J. Math. Soc. Japan
Vol. 34, No. 1, 1982

A generalization of Roberts-Tannaka duality theorem

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(Received May 13, 1980)

1. Introduction.

Let $\{\mathfrak{M}, G, \gamma\}$ be a covariant system, that is, G is a locally compact group and $\gamma: G\rightarrow \mathrm{Aut}(\mathfrak{M})$ is a homomorphism of G into the group of $*$-automorphisms of a von Neumann algebra $\mathfrak{M}$ with the following continuity: $G\ni t\rightarrow\gamma_{t}x\in\mathfrak{M}$ is continuous for each $x\in\mathfrak{M}$ with respect to the σ-weak topology on $\mathfrak{M}$. By definition in [**4**], a *Hilbert space in* $\mathfrak{M}$ is a closed subspace $\mathfrak{K}$ of $\mathfrak{M}$ such that

(i) $y^{*}x$ is a scalar multiple of the identity for every $x, y\in\mathfrak{K}$ and

(ii) for every non-zero $a\in\mathfrak{M}$, there exists an $x\in\mathfrak{K}$ with $ax\neq 0$.

The inner product $(x|y)$ in $\mathfrak{K}$ is given by $y^{*}x$. If a *Hilbert space* $\mathfrak{K}$ *in* $\mathfrak{M}$ is globally invariant under γ, $\gamma_{t}(\mathfrak{K})\subseteq\mathfrak{K}$ for all $t\in G$, we have

$$(\gamma_{t}x|\gamma_{t}y)=\gamma_{t}(y^{*}x)=y^{*}x=(x|y) \qquad \text{for every} \quad x,\ y\in\mathfrak{K},\ t\in G.$$

Hence the restriction of γ to $\mathfrak{K}$ is a unitary representation of G. We denote it by $\pi_{\mathfrak{K}}$. Let $\mathcal{H}_{\gamma}(\mathfrak{M})$ be the collection of all *Hilbert spaces in* $\mathfrak{M}$ globally invariant under γ. Let $\mathfrak{M}^{\gamma}$ denote the fixed point algebra $\{x\in\mathfrak{M}; \gamma_{t}(x)=x$ for all $t\in G\}$ of $\mathfrak{M}$ under γ and $\mathrm{Aut}(\mathfrak{M}|\mathfrak{M}^{\gamma})=\{\rho\in\mathrm{Aut}(\mathfrak{M}); \rho(x)=x$ for all $x\in\mathfrak{M}^{\gamma}\}$.

Under the above situation the following Roberts-Tannaka duality theorem was obtained and was used as a basic tool in [**1**].

THEOREM 1. *Assume that* $\mathfrak{M}^{\gamma}$ *is properly infinite and* G *is compact. If each irreducible subrepresentation of* $\{\gamma, \mathfrak{M}\}$ *is unitarily equivalent to some* $\pi_{\mathfrak{K}}$, $\mathfrak{K}\in\mathcal{H}_{\gamma}(\mathfrak{M})$, *then every* $\sigma\in\mathrm{Aut}(\mathfrak{M}|\mathfrak{M}^{\gamma})$ *leaving every member* $\mathfrak{K}\in\mathcal{H}_{\gamma}(\mathfrak{M})$ *globally invariant must be of the form* γ_{s} *for some* $s\in G$.

In this short note we generalize the above theorem to the case of arbitrary locally compact groups. This problem is suggested in [**3**].

The author would like to thank Dr. Nakagami for his encouragement.

2. A duality theorem.

Before stating the theorem, we show the following lemma.

LEMMA. *If* $\sigma\in\mathrm{Aut}(\mathfrak{M}|\mathfrak{M}^{\gamma})$ *and* $\mathfrak{K}\in\mathcal{H}_{\gamma}(\mathfrak{M})$ *which is globally invariant under* σ, *each globally* γ-*invariant closed subspace* $\mathfrak{K}'$ *of* $\mathfrak{K}$ *is also globally invariant* *under* σ.

