

## Remarks on a theorem of Fujita

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### Introduction.

In [3], Fujita proves some results on certain natural vector bundles arising from a Kähler fiber space (see our (2.2) and (2.14)). His methods are quite direct; the general techniques of variations of Hodge structure and their degeneration, as developed by Griffiths, Schmid, Steenbrink, the author, et al. are for the most part absent. This paper is the outcome of my attempt to discuss Fujita's results from this more general point of view. The advantage of doing this may not be so apparent in the present instance, though it is likely to show itself in future problems. One result in particular ((1.11)), on the Chern forms of Hodge bundles, suggested to us by Griffiths, figures to be of general usefulness.

### § 1. Chern classes for Hodge bundles.

Let  $S$  be a non-singular algebraic variety, and  $V$  a flat complex vector bundle on  $S$  underlying a (real) polarized variation of Hodge structure of weight  $m$ . By this, we mean that we are given the following collection of data:

(1.1) i) A flat real structure on  $V$ ,

ii) A non-degenerate flat bilinear pairing  $(v, w)$  on  $V$ , and hence on sections of  $V$ , which is defined over  $\mathbf{R}$  and is  $(-1)^m$ -symmetric,

iii) An orthogonal  $C^\infty$  bundle decomposition

$$V = \bigoplus_{\substack{p, q \in \mathbf{Z} \\ p+q=m}} H^{p, q},$$

satisfying  $\overline{H^{p, q}} = H^{q, p}$ .

iv) For each  $r$ ,

$$F^r = \bigoplus_{p \geq r} H^{p, q}$$

is a holomorphic sub-bundle of  $V$ ,

v) The flat differentiation in  $V$  with respect to a holomorphic vector field

on  $S$  maps  $\mathcal{E}(F^r)$  into  $\mathcal{E}(F^{r-1})$ ,<sup>1)</sup>

vi) For each  $(p, q)$ ,  $i^{p-q}(v, \bar{v})$  is positive-definite on  $H^{p,q}$ .

Letting  $\pi_{p,q}$  denote orthogonal projection of  $V$  onto  $H^{p,q}$ , and

$$C = \sum i^{p-q} \pi_{p,q} \quad (\text{Weil operator}),$$

we obtain the *Hodge metric*  $\langle v, w \rangle = (Cv, \bar{w})$  on  $V$ , and also on all bundles  $F^r/F^s$  ( $r < s$ ).

The archetypical examples of the above are the systems of cohomology of a family of compact Kähler manifolds (e. g., non-singular projective varieties).

For simplicity of notation, we assume that  $H^{p,q} = 0$  if  $p < 0$  or  $q < 0$ ; moreover, we assume that  $H^{m,0} \neq 0$ .<sup>2)</sup>

There are well-known formulas for the curvature of Hodge bundles with respect to the Hodge metric [4, (5.2)], which yield the assertion:

*The holomorphic vector bundle  $F^0/F^1$  is negative, and  $F^m = H^{m,0}$  is positive (not necessarily strictly in both cases).*

From here on, let  $S$  be a curve. Let  $\bar{S}$  be the smooth completion of  $S$ . We will show that the Chern forms of all bundles  $F^r/F^s$  ( $r < s$ ) are bounded, and hence square-summable, in the Poincaré metric on  $S$ , and therefore represent cohomology classes on  $\bar{S}$ ,<sup>3)</sup> under the following hypothesis. The flat bundle  $V$  is associated to a representation  $\rho$  of  $\pi_1(S)$  on some finite-dimensional vector space  $V$ . Put  $\Sigma = \bar{S} - S$ . Then we assume:

(1.3) HYPOTHESIS. The local monodromy transformations are unipotent, i. e., for the homotopy class  $\gamma$  of any small loop about a point of  $\Sigma$ ,  $\rho(\gamma)$  is a unipotent transformation of  $V$ .

(1.4) REMARK. By a standard argument (see [6, p. 230]) one knows that the eigenvalues of all  $\rho(\gamma)$  are of norm one. In case  $V$  is defined over  $\mathbf{Q}$ , e. g., in the geometric examples (cohomology systems), the eigenvalues must be roots of unity, and thus  $\rho(\gamma^M)$  is unipotent for some positive integer  $M$ .

