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Complex Laplacians on compact complex homogeneous spaces

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Introduction.

M. Berger [1] proposed a problem: Is it possible to determine whether a complex hermitian manifold is Kaehler making use of the data of the spectrum of the (complex) Laplacian?

Let (M, g) be a compact hermitian manifold of complex dimension n. Let $\Delta''_{p,q}$ (resp. $\Delta_{p,q}$) be the complex (resp. real) Laplacian acting on the space $A^{p,q}(M)$ of all complex valued forms of type (p, q) (cf. § 1). Denote the spectrum of $\Delta''_{p,q}$ (resp. $\Delta_{p,q}$) on $A^{p,q}(M)$ by $\mathrm{Spec}(\Delta''_{p,q})$ (resp. $\mathrm{Spec}(\Delta_{p,q})$). It is known that $\Delta''_{p,q}=(1/2)\Delta_{p,q}$ if (M, g) is Kaehler (cf. [11]). Conversely, P. B. Gilkey [6], [7] showed that a compact hermitian manifold is Kaehler if $\mathrm{Spec}(\Delta''_{p,q})$ coincides with $\mathrm{Spec}((1/2)\Delta_{p,q})$ for $(p, q)=(0, 0)$, $(1, 0)$ and $(0, 1)$. Moreover he showed (cf. [6]) that (M, g) is Kaehler if $n\leqq 2$ and $\mathrm{Spec}(\Delta''_{0,0})=\mathrm{Spec}((1/2)\Delta_{0,0})$, and constructed a hermitian metric on a complex torus C^n/Z^{2n} $(n\geqq 3)$ which is not Kaehler and satisfies $\mathrm{Spec}(\Delta''_{0,0})=\mathrm{Spec}((1/2)\Delta_{0,0})$. But it would be perhaps significant to ask which geometric conditions are necessary for the existence of a hermitian metric satisfying the condition $\mathrm{Spec}(\Delta''_{0,0})=\mathrm{Spec}((1/2)\Delta_{0,0})$.

Now it is well known (cf. [12]) that a Kaehler manifold has the positive second Betti number $b_2(M)$. So let us consider the following problem: *Does a compact hermitian manifold with the condition* $\Delta''_{0,0}=(1/2)\Delta_{0,0}$ *have the positive second Betti number* $b_2(M)$?

$$\begin{tikzcd}
\Delta''_{0,0}=\frac{1}{2}\Delta_{0,0} \arrow[rr, dashed] \arrow[dr, "\times" description] & & b_2(M)>0 \\
& \text{Kaehler} \arrow[ur] &
\end{tikzcd}$$

The purpose of this paper is to give a partial answer to this problem.

THEOREM 2.1. *Let* (M, g) *be a compact, simply connected hermitian manifold. Suppose that the group of all holomorphic and isometric transformations of* (M, g) *acts transitively on* M. *If* $\Delta''_{0,0}=(1/2)\Delta_{0,0}$ *or* $\Delta'_{0,0}=(1/2)\Delta_{0,0}$, *then the second Betti*

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number $b_2(M)$ *of* M *is positive.*

Moreover, we shall show that on certain compact complex manifolds admitting no Kaehler metric and the positive second Betti number, there are hermitian metrics satisfying $\Delta''_{0,0}=(1/2)\Delta_{0,0}$ (cf. Theorem 3.1).

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§ 1. Preliminaries.

1.1. Let M be a compact complex manifold of complex dimension n. Let g be a hermitian metric on M. For $0\leqq r\leqq 2n$, let $A^r(M)$ be the space of all complex valued smooth r forms on M. The exterior differentiation of $A^r(M)$ into $A^{r+1}(M)$ is denoted by d. For $0\leqq p, q\leqq n$, let $A^{p,q}(M)$ be the space of complex valued smooth $(p+q)$-forms of type (p, q). Let d' ; $A^{p,q}(M)\rightarrow A^{p+1,q}(M)$ denote the differentiation with respect to complex coordinates, and let d'' ; $A^{p,q}(M)\rightarrow A^{p,q+1}(M)$ denote the differentiation with respect to the conjugates of complex coordinates. Let $\bar{*}$; $A^{p,q}(M)\rightarrow A^{n-p,n-q}(M)$ be the complex star operator on (M, g). Then the inner product (.) on $A^r(M)$ can be defined by

$$(\omega, \eta)=\int_M \omega\wedge{}^*\eta\,,$$

for $\omega, \eta\in A^r(M)$. Then $A^r(M)=\sum_{p+q=r} A^{p,q}(M)$ is the orthogonal decomposition of $A^r(M)$ with respect to this inner product.

Put $\delta=-\bar{*}d\bar{*}$, $\delta'=-\bar{*}d'\bar{*}$ and $\delta''=-\bar{*}d''\bar{*}$. Then it holds that $(d\omega, \eta)=(\omega, \delta\eta)$, $(d'\omega, \eta)=(\omega, \delta'\eta)$, and $(d''\omega, \eta)=(\omega, \delta''\eta)$ for $\omega, \eta\in A^r(M)$, and $\delta=\delta'+\delta''$. We define the operators $\Delta_{p,q}$, $\Delta'_{p,q}$ and $\Delta''_{p,q}$ on $A^{p,q}(M)$ by

$$\Delta_{p,q}=d\delta+\delta d,\quad \Delta'_{p,q}=d'\delta'+\delta'd',\quad\text{and}\quad \Delta''_{p,q}=d''\delta''+\delta''d''\,.$$

The operator $\Delta_{p,q}$ is called the real Laplacian and the operators $\Delta'_{p,q}$ and $\Delta''_{p,q}$ are called complex Laplacians. They are all elliptic operators on $A^{p,q}(M)$ and $\Delta'_{p,q}=\Delta''_{p,q}=(1/2)\Delta_{p,q}$ when a hermitian manifold (M, g) is Kaehler (cf. [12]). In this paper, we will be concerned with the operators $\Delta_{0,0}=\delta d$, $\Delta'_{0,0}=\delta'd'$ and $\Delta''_{0,0}=\delta''d''$ on $A^0(M)=A^{0,0}(M)$, we abbreviate them by Δ, Δ' and Δ'', respectively.

1.2. Let M be a simply connected compact complex manifold of complex dimension n. Let g be a hermitian metric on M. Assume that the group of all holomorphic and isometric transformations of (M, g) acts transitively on M. Let K be the identity component of this group. Let B be the isotropy subgroup of K at some point o of M. Since a maximal semi-simple subgroup of K acts transitively on M (cf. [8], [15]), we may assume that K is a semi-simple Lie group and B is a closed, connected subgroup of K, due to the simply connectedness of M. So we can identify the hermitian manifold (M, g) with a coset space K/B of K admitting an invariant complex structure and an invariant hermitian metric under the transformations τ_k ; $K/B \ni k'\cdot o \mapsto kk'\cdot o \in K/B$, $k \in K$, $o=\{K\} \in K/B$. Then J. Hano and S. Kobayashi [8] obtained the following results (cf. [**15**]):

(1) *B is a C-subgroup of K, that is, its semi-simple part B_s coincides with that of the centralizer $Z_K(T_1)$ of some toral subgroup T_1 of K.*

(2) *T_1 contains the identity component T_0 of the center of B, (so B is contained in $Z_K(T_1)$ by the connectedness of B).*

(3) $\mathrm{rank}(K) - \mathrm{rank}(B_s) = \dim(T_1)$.

(4) *$Z_K(T_1)/B$ is a complex torus.*

(5) *$K/Z_K(T_1)$ admits an invariant complex structure such that the natural projection of K/B onto $K/Z_K(T_1)$ is holomorphic. Hence, K/B is a holomorphic principal fiber bundle over $K/Z_K(T_1)$ with $Z_K(T_1)/B$ as a structure group.*

1.3. Under the above situations, following [**15**] and [8], we prepare some notations to express the above invariant complex structure and the invariant hermitian metric on $M=K/B$.

Let $\mathfrak{k}$ be the Lie algebra of all real left invariant vector fields on K, and let $\mathfrak{b}$ (resp. $\mathfrak{b}_s$, $\mathfrak{t}_1$ and $\mathfrak{t}_0$) be the subalgebra of $\mathfrak{k}$ corresponding to the closed subgroup B, (resp. B_s, T_1 and T_0) of K. Let $\mathfrak{h}_t$ be a maximal abelian subalgebra of $\mathfrak{k}$ containing $\mathfrak{t}_1$. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the complexification of $\mathfrak{k}$ (resp. $\mathfrak{h}_t$). Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. A linear map α of $\mathfrak{h}$ into $\boldsymbol{C}$ is called a root of $(\mathfrak{g}, \mathfrak{h})$ if there exists a non-zero vector X in $\mathfrak{g}$ such that $[H, X]=\alpha(H)X$, for all $H \in \mathfrak{h}$. The vector X is called a root vector for α. Let Δ be the set of all non-zero roots of $(\mathfrak{g}, \mathfrak{h})$. All $\alpha \in \Delta$ are pure imaginary valued on $\mathfrak{h}_t$. For each $\alpha \in \Delta$, we choose a root vector E_α such that $\tau E_\alpha = E_{-\alpha}$ and $\varphi_0(E_\alpha, E_{-\alpha})=-1$. Here τ is the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{k}$ and φ_0 is the Killing form of $\mathfrak{g}$. Put $U_\alpha = E_\alpha + E_{-\alpha}$ and $V_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$ for $\alpha \in \Delta$. For $\alpha \in \Delta$, we choose an element $H_\alpha \in \sqrt{-1}\,\mathfrak{h}_t$ such that $\varphi_0(H_\alpha, H)=\alpha(H)$ for all $H \in \mathfrak{h}$. Then we have $[E_\alpha, E_{-\alpha}]=-H_\alpha$ and

$$\mathfrak{k}=\mathfrak{h}_t+\sum_{\alpha\in\Delta}(\boldsymbol{R}U_\alpha+\boldsymbol{R}V_\alpha). \tag{1.1}$$

