**].

Next we consider the condition $\mathrm{Sign}(X_0/S^1)=\mathrm{Sign}(M/S^1)$ of Proposition 7.2.

If $p>1$ is an integer and if $q_1, q_2, \cdots, q_{2m}$ are integers prime to p, then we define

$$\mathrm{def}(p;q_1,\cdots,q_{2m})=(-1)^m\sum_{j=1}^{p-1}\cot(\pi q_1 j/p)\cdots\cot(\pi q_{2m}j/p).$$

We put

$$d=\sum_{p_j>1}\frac{1}{p_j}\mathrm{def}(p_j;p_1,\cdots,\hat{p}_j,\cdots,p_{2m+1}).$$

By Atiyah [**2**; Corollary 9.12, Theorem 10.3], Atiyah-Bott [**3**; p. 473] and Atiyah-Singer [**4**; §3], we obtain

LEMMA 7.3. *We have*

$$\begin{cases} \mathrm{Sign}(M/S^1)=\langle L(p(TM/S^1)), [M/S^1]\rangle+d\,, \\ \mathrm{Sign}(X_0/S^1)=\langle L(p(TX_0/S^1)), [X_0/S^1]\rangle+d\,, \end{cases}$$

where $L(-)$ *denotes the Hirzebruch's L-polynomial.*

COROLLARY 7.4. *The following twe conditions are equivalent:*

(i) $\mathrm{Sign}(X_0/S^1)=\mathrm{Sign}(M/S^1)$,

(ii) $\langle L(p(TX_0/S^1)), [X_0/S^1]\rangle=\langle L(p(TM/S^1)), [M/S^1]\rangle$.

Moreover we have

COROLLARY 7.5. *The following two conditions are equivalent:*

(i) $\mathrm{Sign}(M/S^1)=\mathrm{Sign}(X_0/S^1)$,

(ii) $\langle L(p(TM/S^1)), [M/S^1]\rangle=\langle L(p((TM-\underline{\underline{W}}+\underline{\underline{V}})/S^1)), [M/S^1]\rangle$.

PROOF. Since $TX_0=\theta_0^*(TM-\underline{\underline{W}}+\underline{\underline{V}})$ in $KO_{S^1}(X_0)$, we have

$$\begin{aligned} p(TX_0/S^1)&=p((\theta_0^*(TM-\underline{\underline{W}}+\underline{\underline{V}}))/S^1) \\ &=(\theta_0/S^1)^*(p((TM-\underline{\underline{W}}+\underline{\underline{V}})/S^1))\,, \end{aligned}$$

where θ_0/S^1: $X_0/S^1\to M/S^1$ denotes the induced map of the orbit spaces. Remark that $(\theta_0/S^1)_*([X_0/S^1])=[M/S^1]$. We obtain

$$\begin{aligned} \langle L(p(TX_0/S^1)), [X_0/S^1]\rangle&=\langle(\theta_0/S^1)^*(L(p((TM-\underline{\underline{W}}+\underline{\underline{V}})/S^1))), [X_0/S^1]\rangle\,, \\ &=\langle L(p((TM-\underline{\underline{W}}+\underline{\underline{V}})/S^1)), (\theta_0/S^1)_*([X_0/S^1])\rangle\,, \\ &=\langle L(p((TM-\underline{\underline{W}}+\underline{\underline{V}})/S^1)), [M/S^1]\rangle\,. \end{aligned}$$

Therefore the result follows from Corollary 7.4. Q. E. D.

8. Proof of the main theorem.

In this section, we use the same notations as in §7. We assume that $m\geqq 3$ and we put $q=\prod_{i=1}^{2m+1} p_i$.

