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A remark on non-enlargable Lie algebras

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Let N be a connected, non-compact, separable, C^{∞} manifold of finite dimension, and let $\boldsymbol{\Gamma}(T_{N})$ be the Lie algebra of all C^{∞} vector fields on N. In this short note, we shall remark the following:

THEOREM. *There is no "infinite dimensional Lie group" with the Lie algebra* $\boldsymbol{\Gamma}(T_{N})$.

The above result shows that Lie's third theorem does not hold in a sense in case of infinite dimensional Lie algebras, but of course it depends how we define the concept of infinite dimensional Lie groups. (See also § 3 below.) Thus to give a precise statement of the above theorem we have to fix at first the meaning of "infinite dimensional Lie groups". However, since the result that we want to obtain is a negative one, we shall fix here the definition as wide as possible.

§ 1. Definition of infinite dimensional Lie groups.

Let G be an abstract group. As usual, $G^{\boldsymbol{R}}$ denotes the group of all mappings of $\boldsymbol{R}$ into G, where the group operations are defined pointwisely. By $G_{e}^{\boldsymbol{R}}$ we denote the subgroup consisting of all $X\in G^{\boldsymbol{R}}$ such that $X(0)=e$, the identity. For each $g\in G$, $X\in G_{e}^{\boldsymbol{R}}$ we denote by $A(g)X$ an element of $G_{e}^{\boldsymbol{R}}$ defined by $(A(g)X)(t)=gX(t)g^{-1}$. A is an action of G on $G_{e}^{\boldsymbol{R}}$, which will be called the *adjoint action*.

A structure of an infinite dimensional Lie group on G is a triple $\{\mathcal{S}, \mathfrak{g}, \pi\}$ of an adjoint invariant subgroup $\mathcal{S}$ of $G_{e}^{\boldsymbol{R}}$ such that if $g(t)\in\mathcal{S}$ then $g(t+s)g(s)^{-1}\in\mathcal{S}$ for any s, an infinite dimensional topological Lie algebra $\mathfrak{g}$ and a homomorphism π of $\mathcal{S}$ *onto* the underlying additive group of $\mathfrak{g}$ satisfying the following:

(a) For every $g\in G$, there is an automorphism $\mathrm{Ad}(g)$ of $\mathfrak{g}$ such that $\pi(A(g)X)=\mathrm{Ad}(g)\pi(X)$.

(b) For every $X\in\mathcal{S}$ and $v\in\mathfrak{g}$, the mapping $t\mapsto\mathrm{Ad}(X(t))v$ is of class C^{∞} such that $d/dt|_{t=0}\ \mathrm{Ad}(X(t))v=[u, v]$, where $u=\pi(X)$ and $[\ ,\ ]$ is the Lie bracket product defined on $\mathfrak{g}$. (See [2], [3] for the definition of differentiability.)

(c) There is a mapping $\exp: \mathfrak{g}\rightarrow G$ such that for every $u\in\mathfrak{g}$, $X(t)=\exp tu$ is an element of $\mathcal{S}$, $\{\exp tu\,;\ t\in\boldsymbol{R}\}$ is a one parameter subgroup of G and $\pi(X)$

$=u$.

An element of $\mathcal{S}$ will be called a *smooth curve* in G, and $\mathfrak{g}$ will be called the *Lie algebra of* G. A group with a structure stated above will be called an *infinite dimensional Lie group*.

§ 2. Proof of Theorem.

Assume for a while that there is an infinite dimensional Lie group G having the Lie algebra $\boldsymbol{\Gamma}(T_N)$. As N is non-compact, there is $u \in \boldsymbol{\Gamma}(T_N)$ which is not a complete vector field on N. Nevertheless by assumption (c), $\exp tu$ is a *smooth* one parameter subgroup of G, and hence $\mathrm{Ad}(\exp tu): \boldsymbol{\Gamma}(T_N) \to \boldsymbol{\Gamma}(T_N)$ is a one parameter automorphism group.

Let $\mathfrak{g}_x$ be the isotropy subalgebra of $\boldsymbol{\Gamma}(T_N)$ at $x \in N$. Then, by Theorem 3 of [**1**], $\mathfrak{g}_x$ is characterized by a maximal finite codimensional subalgebra of $\boldsymbol{\Gamma}(T_N)$, and by Theorem 2 of [**1**] there is a one parameter family ϕ_t of C^∞ diffeomorphisms of N onto itself such that

$$\mathrm{Ad}(\exp tu)v = \mathrm{Ad}(\phi_t)v\,, \tag{1}$$

where $\mathrm{Ad}(\phi_t)v$ is defined by $(\mathrm{Ad}(\phi_t)v)(x) = d\phi_t v(\phi_t^{-1}(x))$. Recall that ϕ_t is defined by

$$\mathrm{Ad}(\exp tu)\mathfrak{g}_x = \mathfrak{g}_{\phi_t(x)}\,. \tag{2}$$

By (2) we get that ϕ_t is a one parameter subgroup of C^∞ diffeomorphisms of N onto itself.