Let $\mathfrak{m}_1$ be the orthogonal complement of $\mathfrak{t}_0$ in $\mathfrak{t}_1$ with respect to the Killing form φ_0. Then there exist linearly independent rational elements $\{H_1, \cdots, H_a\}$ of $\mathfrak{h}$ such that $\mathfrak{m}_1$ (resp. $\mathfrak{t}_0$) is spanned by $\{\sqrt{-1}\,H_i\}_{i=1}^{b}$ (resp. $\{\sqrt{-1}\,H_i\}_{i=b+1}^{a}$) over $\boldsymbol{R}$. Here we put $a=\dim \mathfrak{t}_1$ and $b=\dim \mathfrak{t}_1 - \dim \mathfrak{t}_0$ and an element H of $\mathfrak{h}$ is called rational if all $\alpha \in \Delta$ take rational values at H (cf. [**15**]). In fact, there exist (cf. [**15**] p. 16, 19) linearly independent rational elements $\{H_1', \cdots, H_b', H_{b+1}, \cdots, H_a\}$ of $\mathfrak{h}$ such that $\mathfrak{t}_1$ (resp. $\mathfrak{t}_0$) is spanned by $\{\sqrt{-1}\,H_1', \cdots, \sqrt{-1}\,H_b', \sqrt{-1}\,H_{b+1}, \cdots, \sqrt{-1}\,H_a\}$ (resp. $\{\sqrt{-1}\,H_{b+1}, \cdots, \sqrt{-1}\,H_a\}$) over $\boldsymbol{R}$. Then the images of $\{H_i'\}_{i=1}^{b}$

by the orthogonal projection of $\sqrt{-1}\,\mathfrak{t}_1$ onto $\sqrt{-1}\,\mathfrak{m}_1$ with respect to the decomposition $\sqrt{-1}\,\mathfrak{t}_1=\sqrt{-1}\,\mathfrak{m}_1+\sqrt{-1}\,\mathfrak{t}_0$, constitute a rational basis of $\sqrt{-1}\,\mathfrak{m}_1$.

Let Θ be the set of all elements in Δ which vanish on $\mathfrak{t}_1$. Since $\mathfrak{b}=\mathfrak{t}_0+\sum_{\alpha\in\Theta}(\boldsymbol{R}U_\alpha+\boldsymbol{R}V_\alpha+\boldsymbol{R}\sqrt{-1}\,H_\alpha)$ and $\sum_{\alpha\in\Theta}\boldsymbol{R}\sqrt{-1}\,H_\alpha$ is the orthogonal complement of $\mathfrak{t}_1$ in $\mathfrak{h}_t$ with respect to the Killing form φ_0, we may choose $\{H_{a+1},\cdots,H_l\}$ ($l=\dim\mathfrak{t}_t$) as linearly independent rational elements of $\mathfrak{h}$ belonging to the orthogonal complement of $\sqrt{-1}\,\mathfrak{t}_1$ in $\sqrt{-1}\,\mathfrak{h}_t$. Let $\{\lambda_1,\cdots,\lambda_l\}$ be the set of linear mappings of $\mathfrak{h}$ into $\boldsymbol{C}$ defined by $\lambda_i(H_j)=\delta_{ij}$. Then each $\alpha\in\Delta$ can be expressed as a linear combination of $\lambda_1,\cdots,\lambda_l$ with rational coefficients. So we may define a lexicographic order $>$ on Δ by $\alpha=\sum_{i=1}^{l}r_i\lambda_i>0$ if and only if there exists an integer i ($1\leqq i\leqq l$) such that $r_1=\cdots=r_{i-1}=0$ and $r_i>0$. Let Δ^+ be the set of all positive roots in Δ. Let Θ_+ be the set of all elements in Δ^+ which do not belong to Θ. Put $\Theta_-=\{\alpha\in\Delta;\ -\alpha\in\Theta_+\}$. Then $\Delta=\Theta\cup\Theta_+\cup\Theta_-$ (disjoint union) and they satisfy the conditions (1) $\alpha\in\Theta\cup\Theta_+$, $\beta\in\Theta_+$ and $\alpha+\beta\in\Delta\Rightarrow\alpha+\beta\in\Theta_+$, (2) α, $\beta\in\Theta$ and $\alpha+\beta\in\Delta\Rightarrow\alpha+\beta\in\Theta$, and (3) $\alpha\in\Theta_+\Leftrightarrow-\alpha\Theta_-$. Put $\mathfrak{m}_2=\sum_{\alpha\in\Theta_+}(\boldsymbol{R}U_\alpha+\boldsymbol{R}V_\alpha)$ and $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2$, which is the orthogonal direct sum with respect to the Killing form φ_0. So the algebra $\mathfrak{k}$ decomposes as follows:

$$\mathfrak{k}=\mathfrak{b}+\mathfrak{m}\,,\qquad \mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2\,, \tag{1.2}$$

and the adjoint representation of B acts on $\mathfrak{m}_1$ trivially, and $\mathrm{Ad}(b)\mathfrak{m}_2=\mathfrak{m}_2$ for all $b\in B$.

1.4. We express the complex structure and the hermitian metric on $M=K/B$, given in 1.2 (cf. [8]). Let J be the tensor field on K/B defining the complex structure. Then the restriction of J to the origin $o=\{B\}$ of K/B induces an endomorphism I of $\mathfrak{m}$ with the following properties:

(i) $I^2=-id$,

(ii) $[X, IY]=I[X, Y]$, $X\in\mathfrak{b}$, $Y\in\mathfrak{m}$,

(iii) $I[X, Y]_{\mathfrak{m}}-[IX, Y]_{\mathfrak{m}}-[X, IY]_{\mathfrak{m}}-I[IX, IY]_{\mathfrak{m}}=0$, $X, Y\in\mathfrak{m}$.

Here $Z_{\mathfrak{m}}$ ($Z\in\mathfrak{k}$) denotes the $\mathfrak{m}$-component of Z with respect to the decomposition $\mathfrak{k}=\mathfrak{b}+\mathfrak{m}$. Moreover, the condition (5) in 1.2 induces that the actions R_t; $k\cdot o\in K/B\mapsto kt\cdot o\in K/B$ ($t\in T_1$) are holomorphic. So the endomorphism I satisfies that

(iv) $[X, IY]=I[X, Y]$, $X\in\mathfrak{t}_1$, $Y\in\mathfrak{m}$.

Then, due to the conditions (ii) and (iv), we have

$$I(\mathfrak{m}_1)=\mathfrak{m}_1\,, \tag{1.3}$$

$$IU_\alpha=\varepsilon_\alpha V_\alpha \quad\text{and}\quad IV_\alpha=-\varepsilon_\alpha U_\alpha,\quad \text{i.e.,} \tag{1.4}$$

$$IU_{\varepsilon_\alpha \alpha}=V_{\varepsilon_\alpha \alpha} \quad \text{and} \quad IV_{\varepsilon_\alpha \alpha}=-U_{\varepsilon_\alpha \alpha}\,,$$

where $\varepsilon_\alpha=\pm 1$ $(\alpha\in\Theta_+)$ (cf. [8]). In fact, we have $[X, IY]=I[X, Y]$ for all $X\in\mathfrak{h}_t$ and $Y\in\mathfrak{m}$, so we obtain that $[\sqrt{-1}\,H, IY]=0$ for all $\sqrt{-1}\,H\in\mathfrak{h}_t$, $Y\in\mathfrak{m}_1$, and $[\sqrt{-1}\,H, IU_\alpha]=\alpha(H)IV_\alpha$, $[\sqrt{-1}\,H, IV_\alpha]=-\alpha(H)IU_\alpha$, for all $\sqrt{-1}\,H\in\mathfrak{h}_t$, $\alpha\in\Theta_+$. These equalities, together with (1.2), imply (1.3) and (1.4).

Let $\mathfrak{b}^C$ (resp. $\mathfrak{m}_1^C$, $\mathfrak{m}^C$) be the complexification of $\mathfrak{b}$ (resp. $\mathfrak{m}_1$, $\mathfrak{m}$). We extend I to a complex endomorphism of $\mathfrak{m}^C$ in a natural manner. Let $\mathfrak{m}^+$ (resp. $\mathfrak{m}^-$) be the eigenspace of I belonging to the eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$). Then we have

$$\mathfrak{m}^+=\mathfrak{m}_1^+ + \sum_{\alpha\in\Theta_+} \boldsymbol{C}E_{\varepsilon_\alpha\alpha}\,,$$

$$\mathfrak{m}^-=\mathfrak{m}_1^- + \sum_{\alpha\in\Theta_+} \boldsymbol{C}E_{-\varepsilon_\alpha\alpha}\,.$$

Here $\mathfrak{m}_1^+=\mathfrak{m}_1^C\cap\mathfrak{m}^+$ and $\mathfrak{m}_1^-=\mathfrak{m}_1^C\cap\mathfrak{m}^-$, due to (1.3). The conditions (ii) and (iii) can be expressed as follows (cf. [5]):

(ii′) $\qquad [\mathfrak{b}^C, \mathfrak{m}^+]\subset\mathfrak{m}^+\,, \qquad [\mathfrak{b}^C, \mathfrak{m}^-]\subset\mathfrak{m}^-\,,$

(iii′) $\qquad$ both $\mathfrak{b}^C+\mathfrak{m}^+$ and $\mathfrak{b}^C+\mathfrak{m}^-$ are subalgebras of $\mathfrak{g}$.

