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# Nonlinear ergodic theorems and weak convergence theorems

By Norimichi HIRANO

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## Introduction.

In this paper we study the asymptotic behavior of nonexpansive mappings and of one parameter semigroups of nonexpansive mappings in Banach spaces. In [1], Baillon proved the first nonlinear ergodic theorem for nonexpansive mappings in Hilbert spaces. Reich [16] extended Baillon's result to uniformly convex Banach spaces which have Fréchet differentiable norms and Bruck [8] simplified the original argument of Reich. The weak convergence of trajectories of one parameter semigroups of nonexpansive mappings was studied by Baillon [2], Bruck [7], Pasy [16], Miyadera [12] and Reich [17]. In section 2, we give ergodic theorems for nonexpansive mappings in uniformly convex Banach spaces which satisfy Opial's condition. In section 3, we consider a necessary and sufficient condition for the weak convergence of trajectories of nonexpansive mappings and one parameter semigroups of nonexpansive mappings in Banach spaces.

## 1. Preliminaries and notations.

Let $C$ be a closed convex subset of a Banach space $E$. A mapping $T: C\to E$ is said to be nonexpansive if

$$\|Tx-Ty\|\leqq\|x-y\| \qquad \text{for all} \quad x, y\in C.$$

A one parameter semigroup $S=\{S(t): t\geqq 0\}$ of nonexpansive mappings on $C$ is a family of nonexpansive mappings of $C$ into itself satisfying the following conditions

(1.1) $\quad S(s+t)x=S(s)S(t)x \qquad$ for $s, t\geqq 0$ and $x\in C$;

(1.2) $\quad \|S(t)x-S(t)y\|\leqq\|x-y\| \qquad$ for $t\geqq 0$ and $x, y\in C$;

(1.3) $\quad S(0)x=x \qquad$ for $x\in C$;

(1.4) $\quad \lim_{t\to t_0} S(t)x=S(t_0)x \qquad$ for $t, t_0\geqq 0$ and $x\in C$.

We define $S_n x = \sum_{k=0}^{n-1} T^k x/n$ for $n \geqq 1$ and $x \in C$, and denote by $F(T)$ and $F(S)$ the set of fixed points of $T$ and the set of common fixed points of $S$, respectively. In the following, $\rightarrow$ and $\rightharpoonup$ indicate strong and weak convergence, respectively. Let $E^*$ be the dual of $E$, $\varphi(r)$ a continuous strictly increasing function on $R^1$ with $\varphi(0)=0$ and $\varphi(+\infty)=+\infty$. The duality mapping $J_\varphi$ with respect to $\varphi$ is given by

$$J_\varphi(x) = \{x^* \in E^* : (x, x^*) = \|x\| \|x^*\|, \ \|x^*\| = \varphi(\|x\|)\}.$$

The duality mapping $J_\varphi$ is said to be weakly sequentially continuous if $J_\varphi$ is single valued and $x_n \rightharpoonup x$ in $E$ implies that $\{J_\varphi x_n\}$ converges to $J_\varphi x$ in the weak* topology of $E^*$. A Banach space $E$ satisfies Opial's condition if $x_n \rightharpoonup x_0$ implies that

$$\liminf_n \|x_n - x_0\| < \liminf_n \|x_n - x\| \tag{1.5}$$

for all $x \neq x_0$. It is known that (1.5) is equivalent to the analogous condition obtained by replacing lim inf by lim sup (see [**9**]). Note that if a Banach space $E$ has a weakly sequentially continuous duality mapping $J_\varphi$, then $E$ satisfies Opial's condition [**9**]. Let $E$ be a Banach space, $A$ be a subset of $E \times E$ and $x \in E$. Then we define $Ax = \{y \in E : [x, y] \in A\}$, and set $D(A) = \{x \in E : Ax \neq \emptyset\}$. A subset $A \subset E \times E$ is said to be accretive if for any $[x_i, y_i] \in A$, $i=1, 2$, there exists $j \in J_\varphi(x_1 - x_2)$ such that

$$(y_1 - y_2, j) \geqq 0.$$

An accretive set $A$ with $D(A) \subset C$ is said to be maximal accretive in $C$ if it is not properly contained in any accretive set $B$ of $E \times E$ with $D(B) \subset C$. A sequence $\{x_n\} \subset E$ is said to be (weakly) almost convergent to a point $x$ in $E$ if

$$\text{(weak-)}\lim_n \sum_{k=0}^{n-1} x_{k+i}/n = x \quad \text{uniformly in } i=1, 2, \cdots.$$

## 2. Nonlinear ergodic theorems.

THEOREM 2.1. *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ which satisfies Opial's condition, $T: C \rightarrow C$ be a nonexpansive mapping with a fixed point, and $x \in C$. Then $\{T^n x\}$ is weakly almost convergent to a fixed point of $T$.*

To prove Theorem 2.1, we need some lemmas.