LEMMA 8.1. *Let* ξ *be an arbitrary element of* $KO(CP^{2m})$. *Then there is* $\eta\in\widetilde{KO}(CP^{2m})$ *such that* $n\eta$ $(n=1, 2, \cdots)$ *satisfy the following:*

(i) $J(n\eta)=0$ *in* $J(CP^{2m})$,

(ii) $\langle L(p(\xi\oplus n\eta)), [CP^{2m}]\rangle=\langle L(p(\xi)), [CP^{2m}]\rangle$,

(iii) *the total Pontrjagin classes* $p(\xi\oplus n\eta)$ $(n=1, 2, \cdots)$ *are different from one another.*

PROOF. By the same argument as in §4 of Hsiang [**7**], there exists $\eta\in\widetilde{KO}(CP^{2m})$ such that

(a) $J(\eta)=0$ in $J(CP^{2m})$,

(b) $\langle L(p(\xi\oplus\eta)), [CP^{2m}]\rangle=\langle L(p(\xi)), [CP^{2m}]\rangle$,

(c) $\begin{cases} p_i(\eta)=0 \text{ for } 1\leqq i\leqq [m/2], \\ p_{i_0}(\eta)\neq 0 \text{ for some } [m/2]<i_0\leqq m. \end{cases}$

It is easy to see that $n\eta$ $(n=1, 2, \cdots)$ satisfy the conditions (i) and (ii). Let j_0 $([m/2]<j_0\leqq m)$ be an integer such that $p_j(\eta)=0$ for $1\leqq j<j_0$ and $p_{j_0}(\eta)\neq 0$. Then we have

$$p_{j_0}(\xi\oplus n\eta)=p_{j_0}(\xi)+np_{j_0}(\eta)\in H^{4j_0}(CP^{2m}; \boldsymbol{Z}).$$

Therefore $p(\xi\oplus n\eta)$ $(n=1, 2, \cdots)$ are different from one another. Q. E. D.

Let $R(G)$ (resp. $RO(G)$) denote the complex representation ring (resp. the real representation ring) of G. Let V be a complex S^1-representation space; then $S^{4m+1}(1, \cdots, 1)\times_{S^1} V$ is a complex vector bundle over CP^{2m}. The assignment $V\mapsto S^{4m+1}(1, \cdots, 1)\times_{S^1} V$ is additive, so it induces a homomorphism

$$\alpha_0: R(S^1) \longrightarrow K(CP^{2m}).$$

Similarly, we obtain a homomorphism

$$\alpha_q: R(\boldsymbol{Z}_q) \longrightarrow K(L^{2m}(q)),$$

where $L^{2m}(q)$ denotes the lens space $S^{4m+1}(1, \cdots, 1)/\boldsymbol{Z}_q$. It is well-known that the homomorphisms α_0 and α_q are surjective (see Atiyah [**1**: § 2.7], Mahammed [**12**; Lemma 3.3]).

LEMMA 8.2. *Let η be an arbitrary element of $\tilde{K}(CP^{2m})$. Then there are $x\in R(S^1)$ and an integer $n>0$ such that $\alpha_0(x)=n\eta$ and $i^*(x)=0$ in $R(\boldsymbol{Z}_q)$ where $i: \boldsymbol{Z}_q\to S^1$ is the natural inclusion.*

PROOF. Let $\pi: L^{2m}(q)\to CP^{2m}$ be the natural projection. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} i^* & \longrightarrow & R(S^1) & \xrightarrow{i^*} & R(\boldsymbol{Z}_q) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_0|} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_q} & & \\
0 & \longrightarrow & \operatorname{Ker} \pi^* & \longrightarrow & K(CP^{2m}) & \xrightarrow{\pi^*} & K(L^{2m}(q)) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It is easy to see that $\alpha_0(\operatorname{Ker} i^*)=\operatorname{Ker} \pi^*$. Hence we have

$$\tilde{K}(CP^{2m})/\alpha_0(\operatorname{Ker} i^*)=\tilde{K}(CP^{2m})/\operatorname{Ker} \pi^*\cong\tilde{K}(L^{2m}(q)).$$

Since it is well-known that $\tilde{K}(L^{2m}(q))$ is a finite group, there exists an integer

$n>0$ such that $n\eta\in\alpha_0(\mathrm{Ker}\, i^*)$. This shows the result. Q. E. D.