It suffices to restrict the situation to a small punctured disc  $\Delta^*$  in  $S$  centered at a point  $s \in \Sigma$ . We pull back  $V|_{\Delta^*}$  to the upper half-plane  $\mathfrak{h}$  via the covering map  $\phi: \mathfrak{h} \rightarrow \Delta^*$ ,  $\phi(z) = \exp(2\pi iz)$ , and recall the asymptotic analysis of the Hodge metric given in [6, § 6]. Choose a local basis for the *canonical extension*  $\bar{V}$  of  $V$  to  $\bar{S}$  [2, p. 91], flagged so as to induce bases for all  $\bar{F}^r/\bar{F}^s$  (where  $\bar{F}^r$  is the closure of  $F^r$  in  $\bar{V}$ , which turns out to be a sub-bundle, cf. [8, p. 190]). Elements of the basis can be taken of the form (in terms of the coordinate on  $\mathfrak{h}$ )

1) Our notation is: for a  $C^\infty$  vector bundle  $E$ ,  $\mathcal{E}(E)$  denotes the sheaf of germs of  $C^\infty$  sections; for a holomorphic bundle  $E$ ,  $\mathcal{O}(E)$  is the analogous analytic object.

2) These normalizations can always be achieved by a shift of weight.

3) To see this, one can use either the zero-residue argument of [5, (1.2)], or the  $L_2$  representability of the cohomology of  $\bar{S}$  [9, § 6].

$$(1.5) \quad \tilde{v}(z) = \exp(zN)v \quad v \in V,$$

where  $N$  is the nilpotent logarithm of  $\rho(\gamma)$ .

We will carry out the discussion only for the total bundle  $\bar{V} = \bar{F}^0$ , the other cases being similar (compare [9, §5]). Since the Hodge metric is not flat, it is best to move  $\tilde{v}(z)$  by a family of isometries  $T(z)$  so that the resulting function

$$T \cdot \tilde{v} : \mathfrak{h} \rightarrow V$$

registers the Hodge norm. It follows from [6, p. 253] that

$$(1.6) \quad T(z)\tilde{v}(z) = \lambda(z)^{-1} [y^{l/2}(v + \phi(y))],$$

where  $y = \text{Im } z$ ,  $l \in \mathbf{Z}$ ,  $\phi(y)$  is a series in positive powers of  $y^{-1/2}$ , and  $\lambda : \mathfrak{h} \rightarrow \text{Aut}(V)$  is “close” to the identity matrix as  $y \rightarrow \infty$ .

We first assume that  $\lambda \equiv I$ . The main point is that  $T(z)\tilde{v}(z)$  is dominated by its leading term  $y^{l/2}v$ . The integers  $l$  which occur fall into certain patterns: by decomposing the bundle into  $SL_2$ -components (see [6, (6.24)]), it suffices to assume that  $V$  is irreducible; and then, to simplify notation, we assume  $V$  is isomorphic to the basic structure  $S(m)$ —cf. [9, p. 435]. There then exists a basis  $\{\tilde{v}_0, \dots, \tilde{v}_m\}$  of  $\bar{V}$  such that the matrix of the Hodge metric  $h = [h_{ij}]_{i,j=0}^m$ , is asymptotically

$$(1.7) \quad h_{ij} = \langle \tilde{v}_i, \tilde{v}_j \rangle \sim y^{m - (1/2)(i+j)}.$$

To estimate the metric connection form  $\theta = (\partial h)h^{-1}$ , we note that

$$h^{-1} = (\det h)^{-1} \text{Adj}(h),$$

and therefore one readily sees that

$$(h^{-1})_{ij} = (\det h)^{-1} \text{Adj}(h)_{ij} \sim y^{-[m - (1/2)(i+j)]}.$$

Of course

$$(\partial h)_{ij} \sim y^{(m-1) - (1/2)(i+j)} dz$$

and so

$$(1.8) \quad \theta_{ij} = \sum_k (\partial h)_{ik} (h^{-1})_{kj} \sim \sum_k y^{m-1 - (1/2)(i+k)} y^{-[m - (1/2)(k+j)]} dz \sim y^{-1 - (1/2)(i-j)} dz.$$

The curvature  $\kappa$  is then given by

$$\kappa_{ij} = \bar{\partial}(\theta)_{ij} \sim y^{-2 - (1/2)(i-j)} dz \wedge d\bar{z},$$

and we have for the Chern form

$$(1.9) \quad \Phi = \text{Trace}(\kappa) \sim y^{-2} dz \wedge d\bar{z}.$$

The right-hand side of the above is, up to a constant, the Poincaré area element, which is, of course, of bounded Poincaré norm.