LEMMA 2.1. *Let $F$ be a closed convex subset of a reflexive Banach space $E$ and $\{x_n\} \subset E$ be a bounded sequence such that for each $y \in F$, $\lim_n \|x_n - y\|$ exists. Then there exists $y_0 \in F$ such that*

$$\lim_n \|x_n - y_0\| = \min\{\lim_n \|x_n - y\| : y \in F\}. \tag{2.1}$$

PROOF. Let $r = \inf\{\lim_n \|x_n - y\| : y \in F\}$ and $D_k = \{y \in F : \lim_n \|x_n - y\| \leqq r + 1/k\}$ for $k \geqq 1$. Then for each $k \geqq 1$, $D_k$ is weakly compact convex and $D_{k+1} \subset D_k$. Therefore $\bigcap_k D_k \neq \emptyset$ and this completes the proof.

Let $F$, $E$ and $\{x_n\}$ be as in Lemma 2.1. Then we define

$$r(\{x_n\}, y) = \lim_n \|x_n - y\|, \quad \text{for} \quad y \in F,$$

$$r(\{x_n\}, F) = \min\{r(\{x_n\}, y) : y \in F\}.$$

LEMMA 2.2. *Let $F$ be a closed convex subset of a uniformly convex Banach space $E$ and $\Lambda$ be a set of bounded sequences in $E$ which satisfies the following conditions:*

(2.2) *If $\{x_n\} \in \Lambda$, then for each $y \in F$, $\lim_n \|x_n - y\|$ exists;*

(2.3) *if $\{x_n\}$, $\{y_n\} \in \Lambda$, then there exists $\{z_n\} \in \Lambda$ such that*

$$r(\{z_n\}, y) \leqq r(\{x_n\}, y)$$

*and*

$$r(\{z_n\}, y) \leqq r(\{y_n\}, y) \qquad \text{for all } y \in F.$$

*Let $r = \inf\{r(\{x_n\}, F) : \{x_n\} \in \Lambda\}$ and $\{\{x_n^{(i)}\} : i \geqq 1\}$ be a sequence in $\Lambda$ such that $\lim_i r(\{x_n^{(i)}\}, F) = r$. Then there exists a sequence $\{z_i\} \subset F$ such that $r(\{x_n^{(i)}\}, F) = r(\{x_n^{(i)}\}, z_i)$ for all $i \geqq 1$ and it follows that $\{z_i\}$ converges to a point in $F$.*

PROOF. The existence of $\{z_i\}$ is a direct consequence of Lemma 2.1. Now we shall show that $\{z_i\}$ is a Cauchy sequence and hence $\{z_i\}$ converges to a point in $F$. If $r=0$, then for each $i, j \geqq 1$, there exists $\{y_n\} \in \Lambda$ such that $r(\{x_n^{(i)}\}, z_i) \geqq r(\{y_n\}, z_i)$ and $r(\{x_n^{(j)}\}, z_j) \geqq r(\{y_n\}, z_j)$ and hence we have

$$\begin{aligned} \|z_i - z_j\| &\leqq \lim_n \|y_n - z_i\| + \lim_n \|y_n - z_j\| \\ &= r(\{y_n\}, z_i) + r(\{y_n\}, z_j) \\ &\leqq r(\{x_n^{(i)}\}, z_i) + r(\{x_n^{(j)}\}, z_j). \end{aligned} \tag{2.4}$$