Define $r: \tilde{K}(CP^{2m})\to\widetilde{KO}(CP^{2m})$ to be induced by taking the underlying real structure of a complex vector bundle. Then it is well-known that the homomorphism r is surjective (see Sanderson [**20**; Theorem 3.9]). Define an S^1-map

$$h: S^{4m+1}(1, \cdots, 1) \longrightarrow M$$

by

$$h(z_1, \cdots, z_{2m+1})=\frac{(z_1^{p_1}, \cdots, z_{2m+1}^{p_{2m+1}})}{\|(z_1^{p_1}, \cdots, z_{2m+1}^{p_{2m+1}})\|}.$$

Let $\mu: KO_{S^1}(S^{4m+1}(1, \cdots, 1))\to KO(CP^{2m})$ be the isomorphism defined by Segal [**21**; Proposition 2.1].

LEMMA 8.3. *There is $x\in R(S^1)$ such that nx $(n=1, 2, \cdots)$ satisfy the following:*

(i) $nx|\boldsymbol{Z}_q=0$ *in* $RO(\boldsymbol{Z}_q)$,

(ii) $J(\mu(h^*(M\times nx)))=0$ *in* $J(CP^{2m})$,

(iii) $\langle L(p(\mu(h^*(TM-\underline{\underline{nx}})))), [CP^{2m}]\rangle=\langle L(p(\mu(h^*(TM)))), [CP^{2m}]\rangle$,

(iv) *the total Pontrjagin classes* $p(\mu(h^*(TM-\underline{\underline{nx}})))$ $(n=1, 2, \cdots)$ *are different from one another.*

PROOF. Since the homomorphism $r: \tilde{K}(CP^{2m})\to\widetilde{KO}(CP^{2m})$ is surjective, the result follows from Lemmas 8.1 and 8.2.

PROOF OF THE MAIN THEOREM. In order to prove the main theorem, by Proposition 7.2 and Corollary 7.5, it suffices to show that there are infinitely many $y\in R(S^1)$ such that

(8.4) $$y|\boldsymbol{Z}_{p_i}=0 \quad \text{in} \quad RO(\boldsymbol{Z}_{p_i}) \quad \text{for} \quad 1\leqq i\leqq 2m+1,$$

(8.5) $$M\times y\geqq 0,$$

(8.6) $$\langle L(p((TM-\underline{\underline{y}})/S^1)), [M/S^1]\rangle=\langle L(p(TM/S^1)), [M/S^1]\rangle,$$

(8.7) the total Pontrjagin classes $p(ES^1\times_{S^1}(TM-\underline{\underline{y}}))$ are different from one another.

Let $x\in R(S^1)$ be as in Lemma 8.3. We shall show that nx $(n=1, 2, \cdots)$ satisfy (8.4), (8.5), (8.6) and (8.7). It is obvious that nx satisfies (8.4). It follows from Kakutani [**9**; Theorem 6.2] that nx satisfies (8.5). Therefore we shall show that nx satisfies (8.6) and (8.7). Let $h/S^1: CP^{2m}\to M/S^1$ be the induced map of the orbit spaces. We remark that

$$p(\mu(h^*(TM-\underline{\underline{nx}})))=(h/S^1)^*(p((TM-\underline{\underline{nx}})/S^1)).$$

Hence (8.7) follows from the condition (iv) of Lemma 8.3. Moreover it follows from Kawasaki [**10**] that $(h/S^1)_*([CP^{2m}])=q[M/S^1]$. Hence we have

$$\begin{aligned}\langle L(p(\mu(h^*(TM)))), [CP^{2m}]\rangle &= \langle (h/S^1)^*(L(p(TM/S^1))), [CP^{2m}]\rangle \\ &= \langle L(p(TM/S^1)), (h/S^1)_*([CP^{2m}])\rangle \\ &= q\langle L(p(TM/S^1)), [M/S^1]\rangle .\end{aligned}$$

Similarly, we have

$$\langle L(p(\mu(h^*(TM-\underline{\underline{nx}})))), [CP^{2m}]\rangle = q\langle L(p((TM-\underline{\underline{nx}})/S^1)), [M/S^1]\rangle .$$

Thus (8.6) follows from the condition (iii) of Lemma 8.3. This completes the proof of the main theorem.