(1.10) REMARK. Observe that (1.8) shows that  $\theta$  is a bounded local section of  $\Omega_{\mathbb{S}}^1(\text{End } V)$ .

Let now  $\lambda$  be arbitrary. In order that the preceding calculations be adaptable, it suffices to know that  $\lambda(z)I$  and its first and second derivatives are of negative exponential growth in  $y$ . Schmid needed the estimate only for the matrix itself in [6]. However, if one adopts the point of view that  $\bar{V}$  is the central object, one can see readily via use of the frame given by (1.5) and by [6, pp. 252-253] that we may choose  $\lambda$  so that the estimate holds for all derivatives of  $\lambda$ ; cf. [1, pp. 89-91].

We summarize.

(1.11) PROPOSITION. *Under Hypothesis (1.3), the Chern forms of the bundles  $\mathbf{F}^r/\mathbf{F}^s$  with respect to the Hodge metric are bounded in the Poincaré metric.*

The above result is interesting because of the following corollary.

(1.12) COROLLARY. *Under Hypothesis (1.3), the Chern form of the bundle  $\mathbf{F}^r/\mathbf{F}^s$  with respect to the Hodge metric represents in  $H^2(\bar{S}, \mathbf{C})$  the Chern class of the canonical extension  $\bar{\mathbf{F}}^r/\bar{\mathbf{F}}^s$ .*

PROOF. See [5, Theorem 1.4], and recall that we have computed relative to a basis of  $\bar{V}$ .

(1.13) REMARK. It is clear that when the local monodromy fails to be unipotent, the conclusion of (1.11) no longer holds. To see this, some generator of  $V$  can be taken to be of the form  $\exp(2\pi i\alpha z)\tilde{v}(z)$  where  $\alpha$  is a real number with  $0 < \alpha < 1$  (constant in each  $SL_2$  component), and in the expression (1.5) for  $\tilde{v}$ ,  $N$  is the nilpotent logarithm of the unipotent part of  $\rho(\gamma)$ . This changes the matrix  $h$  (from before) by a factor of  $g(z) = \exp(4\pi i\alpha z)$ . Put

$$h_\alpha = gh,$$

the matrix of the metric in the present situation. The connection matrix becomes

$$\theta_\alpha = \theta + \partial(\log g)I = \theta + (4\pi i\alpha dz)I.$$

If  $t$  is the parameter on  $\mathcal{A}^*$ ,  $2\pi i dz = \phi^*(dt/t)$ , and  $dt/t$  is not  $L_2$  in the Poincaré metric.

(1.14) REMARK. The results of this section remain valid for higher-dimensional  $S \subset \bar{S}$ , so long as the compactifying locus  $\Sigma = \bar{S} - S$  is a smooth hypersurface. When  $\dim S > 1$ , this is no longer a general situation.

## § 2. Application to the result of Fujita.

Let  $\bar{f}: \bar{X} \rightarrow \bar{S}$  be a Kähler fiber space over the smooth curve  $\bar{S}$ . That is,  $\bar{X}$  is a compact Kähler manifold and  $\bar{f}$  is surjective. Then  $\bar{S}$  is compact, and the general fiber of  $\bar{f}$  is non-singular. Let  $f: X \rightarrow S$  denote the mapping obtained

by deleting the singular fibers from  $\bar{X}$  and their images from  $\bar{S}$ . Set  $\Sigma = \bar{S} - S$  as before, and put  $n = \dim X$ . One defines the *relative dualizing sheaf*

$$(2.1) \quad \omega_{\bar{X}/\bar{S}} = \Omega_{\bar{X}}^n \otimes \bar{f}^*(\Omega_{\bar{S}}^1)^\vee.$$

The theorem of Fujita in question is

(2.2) THEOREM [3].  $\bar{\mathcal{E}} = \bar{f}_* \omega_{\bar{X}/\bar{S}}$  is numerically semi-positive.