Since $\lim_i r(\{x_n^{(i)}\}, z_i) = \lim_i r(\{x_n^{(i)}\}, F) = 0$, $\{z_i\}$ is a Cauchy sequence. Let $r \neq 0$. If $\{z_i\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ such that for any $k \geqq 1$, there exist $j, j' \geqq k$ with $\|z_j - z_{j'}\| > \varepsilon$. Choose $c$ so small that $r > (r+c) \cdot (1 - \delta(\varepsilon/(r+c)))$, where $\delta$ is the modulus of convexity of the norm of $E$. Let $i, j$ be positive integers such that $\|z_i - z_j\| > \varepsilon$, $r(\{x_n^{(i)}\}, F) \leqq r + c$, and $r(\{x_n^{(j)}\}, F) \leqq r + c$ and let $\{y_n\}$ be a sequence in $\Lambda$ such that $r(\{y_n\}, z_i) \leqq r(\{x_n^{(i)}\}, z_i)$ and $r(\{y_n\}, z_j) \leqq r(\{x_n^{(j)}\}, z_j)$. Then, by the definition of the modulus of the convexity, we obtain

$$r(\{y_n\},\ F) \leqq r(\{y_n\},\ (z_i+z_j)/2) = \lim_n \|y_n-(z_i+z_j)/2\| \leqq (r+c)(1-\delta(\varepsilon/(r+c))) < r. \tag{2.5}$$

This contradicts the definition of $r$.

LEMMA 2.3. *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ which satisfies Opial's condition, $T: C \to C$ be a nonexpansive mapping with a fixed point, and $\{x_n\} \subset C$ be a sequence such that $\lim_n \|Tx_n - x_n\| = 0$ and $\lim_n \|x_n - y\|$ exists for all $y \in F(T)$. Then $\{x_n\}$ converges weakly to a point $z \in F(T)$ such that $r(\{x_n\}, z) = r(\{x_n\}, F(T))$.*

PROOF. By Theorem 8.4 of Browder [**6**], any weak subsequential limit of $\{x_n\}$ is a fixed point of $T$. Now we show that the conditions $x_{m_i} \rightharpoonup u$, $x_{n_i} \rightharpoonup v$ imply $u = v \in F(T)$. If $u \neq v$, then by Opial's condition,

$$\lim_i \|x_{n_i} - u\| = \lim_i \|x_{m_i} - u\| < \lim_i \|x_{m_i} - v\| = \lim_i \|x_{n_i} - v\| < \lim_i \|x_{n_i} - u\|. \tag{2.6}$$

This is impossible. Therefore $\{x_n\}$ converges weakly to a point $z \in F(T)$. Also by using Opial's condition, we can see that $r(\{x_n\}, z) = r(\{x_n\}, F(T))$.

LEMMA 2.4 (Lemma 4, [**10**]). *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$, $T: C \to C$ be a nonexpansive mapping with a fixed point, and $x \in C$. Then for each $n \geqq 0$,*

$$\lim_i \|S_n T^k T^i x - T^k S_n T^i x\| = 0, \text{ uniformly in } k \geqq 0. \tag{2.7}$$

LEMMA 2.5. *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ which satisfies Opial's condition, $T: C \to C$ be a nonexpansive mapping with a fixed point, and $x \in C$. Let $\{S_{2n} T^{k_n} x\}_{n \geq 0}$ satisfy that*

$$k_{n+1} \geqq k_n \text{ for all } n \geqq 1 \text{ and } \lim_n \|T^k S_{2n} T^{k_n} x - S_{2n} T^{k_n + k} x\| = 0 \text{ uniformly in } k \geqq 0. \tag{2.8}$$

*Then for each* $y \in F(T)$, $\lim_n \|S_{2^n} T^{k_n} x - y\|$ *exists and* $\{S_{2^n} T^{k_n} x\}$ *converges weakly to a fixed point of* $T$.

PROOF. Let $y \in F(T)$ and $r = \liminf_n \|S_{2^n} T^{k_n} x - y\|$. For arbitrary $\varepsilon > 0$, there exists $n_0 \geqq 1$ such that for any $n \geqq n_0$,