The condition (iii′) is equivalent to the condition

(iii″) $\qquad \Theta\cup\{\varepsilon_\alpha\alpha\,;\ \alpha\in\Theta_+\}$ is closed, i.e.,

it contains the sum of any two of its elements whenever this sum belongs to Δ. Then there exists a suitable order $\succ$ on Δ such that $P\cup\{\varepsilon_\alpha\alpha\,;\ \alpha\in\Theta_+\}=\{\alpha\in\Delta\,;\ \alpha\succ 0\}$, where P is the set of all positive roots in Θ with respect to the order $>$ (cf. [4], [9]). Put $\Psi_+=\{\varepsilon_\alpha\alpha\,;\ \alpha\in\Theta_+\}$, $\Psi_-=\{-\varepsilon_\alpha\alpha\,;\ \alpha\in\Theta_+\}$. Then these satisfy the condition:

(1.5) $\qquad \alpha\in\Theta\cup\Psi_+,\ \beta\in\Psi_+ \ \text{and}\ \alpha+\beta\in\Delta \ \Rightarrow\ \alpha+\beta\in\Psi_+\,.$

So let $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ $(\alpha_1\succ\cdots\succ\alpha_l)$ be the fundamental root system of Δ with respect to this order $\succ$. Then, due to the above condition and the closedness of Θ, a subset $\Pi_0=\{\alpha_{a+1}, \cdots, \alpha_l\}$ is a fundamental system of Θ (cf. [8] Proposition 7.5). Let $\{H_i'\}_{i=1}^l$ be a basis of $\sqrt{-1}\,\mathfrak{h}_t$ such that $\varphi_0(H_{\alpha_i}, H_j')=\alpha_i(H_j')=\delta_{ij}$ $(1\leqq i, j\leqq l)$. Then each element H_i' $(1\leqq i\leqq l)$ is rational, $\{H_i'\}_{i=1}^a$ spans $\sqrt{-1}\,\mathfrak{t}_1$ and the images of H_i' $(a+1\leqq i\leqq l)$ by the projection of $\sqrt{-1}\,\mathfrak{h}_t$ onto $\sum_{\alpha\in\Theta}\boldsymbol{R}\sqrt{-1}\,H_\alpha$ with respect to the decomposition $\sqrt{-1}\,\mathfrak{h}_t=\sqrt{-1}\,\mathfrak{t}_1+\sum_{\alpha\in\Theta}\boldsymbol{R}\sqrt{-1}\,H_\alpha$ are linearly independent over $\boldsymbol{R}$ and rational. We denote these images by the same letter H_i' $(a+1\leqq i\leqq l)$. Then we have a basis $\{H_i'\}_{i=1}^l$ of $\mathfrak{h}$ such that each H_i' $(i=1, \cdots, l)$ is rational and

(1.6) $\mathfrak{t}_1$ (resp. $\sum_{\alpha\in\Theta}\boldsymbol{R}\sqrt{-1}\,H_\alpha$) is spanned by $\{H'_i\}_{i=1}^{a}$ (resp. $\{H'_i\}_{i=a+1}^{l}$).

Moreover, let $\{\lambda'_i\}_{i=1}^{l}$ be the linear mappings of $\mathfrak{h}$ into $\boldsymbol{C}$ defined by $\lambda'_i(H'_j)=\delta_{ij}$ $(1\leqq i,\ j\leqq l)$, then each $\alpha\in\Delta$ can be expressed as $\alpha=\sum_{i=1}^{l} r_i\lambda'_i$ $(r_i\in\boldsymbol{R})$ and

$$\alpha\succ 0\Leftrightarrow r_1=\cdots=r_{i-1}=0 \quad\text{and}\quad r_i>0 \qquad \text{for some } i\ (1\leqq i\leqq l). \tag{1.7}$$

In fact, this is immediate from the definition of $\{H'_i\}_{i=1}^{l}$ and the choice of $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ $(\alpha_1\succ\cdots\succ\alpha_l)$.

Lastly, we mention that the hermitian metric g on K/B admits the transformations τ_k $(k\in K)$ as isometries. This metric is given (cf. [**11**] p. 200) by

$$g_{k\cdot o}(\tau_{k*}X_0,\ \tau_{k*}Y_0)=\varphi(X,\ Y), \qquad \text{for } k\in K,\ X,\ Y\in\mathfrak{m}. \tag{1.8}$$

Here τ_{k*} $(k\in K)$ is the differential of τ_k at the origin o, a tangent vector $X_0\in T_o(K/B)$ is identified usually with an element $X\in\mathfrak{m}$, and $\varphi(X,\ Y)$, $X,\ Y\in\mathfrak{m}$, is an $\mathrm{Ad}(B)$-invariant inner product on $\mathfrak{m}$. Since the metric g is hermitian with respect to the above complex structure J, it satisfies that

$$\varphi(IX,\ IY)=\varphi(X,\ Y), \qquad X,\ Y\in\mathfrak{m}. \tag{1.9}$$

§ 2. Complex Laplacians.

In this section, we preserve the notations and situations as in § 1. We consider the (complex) Laplacians $\boldsymbol{\Delta}$, $\boldsymbol{\Delta}'$, $\boldsymbol{\Delta}''$ acting on $A^0(K/B)$ for the above hermitian manifold $(K/B,\ J,\ g)$. Since the transformations τ_k $(k\in K)$ act holomorphically and isometrically, $\boldsymbol{\Delta}$ (resp. $\boldsymbol{\Delta}'$, $\boldsymbol{\Delta}''$) is τ_k $(k\in K)$ invariant, i.e.,

$$\boldsymbol{\Delta}\circ\tau_k=\tau_k\circ\boldsymbol{\Delta}\ (\text{resp. } \boldsymbol{\Delta}'\circ\tau_k=\tau_k\circ\boldsymbol{\Delta}',\ \boldsymbol{\Delta}''\circ\tau_k=\tau_k\circ\boldsymbol{\Delta}''),$$

for $k\in K$, they can be expressed using the terms of the Lie algebras (cf. Proposition 2.2).

2.1. Firstly, notice that the inner product $(\cdot,\ \cdot)$ on $A^1(K/B)$ is given by

$$(\omega,\ \eta)=\int_{K/B}(\omega\,|\,\eta)dv_g, \qquad \eta,\ \omega\in A^1(K/B), \tag{2.1}$$

where dv_g is the volume element of $(K/B,\ g)$ and $(\omega|\eta)$ is the pointwise inner product of ω and η. That is, $(\omega|\eta)(x)=(\omega_x,\ \eta_x)$, $x\in K/B$, where ω_x, η_x are elements of the complexified cotangent space $T^*_x(K/B)^C$ of K/B at x and the right hand side is the hermitian inner product in $T^*_x(K/B)^C$ induced from the metric g.

Now we define a left invariant Riemannian metric $\tilde{g}$ on K by

$$\tilde{g}_k(X_k, Y_k)=\tilde{\varphi}(X, Y), \qquad X, Y\in\mathfrak{k},$$

where X_k, $Y_k\in T_k(K)$ are tangent vectors of K at k corresponding to $X, Y\in\mathfrak{k}$ and the inner product $\tilde{\varphi}$ on $\mathfrak{k}$ is given by

$$\tilde{\varphi}(Z_1+Y_1, Z_2+Y_2)=(Z_1, Z_2)+\varphi(Y_1, Y_2),$$

$Z_i\in\mathfrak{b}$, $Y_i\in\mathfrak{m}$ $(i=1, 2)$. Here $(\cdot, \cdot)$ is an arbitrary inner product on $\mathfrak{b}$ and φ is the $\mathrm{Ad}(B)$-invariant inner product on $\mathfrak{m}$ in (1.8). Then the natural projection π of K onto K/B is a Riemannian submersion of (K, g) onto $(K/B, g)$, i.e., the differential π_{*k} of π at $k\in K$ maps the subspace $\{X_k\in T_k(K); X\in\mathfrak{m}\}$ of $T_k(K)$ onto $T_{k\cdot o}(K/B)$ isometrically, π_{*k} vanishes on the subspace $\{X_k\in T_k(K); X\in\mathfrak{b}\}$ and the decomposition $T_k(K)=\{X_k; X\in\mathfrak{b}\}\oplus\{X_k; X\in\mathfrak{m}\}$ is an orthogonal direct sum with respect to the Riemannian metric $\tilde{g}$. Then we have the following lemma.

LEMMA 2.1. *For $f\in A^0(K)$, we have*

$$\int_K f(k)dv_{\tilde{g}}(k)=\int_{K/B}\left[\int_B f(kb)dv_{g'}(b)\right]dv_g(k\cdot o),$$

where g' is the Riemannian metric on B corresponding to the inner product $(\cdot, \cdot)$ on $\mathfrak{b}$ and $dv_{\tilde{g}}$ (resp. $dv_{g'}$) is the volume element on K (resp. B) corresponding to the Riemannian metric $\tilde{g}$ (resp. g'). In particular, if $f\in A^0(K)$ satisfies $f(kb)=f(k)$, $k\in K$, $b\in B$, then

$$\int_K f(k)dv_{\tilde{g}}(k)=\mathrm{vol}(B, g')\int_{K/B}f(k\cdot o)dv_g(k\cdot o).$$

Here $\mathrm{vol}(B, g')$ *is the volume of (B, g') and we regard $f\in A^0(K)$ satisfying $f(kb)=f(k)$, $b\in B$, as a function on K/B.*

PROOF. For each $x=k\cdot o\in K/B$ $(k\in K)$, the Riemannian metric g_x on the fiber $\pi^{-1}(x)$ induced from $\tilde{g}$ coincides with $L_{k^{-1}}{}^*g'$, where L_k is the left translation by $k\in K$. Then, for $f\in A^0(K)$, we have

$$\int_{\pi^{-1}(x)}(f|_{\pi^{-1}(x)})dv_{g_x}=\int_B f(kb)dv_{g'}(b),$$

where dv_{g_x} is the volume element of $(\pi^{-1}(x), g_x)$ and $f|_{\pi^{-1}(x)}$ is the restriction to $\pi^{-1}(x)$ of f. Together with Proposition A. III. 5. in [2] p. 16, Lemma 2.1 is proved.

LEMMA 2.2. *For $\eta, \omega\in A^1(K/B)$, we have*

$$\pi^*(\omega|\eta)=(\pi^*\omega|\pi^*\eta), \qquad \text{i.e.,}$$

$$(\omega_{k\cdot o}, \eta_{k\cdot o})=((\pi^*\omega)_k, (\pi^*\eta)_k), \qquad k\in K.$$

Here the right hand side is the hermitian inner product in $A^1(K)$ induced from *the metric* $\tilde{g}$ *on* K.

PROOF. This is clear from the fact that the projection π of $(K, \tilde{g})$ onto $(K/B, g)$ is a Riemannian submersion. Q. E. D.