The proof of (2.2) is reduced to showing that for any finite morphism  $\bar{\pi} : \bar{T} \rightarrow \bar{S}$ ,  $\bar{\pi}^* \bar{\mathcal{E}}$  is pseudo-semipositive, i. e., every quotient line bundle of  $\bar{\pi}^* \bar{\mathcal{E}}$  has non-negative degree. We will aim to show that this criterion is satisfied by a different technique, using results on degenerating Hodge structures and Corollary (1.12). Throughout this section,  $V$  will be the cohomology bundle on  $H^{n-1}$  of a fiber of  $f$ .

We begin by remarking that

$$(2.3) \quad \bar{\mathcal{E}} \cong (\Omega_{\bar{S}}^1)^\vee \otimes \bar{f}_* \Omega_{\bar{X}}^n.$$

(2.4) LEMMA. Let  $\tilde{f} : \tilde{X} \rightarrow \bar{S}$  be another completion of  $f : X \rightarrow S$ . Then

$$\tilde{f}_* \Omega_{\tilde{X}}^n \cong \bar{f}_* \Omega_{\bar{X}}^n,$$

i. e., the direct image sheaf is independent of the choice of smooth compactification of  $X$ .

PROOF. The assertion is no more than the bimeromorphic invariance of the space of holomorphic differential forms.

(2.5) REMARK. The isomorphism in (2.4) is given by the equality of the images of each sheaf inside  $j_* f_* \Omega_X^n$  ( $j : S \hookrightarrow \bar{S}$ ).

(2.6) COROLLARY. With notation as in (2.4), and additionally  $\tilde{\mathcal{E}} = \tilde{f}_* \omega_{\tilde{X}/\bar{S}}$ , there is a canonical isomorphism  $\tilde{\mathcal{E}} \cong \bar{\mathcal{E}}$ .

Thus, we may freely replace the given compactification  $\bar{X}$  by a manifold obtained by applying Hironaka's resolution of singularities, and assume therefore that  $Y = \bar{f}^{-1}(\Sigma)$  is a union of smooth divisors (not necessarily reduced) with normal crossings. We then have

(2.7) LEMMA. i)  $\Omega_{\bar{X}}^n(\log Y) \otimes \bar{f}^* \Omega_{\bar{S}}^1(\log \Sigma)^\vee \subset \omega_{\bar{X}/\bar{S}}$  (again, we regard both sides as subsheaves of  $i_*(\Omega_X^n \otimes f^* \Omega_S^1)^\vee$ , where  $i : X \hookrightarrow \bar{X}$ ), with quotient supported on  $Y$ . Moreover, we have equality if and only if  $Y$  is reduced.

ii) We also have

$$\omega_{\bar{X}/\bar{S}}(-Y) \subset \Omega_{\bar{X}}^n(\log Y) \otimes \bar{f}^* \Omega_{\bar{S}}^1(\log \Sigma)^\vee.$$

PROOF. This is clear, since in the top dimension al logarithmic pole on  $Y$  (resp.  $\Sigma$ ) is the same as a first-order pole on  $Y^{\text{red}}$  (resp.  $\Sigma$ ); and also, as divisors on  $\bar{X}$ , we have

$$Y^{\text{red}} \leq Y = \bar{f}^* \Sigma,$$

with equality precisely when  $Y$  is reduced.

(2.8) COROLLARY.  $\Omega_{\bar{X}/\bar{S}}^{n-1}(\log Y) \subset \omega_{\bar{X}/\bar{S}}$ , with equality if and only if  $Y$  is reduced. Moreover,  $\omega_{\bar{X}/\bar{S}}(-Y) \subset \Omega_{\bar{X}/\bar{S}}^{n-1}(\log Y)$ .

PROOF. One need only observe that the right-hand side in (2.7, ii) is naturally isomorphic to  $\Omega_{\bar{X}/\bar{S}}^{n-1}(\log Y)$ , since we are in the top dimension.