$$\|T^k S_{2^n} T^{k_n} x - S_{2^n} T^{k_n + k} x\| < \varepsilon/2$$

uniformly in $k \geqq 0$. Hence we choose $n \geqq n_0$ such that

$$\|S_{2^n} T^{k_n} x - y\| < r + \varepsilon/2 .$$

Then

$$
\begin{aligned}
(2.9)\qquad & \|S_{2^{n+1}} T^{k_{n+1}} x - y\| \\
&= \|(T^{k_{n+1}} x + T^{k_{n+1}+1} x + \cdots + T^{k_{n+1}+2^{n+1}-1} x)/2^{n+1} - y\| \\
&= \|\{(T^{k_{n+1}} x + \cdots + T^{k_{n+1}+2^n-1} x)/2^n \\
&\qquad + (T^{k_{n+1}+2^n} x + \cdots + T^{k_{n+1}+2^{n+1}-1} x)/2^n\}/2 - y\| \\
&= \|(S_{2^n} T^{k_{n+1}} x + S_{2^n} T^{k_{n+1}+2^n} x)/2 - y\| \\
&\leqq (\|S_{2^n} T^{k_{n+1}} x - T^{k_{n+1}-k_n} S_{2^n} T^{k_n} x\| + \|T^{k_{n+1}-k_n} S_{2^n} T^{k_n} x - y\|)/2 \\
&\quad + (\|S_{2^n} T^{k_{n+1}+2^n} x - T^{k_{n+1}-k_n+2^n} S_{2^n} T^{k_n} x\| \\
&\qquad + \|T^{k_{n+1}-k_n+2^n} S_{2^n} T^{k_n} x - y\|)/2 \\
&\leqq (\varepsilon/2 + r + \varepsilon/2)/2 + (\varepsilon/2 + r + \varepsilon/2)/2 \\
&= r + \varepsilon .
\end{aligned}
$$

Similarly, we obtain $\|S_{2^{n+i}} T^{k_{n+i}} x - y\| < r + \varepsilon$ for all $i \geqq 0$. Therefore

$$\lim_n \|S_{2^n} T^{k_n} x - y\|$$

exists. While the condition (2.8) implies

$$
\begin{aligned}
& \limsup_n \|T S_{2^n} T^{k_n} x - S_{2^n} T^{k_n} x\| \\
&\leqq \lim_n \|T S_{2^n} T^{k_n} x - S_{2^n} T^{k_n+1} x\| + \lim_n \|S_{2^n} T^{k_n+1} x - S_{2^n} T^{k_n} x\| \\
&= \lim_n \|T^{k_n+2^n} x - T^{k_n} x\|/2^n \\
&= 0 .
\end{aligned}
$$

Hence by Lemma 2.3, we obtain that $\{S_{2^n} T^{k_n} x\}$ converges weakly to a fixed point of $T$.

PROOF OF THEOREM 2.1. By Lemma 2.4, there exists a sequence $\{S_{2n}T^{k_n}x\}$ in $E$ which satisfies the condition (2.8). We set $\Lambda=\{S_{2n}T^{h_n}x : h_n \geqq k_n, h_{n+1}\geqq h_n$ for all $n\geqq 1\}$. Then each element of $\Lambda$ satisfies the condition (2.8). For simplicity, we set $A_{h_n}x=S_{2n}T^{h_n}x$ for each sequence $\{h_n\}_{n\geq 1}$ of integers. By Lemma 2.5, we have that for any $\{A_{h_n}x\}\in\Lambda$ and $y\in F(T)$, $\lim_n\|A_{h_n}x-y\|$ exists and $\{A_{h_n}x\}$ converges weakly to a fixed point of $T$. While if $\{A_{h_n}x\}$, $\{A_{m_n}x\}\in\Lambda$ and $h_n\geqq m_n$ for all $n\geqq 1$, then the condition (2.8) implies that for each $y\in F(T)$,

$$\begin{aligned}(2.10)\qquad &\lim_n\|A_{h_n}x-y\|\\ &\leqq\lim_n\|A_{h_n}x-T^{h_n-m_n}A_{m_n}x\|+\lim_n\|T^{h_n-m_n}A_{m_n}x-y\|\\ &\leqq\lim_n\|S_{2n}T^{m_n+(h_n-m_n)}x-T^{h_n-m_n}S_{2n}T^{m_n}x\|+\lim_n\|A_{m_n}x-y\|\\ &=\lim_n\|A_{m_n}x-y\|\,.\end{aligned}$$