9. Appendix.

In this section, we give a simple proof of Proposition 7.2 by making use of the Browder-Novikov theory. Before beginning the proof of Proposition 7.2, we require some notations and lemmas.

Let M, X_0, θ_0, V and W be as in Proposition 7.2. Denote by N a closed invariant tubular neighborhood of the singular orbits in M. Then, by the discussion in § 6, a closed invariant tubular neighborhood of the singular orbits in X_0 is identified with N and we see that

$$\begin{cases} \theta_0 | N = \mathrm{id}: N \longrightarrow N \subset M, \\ \theta_0^{-1}(N) = N. \end{cases}$$

We put

$$P = M - \mathrm{Int}\, N, \quad Q = X_0 - \mathrm{Int}\, N .$$

Then P and Q are oriented compact smooth manifolds with free S^1-actions and $\partial P = \partial Q = \partial N$. Since $\theta_0^{-1}(N) = N$, we can define an S^1-map

$$f_0 = \theta_0 | Q : Q \longrightarrow P .$$

Then $f_0 | \partial Q : \partial Q \to \partial P$ is the identity and $\deg f_0 = +1$.

Moreover we have

$$TQ = f_0^*(TP - \xi | P) \quad \text{in} \quad KO_{S^1}(Q),$$

where TQ (resp. TP) denotes the tangent bundle of Q (resp. P) and $\xi = \underline{V} - \underline{\underline{W}} \in KO_{S^1}(M)$.

We put

$$\bar{P} = P/S^1 , \quad \bar{Q} = Q/S^1$$

and

$$\bar{f}_0 = f_0/S^1 : \bar{Q} \longrightarrow \bar{P} ,$$

where f_0/S^1 denotes the induced map of the orbit spaces. Then $\bar{P}$ and $\bar{Q}$ are compact smooth manifolds and have canonical orientations determined by those of P, Q (see §7), and $\bar{f}_0$ is a smooth map such that

$$\deg \bar{f}_0 = +1, \quad \bar{f}_0 | \partial\bar{Q} = \text{the identity}.$$

and

$$(TQ)/S^1 = \bar{f}_0^*((TP)/S^1 - (\xi | P)/S^1) \quad \text{in} \quad KO(\bar{Q}).$$

The following lemma is well-known.

LEMMA 9.1. *Let B be a compact smooth manifold with a free S^1-action. Then there is an isomorphism of vector bundles:*

$$(TB)/S^1 \cong T(B/S^1) \oplus \underline{\boldsymbol{R}}^1,$$

where TB (resp. $T(B/S^1)$) denotes the tangent bundle of B (resp. B/S^1).

It follows from Segal [**21**; Proposition 2.1] that the natural projection $\mathrm{pr}: B \to B/S^1$ induces an isomorphism of rings:

$$\mathrm{pr}^*: KO(B/S^1) \longrightarrow KO_{S^1}(B).$$

By Lemma 9.1, we have

$$\begin{cases} \mathrm{pr}^*(T(B/S^1)) = TB - \underline{\boldsymbol{R}}^1 & \text{in} \quad KO_{S^1}(B), \\ (TB)/S^1 = T(B/S^1) + \underline{\boldsymbol{R}}^1 & \text{in} \quad KO(B/S^1). \end{cases}$$

Therefore we obtain

$$T\bar{Q} = \bar{f}_0^*(T\bar{P} - \xi') \quad \text{in} \quad KO(\bar{Q}),$$

where $\xi' = (\xi | P)/S^1$.