By the assumed property (b), we see easily

$$\frac{d}{dt}\mathrm{Ad}(\exp tu)v = [u, \mathrm{Ad}(\exp tu)v]\,. \tag{3}$$

Using (1) and the assumption (b), we see that $\mathrm{Ad}(\phi_t)v$ is C^∞ in t such that

$$\frac{d}{dt}\mathrm{Ad}(\phi_t)v = [u, \mathrm{Ad}(\phi_t)v] \tag{4}$$

for every $v \in \boldsymbol{\Gamma}(T_N)$. Remark that the above equality makes sense on every open subset of N.

For a relatively compact open subset U of N, we denote by ψ_t a local one parameter group on U generated by u. We assume ψ_t is defined for t such that $|t| < \varepsilon$, $\varepsilon > 0$. For every $v \in \boldsymbol{\Gamma}(T_N)$, $\mathrm{Ad}(\psi_t)v$ is well-defined as a local vector field on U, and it is easy to see that

$$\frac{d}{dt}\mathrm{Ad}(\psi_t^{-1})v = -[u, \mathrm{Ad}(\psi_t^{-1})v]\,, \qquad |t| < \varepsilon\,, \tag{5}$$

on U. Remark also that $\mathrm{Ad}(\psi_t^{-1})u=u$ on U.

Now, on U

$$\frac{d}{dt}\mathrm{Ad}(\psi_t^{-1})\,\mathrm{Ad}(\phi_t)v=-[u,\ \mathrm{Ad}(\psi_t^{-1})\,\mathrm{Ad}(\phi_t)v]+\mathrm{Ad}(\psi_t^{-1})[u,\ \mathrm{Ad}(\phi_t)v]\,.$$

Since $\mathrm{Ad}(\psi_t^{-1})[u, w]=[\mathrm{Ad}(\psi_t^{-1})u, \mathrm{Ad}(\psi_t^{-1})w]$ on U, we see that the above quantity is 0 on U for every $v\in\boldsymbol{\Gamma}(T_N)$. Hence considering at each point on U, we get $\mathrm{Ad}(\phi_t)v=\mathrm{Ad}(\psi_t)v$ on U. Since v is arbitrary we get $\phi_t=\psi_t$ on U, hence $\phi_t(x)$ is an integral curve of u for every $x\in U$. Note that U can be chosen arbitrary. Thus, one can conclude that $\phi_t(x)$ is an integral curve of u for every $x\in N$ and for all t. This contradicts the incompleteness of u.

§ 3. Several remarks.

There is another definition of infinite dimensional Lie groups. By using the notion of differentiability defined in [**2**], [**3**], one can define a concept of C^∞ manifolds modeled on a topological vector space. Thus, G is an infinite dimensional Lie group modeled on a topological vector space E, if G is a C^∞ manifold and a topological group such that the group operations are C^∞. If E is a Banach space (resp. Hilbert space, Fréchet space), then G is called a Banach-Lie group (resp. Hilbert-Lie group, Fréchet-Lie group). In all such Lie groups, one can define naturally the notion of smooth curves and the Lie algebra of G using the tangent space at the identity.

It is well-known that every Banach-Lie group satisfies (a)-(c) in the previous section. Moreover, every strong ILB-Lie group defined in [**5**] also satisfies the same properties. It is not hard to see that every Fréchet-Lie group satisfies conditions (a) and (b), but it is not known yet whether there exists an exponential mapping exp.

Recall also the result of Van Est and Korthagen [**6**]. They have proved that there exists a Banach-Lie algebra which is not a Lie algebra of any Banach-Lie group, although their example is made by a pathological manner. Our Lie algebra $\boldsymbol{\Gamma}(T_N)$ is a very concrete one, but we can not make $\boldsymbol{\Gamma}(T_N)$ a Banach-Lie algebra (cf. Theorem III in [**4**]). It can only be a Fréchet-Lie algebra under a standard topology. Therefore, if $\boldsymbol{\Gamma}(T_N)$ could be a Lie algebra of a Fréchet-Lie group, then it would follow the existence of Fréchet-Lie groups without exponential mappings. Even if this is true, the author hesitates to call such a group an infinite dimensional Lie group.

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