PROOF. Let $\{e_i\}_{i\in I'}$ be an orthonormal basis of $\mathfrak{K}'$ which is extended to those $\{e_i\}_{i\in I}$ $(I'\subseteqq I)$, of $\mathfrak{K}$, i.e., $e_i^*e_j=\delta_{ij}\cdot 1$ and $\sum_{i\in I} e_ie_i^*=1$. Put $p=\sum_{i\in I'} e_ie_i^*$.

Since $\mathfrak{K}'$ is globally γ-invariant, we have

$$\begin{aligned}\gamma_t(p)\pi_{\mathfrak{K}}(t)a&=\gamma_t(pa)\\&=\pi_{\mathfrak{K}}(t)(pa)\\&=p\pi_{\mathfrak{K}}(t)a\qquad\text{for every}\quad a\in\mathfrak{K},\ t\in G.\end{aligned}$$

This implies that $\gamma_t(p)=p$ for every $t\in G$, i.e., $p\in\mathfrak{M}^\gamma$. Hence we have

$$\sigma(\mathfrak{K}')=\sigma(p\mathfrak{K})=p\sigma(\mathfrak{K})\subseteqq p\mathfrak{K}=\mathfrak{K}'. \qquad \text{Q.E.D.}$$

THEOREM 2. *Let* $\{\mathfrak{M}, G, \gamma\}$ *be a covariant system such that* $\mathfrak{M}^\gamma$ *is properly infinite. If there exists* $\mathfrak{K}_0$ *in* $\mathcal{H}_\gamma(\mathfrak{M})$ *such that* $\pi_{\mathfrak{K}_0}$ *is unitarily equivalent to the left regular representation* λ *of* G, *every* $\sigma\in\mathrm{Aut}(\mathfrak{M}\,|\,\mathfrak{M}^\gamma)$ *leaving* $\mathfrak{K}_0$ *globally invariant must be of the form* γ_s *for some* $s\in G$.

PROOF. For each $a, b\in\mathfrak{K}$, $\mathfrak{K}\in\mathcal{H}_\gamma(\mathfrak{M})$, let $f_{a,b}$ be the function on G given by

(1) $$f_{a,b}(t)=(a\,|\,\pi_{\mathfrak{K}}(t)b)=\gamma_t(b^*)a\qquad\text{for every}\quad t\in G.$$

Since $\pi_{\mathfrak{K}_0}$ is unitarily equivalent to λ, the set $\{f_{a,b}\,;\ a, b\in\mathfrak{K}_0\}$ is nothing but the Fourier algebra $A(G)$ of G. [2]. Since $\mathfrak{M}^\gamma$ is properly infinite, there exist isometries w_1, w_2 in $\mathfrak{M}^\gamma$ with $w_1w_1^*+w_2w_2^*=1$. Let $a, b, c, d\in\mathfrak{K}_0$, $\alpha\in\boldsymbol{C}$. By direct computation we have the followings;

(2) $$w_1\mathfrak{K}_0+w_2\mathfrak{K}_0,\quad \mathfrak{K}_0\cdot\mathfrak{K}_0\in\mathcal{H}_\gamma(\mathfrak{M}),$$

(3) $$f_{a,b}(t)+f_{c,d}(t)=f_{w_1a+w_2c,\,w_1b+w_2d}(t),$$

(4) $$f_{a,b}(t)\cdot f_{c,d}(t)=f_{ac,bd}(t)\quad\text{and}$$

(5) $$\alpha f_{a,b}(t)=f_{\alpha a,b}(t)\qquad\text{for every}\quad t\in G.$$

Hence the sets $\{f_{x,y}\,;\ x, y\in w_1\mathfrak{K}_0+w_2\mathfrak{K}_0\}$ and $\{f_{x,y}\,;\ x, y\in\mathfrak{K}_0\cdot\mathfrak{K}_0\}$ are both subsets of $A(G)$.