Therefore $\Lambda$ satisfies the conditions (2.2), (2.3) for $F=F(T)$. Now we set $r=\inf\{r(\{A_{h_n}x\}, F(T)) : \{A_{h_n}x\}\in\Lambda\}$. Then there exists a sequence $\{\{A_{h_n^{(i)}}x\} : i\geqq 1\}$ in $\Lambda$ such that $\lim_i r(\{A_{h_n^{(i)}}x\}, F(T))=r$. By Lemma 2.2, there exists a sequence $\{y_i\}\subset F(T)$ such that $r(\{A_{h_n^{(i)}}x\}, F(T))=r(\{A_{h_n^{(i)}}x\}, y_i)$ for all $i\geqq 1$. Also by Lemma 2.2, it follows that $\{y_i\}$ converges to a point $y$ in $F(T)$. If we set $h_n=\max\{h_n^{(i)} : 1\leqq i\leqq n\}$ for all $n\geqq 1$, then it follows that $\{A_{h_n}x\}\in\Lambda$ and

$$\begin{aligned}r(\{A_{h_n}x\}, y)&=\lim_i r(\{A_{h_n}x\}, y_i)\\ &\leqq\lim_i r(\{A_{h_n}x\}, y_i)\\ &=r\,.\end{aligned}$$

Therefore $r(\{A_{h_n}x\}, F(T))=r(\{A_{h_n}x\}, y)=r$ and $\{A_{h_n}x\}$ converges weakly to $y$. Moreover we obtain that each $\{A_{m_n}x\}\in\Lambda$ such that $m_n\geqq h_n$ for all $n\geqq 1$ converges weakly to $y$. In fact, if $m_n\geqq h_n$ for all $n\geqq 1$ and $A_{m_n}x\rightharpoonup z\ (\neq y)$, then

$$\begin{aligned}(2.11)\qquad \lim_n\|A_{m_n}x-z\|&<\lim_n\|A_{m_n}x-y\|\\ &\leqq\lim_n\|A_{h_n}x-y\|\\ &=r\,.\end{aligned}$$

This contradicts the definition of $r$. Also, we can see that $A_{h_n+k2^n+i}x\rightharpoonup y$ as

$n\to\infty$ uniformly in $k\geqq 0$ and $i\geqq 0$, since $r(\{A_{h_n+k_n 2^n+i_n}\dot{x}\}, y)=r$ for all sequences $\{k_n\}$ and $\{i_n\}$ of integers. Now we show that $\{S_m T^i x\}_{m\geq 1}$ converges weakly to $y$ uniformly in $i\geqq 0$. For $n$ and $m$ with $m>h_n$,

$$(2.12)\qquad S_m T^i x=\sum_{k=0}^{m-1} T^{k+i}x/m$$

$$=\left(\sum_{k=0}^{h_n-1} T^{k+i}x+2^n\left(\sum_{k=0}^{j-1} S_{2^n}T^{h_n+k2^n+i}x\right)+\sum_{k=h_n+j2^n}^{m-1} T^{k+i}x\right)\Big/ m$$

where $m=j\cdot 2^n+h_n+r$, $r<2^n$. Since $\{S_{2^n}T^{h_n+k2^n+i}x\}_{n\geq 1}$ converges weakly to $y$ uniformly in $k$ and $i$, we obtain that $\{S_m T^i x\}_{m\geq 1}$ converges weakly to $y$, uniformly in $i\geqq 0$.

REMARK. In Theorem 2.1, we do not know if 'Opial's condition' is essential. But if $C$ is compact, it is easy to see that we need not 'Opial's condition'.

COROLLARY 2.1. *Let $C$ be a compact convex subset of a uniformly convex Banach space $E$, $T: C\to C$ be a nonexpansive mapping and $x\in C$. Then $\{T^n x\}$ is almost convergent to a fixed point of $T$.*

Let $E$, $C$, and $T$ be as in Theorem 2.1, and $P$ be the metric projection on $F(T)$. It is known that if $E$ is a Hilbert space and $x\in C$, then $\{PT^n x\}$ converges to a point in $F(T)$ and which coincides with the weak limit point of $\left\{\sum_{k=0}^{n-1} T^k x/n\right\}_{n\geq 1}$ (cf. [1]). In Banach spaces, we do not know whether the result above holds. But if $F(T)$ is compact, we have the following proposition.