LEMMA 2.3. *For each* $E=X+\sqrt{-1}\,Y\in\mathfrak{g}$ $(X, Y\in\mathfrak{k})$, *we have*

$$\int_K (Ef_1)\bar{f}_2 dv_{\tilde{g}} = -\int_K f_1 \overline{\tau(E) f_2}\, dv_{\tilde{g}}\,, \qquad f_1, f_2\in A^0(K)\,,$$

where τ *is the conjugation of* $\mathfrak{g}$ *with respect to* $\mathfrak{k}$ *and* $\bar{f}$ *is the complex conjugation of* $f\in A^0(K)$.

PROOF. It holds that

$$\int_K (Xf_1)\bar{f}_2 dv_{\tilde{g}} = -\int_K f_1 \overline{(Xf_2)}\, dv_{\tilde{g}}\,, \qquad X\in\mathfrak{k}\,.$$

In fact, the volume element $dv_{\tilde{g}}$ on K is invariant by the isometries L_k $(k\in K)$ of $(K, \tilde{g})$. Since K is unimodular, $dv_{\tilde{g}}$ is also invariant by the right translations R_k $(k\in K)$. Then the equality holds. Lemma 2.3 is immediate from the one. Q. E. D.

For the space $A^1(K)$ of all complex valued smooth 1-forms on K, put

$$A^1(K, B)=\{\pi^*\omega\in A^1(K)\,;\ \omega\in A^1(K/B)\}\,,$$

$$A^{1,0}(K, B)=\{\pi^*\omega\in A^1(K)\,;\ \omega\in A^{1,0}(K/B)\}\,,$$

and

$$A^{0,1}(K, B)=\{\pi^*\omega\in A^1(K)\,;\ \omega\in A^{0,1}(K/B)\}\,.$$

Then it is known (cf. [13]) that the space $A^1(K, B)$ coincides with the space of all $\eta\in A^1(K)$ such that $R_b^*\eta=\eta$ for all $b\in B$ and $i(X)\eta=0$ for all $X\in\mathfrak{b}$, where $i(X)$ is the operator of interior product by a vector field $X\in\mathfrak{b}$ on K.

Let $\{X_i\}_{i=1}^n$ (resp. $\{Y_i\}_{i=1}^n$) be a basis of $\mathfrak{m}^+$ (resp. $\mathfrak{m}^-$). Let $T^+_{k\cdot o}(K/B)$ (resp. $T^-_{k\cdot o}(K/B)$) be the space of all holomorphic (resp. anti-holomorphic) vectors of the complexification $T^C_{k\cdot o}(K/B)$ of $T_{k\cdot o}(K/B)$. Then we have

$$T^+_{k\cdot o}(K/B)=\sum_{i=1}^n \boldsymbol{C}\tau_{k*}(X_i)_o\,, \qquad T^-_{k\cdot o}(K/B)=\sum_{i=1}^n \boldsymbol{C}\tau_{k*}(Y_i)_o\,,$$

where $(X_i)_o$, $(Y_i)_o\in T^C_o(K/B)$ $(i=1, \cdots, n)$. We define complex valued left invariant 1-forms $\{\eta_i, \xi_i\}_{i=1}^n$ on K by

$$\eta_i(X_j)=\xi_i(Y_j)=\delta_{ij}\,, \qquad \eta_i(Y_j)=\xi_i(X_j)=0\,,$$

and

$$\eta_i(X)=\xi_i(X)=0 \qquad (X\in\mathfrak{b}^C)\,.$$

Then we have the following lemma.

LEMMA 2.4. *For* $\eta\in A^{1,0}(K, B)$, (*resp.* $A^{0,1}(K, B)$), *we have*

$$\eta=\sum_{i=1}^{n}\eta(X_i)\eta_i \qquad (resp.\ \eta=\sum_{i=1}^{n}\eta(Y_i)\xi_i).$$

PROOF. For $\eta=\pi^*\omega$, $\omega\in A^{1,0}(K/B)$, we have $\eta(X)=0$ $(X\in\mathfrak{b}^C)$ and $\eta(Y_i)=0$ $(i=1,\cdots,n)$. Then $\eta=\sum_{i=1}^{n}\eta(X_i)\eta_i$. For $\eta\in A^{0,1}(K,B)$, we have $\eta=\sum_{i=1}^{n}\eta(Y_i)\xi_i$, similarly. Q.E.D.

Under the above preparations, we have the following proposition.

PROPOSITION 2.1. *For $f_1, f_2\in A^0(K/B)$, we have*

$$(\Delta' f_1, f_2)=\mathrm{vol}(B, g')^{-1}\int_K D'(f_1\circ\pi)\overline{f_2\circ\pi}\,dv_{\tilde g},$$

$$(\Delta'' f_1, f_2)=\mathrm{vol}(B, g')^{-1}\int_K D''(f_1\circ\pi)\overline{f_2\circ\pi}\,dv_{\tilde g},$$

and

$$(\Delta f_1, f_2)=\mathrm{vol}(B, g')^{-1}\int_K D(f_1\circ\pi)\overline{f_2\circ\pi}\,dv_{\tilde g}.$$

Here D', D'' and D are the differential operators on K given as follows:

$$-D'=\sum_{i,j=1}^{n}(\eta_i|\eta_j)\tau(X_j)X_i,$$

$$-D''=\sum_{i,j=1}^{n}(\xi_i|\xi_j)\tau(Y_j)Y_i,$$

and

$$D=D'+D'',$$

where τ is the conjugation of $\mathfrak{m}^C$ with respect to $\mathfrak{m}$, both $(\eta_i|\eta_j)$ and $(\xi_i|\xi_j)$ are the pointwise hermitian inner product on $A^1(K)$ of $(K, \tilde g)$ and these are constant functions on K.

PROOF. For $f_1, f_2\in A^0(K/B)$, we have

$$(\Delta' f_1, f_2)=\int_{K/B}(d'f_1|d'f_2)dv_g$$

$$=\mathrm{vol}(B, g')^{-1}\int_K(\pi^*d'f_1|\pi^*d'f_2)dv_{\tilde g},$$

by Lemmas 2.1 and 2.2. Since $\pi^*d'f_j$ $(j=1, 2)$ belong to $A^{1,0}(K, B)$, we have

$$\pi^*d'f_j=\sum_{i=1}^{n}(\pi^*d'f_j)(X_i)\eta_i, \qquad (j=1, 2),$$

by Lemma 2.4. Moreover, since $\pi_*(X)_k\in T^+_{k\cdot o}(K/B)$, for $X=X_i$ $(i=1,\cdots,n)$, and

$k\in K$, we have

$$(\pi^*d'f)(X)_k=(d'f)_{k\cdot o}(\pi_*(X)_k)=(df)_{k\cdot o}(\pi_*(X)_k)$$
$$=X(f\circ\pi)(k)\,,$$

for $f\in A^0(K/B)$. Therefore we have

$$(\pi^*d'f_1|\pi^*d'f_2)=\sum_{i,j=1}^{n}X_i(f_1\circ\pi)\overline{X_j(f_2\circ\pi)}(\eta_i|\eta_j)\,.$$

Since $(\eta|\omega)$ is constant on K for η, $\omega=\eta_i$ $(i=1,\cdots,n)$, we obtain the desired result due to Lemma 2.3. The remain can be proved similarly. Q. E. D.

We take the above basis $\{X_i, Y_i\}_{i=1}^{n}$ of $\mathfrak{m}^C$ so that

$$X_i=2^{-1}(A_i-\sqrt{-1}\,B_i) \quad\text{and}\quad Y_i=2^{-1}(A_i+\sqrt{-1}\,B_i) \qquad (i=1,\cdots,n)\,,$$

where $\{A_i, B_i\}_{i=1}^{n}$ is a basis of $\mathfrak{m}$ and satisfies $IA_i=B_i$ and $IB_i=-A_i$ $(i=1,\cdots,n)$. Then $\tau(X_i)=Y_i$ and $\tau(Y_i)=X_i$, so we have $(\xi_i|\xi_j)=\overline{(\eta_i|\eta_j)}$ by the definition of the inner product $(\cdot|\cdot)$ on $A^1(K)$ and the choice of $\{\eta_i, \xi_i\}_{i=1}^{n}$. Thus we have

$$-D'=\sum_{i,j=1}^{n}(\eta_i|\eta_j)Y_jX_i\,,$$

and

$$-D''=\sum_{i,j=1}^{n}\overline{(\eta_i|\eta_j)}X_jY_i=\sum_{i,j=1}^{n}(\eta_i|\eta_j)X_iY_j\,.$$

So we obtain

$$-D'=-2^{-1}D-2^{-1}F\,, \quad\text{and}\quad -D''=-2^{-1}D+2^{-1}F\,, \tag{2.2}$$

where

$$D=\sum_{i,j=1}^{n}(\eta_i|\eta_j)(X_iY_j+Y_jX_i)\,, \tag{2.3}$$

and

$$F=\sum_{i,j=1}^{n}(\eta_i|\eta_j)[X_i, Y_j]=\sum_{i,j=1}^{n}(\eta_i|\eta_j)[X_i, \tau(X_j)]\,. \tag{2.4}$$

DEFINITION 2.1 (cf. [**10**], [**14**]). Let $\boldsymbol{D}(K/B)$ be the set of all τ_k $(k\in K)$ invariant differential operators on K/B. Let $S(\mathfrak{m})$ be the symmetric algebra over $\mathfrak{m}$, considered as a B-module by the adjoint action of B on $\mathfrak{m}$. Let $S(\mathfrak{m})_B$ be the set of all elements in $S(\mathfrak{m})$ which are invariant by the action $\mathrm{Ad}\,(b)$, $b\in B$, and let $S(\mathfrak{m})_B^C$ be the complexification of $S(\mathfrak{m})_B$. Then, for $P(Z_1,\cdots,Z_{2n})\in S(\mathfrak{m})_B^C$, we define $\lambda(P)\in\boldsymbol{D}(K/B)$ by

$$[\lambda(P)f](k\cdot o)=\Big[P\Big(\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_{2n}}\Big)f(k\exp\sum_{i=1}^{2n}y_iZ_i)\cdot o)\Big](0)\,, \tag{2.5}$$

for $f\in A^0(K/B)$. Here, in the right hand side, $\{Z_i\}_{i=1}^{2n}$ is a basis of $\mathfrak{m}$, we regard $f(k\exp(\sum_{i=1}^{2n} y_iZ_i)\cdot o)$ as a function in $(y_1,\cdots,y_{2n})$ and $P(\partial/\partial y_1,\cdots,\partial/\partial y_{2n})$ expresses the differential operator given by substituting $\partial/\partial y_1,\cdots,\partial/\partial y_{2n}$ into the polynomial $P(Z_1,\cdots,Z_{2n})$.