LEMMA 9.2. *The following two conditions are equivalent:*

i) $\mathrm{Sign}(X_0/S^1) = \mathrm{Sign}(M/S^1)$,

ii) $\mathrm{Sign}(\bar{Q}, \partial\bar{Q}) = \mathrm{Sign}(\bar{P}, \partial\bar{P})$.

The proof is easy.

PROOF OF PROPOSITION 7.2. Let us assume that $n \gg 4m$, and that $(\bar{Q}, \partial\bar{Q})$ is embedded in (D^n, S^{n-1}) with normal bundle ν_0. Then we have

$$\begin{aligned} \nu_0 = \underline{\boldsymbol{R}}^n - T\bar{Q} &= \underline{\boldsymbol{R}}^n - \bar{f}_0^*(T\bar{P} - \xi') \\ &= \bar{f}_0^*(\underline{\boldsymbol{R}}^n - T\bar{P} + \xi') \quad \text{in} \quad KO(\bar{Q}). \end{aligned}$$

Then there exist an integer $k > 0$ and a vector bundle η_1 over $\bar{P}$ such that

$$\eta_1 = \underline{\boldsymbol{R}}^n - T\bar{P} + \xi' + \underline{\boldsymbol{R}}^k \quad \text{in} \quad KO(\bar{P})$$

and

$$\nu_0 \oplus \underline{\boldsymbol{R}}^k \cong \bar{f}_0^*(\eta_1).$$

Hence $\bar{f}_0: (\bar{Q}, \partial\bar{Q}) \to (\bar{P}, \partial\bar{P})$ is a normal map in the sense of Browder [**6**]. We remark that $\pi_1(\bar{P}) = \{0\}$ and $\dim \bar{P} > 5$. Moreover, by the assumption of Proposition 7.2 and Lemma 9.2, we have

$$\mathrm{Sign}(\bar{Q}, \partial\bar{Q})=\mathrm{Sign}(\bar{P}, \partial\bar{P}).$$

Therefore it follows from Fundamental Surgery Theorem of Browder [**6**; II. §1] that $(\bar{Q}, \bar{f}_0)$ is normally cobordant rel$\partial\bar{Q}$ to a homotopy equivalence

$$\bar{f}_1: \bar{Z} \longrightarrow \bar{P}.$$

Here we remark that $\bar{Z}$ is an oriented compact smooth manifold and $\bar{f}_1$ is a smooth map such that

$$\begin{cases} \partial\bar{Z}=\partial\bar{Q} \ (=\partial N/S^1), \\ \bar{f}_1|\partial\bar{Z}=\bar{f}_0|\partial\bar{Q}=\text{the identity}, \\ \deg\bar{f}_1=+1. \end{cases}$$

Denote by ν_1 the normal bundle of $(\bar{Z}, \partial\bar{Z})$ in (D^{n+k}, S^{n+k-1}). Then we have

$$\nu_1\cong\bar{f}_1^*(\eta_1).$$

Thus we see that

$$\begin{aligned} T\bar{Z}&=\underline{\boldsymbol{R}}^{n+k}-\nu_1=\underline{\boldsymbol{R}}^{n+k}-\bar{f}_1^*(\eta_1) \\ &=\bar{f}_1^*(T\bar{P}-\xi') \qquad \text{in} \quad KO(\bar{Z}), \end{aligned}$$

where $T\bar{Z}$ denotes the tangent bundle of $\bar{Z}$.