Let $\sigma\in\mathrm{Aut}(\mathfrak{M}\,|\,\mathfrak{M}^\gamma)$ leaving $\mathfrak{K}_0$ globally invariant, then it is easily seen that σ leaves also $w_1\mathfrak{K}_0+w_2\mathfrak{K}_0$ and $\mathfrak{K}_0\cdot\mathfrak{K}_0$ globally invariant.

Let $\mathfrak{K}_i\in\mathcal{H}_\gamma(\mathfrak{M})$ such that $\sigma(\mathfrak{K}_i)\subseteqq\mathfrak{K}_i$ $(i=1, 2)$ and $a, b\in\mathfrak{K}_1$, $c, d\in\mathfrak{K}_2$.

If $f_{a,b}(t)=f_{c,d}(t)$ for all $t\in G$, then

$$\begin{aligned}&(\gamma_{t^{-1}}(w_1a-w_2c)\,|\,w_1b+w_2d)\\&=f_{w_1a-w_2c,\,w_1b+w_2d}(t)\\&=f_{a,b}(t)-f_{c,d}(t)=0\qquad\text{for all}\quad t\in G.\end{aligned}$$

Put $\mathfrak{K}'=[\{\gamma_t(w_1a-w_2c)\,;\ t\in G\}]$, then $\mathfrak{K}'$ is a globally γ-invariant closed subspace of $w_1\mathfrak{K}_1+w_2\mathfrak{K}_2$. Since $w_1\mathfrak{K}_1+w_2\mathfrak{K}_2$ is an element of $\mathcal{H}_\gamma(\mathfrak{M})$ which is globally invariant under σ, it follows from Lemma that $\sigma(\mathfrak{K}')\subseteqq\mathfrak{K}'$, especially $\sigma(w_1a-w_2c)\in\mathfrak{K}'$. Hence we have

$$\begin{aligned} b^*\sigma(a)-d^*\sigma(c) &= (w_1b+w_2d)^*\sigma(w_1a-w_2c)\\ &= (\sigma(w_1a-w_2c)\,|\,w_1b+w_2d)\\ &= 0\,. \end{aligned}$$

If $f\in A(G)$ is of the form $f_{a,b}$ for some a, $b\in\mathfrak{K}$, $\mathfrak{K}\in\mathcal{H}_\gamma(\mathfrak{M})$ with $\sigma(\mathfrak{K})\subseteqq\mathfrak{K}$, by the above argument we can define a functional $\hat{\sigma}$ on $A(G)$ as follows;

$$\hat{\sigma}(f)=\hat{\sigma}(f_{a,b})=b^*\sigma(a)\,. \tag{6}$$

Since $\mathfrak{K}_0$ is globally invariant under σ, the domain of $\hat{\sigma}$ is the whole space $A(G)$. Let a, b, c, $d\in\mathfrak{K}_0$, $\alpha\in\boldsymbol{C}$. By (1)~(6) we have the followings;

$$\begin{aligned} \hat{\sigma}(f_{a,b}+f_{c,d}) &= \hat{\sigma}(f_{w_1a+w_2c,\,w_1b+w_2d})\\ &= (w_1b+w_2d)^*(w_1\sigma(a)+w_2\sigma(c))\\ &= b^*\sigma(a)+d^*\sigma(c)\\ &= \hat{\sigma}(f_{a,b})+\hat{\sigma}(f_{c,d})\,,\\ \hat{\sigma}(\alpha f_{a,b}) &= \hat{\sigma}(f_{\alpha a,b})\\ &= b^*\sigma(\alpha a)\\ &= \alpha\hat{\sigma}(f_{a,b}) \quad \text{and}\\ \hat{\sigma}(f_{a,b}\cdot f_{c,d}) &= \hat{\sigma}(f_{ac,bd})\\ &= (bd)^*\sigma(ac)\\ &= \hat{\sigma}(f_{a,b})\cdot\hat{\sigma}(f_{c,d})\,. \end{aligned}$$