LEMMA 2.5. *Both the operators D and $F_{\mathfrak{m}}$ belong to $S(\mathfrak{m})_B^C$, where $F_{\mathfrak{m}}$ is the $\mathfrak{m}^C$-component of $F\in\mathfrak{k}^C$ with respect to the decomposition $\mathfrak{k}^C=\mathfrak{b}^C+\mathfrak{m}^C$.*

PROOF. We notice that the decomposition $\mathfrak{m}^C=\mathfrak{m}^++\mathfrak{m}^-$ is the orthogonal one with respect to the hermitian inner product φ, and both $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are invariant under $\mathrm{Ad}(b)$, $b\in B$. We show that $\mathrm{Ad}(b)F_{\mathfrak{m}}=F_{\mathfrak{m}}$, $b\in B$. For $b\in B$, let $X_j'=\mathrm{Ad}(b)X_j$ $(j=1,\cdots,n)$. Since $\{X_j'\}_{j=1}^n$ is also a basis of $\mathfrak{m}^+$, we may put $X_j'=\sum_{i=1}^n u_{ij}X_i$, for some unitary matrix $U=(u_{ij})$ of degree n. Let $\{\eta_i'\}_{i=1}^n$ be the dual basis of $\{X_i'\}_{i=1}^n$. Since the matrix $(\varphi(X_i',X_j'))_{1\leq i,j\leq n}$ (resp. $((\eta_i'|\eta_j'))_{1\leq i,j\leq n}$) coincides with ${}^tU(\varphi(X_k,X_l))_{1\leq k,l\leq n}\bar{U}$ (resp. ${}^t\bar{U}((\eta_k|\eta_l))_{1\leq k,l\leq n}U$), we have

$$\sum_{i,j=1}^n(\eta_i'|\eta_j')[X_i',\tau(X_j')]_{\mathfrak{m}}=\sum_{i,j=1}^n(\eta_i|\eta_j)[X_i,\tau(X_j)]_{\mathfrak{m}},$$

which is the desired. By the same manner, we may prove that $D\in S(\mathfrak{m})_B^C$.

Q. E. D.

Then we have

$$D(f\circ\pi)=(\hat{\lambda}(D)f)\circ\pi\,,$$

$$D'(f\circ\pi)=(2^{-1}(\hat{\lambda}(D)+\hat{\lambda}(F_{\mathfrak{m}}))f)\circ\pi\,,$$

and

$$D''(f\circ\pi)=(2^{-1}(\hat{\lambda}(D)-\hat{\lambda}(F_{\mathfrak{m}}))f)\circ\pi\,.$$

Thus, together with Proposition 2.1, we obtain the following proposition.

PROPOSITION 2.2. *We have*

$$\boldsymbol{\Delta}=\hat{\lambda}(D),\ \boldsymbol{\Delta}'=2^{-1}(\hat{\lambda}(D)+\hat{\lambda}(F_{\mathfrak{m}})),\ \text{and}\ \ \boldsymbol{\Delta}''=2^{-1}(\hat{\lambda}(D)-\hat{\lambda}(F_{\mathfrak{m}}))\,,$$

where $\hat{\lambda}(D)$ and $\hat{\lambda}(F_{\mathfrak{m}})$ are the differential op3rators in $\boldsymbol{D}(K/B)$ given by (2.3), (2.4) *and* (2.5).

Now we make use of the facts in §1. We may take $\{X_i\ (i=1,\cdots,u),\ E_\alpha\ (\alpha\in\Psi_+)\}$ (resp. $\{Y_i\ (i=1,\cdots,u),\ E_{-\alpha}\ (\alpha\in\Psi_+)\}$) as the basis of $\mathfrak{m}^+$ (resp. $\mathfrak{m}^-$). Here $\{X_i\}_{i=1}^u$, (resp. $\{Y_i\}_{i=1}^u$) $(u=2^{-1}b)$ is a basis of $\mathfrak{m}_1^+$ (resp. $\mathfrak{m}_1^-$) given by $X_i=2^{-1}(A_i-\sqrt{-1}\,B_i)$ (resp. $Y_i=2^{-1}(A_i+\sqrt{-1}\,B_i)$. $\{A_i,B_i\}_{i=1}^u$ is a basis of $\mathfrak{m}_1$ such that $IA_i=B_i$ and $IB_i=-A_i$ $(i=1,\cdots,u)$. For $\alpha\in\Delta$, define complex valued left invariant 1-forms ω_α on K by

$$\omega_\alpha(E_\beta)=\delta_{\alpha\beta},\ \text{ and }\ \omega_\alpha(X)=0\quad(X\in\mathfrak{b}^C+\mathfrak{m}_1^C)\,, \tag{2.6}$$

and define complex valued left invariant 1-forms $\{\eta_i,\xi_i\}_{i=1}^u$ on K by

$$\eta_i(X_j)=\xi_i(Y_j)=\delta_{ij}\,, \qquad \eta_i(Y_j)=\xi_i(X_j)=0\,, \tag{2.7}$$

and

$$\eta_i(X)=\xi_i(X)=0 \qquad (X\in \mathfrak{b}^C+\mathfrak{m}_2^C)\,.$$

Then both D and F can be expressed by

$$\begin{aligned} D= &\sum_{i,j=1}^{u}(\eta_i|\eta_j)(X_iY_j+Y_jX_i)+\sum_{\substack{i=1,\\ \alpha\in\Psi_+}}^{u}(\eta_i|\omega_\alpha)(X_iE_{-\alpha}+E_{-\alpha}X_i) \\ &+\sum_{\substack{i=1,\\ \alpha\in\Psi_+}}^{u}(\omega_\alpha|\eta_i)(E_\alpha Y_i+Y_iE_\alpha)+\sum_{\alpha,\beta\in\Psi_+}(\omega_\alpha|\omega_\beta)(E_\alpha E_{-\beta}+E_{-\beta}E_\alpha)\,, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} F= &\sum_{\substack{i=1,\\ \alpha\in\Psi_+}}^{u}(\eta_i|\omega_\alpha)[X_i, E_{-\alpha}]+\sum_{\substack{i=1,\\ \alpha\in\Psi_+}}^{u}(\omega_\alpha|\eta_i)[E_\alpha, Y_i] \\ &+\sum_{\alpha,\beta\in\Psi_+}(\omega_\alpha|\omega_\beta)[E_\alpha, E_{-\beta}]\,, \end{aligned} \tag{2.9}$$

due to (2.3), (2.4), $(\xi_i|\xi_j)=\overline{(\eta_i|\eta_j)}$, $(\xi_i|\omega_{-\alpha})=\overline{(\eta_i|\omega_\alpha)}$ and $(\omega_{-\alpha}|\omega_{-\beta})=\overline{(\omega_\alpha|\omega_\beta)}$.

2.2.

PROPOSITION 2.3. *Under the above situation, we assume $\Delta''=(1/2)\Delta$ (or $\Delta'=(1/2)\Delta$). Then $\mathfrak{m}_1=\{0\}$ (i.e.,* rank(K)=rank(B)*), or the $\mathfrak{t}_0$-component of the element $\sqrt{-1}\,H_1'$ in* (1.6) *is not zero, where $\mathfrak{t}_1=\mathfrak{m}_1+\mathfrak{t}_0$.*

PROOF. Assume that $\Delta''=(1/2)\Delta$ (or $\Delta'=(1/2)\Delta$). We suppose that the conclusion would be false. Then $\mathfrak{m}_1\neq\{0\}$ and the $\mathfrak{t}_0$-component of $\sqrt{-1}\,H_1'$ is zero. So we have $\sqrt{-1}\,H_1'\neq 0$ and it belongs to $\mathfrak{m}_1$. Then

$$\alpha(H_1')\geqq 0\,, \qquad \text{for all } \alpha\in\Psi_+\,, \tag{2.10}$$

by (1.6). The assumption $\Delta''=(1/2)\Delta$ (or $\Delta'=(1/2)\Delta$) implies

$$F_{\mathfrak{m}_1}=-\sum_{\alpha\in\Psi_+}(\omega_\alpha|\omega_\alpha)(H_\alpha)_{\mathfrak{m}_1}=0\,,$$

where $(H_\alpha)_{\mathfrak{m}_1}$ is the $\sqrt{-1}\,\mathfrak{m}_1$-component of H_α with respect to the decomposition $\sqrt{-1}\,\mathfrak{t}_1=\sqrt{-1}\,\mathfrak{m}_1+\sqrt{-1}\,\mathfrak{t}_0$. Therefore

$$\sum_{\alpha\in\Psi_+}(\omega_\alpha|\omega_\alpha)\alpha(H_1')=\varphi_0\Big(\sum_{\alpha\in\Psi_+}(\omega_\alpha|\omega_\alpha)(H_\alpha)_{\mathfrak{m}_1}, H_1'\Big)=0\,,$$

so we have $\alpha(H_1')=0$ for all $\alpha\in\Psi_+$, due to (2.10). On the other hand, $\alpha(H_1')=0$ for all $\alpha\in\Theta$, by the definition of Θ. Then we have $H_1'=0$, which is a contradiction. Q.E.D.

THEOREM 2.1. *Let (M, g) be a compact, simply connected hermitian manifold. Suppose that the group of all holomorphic and isometric transformations of (M, g) acts transitively on M. If $\Delta''=(1/2)\Delta$ (or $\Delta'=(1/2)\Delta$), then the second Betti number $b_2(M)$ of M is positive.*

PROOF. In general, if K is a compact, semi-simple Lie group and B is a closed connected subgroup of K, then $b_2(K/B)=0$ if and only if B is semi-simple (cf. [4] p. 499). Assume that $\Delta''=(1/2)\Delta$ (or $\Delta'=(1/2)\Delta$). We suppose that the conclusion would be false. Then B is semi-simple, i.e., the center $\mathfrak{t}_0$ of $\mathfrak{b}$ is zero. Then, due to Proposition 2.3, we have $\mathfrak{m}_1=\{0\}$. Thus the complex homogeneous space K/B has to satisfy that B is semi-simple and rank(K)=rank(B). But it never happens (cf. [4] p. 500 or [15] p. 14 Corollary). Q.E.D.