We can now apply a theorem of Steenbrink [7, (2.11)], which implies (among other things) that for  $\mathcal{F}^{n-1} = : \mathcal{O}(\mathbf{H}^{n-1,0}) \cong f_* \Omega_{\bar{X}/\bar{S}}^{n-1}$ ,

$$(2.9) \quad \bar{\mathcal{F}}^{n-1} = : \mathcal{O}(\bar{\mathbf{H}}^{n-1,0}) \cong \bar{f}_* \Omega_{\bar{X}/\bar{S}}^{n-1}(\log Y).$$

This yields the following assertion, which gives the key comparison of the locally free sheaf  $\bar{\mathcal{E}}$  and the canonical extension bundles discussed in §1.

(2.10) PROPOSITION.  $\bar{\mathcal{F}}^{n-1} \subset \bar{\mathcal{E}}$  with equality if  $Y$  is reduced. Also,  $\bar{\mathcal{E}} \subset \bar{\mathcal{F}}^{n-1}(\Sigma)$ .

(2.11) DEFINITION. We say that a locally-free sheaf  $\mathcal{A}$  on  $\bar{S}$  satisfies condition  $P$  if for every finite morphism  $\bar{\pi} : \bar{T} \rightarrow \bar{S}$ , every quotient line bundle of  $\bar{\pi}^* \mathcal{A}$  has non-negative degree.

We are now in a position to prove:

(2.12) THEOREM ([3, pp. 785-786]).  $\bar{\mathcal{E}}$  satisfies condition  $P$ .

As it was mentioned earlier, (2.2) is a consequence of (2.12).

PROOF. As it will become apparent during the course of the argument, the discussion is unchanged if  $\bar{\mathcal{E}}$  is replaced by  $\bar{\pi}^* \bar{\mathcal{E}}$ , with  $\bar{\pi}$  as in (2.11). Therefore, we may as well assume that  $\bar{\pi}$  is the identity mapping of  $\bar{S}$ .

Let  $\bar{\mathcal{L}}$  be a quotient line bundle of  $\bar{\mathcal{E}}$ . We first assume that Hypothesis (1.3) holds (unipotent local monodromy), so that we have  $\bar{\mathcal{F}}^{n-1} \subset \bar{\mathcal{E}}$  by (2.10). Let  $\bar{\mathcal{M}}$  denote the image of  $\bar{\mathcal{F}}^{n-1}$  under

$$\bar{\mathcal{F}}^{n-1} \rightarrow \bar{\mathcal{E}} \rightarrow \bar{\mathcal{L}}.$$

It is enough to show that  $\deg \bar{\mathcal{M}} \geq 0$ . We claim that the Chern form for  $\bar{\mathcal{M}}$  in the induced metric is non-negative and is  $L_2$  in the Poincaré metric. The first assertion is a standard fact about curvatures [4, §4], whereas the second follows from the calculations that led to (1.9) and (1.11). Since the degree of  $\bar{\mathcal{M}}$  can be computed by integrating the Chern form over  $\bar{S}$ , we see that

$$\deg \bar{\mathcal{M}} \geq 0.$$

For the general case, we must prove first a refinement of (2.10). At each point  $s \in \Sigma$ , we may decompose  $\bar{\mathcal{V}}$  as

$$\bar{\mathcal{V}} = \bar{\mathcal{V}}_u \oplus \bar{\mathcal{V}}_n,$$

into a summand  $\bar{\mathcal{V}}_u$  with unipotent monodromy, and  $\bar{\mathcal{V}}_n$  where the eigenvalues of the monodromy are non-trivial roots of unity. Let  $\bar{\mathcal{G}}$  denote the extension

to  $\bar{S}$  of  $\mathcal{F}^{n-1}$  which agrees with  $j_*\mathcal{F}^{n-1} \cap [\mathcal{O}(\bar{V}_u) + t^{-1}\mathcal{O}(\bar{V}_n)]$  ( $j: S \hookrightarrow \bar{S}$ ) on  $\Sigma$ . Then:

(2.13) PROPOSITION.  $\bar{\mathcal{G}} \subset \bar{\mathcal{E}}$ .