Let $\mathfrak{B}$ denote the smooth principal S^1-bundle:

$$S^1 \longrightarrow P \longrightarrow \bar{P}.$$

Then, the total space Z of $\bar{f}_1^*\mathfrak{B}$ is an oriented compact smooth manifold with a free S^1-action such that

$$Z/S^1=\bar{Z}, \quad \partial Z=\partial Q=\partial N.$$

Let $f_1: Z\to P$ denote the induced bundle map covering $\bar{f}_1$; that is, f_1 is an S^1-map of Z to P such that $f_1/S^1=\bar{f}_1$. It is easy to see that f_1 is a homotopy equivalence such that

$$\deg f_1=+1, \quad f_1|\partial Z=\text{the identity}.$$

On the other hand, by Lemma 9.1, we have

$$TZ=f_1^*(TP-\xi|P) \qquad \text{in} \quad KO_{S^1}(Z).$$

Since $\partial Z=\partial N$, we can construct a smooth manifold

$$\Sigma=Z\cup N,$$

by identifying points of ∂Z and ∂N under the identity, where Σ has a smoothness structure so that each inclusion $Z\to\Sigma$, $N\to\Sigma$ is a diffeomorphism onto its image. Then Σ is an oriented closed smooth manifold with a pseudofree S^1-action, and we see that $I(\Sigma)=I(M)$.

We also define an S^1-map

$$\theta_1 : \Sigma \longrightarrow M$$

by

$$\theta_1(x)=\begin{cases} f_1(x) & \text{if } x\in Z, \\ x & \text{if } x\in N. \end{cases}$$

Then θ_1 is a well-defined S^1-map such that

$$\deg\theta_1=+1, \quad \theta_1|N=\text{the identity}.$$

Moreover we have

ASSERTION 9.3. *$\theta_1 : \Sigma\to M$ is an S^1-homotopy equivalence.*

PROOF. Remark that $\theta_1|N=$the identity. If θ_1 is a homotopy equivalence, then it follows from Iberkleid [8; Corollary 3.5] that θ_1 is an S^1-homotopy equivalence. Therefore we shall show that θ_1 is a homotopy equivalence.

It is easy to see that

$$\theta_{1*} : H_q(\Sigma ; \boldsymbol{Z}) \longrightarrow H_q(M ; \boldsymbol{Z})$$

is an isomorphism for all q.

On the other hand, we see that

$$\pi_1(\Sigma)=\pi_1(M)=\{0\}.$$

Since Σ and M are finite CW-complexes, it follows from the theorem of J. H. C. Whitehead that θ_1 is a homotopy equivalence. This completes the proof of Assertion 9.3.

In order to complete the proof of Proposition 7.2, in the remainder of this section, we shall show that

$$T\Sigma=\theta_1^*(TM-\xi) \quad \text{in} \quad KO_{S^1}(\Sigma). \tag{9.4}$$

Let $\theta_2 : M\to\Sigma$ be an S^1-homotopy inverse of θ_1. Then (9.4) is equivalent to the following:

$$\theta_2^*(T\Sigma)=TM-\xi \quad \text{in} \quad KO_{S^1}(M). \tag{9.5}$$

Therefore we shall show (9.5) instead of (9.4). Consider the following two conditions:

$$(\theta_2^*(T\Sigma))_{e_i}=(TM-\xi)_{e_i} \quad \text{in} \quad RO(\boldsymbol{Z}_{p_i}) \quad \text{for} \quad 1\leqq i\leqq 2m+1, \tag{9.6}$$

$$p_i(ES^1\underset{S^1}{\times}\theta_2^*(T\Sigma))=p_i(ES^1\underset{S^1}{\times}(TM-\xi)) \quad \text{in} \quad H^{4i}(ES^1\underset{S^1}{\times}M ; \boldsymbol{Z}) \quad \text{for} \quad 1\leqq i\leqq m, \tag{9.7}$$

where $e_i \in M$ $(=S^{4m+1}(p_1, p_2, \cdots, p_{2m+1}))$ is the i-th unit vector.

It follows from Kakutani [**9**; Theorem 4.12] that (9.6) and (9.7) imply (9.5). Thus we shall show (9.6) and (9.7).

Since $I(\Sigma)=I(M)$ and ξ $(=\underline{\underline{V}}-\underline{\underline{W}})$ satisfies (6.2), we obtain (9.6).