§3. A construction of examples for $\Delta'=\Delta''=(1/2)\Delta$.

Conversely, let us consider a problem to construct a hermitian metric g on compact homogeneous complex manifolds satisfying $\Delta'=\Delta''=(1/2)\Delta$. In this section, on some compact complex manifolds admitting no Kaehler metric and the positive second Betti number, we construct hermitian metrics satisfying $\Delta'=\Delta''=(1/2)\Delta$ (cf. Theorem 3.1).

3.1. Let K be a compact, connected, semi-simple Lie group, and assume that a closed, connected subgroup B of K is a C-subgroup and satisfies the conditions (1), (2) and (3) in 1.2. We preserve the notations and situations in 1.3.

We define an endomorphism I on $\mathfrak{m}$ by

$$I(\sqrt{-1}\,H_{2i})=\sqrt{-1}\,H_{2i-1}\,,\quad I(\sqrt{-1}\,H_{2i-1})=-\sqrt{-1}\,H_{2i}\quad (1\leqq i\leqq u)\,, \tag{3.1}$$

and

$$\begin{aligned} &I(U_\alpha)=\varepsilon_\alpha V_\alpha\,,\qquad I(V_\alpha)=-\varepsilon_\alpha U_\alpha\,,\qquad \text{i.e.,}\\ &I(U_{\varepsilon_\alpha\alpha})=V_{\varepsilon_\alpha\alpha}\,,\qquad I(V_{\varepsilon_\alpha\alpha})=-U_{\varepsilon_\alpha\alpha}\qquad (\alpha\in\Theta_+)\,, \end{aligned} \tag{3.2}$$

where $u=(1/2)b$ and $\varepsilon_\alpha=\pm 1$ $(\alpha\in\Theta_+)$ are defined as follows: For a permutation s of $\{1,\cdots,a\}$ and $\varepsilon(i)=\pm 1$ $(i=1,\cdots,a)$, $(a=\dim(\mathfrak{t}_1))$, we choose an order $\succ$ on the root system Δ in such a way that

$$\alpha=\sum_{i=1}^{a} r_i\varepsilon(i)\lambda_{s(i)}+\sum_{i=a+1}^{l} r_i\lambda_i\succ 0\qquad (r_1,\cdots,r_l\in\boldsymbol{R})$$

if and only if there exists an integer i $(1\leqq i\leqq l)$ satisfying $r_1=\cdots=r_{i-1}=0$ and $r_i>0$. For this order $\succ$, put $\Psi_+=\{\alpha\in\Delta\backslash\Theta\,;\,\alpha\succ 0\}$ and $\Psi_-=\{-\alpha\in\Delta\backslash\Theta\,;\,\alpha\succ 0\}$. Then it holds that

$$\alpha\in\Theta\cup\Psi_+,\ \beta\in\Psi_+,\ \alpha+\beta\in\Delta\ \Rightarrow\ \alpha+\beta\in\Psi_+\,. \tag{1.5}$$

So we define ε_α $(\alpha\in\Theta_+)$ by $\{\varepsilon_\alpha\alpha\,;\,\alpha\in\Theta_+\}=\Psi_+$. We extend I complex linearly to the complexification $\mathfrak{m}^C$ of $\mathfrak{m}$ usually. Then the $\sqrt{-1}$ eigenspace $\mathfrak{m}^+$ (resp. $\sqrt{-1}$ eigenspace $\mathfrak{m}^-$) of I on $\mathfrak{m}^C$ is given by

$$\mathfrak{m}^+=\mathfrak{m}_1^+ +\sum_{\alpha\in\Psi_+} CE_\alpha\quad (\text{resp. } \mathfrak{m}^-=\mathfrak{m}_1^- +\sum_{\alpha\in\Psi_+} CE_{-\alpha})\,,$$

where $\mathfrak{m}_1^+$ (resp. $\mathfrak{m}_1^-$) is spanned by $\{X_i\}_{i=1}^u$ (resp. $\{Y_i\}_{i=1}^u$) and $X_i=(1/2)(\sqrt{-1}\,H_{2i}-\sqrt{-1}\,H_{2i-1})$ (resp. $Y_i=(1/2)(\sqrt{-1}\,H_{2i}+\sqrt{-1}\,H_{2i-1})$), $(i=1, \cdots, u)$. These subspaces $\mathfrak{m}^+$ and $\mathfrak{m}^-$ satisfy the conditions (ii′) and (iii′), due to (1.5). Thus the endomorphism I on $\mathfrak{m}$ induces a tensor field J which defines a complex structure on K/B.

In connection with this complex structure on K/B, we define a hermitian inner product φ on $\mathfrak{m}$ by

$$\varphi(\sqrt{-1}\,H_i, \sqrt{-1}\,H_j)=\delta_{ij} \qquad (1\leqq i, j\leqq b), \tag{3.3}$$

$$\varphi(U_\alpha, U_\beta)=\varphi(V_\alpha, V_\beta)=2a_\alpha^{-2}\delta_{\alpha\beta}, \ \varphi(U_\alpha, V_\beta)=0 \qquad (\alpha, \beta\in\Psi_+), \tag{3.4}$$

$$\varphi(X, Y)=0 \qquad (X\in\mathfrak{m}_1, Y\in\mathfrak{m}_2). \tag{3.5}$$

Then $\{(\sqrt{-1}/\sqrt{2})H_i\ (i=1, \cdots, b), (1/\sqrt{2})\,a_\alpha U_\alpha, (1/\sqrt{2})\,a_\alpha V_\alpha\ (\alpha\in\Psi_+)\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to φ. Here a_α $(\alpha\in\Psi_+)$ have to be positive constants satisfying that

$$a_{\alpha+\beta}=a_\alpha, \qquad \text{for } \alpha\in\Psi_+, \ \beta\in\Theta. \tag{3.6}$$

We notice that φ is $\mathrm{Ad}(B)$-invariant by (3.6). Due to the definition of φ, it holds that

$$\varphi(IX, IY)=\varphi(X, Y), \qquad X, Y\in\mathfrak{m}. \tag{3.7}$$

So φ induces a hermitian metric g on the complex manifold $(K/B, J)$. Let $\tilde{g}$ be a Riemannian metric on K such that the natural projection π of K onto K/B is a Riemannian submersion, as in 2.1. We extend φ to a hermitian inner product on $\mathfrak{m}^C$, denoted by the same letter φ. Let $\{\eta_i, \xi_i\ (i=1, \cdots, u), \omega_\alpha, \omega_{-\alpha}\ (\alpha\in\Psi_+)\}$ be complex valued 1-forms on K defined by (2.6) and (2.7). Then we have

$$(\eta_i|\omega_\alpha)=0 \qquad (i=1, \cdots, u, \ \alpha\in\Psi_+), \tag{3.8}$$

and

$$(\omega_\alpha|\omega_\beta)=a_\alpha^2\delta_{\alpha\beta} \qquad (\alpha, \beta\in\Psi_+),$$

since $\{X_i, Y_i\ (i=1, \cdots, u), a_\alpha E_\alpha, a_\alpha E_{-\alpha}\ (\alpha\in\Psi_+)\}$ is an orthonormal basis of $\mathfrak{m}^C$ with respect to φ. Here $(\cdot|\cdot)$ is the pointwise inner product on $A^1(K)$ induced by the Riemannian metric $\tilde{g}$ on K. Hence F $(\in\mathfrak{k}^C)$, in (2.9), is given by

$$F=-\sum_{\alpha\in\Psi_+} a_\alpha^2 H_\alpha.$$

Therefore we have

$$\Delta''=\frac{1}{2}\Delta \ \left(\text{or } \Delta'=\frac{1}{2}\Delta\right) \iff \sum_{\alpha\in\Psi_+} a_\alpha^2\alpha(H_i)=0, \qquad (i=1, \cdots, b). \tag{3.9}$$

REMARK. In case of $\mathfrak{m}_1=\{0\}$, i.e., $\mathrm{rank}(K)=\mathrm{rank}(B)$, the right hand side of (3.9) holds always without the d-closedness condition of the fundamental Kaehler form associated to the hermitian metric g on K/B.

3.2. In particular, let $B=T_0$ be a toral subgroup of K such that $\dim(K/T_0)$ $=$even, i.e., $b=\mathrm{rank}(K)-\dim(T_0)=$even. Then $B=T_0$ is a C-subgroup of K. In fact, let T_1 be a maximal toral subgroup of K containing T_0. Then the centralizer $Z_K(T_1)$ of T_1 in K coincides with T_1 and the semi-simple part of $B=T_0$ consists of only the identity. In this case, $\mathfrak{h}_t=\mathfrak{t}_1$, $\mathfrak{b}=\mathfrak{t}_0$, $\mathfrak{b}_s=\{0\}$, $\Theta=\emptyset$ and $\Delta\backslash\Theta=\Delta$. We preserve the notations and situations in 3.1. We give a complex structure J on K/T_0 by (3.1) and (3.2), and also a hermitian inner product φ on $\mathfrak{m}$ by (3.3), (3.4) and (3.5). Then φ satisfies (3.6) and (3.7), so it gives a hermitian metric g on the complex manifold $(K/T_0, J)$. Put $\Delta'_+=\{\alpha\in\Delta^+ ; \alpha\succ 0\}$ $=\{\alpha\in\Delta^+ ; \varepsilon_\alpha=1\}$, $\Delta''_+=\{\alpha\in\Delta^+ ; \alpha\prec 0\}=\{\alpha\in\Delta^+ ; \varepsilon_\alpha=-1\}$. Then the right hand side of (3.9) is

$$(3.9') \qquad \sum_{\alpha\in\Delta'_+} a_\alpha^2\alpha(H_i)-\sum_{\alpha\in\Delta''_+} a_\alpha^2\alpha(H_i)=0 \qquad (i=1, \cdots, b).$$

For classical groups $K=SU(l+1)$, $SO(2l+1)$, $SO(2l)$ or $Sp(l)$ $(l\geqq 3)$, we will construct a $(l-2)$-dimensional toral subgroup T_0 of K, $\{H_i\}_{i=1}^l$, φ and I satisfying (3.9').