PROOF. Fujita shows by calculation that generating sections of  $\bar{\mathcal{E}}$  have Hodge norm bounded away from zero. From (1.7) and the definition of  $\bar{V}$ , alluded to in (1.13), the Hodge norm of a section  $\sigma$  of  $\bar{V}_n$  vanishes like  $t^\alpha$  ( $0 < \alpha < 1$ ) at the origin, but that of  $t^{-1}\sigma$  blows up. The inclusion of (2.13) follows immediately.

We will prove (2.12) by means of a base change  $\bar{g}: \bar{U} \rightarrow \bar{S}$ , such that (1.3) holds on  $\bar{U}$ . With  $g: U \rightarrow S$  the restriction of  $\bar{g}$ , let  $\bar{\mathcal{H}}^{n-1}$  denote the canonical extension of the Hodge bundle  $g^*\mathcal{F}^{n-1}$  on  $U$ . Then by (2.13), coupled with the definition of  $\bar{V}$ ,

$$\bar{\mathcal{H}}^{n-1} \subset \bar{g}^*\bar{\mathcal{G}} \subset \bar{g}^*\bar{\mathcal{E}}.$$

For  $\bar{\mathcal{L}}$  a quotient line bundle of  $\bar{\mathcal{E}}$ , let  $\bar{\mathcal{N}}$  be the image of  $\bar{\mathcal{H}}^{n-1}$  in  $\bar{g}^*\bar{\mathcal{L}}$ . Then, since (1.3) holds on  $U$ ,

$$0 \leq \deg \bar{\mathcal{N}} \leq \deg \bar{g}^*\bar{\mathcal{L}} = (\deg \bar{g})(\deg \bar{\mathcal{L}}),$$

hence  $\deg \bar{\mathcal{L}} \geq 0$ , and we are finished.

We can also reprove Theorem (3.1) of [3] along similar lines:

(2.14) THEOREM. *There is a bundle decomposition*

$$\bar{\mathcal{E}} \cong \bar{\mathcal{K}} \oplus \bar{\mathcal{B}},$$

where  $\bar{\mathcal{K}}$  is trivial and  $\bar{\mathcal{B}}^\vee$  has no sections.

PROOF. We use Proposition (2.13) again. At each  $s \in \Sigma$ , we have

$$(2.15) \quad j_*\mathcal{F}^{n-1} \cap (\mathcal{O}(\bar{V}_u) \oplus t^{-1}\mathcal{O}(\bar{V}_n)) \subset \bar{\mathcal{E}}.$$

There are natural dual pairings

$$\begin{aligned} [\mathcal{O}(\bar{V}_u) \oplus t^{-1}\mathcal{O}(\bar{V}_n)] \otimes \mathcal{O}(\bar{V}) &\rightarrow \mathcal{O}_{\bar{S}} \\ \mathcal{F}^{n-1} \otimes \mathcal{F}^0 / \mathcal{F}^1 &\rightarrow \mathcal{O}_S, \end{aligned}$$

the first from the definition of the canonical extension, and the second by Hodge theory. Combining them with (2.15), we obtain

$$\mathcal{A} \rightarrow (\bar{\mathcal{E}})^\vee \subset \bar{\mathcal{F}}^0 / \bar{\mathcal{F}}^1 =: \bar{\mathcal{G}}_i^0$$

where  $\mathcal{A} = t\mathcal{O}(\bar{V}_u) \oplus \mathcal{O}(\bar{V}_n)$ .

By [9, (10.1)], every section of  $\bar{\mathcal{G}}_i^0$  is induced by a flat section of  $\mathcal{O}(\bar{V})$ , which may be taken to be everywhere of type  $(0, n-1)$ . Let  $\bar{\mathcal{K}}^\vee$  denote the free sheaf on the space of flat sections of type  $(0, n-1)$  that map into  $(\bar{\mathcal{E}})^\vee$ , and let

$$\phi: \bar{\mathcal{K}}^\vee \rightarrow (\bar{\mathcal{E}})^\vee$$

be the obvious mapping. Since  $\bar{\mathcal{K}}^\vee$  is well-known to be a summand of  $\overline{\mathcal{G}_i^0}$ , it necessarily also splits off of  $\bar{\mathcal{E}}^\vee$ . By construction, every section of  $\bar{\mathcal{E}}^\vee$  is actually a section of  $\bar{\mathcal{K}}^\vee$ , so the complementary bundle has no sections. This proves (2.14).

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