PROPOSITION 2.1. *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ which satisfies Opial's condition, $T: C\to C$ be a nonexpansive mapping with a fixed point and $x\in C$. If $F(T)$ is compact, then $\left\{P\left(\sum_{k=0}^{n-1} T^k x/n\right)\right\}_{n\geq 1}$ converges to the weak limit point of $\left\{\sum_{k=0}^{n-1} T^k x/n\right\}_{n\geq 1}$.*

PROOF. Let $\{PS_{n_i}x\}_{i\geq 1}$ be a convergent subsequence of $\{PS_n x\}$. If we set $y=\text{weak-}\lim_n S_n x$ and $z=\lim_i PS_{n_i}x$, then

$$\liminf_i \|S_{n_i}x-z\|=\liminf_i \|S_{n_i}x-PS_{n_i}x\|$$

$$\leqq \liminf_i \|S_{n_i}x-y\|\,.$$

Therefore by Opial's condition, we have that $y=z$ and this completes the proof.

## 3. Weak convergence theorems.

In this section, we study the weak convergence of trajectory $\{T^n x\}$ of a nonexpansive mapping $T$ and the trajectory $\{S(t)x\}_{t\geq 0}$ of a one parameter semigroup $S$ of nonexpansive mappings. First we show the following theorem due to Miyadera [**14**] by using the result of section 2.

THEOREM 3.1. *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ which satisfies Opial's condition, $T: C \to C$ be a nonexpansive mapping with a fixed point, and $x \in C$. Then $\{T^n x\}$ converges weakly to a fixed point of $T$ if and only if* weak-$\lim_n (T^{n+1}x - T^n x) = 0$.

PROOF. By Theorem 2.1, $\{T^n x\}$ is weakly almost convergent to a fixed point of $T$. Then Theorem 3.1 follows from easy Tauberian condition (cf. Lorentz [**11**]).

PROPOSITION 3.1. *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ which satisfies Opial's condition, $T: C \to C$ be a nonexpansive mapping, and $x \in C$. If $\{T^n x\}$ converges weakly to $y \in F(T)$, then $\{PT^n x\}$ converges to $y$, where $P$ is the metric projection on $F(T)$.*

PROOF. The strong convergence of $\{PT^n x\}$ is known [**18**]. So we are sufficient to show that $\lim_n PT^n x = y$. If we set $z = \lim_n PT^n x$, then

$$\begin{aligned}\lim_n \|T^n x - z\| &\leqq \lim_n \|T^n x - PT^n x\| + \lim_n \|PT^n x - z\| \\ &= \lim_n \|T^n x - PT^n x\| \\ &\leqq \lim_n \|T^n x - y\| .\end{aligned}$$

Then by Opial's condition, we have $z = y$.

THEOREM 3.2. *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ which satisfies Opial's condition, $S = \{S(t) : t > 0\}$ be a semigroup of nonexpansive mappings on $C$ which has a conmmon fixed point and satisfies $S(s+t) = S(s)S(t)$ for all $t$, $s > 0$, and let $x \in C$. Then $\{S(t)x\}_{t>0}$ converges weakly to a common fixed point of $S$ if and only if* weak-$\lim_{t\to\infty}(S(t+h)x - S(t)x) = 0$ *for all $h > 0$.*

PROOF. We are sufficient to prove 'if part'. First we show that if $S(t_k)x \rightharpoonup u$ where $t_k \to \infty$ as $k \to \infty$, then $u \in F(S) = \bigcap_{t>0} F(S(t))$. We use the same argument as in the proof of Proposition in [**13**]. Since weak-$\lim_{t\to\infty}(S(t+s)x - S(t)x) = 0$, for all $s > 0$, we have that weak-$\lim_k S(t_k + s)x = u$ for all $s \geqq 0$. By Opial's condition it follows that

$$\begin{aligned}r_{s+t} &= \lim_k \sup \|S(t_k + s + t)x - u\| \\ &\leqq \lim_k \sup \|S(t_k + s + t)x - S(t)u\| \\ &\leqq \lim_k \sup \|S(t_k + s)x - u\| = r_s\end{aligned}$$