3.3. Case 1. Let $K=SU(l+1)$ $(l\geqq 3)$. Then $k=\mathfrak{su}(l+1)$ and $\mathfrak{g}=\mathfrak{sl}(l+1, \boldsymbol{C})$. Let $\mathfrak{h}_t=\left\{\begin{pmatrix}\sqrt{-1}\,\theta_1 & & 0\\ & \ddots & \\ 0 & & \sqrt{-1}\,\theta_{l+1}\end{pmatrix} ; \theta_i\in\boldsymbol{R}\ (i=1, \cdots, l+1),\ \sum_{i=1}^{l+1}\theta_i=0\right\}$ and $\mathfrak{h}=\mathfrak{h}_t^C$. Put

$$H_1=\begin{pmatrix}1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 & \\ & & & & -1\end{pmatrix} \quad\text{and}\quad H_i=\begin{pmatrix}\left.\begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix}\right\}{\scriptstyle i-1} & & & & \\ & \overset{i}{0} & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -1\end{pmatrix} \quad (2\leqq i\leqq l).$$

Then $\{H_1, \cdots, H_l\}$ are linearly independent, rational elements of $\mathfrak{h}$, which are mutually orthogonal with respect to the Killing form $\varphi_0(X, Y)=2(l+1)\,\mathrm{Trace}\,(XY)$, $X, Y\in\mathfrak{g}$. Let $\{\lambda_i\}_{i=1}^l$ be the mappings of $\mathfrak{h}$ into $\boldsymbol{C}$ defined by $\lambda_i(H_j)=\delta_{ij}$ $(1\leqq i, j\leqq l)$. We have

$$\mu_i=i\lambda_i-\lambda_{i+1}-\cdots-\lambda_l \quad (i=1, \cdots, l-1), \quad\text{and}\quad \mu_l=l\lambda_l,$$

where μ_i is a mapping of $\mathfrak{h}$ into $\boldsymbol{C}$ defined by $\mathfrak{h}\ni\begin{pmatrix}\alpha_1 & & 0\\ & \ddots & \\ 0 & & \alpha_{l+1}\end{pmatrix}\mapsto\alpha_i$ $(i=1, \cdots, l+1)$.

Let $\mathfrak{b}=\mathfrak{t}_0$ be an abelian subalgebra of $\mathfrak{k}$ spanned by $\{\sqrt{-1}\,H_3, \cdots, \sqrt{-1}\,H_l\}$. $\mathfrak{b}$ generates a $(l-2)$-dimensional toral subgroup $B=T_0$ of K. The root system Δ

of $(\mathfrak{g}, \mathfrak{h})$ is $\{\mu_i-\mu_j;\ 1\leqq i,\ j\leqq l+1,\ i\neq j\}$. The lexicographic order $>$ of Δ given by $\lambda_1>\cdots>\lambda_l>0$ induces that $\mu_1>\cdots>\mu_l>0>\mu_{l+1}$ $(\mu_{l+1}=-\mu_1-\cdots-\mu_l)$. Then the set Δ^+ of positive roots is $\{\mu_i-\mu_j;\ 1\leqq i<j\leqq l+1\}$. Let $\mathfrak{m}_1$ be a subspace of $\mathfrak{h}_{\mathfrak{k}}$ spanned by $\{\sqrt{-1}\,H_1, \sqrt{-1}\,H_2\}$. For $\alpha\in\Delta^+$, define elements U_α and V_α in $\mathfrak{k}$, as in 1.3. The complex structure J on K/T_0 is induced from the endomorphism I defined by

$$\text{(3.10)}\qquad \begin{cases} I(\sqrt{-1}\,H_2)=\sqrt{-1}\,H_1\,, & I(\sqrt{-1}\,H_1)=-\sqrt{-1}\,H_2\,, \quad \text{and} \\ I(U_\alpha)=\varepsilon_\alpha V_\alpha\,, & I(V_\alpha)=-\varepsilon_\alpha U_\alpha \quad (\alpha\in\Delta^+)\,, \end{cases}$$

as in 3.1. The numbers $\varepsilon_\alpha=\pm 1$ $(\alpha\in\Delta^+)$ are defined in such a way that $\{\varepsilon_\alpha\alpha;\ \alpha\in\Delta^+\}=\{\alpha\in\Delta;\ \alpha\succ 0\}$ with respect to the following order $\succ$ on Δ: We define the one $\succ$ on Δ by

$$\text{(3.11)}\qquad \lambda_3\succ\cdots\succ\lambda_l\succ\lambda_1\succ\lambda_2\succ 0\,.$$

That is, $\alpha=r_1\lambda_3+\cdots+r_{l-2}\lambda_l+r_{l-1}\lambda_1+r_l\lambda_2\succ 0$ if and only if

$$r_1=\cdots=r_{i-1}=0 \quad\text{and}\quad r_i>0\,, \qquad \text{for some } 1\leqq i\leqq l\,.$$

Then $\Delta'_+=\{\alpha\in\Delta_+;\ \alpha\succ 0\}$ is $\{\mu_i-\mu_j;\ 3\leqq i<j\leqq l+1\}\cup\{\mu_1-\mu_{l+1},\ \mu_2-\mu_{l+1},\ \mu_1-\mu_2\}$ and $\Delta''_+=\{\alpha\in\Delta_+;\ \alpha\prec 0\}$ is $\{\mu_1-\mu_j,\ \mu_2-\mu_j;\ 3\leqq j\leqq l+1\}$. Then we have

$$\text{(3.12)}\qquad \sum_{\alpha\in\Delta^+} a_\alpha{}^2\varepsilon_\alpha\alpha(H_1)=\{-a_{1,l+1}{}^2+a_{2,l+1}{}^2-2a_{12}{}^2\}-\{-\sum_{i=3}^{l}(a_{1i}{}^2+a_{2i}{}^2)\}\,,$$

$$\text{(3.12}'\text{)}\qquad \sum_{\alpha\in\Delta^+} a_\alpha{}^2\varepsilon_\alpha\alpha(H_2)=\{2\sum_{j=4}^{l}a_{3j}{}^2-a_{1,l+1}{}^2-a_{2,l+1}{}^2\}$$

$$-\{-3(a_{13}{}^2+a_{23}{}^2)-\sum_{j=4}^{l}(a_{1j}{}^2+a_{2j}{}^2)\}\,,$$

where we denote $a_{ij}=a_{\mu_i-\mu_j}$ $(1\leqq i<j\leqq l+1)$. Therefore we may give a_α $(\alpha\in\Delta^+)$ such that both (3.12) and (3.12′) are zero. Thus such $\{a_\alpha;\ \alpha\in\Delta^+\}$ give a hermitian metric on this complex manifold $(K/T_0, J)$ of complex dimension $(1/2)(l^2+l+2)$, which satisfies $\boldsymbol{\Delta}'=\boldsymbol{\Delta}''=(1/2)\,\boldsymbol{\Delta}$.

Case 2. Let $K=SO(2l)$ (resp. $SO(2l+1)$) $(l\geqq 3)$. Then $\mathfrak{k}=\mathfrak{o}(2l)$ (resp. $\mathfrak{o}(2l+1)$), and $\mathfrak{g}=\mathfrak{o}(2l, \boldsymbol{C})$ (resp. $\mathfrak{o}(2l+1, \boldsymbol{C})$). For $\alpha_i\in\boldsymbol{C}$ $(i=1, \cdots, l)$, let $H(\alpha_1, \cdots, \alpha_l)$ $=\begin{bmatrix} R(\alpha_1) & & 0 & \\ & \ddots & & \\ 0 & & R(\alpha_l) & \\ & & & (0) \end{bmatrix}\in\mathfrak{g}$, where $R(\alpha)=\begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$, for $\alpha\in\boldsymbol{C}$. Let $\mathfrak{h}_{\mathfrak{k}}=\{H(\theta_1, \cdots, \theta_l);\ \theta_i\in\boldsymbol{R}$ $(i=1, \cdots, l)\}$ and $\mathfrak{h}=\{H(\alpha_1, \cdots, \alpha_l);\ \alpha_i\in\boldsymbol{C}$ $(i=1, \cdots, l)\}$. Then $\mathfrak{h}_{\mathfrak{k}}$ is a maximal abelian subalgebra of $\mathfrak{k}$ and $\mathfrak{h}=\mathfrak{h}_{\mathfrak{k}}^{C}$. Let $H_i=-\sqrt{-1}\,H(0, \cdots, 0, \overset{\overset{i}{\smile}}{1}, 0, \cdots, 0)$ $(1\leqq i\leqq l)$. Then $\{H_1, \cdots, H_l\}$ are linearly independent, rational elements of $\mathfrak{h}$ which are mutually orthogonal with respect to the Killing form $\varphi_0(X, Y)=(m-2)$