Let $j: P \to M$ be the natural inclusion.

ASSERTION 9.8. *There exists an S^1-map*

$$\tilde{h}: S^{4m-3}(1, \cdots, 1) \longrightarrow P$$

such that the following diagram is S^1-homotopy commutative:

$$\begin{array}{ccc} S^{4m-3}(1, \cdots, 1) & \xrightarrow{\quad i \quad} & S^{4m+1}(1, \cdots, 1) \\ \downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle h} \\ P & \xrightarrow{\quad j \quad} & M, \end{array}$$

where i is the natural inclusion and h is as in § 8.

PROOF. (cf. Iberkleid [8; Theorem 3.4].) The obstructions to constructing an S^1-map $\tilde{h}$ lie in

$$H^q(CP^{2m-2}; \pi_{q-1}(P)) \qquad \text{for} \quad 1 \leqq q \leqq 4m-4.$$

Since P is $(4m-2)$-connected, all groups are zero. Hence we have an S^1-map $\tilde{h}$.

The obstructions to constructing an S^1-homotopy between $h \circ i$ and $j \circ \tilde{h}$ lie in

$$H^q(CP^{2m-2}; \pi_q(M)) \qquad \text{for} \quad 1 \leqq q \leqq 4m-4.$$

Since M is $4m$-connected, all groups are zero. Hence we have an S^1-homotopy between $h \circ i$ and $j \circ \tilde{h}$. This completes the proof of Assertion 9.8.

ASSERTION 9.9. *Let $\xi, \eta \in KO_{S^1}(M)$. If $j^*\xi = j^*\eta$ in $KO_{S^1}(P)$, then we have*

$$p_i(ES^1 \underset{S^1}{\times} \xi) = p_i(ES^1 \underset{S^1}{\times} \eta) \qquad \textit{for} \quad 1 \leqq i \leqq m-1.$$

PROOF. Assertion 9.9 will follow from Assertion 9.8 by the same argument as in the proof of Lemma 4.11 in [**9**].

It is easy to see that

$$j^*(\theta_2^*(T\Sigma)) = j^*(TM-\xi) \qquad \text{in} \quad KO_{S^1}(P).$$

Hence it follows from Assertion 9.9 that

$$(9.10) \qquad p_i(ES^1 \underset{S^1}{\times} \theta_2^*(T\Sigma)) = p_i(ES^1 \underset{S^1}{\times} (TM-\xi))$$
$$\text{in} \quad H^{4i}(ES^1 \underset{S^1}{\times} M; \boldsymbol{Z}) \qquad \text{for} \quad 1 \leqq i \leqq m-1.$$

ASSERTION 9.11.

$$p_m(ES^1 \times_{S^1} \theta_2^*(T\Sigma)) = p_m(ES^1 \times_{S^1} (TM-\xi)) \quad in \quad H^{4m}(ES^1 \times_{S^1} M; \boldsymbol{Z}).$$

PROOF. By the assumption of Proposition 7.2 and Corollary 7.5, we have

$$\langle L(p((TM-\xi)/S^1)), [M/S^1]\rangle = \langle L(p((TM)/S^1)), [M/S^1]\rangle .$$

On the other hand, by the same argument as in §7, it is easy to see that

$$\langle L(p((T\Sigma)/S^1)), [\Sigma/S^1]\rangle = \langle L(p((TM)/S^1)), [M/S^1]\rangle .$$

Thus we have

$$\begin{aligned}\langle L(p((TM-\xi)/S^1)), [M/S^1]\rangle &= \langle L(p((T\Sigma)/S^1)), [\Sigma/S^1]\rangle \\ &= \langle L(p((T\Sigma)/S^1)), (\theta_2/S^1)_*([M/S^1])\rangle \\ &= \langle (\theta_2/S^1)^*(L(p((T\Sigma)/S^1))), [M/S^1]\rangle \\ &= \langle L(p((\theta_2^*(T\Sigma))/S^1)), [M/S^1]\rangle .\end{aligned}$$