for all $s, t>0$. Therefore $\{r_s\}$ is convergent to $r=\inf\{r_s: s>0\}$. If $r=0$, then there exists a sequence $\{S(s_k)x\}$ with $s_k \uparrow \infty$ which converges strongly to $u$. Since $\lim_{k\to\infty} S(s_k+t)x=S(t)u$ for all $t>0$, we have that $S(t)u=u$ for all $t>0$. Let $r\neq 0$ and suppose that $\|S(t_0)u-u\|\geqq\varepsilon$ for some $\varepsilon>0$ and $t_0>0$. We choose an $\varepsilon_0>0$ such that $(r+\varepsilon_0)[1-\delta(\varepsilon/(r+\varepsilon_0)]<r$ where $\delta$ is the modulus of convexity of the norm, and choose $s_0>0$ such that $r_{s-t_0}\leqq r+\varepsilon_0$ for all $s\geqq s_0$. Then

$$\limsup_k \|S(t_k+s)x-S(t_0)u\|\leqq r+\varepsilon_0$$

and

$$\limsup_k \|S(t_k+s)x-u\|\leqq r+\varepsilon_0$$

for $s\geqq s_0$. Therefore we have that for each $s\geqq s_0$,

$$\begin{aligned} r&\leqq\limsup_k \|S(t_k+s)x-u\| \\ &<\limsup_k \|S(t_k+s)x-(S(t_0)u+u)/2\| \\ &\leqq(r+\varepsilon_0)[1-\delta(\varepsilon/(r+\varepsilon_0)]<r\,. \end{aligned}$$

This is a contradiction. Therefore we obtain that $u$ is a common fixed point of $S$. Next we show that there exists a $y\in F(S)$ such that $\lim_{t\to\infty} P_sS(t)x=y$, where $P_s$ is the metric projection on $F(S)$. Since

$$\begin{aligned} d(t+s)&=\|S(t+s)x-P_sS(t+s)x\| \\ &\leqq\|S(t+s)x-P_sS(t)x\| \\ &\leqq\|S(t)x-P_sS(t)x\|=d(t) \end{aligned}$$

for $s, t>0$, $\{d(t)\}_{t>0}$ is convergent to $d=\inf\{d(t): t>0\}$. First, let $d=0$. For $s, t>0$, we have

$$\begin{aligned} &\|P_sS(t+s)x-P_sS(t)x\| \\ &\leqq\|P_sS(t+s)x-S(t+s)x\|+\|S(t+s)x-P_sS(t)x\| \\ &\leqq\|P_sS(t+s)x-S(t+s)x\|+\|S(t)x-P_sS(t)x\| \\ &=d(t+s)+d(t)\,. \end{aligned}$$

Since $\lim_{t\to\infty} d(t)=d=0$, it follows that $\{P_sS(t)x\}$ is convergent to a point $y\in F(S)$. Next, let $r>0$. If $\{P_sS(t)x\}$ does not converges strongly, then there exists a sequence $\{P_sS(t_k)x\}$ with $t_k\uparrow\infty$ which satisfies that for some $\varepsilon>0$,

$$\|P_sS(t_j)x-P_sS(t_k)x\|\geqq\varepsilon \qquad \text{for all} \quad j,\ k\geqq 1\ (j\neq k).$$

We choose $\varepsilon'>0$ such that $(d+\varepsilon')[1-\delta(\varepsilon/(d+\varepsilon'))]<d$, and $t'>0$ such that $d(t)\leqq d+\varepsilon'$ for all $t\geqq t'$. Then by the same argument again, we have that for all $t_j>t_k\geqq t'$,

$$d\leqq\|S(t_j)x-(P_sS(t_j)x+P_sS(t_k)x)/2\|$$
$$(d+\varepsilon')[1-\delta(\varepsilon/(d+\varepsilon'))]<d\,.$$

This is a contradiction. Therefore $\{P_sS(t)x\}$ converges strongly to a point $y\in F(S)$. Now we prove that $\{S(t)x\}$ converges weakly to $y=\lim_{t\to\infty}P_sS(t)x$. Let $u=\text{weak-}\lim_k S(t_k)x$ where $t_k\to\infty$ as $k\to\infty$. Then $u\in F(S)$. If $u\neq y$, then

$$\begin{aligned}\lim_k\|S(t_k)x-y\|&=\lim_k\|S(t_k)x-P_sS(t_k)x\|\\&\leqq\lim_k\|S(t_k)x-u\|\\&<\lim_k\|S(t_k)x-y\|\,.\end{aligned}$$

Therefore we have that $u=y$ and this completes the proof.