Therefore $\hat{\sigma}$ is a non-zero continuous character of $A(G)$. From Eymard duality theorem [2] it follows that there exists uniquely $s\in G$ such that

$$\hat{\sigma}(f_{a,b})=f_{a,b}(s^{-1}) \qquad \text{for every} \quad a,\ b\in\mathfrak{K}_0\,.$$

Since it holds that

$$(\sigma(a)\,|\,b)=(\gamma_s a\,|\,b) \qquad \text{for every} \quad a,\ b\in\mathfrak{K}_0\,,$$

we have

$$\sigma=\gamma_s \quad \text{on } \mathfrak{K}_0\,. \tag{7}$$

Finally we shall show that $\mathfrak{M}^{\gamma}$ and $\mathfrak{K}_0$ generate $\mathfrak{M}$. For each $k\in K(G)$, $x\in\mathfrak{M}$ put

$$\gamma_k(x)=\int_G k(t)\gamma_t(x)dt\,, \tag{8}$$

where $K(G)$ denotes the set of all continuous functions on G with compact support and dt denotes a left invariant Haar measure on G. Let $\{V_i\}_i$ be a fundamental system of neighbourhoods of the unit of G and $\{k_i\}_i$ a family of functions such that

a) $k_i\in A(G)\cap K(G)_+$, where $K(G)_+=\{k\in K(G)\,;\ k(t)\geqq 0$ for all $t\in G\}$,
b) $\mathrm{supp}(k_i)\subseteqq V_i$ and
c) $\int_G k_i(t)dt=1$, for all i.

Then for every σ-weakly continuous linear functional ϕ of $\mathfrak{M}$ it holds that

$$|\phi(x-\gamma_{k_i}(x))|\leqq\int_G|\phi(x-\gamma_t(x))|\,k_i(t)dt \qquad \text{for each } x\in\mathfrak{M}. \tag{9}$$

Since each k_i belongs to $A(G)$, there exist $a_i,\ b_i\in\mathfrak{K}_0$ such that

$$k_i(t)=f_{a_i,b_i}(t) \qquad \text{for all} \quad t\in G.$$

Then we have

$$\gamma_{k_i}(x)=\gamma_{f_{a_i,b_i}}(x)=\int_G f_{a_i,b_i}(t)\gamma_t(x)dt \tag{10}$$

$$=\int_G\gamma_t(xb_i^*)dt\cdot a_i\in\mathfrak{M}^{\gamma}\cdot\mathfrak{K}_0.$$

Hence it follows from (9) that each $x\in\mathfrak{M}$ is approximated σ-weakly by the elements of $\mathfrak{M}^{\gamma}\cdot\mathfrak{K}_0$ and consequently $\mathfrak{M}^{\gamma}$ and $\mathfrak{K}_0$ generate $\mathfrak{M}$.

Since σ coincides with γ_s on $\mathfrak{M}^{\gamma}$ and $\mathfrak{K}_0$ by (7), we conclude that $\sigma=\gamma_s$ on $\mathfrak{M}$. Q. E. D.

Remark. It should be noticed that if the action γ is faithful, Theorem 1 is reduced to the compact case of Theorem 2.

References

[1] H. Araki, R. Haag, D. Kastler and M. Takesaki, Extension of KMS states and chemical potential, Comm. Math. Phys., **53** (1977), 97-134.

[2] P. Eymard, L'algebre de Fourier d'un group localement compact, Bull. Soc. Math. France, **92** (1964), 181-236.

[3] Y. Nakagami and M. Takesaki, Duality for crossed products of von Neumann algebras, Lecture Notes in Math., **731**, Springer, Berlin, 1979.

[4] J.E. Roberts, Cross product of von Neumann algebras by group duals, Symposia Math., **20** (1976), 335-363.

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