We remark that the coefficient of p_m in the polynomial $L_m(p_1, \cdots, p_m)$ is non-zero. Hence it follows from (9.10) that

$$p_m((\theta_2^*(T\Sigma))/S^1) = p_m((TM-\xi)/S^1) \quad \text{in} \quad H^{4m}(M/S^1; \boldsymbol{Q}) .$$

Therefore we have

$$p_m(ES^1 \times_{S^1} \theta_2^*(T\Sigma)) = p_m(ES^1 \times_{S^1} (TM-\xi)) \quad \text{in} \quad H^{4m}(ES^1 \times_{S^1} M; \boldsymbol{Z}) .$$

This completes the proof of Assertion 9.11.

Combining (9.10) and Assertion 9.11, we have (9.7). This makes the proof of Proposition 7.2 complete.

References

[1] M.F. Atiyah, K-theory, Benjamin, 1967.
[2] M.F. Atiyah, Elliptic operators and compact groups, Lecture Notes in Math., 401, Springer-Verlag, 1974.
[3] M.F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes, II, Applications, Ann. of Math., **88** (1968), 451-491.
[4] M.F. Atiyah and I.M. Singer, The index of elliptic operators, III, Ann. of Math., **87** (1968), 546-604.
[5] G.E. Bredon, Introduction to compact transformation groups, Academic Press, 1972.
[6] W. Browder, Surgery on simply-connected manifolds, Ergebn. Math., **65**, Springer-Verlag, 1972.
[7] W.C. Hsiang, A note on free differentiable actions of S^1 and S^3 on homotopy spheres, Ann. of Math., **83** (1966), 266-272.
[8] W. Iberkleid, Pseudo-linear spheres, Michigan Math. J., **25** (1978), 359-370.

[9] S. Kakutani, Equivariant KO-rings and J-groups of spheres which have linear pseudofree S^1-actions, Osaka J. Math., **18** (1981), 533-554.

[10] T. Kawasaki, Cohomology of twisted projective spaces and lens complexes, Math. Ann., **206** (1973), 243-248.

[11] C.N. Lee and A.G. Wasserman, On the groups $JO(G)$, Mem. Amer. Math. Soc., **159** (1975).

[12] M.N. Mahammed, A propos de la K-théorie des espaces lenticulaires, C. R. Acad. Sci. Paris, **271** (1970), 639-642.

[13] A. Meyerhoff and T. Petrie, Quasi equivalence of G modules, Topology, **15** (1976), 69-75.

[14] J. Milnor, Differential topology, Lecture notes, Princeton Univ., 1958.

[15] D. Montgomery and C.T. Yang, Differentiable pseudo-free circle actions, Proc. Nat. Acad. Sci. U. S. A., **68** (1971), 894-896.

[16] D. Montgomery and C.T. Yang, Differentiable pseudo-free circle actions on homotopy seven spheres, Proc. of the Second Conf. on Compact Transformation Groups, Amherst, Lecture Notes in Math., 298, Springer-Verlag (1971), 41-101.

[17] H. Ozeki and F. Uchida, Principal circle actions on a product of spheres, Osaka J. Math., **9** (1972), 379-390.

[18] T. Petrie, Equivariant quasi-equivalence, transversality and normal cobordism, Proc. Internat. Congr. Math. (Vancouver 1974), 537-541.

[19] T. Petrie, Pseudoequivalences of G-manifolds, Proc. Sympos. Pure Math., **32** (1978), 169-210.

[20] B.J. Sanderson, Immersions and embeddings of projective spaces, Proc. London Math. Soc., **14** (1964), 137-153.

[21] G. Segal, Equivariant K-theory, Publ. Math. Inst. des Hautes Études Scient., **34** (1968), 129-151.

[22] A.G. Wasserman, Equivariant differential topology, Topology, **8** (1969), 127-150.

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