THEOREM 3.3. *Let $C$ be a closed convex subset of a reflexive Banach space $E$ which has a weakly sequentially continuous duality mapping $J_\varphi$, $S=\{S(t):t\geqq 0\}$ be a one parameter semigroup on $C$ such that $F(S)\neq\emptyset$ and*

$$D=\{x\in C:\ \lim_{t\to 0}(x-S(t)x)/t\ \text{exists}\}$$

*is dense in $C$, and let $x\in C$. Then $\{S(t)x\}_{t\geqq 0}$ converges weakly to a common fixed point of $S$ as $t\to\infty$ if and only if* $\text{weak-}\lim_{t\to\infty}(S(t+h)x-S(t)x)=0$ *for all* $h>0$.

PROOF. By the assumption, there exists an accretive set $A\subset E\times E$ such that $\overline{D(A)}=C$, $A$ is maximal in $C$, and

$$\frac{d}{dt}S(t)z+AS(t)z\ni 0\ \text{(a. e. } t) \qquad \text{for}\quad z\in D(A)\,. \tag{3.1}$$

Let $v\in D(A)$ and $w\in Av$. If we set $\Phi(t)=\int_0^t\varphi(s)ds$ for $t\geqq 0$, then (3.1) implies that for any $z\in D(A)$,

$$\Phi(\|S(t)x-v\|)-\Phi(\|S(s)z-v\|)\leqq\int_s^t(w,\ J_\varphi(v-S(\tau)z))d\tau$$

for $t\geqq s\geqq 0$. Since $\overline{D(A)}=C$, the inequality above holds for all $z\in C$. In particular,

$$\Phi(\|S(t)x-v\|)-\Phi(\|S(s)x-v\|)\leqq\int_s^t(w,\ J_\varphi(v-S(\tau)x))d\tau \tag{3.2}$$

for $t \geqq s \geqq 0$. Suppose that $S(t_k)x \rightharpoonup y$ with $t_k \uparrow \infty$. Then by (3.2), we have that

$$-\Phi(r) \leqq \Phi(\|S(t_k+T)x-v\|)-\Phi(\|S(t_k)x-v\|) \leqq \int_0^T (w, J_\varphi(v-S(t_k+\tau)x))d\tau \tag{3.3}$$

for $T>0$, where $r=\sup\{\|S(t)x-v\| : t>0\}$. Since $\text{weak-}\lim_{t\to\infty}(S(t+\tau)x-S(t)x)=0$ for each $\tau>0$, we have that $\text{weak-}\lim_k S(t_k+\tau)x=y$ for all $\tau>0$. Therefore, if $k\to\infty$, (3.3) implies that $-\Phi(r)/T \leqq (w, J_\varphi(v-y))$. Since $T$ is arbitrary, we obtain

$$(w, J_\varphi(v-y)) \geqq 0 \quad \text{for all } v \in D(A) \text{ and } w \in Av. \tag{3.4}$$

(3.4) implies that $0 \in Ay$ because $A$ is maximal accretive in $C$. Therefore $S(t)y=y$ for all $t \geqq 0$. Now it is enough to show that if $S(t_m)x \rightharpoonup y \in F(S)$ and $S(t_n)x \rightharpoonup z \in F(S)$ for sequences $\{t_m\}$ and $\{t_n\}$ with $t_m \uparrow \infty$ and $t_n \uparrow \infty$, then $y=z \in F(S)$. Since $E$ satisfies Opial's condition, this follows from the same argument as in the proof of Lemma 2.3.

COROLLARY 3.2. *Let $C$ be a closed convex subset of a uniformly smooth Banach space $E$ which has a weakly sequentially continuous duality mapping $J_\varphi$, $S=\{S(t) : t \geqq 0\}$ be a one parameter semigroup of nonexpansive mappings on $C$ with a common fixed point of $S$, and $x \in C$. Then $\{S(t)x\}_{t\geqq 0}$ converges weakly to a common fixed point of $S$ as $t\to\infty$ if and only if* $\text{weak-}\lim_{t\to\infty}(S(t+h)x-S(t)x)=0$ *for all $h>0$.*

PROOF. If $E$ is uniformly smooth, then the generator $A_0$ of $S(t)$ has a domain dense in $C$ (cf. Baillon [3]). Therefore Corollary 3.2 follows from Theorem 3.3.

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Norimichi HIRANO
Department of Information Sciences
Tokyo Institute of Technology
Meguro, Tokyo 152
Japan