$\cdot\mathrm{Trace}(XY)$, $X, Y\in\mathfrak{g}$, where $m=2l$ (resp. $2l+1$). Let $\{\lambda_i\}_{i=1}^{l}$ be the mappings of $\mathfrak{h}$ into $\boldsymbol{C}$ defined by $\lambda_i(H_j)=\delta_{ij}$ $(1\leqq i, j\leqq l)$. Then it holds that

$$\lambda_i=\sqrt{-1}\,\mu_i\,,$$

where μ_i is a mapping of $\mathfrak{h}$ into $\boldsymbol{C}$ defined by $\mathfrak{h}\ni H(\alpha_1,\cdots,\alpha_l)\mapsto\alpha_i$ $(1\leqq i\leqq l)$. Let $\mathfrak{b}=\mathfrak{t}_0$ be an abelian subalgebra of $\mathfrak{k}$ spanned by $\{\sqrt{-1}\,H_3,\cdots,\sqrt{-1}\,H_l\}$. Then $\mathfrak{b}$ generates a $(l-2)$-dimensional toral subgroup $B=T_0$ of K. The root system Δ of $(\mathfrak{g},\mathfrak{h})$ is $\{\pm\sqrt{-1}\,\mu_i\pm\sqrt{-1}\,\mu_j\,;\,1\leqq i<j\leqq l\}$ (resp. $\{\pm\sqrt{-1}\,\mu_i\pm\sqrt{-1}\,\mu_j$ $(1\leqq i<j\leqq l)$, $\pm\sqrt{-1}\,\mu_i$ $(1\leqq i\leqq l)\}$). With respect to the lexicographic order $>$ of Δ given by $\lambda_1>\cdots>\lambda_l>0$, the set Δ^+ of positive roots is $\{\sqrt{-1}\,\mu_i\pm\sqrt{-1}\,\mu_j\,;\,1\leqq i<j\leqq l\}$ (resp. $\{\sqrt{-1}\,\mu_i\pm\sqrt{-1}\,\mu_j$ $(1\leqq i<j\leqq l)$, $\sqrt{-1}\,\mu_i$ $(1\leqq i\leqq l)\}$). Let $\mathfrak{m}_1$ be a subspace of $\mathfrak{h}_t$ spanned by $\{\sqrt{-1}\,H_1,\sqrt{-1}\,H_2\}$. For $\alpha\in\Delta^+$, define elements U_α and V_α in $\mathfrak{k}$ as in 1.3. The complex structure J on K/T_0 is induced from an endomorphism I defined by the same manner as (3.10), and the numbers $\varepsilon_\alpha=\pm1$ $(\alpha\in\Delta^+)$ are also given by the order $\succ$ on Δ similar to (3.11). Then $\Delta'_+=\{\alpha\in\Delta_+\,;\,\alpha\succ0\}$ is $\{\sqrt{-1}\,\mu_i+\sqrt{-1}\,\mu_j$ $(1\leqq i<j\leqq l)$, $\sqrt{-1}\,\mu_i-\sqrt{-1}\,\mu_j$ $(3\leqq i<j\leqq l)$, $\sqrt{-1}\,\mu_1-\sqrt{-1}\,\mu_2\}$ (resp. $\{\sqrt{-1}\,\mu_i+\sqrt{-1}\,\mu_j$ $(1\leqq i<j\leqq l)$, $\sqrt{-1}\,\mu_i-\sqrt{-1}\,\mu_j$ $(3\leqq i<j\leqq l)$, $\sqrt{-1}\,\mu_1-\sqrt{-1}\,\mu_2$, $\sqrt{-1}\,\mu_i$ $(1\leqq i\leqq l)\}$), and $\Delta''_+=\{\alpha\in\Delta_+\,;\,\alpha\prec0\}$ is $\{\sqrt{-1}\,\mu_1-\sqrt{-1}\,\mu_j,\ \sqrt{-1}\,\mu_2-\sqrt{-1}\,\mu_j$ $(3\leqq j\leqq l)\}$. Thus we have

$$\sum_{\alpha\in\Delta^+}a_\alpha^2\varepsilon_\alpha\alpha(H_1)=\{\sum_{j=2}^{l}a_{1j}{}^2+b_{12}{}^2\}-\sum_{j=3}^{l}b_{1j}{}^2\,,\tag{3.13}$$

$$(\text{resp. }\{\sum_{j=2}^{l}a_{1j}{}^2+b_{12}{}^2+c_1{}^2\}-\sum_{j=3}^{l}b_{1j}{}^2)\,,$$

$$\sum_{\alpha\in\Delta^+}a_\alpha{}^2\varepsilon_\alpha\alpha(H_2)=\{\sum_{j=3}^{l}a_{2j}{}^2+a_{12}{}^2-b_{12}{}^2\}-\sum_{j=3}^{l}b_{2j}{}^2\,,\tag{3.13′}$$

$$(\text{resp. }\{\sum_{j=3}^{l}a_{2j}{}^2+a_{12}{}^2-b_{12}{}^2+c_2{}^2\}-\sum_{j=3}^{l}b_{2j}{}^2)\,,$$

where we denote $a_{ij}=a_{\sqrt{-1}\,\mu_i+\sqrt{-1}\,\mu_j}$, $b_{ij}=a_{\sqrt{-1}\,\mu_i-\sqrt{-1}\,\mu_j}$ $(1\leqq i<j\leqq l)$, $c_i=a_{\sqrt{-1}\,\mu_i}$ $(1\leqq i\leqq l)$. Therefore we may give a_α $(\alpha\in\Delta^+)$ satisfying that both (3.13) and (3,13′) are zero. Thus such $\{a_\alpha\,;\,\alpha\in\Delta^+\}$ give a hermitian metric on this complex manifold $(K/T_0, J)$ of complex dimension l^2-l+1 (resp. l^2+1), which satisfies $\boldsymbol{\Delta}'=\boldsymbol{\Delta}''=(1/2)\,\boldsymbol{\Delta}$.

Case 3. Let $K=Sp(l)=\{x\in U(2l)\,;\,{}^txJ_lx=J_l\}$ $(l\geqq3)$ where $J_l=\begin{pmatrix}0 & I_l\\ -I_l & 0\end{pmatrix}$, and I_l is the unit matrix of order l. Then $\mathfrak{k}=\mathfrak{sp}(l)=\left\{\begin{pmatrix}Z & Y\\ -{}^t\bar{Y} & -{}^tZ\end{pmatrix};\,{}^t\bar{Z}+Z=0,\ Y={}^tY\right\}$ and $\mathfrak{g}=\mathfrak{sp}(l,\boldsymbol{C})$. For $\alpha_i\in\boldsymbol{C}$ $(i=1,\cdots,l)$, let

$$H(\alpha_1, \cdots, \alpha_l)=\begin{pmatrix} \alpha_1 & & & & & \\ & \ddots & & & 0 & \\ & & \alpha_l & & & \\ & & & -\alpha_1 & & \\ & 0 & & & \ddots & \\ & & & & & -\alpha_l \end{pmatrix} \in \mathfrak{g}.$$

Let $\mathfrak{h}_t=\{H(\theta_1, \cdots, \theta_l);\ \theta_j\in \boldsymbol{R}\ (i=1, \cdots, l)\}$ and $\mathfrak{h}=\{H(\alpha_1, \cdots, \alpha_l);\ \alpha_i\in \boldsymbol{C}\ (i=1, \cdots, l)\}$. Then $\mathfrak{h}_t$ is a maximal abelian subalgebra of $\mathfrak{k}$ and $\mathfrak{h}=\mathfrak{h}_t^{\boldsymbol{C}}$. Let $H_i=H(0, \cdots, 0, \overset{i}{\check{1}}, 0, \cdots, 0)$ $(1\leqq i\leqq l)$. Then $\{H_1, \cdots, H_l\}$ are linearly independent, rational elements of $\mathfrak{h}$, which are mutually orthogonal with respect to the Killing form $\varphi_0(X, Y)=(2l+2)\operatorname{Trace}(XY)$, $X, Y\in\mathfrak{g}$. Let $\{\lambda_i\}_{i=1}^{l}$ be the mappings of $\mathfrak{h}$ into $\boldsymbol{C}$ defined by $\lambda_i(H_j)=\delta_{ij}$ $(1\leqq i, j\leqq l)$. Then it holds that

$$\lambda_i=\mu_i,$$

where μ_i is a mapping of $\mathfrak{h}$ into $\boldsymbol{C}$ defined by $\mathfrak{h}\ni H(\alpha_1, \cdots, \alpha_l)\mapsto\alpha_i$ $(1\leqq i\leqq l)$. Let $\mathfrak{b}=\mathfrak{t}_0$ be an abelian subalgebra of $\mathfrak{k}$ spanned by $\{\sqrt{-1}\,H_3, \cdots, \sqrt{-1}\,H_l\}$. Then $\mathfrak{b}$ generates a $(l-2)$-dimensional toral subgroup $B=T_0$ of K. The root system Δ of $(\mathfrak{g}, \mathfrak{h})$ is $\{\pm\mu_i\pm\mu_j\ (1\leqq i<j\leqq l),\ \pm2\mu_i\ (1\leqq i\leqq l)\}$. With respect to the lexicographic order $>$ of Δ given by $\lambda_1>\cdots>\lambda_l>0$, the set Δ^+ of positive roots is $\{\mu_i\pm\mu_j\ (1\leqq i<j\leqq l),\ 2\mu_i\ (1\leqq i\leqq l)\}$. Let $\mathfrak{m}_1$ be a subspace of $\mathfrak{h}_t$ spanned by $\{\sqrt{-1}\,H_1, \sqrt{-1}\,H_2\}$. For $\alpha\in\Delta^+$, define elements U_α and V_α in $\mathfrak{k}$, as in 1.3. The complex structure J on K/T_0 is induced from an endomorphism I defined by the same manner as (3.10), and the numbers $\varepsilon_\alpha=\pm1$ $(\alpha\in\Delta^+)$ are also given by the similar order $\succ$ on Δ as (3.11). Then $\Delta'_+=\{\alpha\in\Delta_+;\ \alpha\succ0\}$ is $\{\mu_i+\mu_j\ (1\leqq i<j\leqq l),\ \mu_i-\mu_j\ (3\leqq i<j\leqq l),\ \mu_1-\mu_2,\ 2\mu_i\ (1\leqq i\leqq l)\}$, and $\Delta''_+=\{\alpha\in\Delta_+;\ \alpha\prec0\}$ is $\{\mu_1-\mu_j, \mu_2-\mu_j\ (3\leqq j\leqq l)\}$. Thus we have

$$\sum_{\alpha\in\Delta^+} a_\alpha{}^2\varepsilon_\alpha\alpha(H_1)=\{\sum_{j=2}^{l} a_{1j}{}^2+b_{12}{}^2+c_1{}^2\}-\sum_{j=3}^{l} b_{1j}{}^2, \tag{3.14}$$

$$\sum_{\alpha\in\Delta^+} a_\alpha{}^2\varepsilon_\alpha\alpha(H_2)=\{\sum_{j=3}^{l} a_{2j}{}^2+a_{12}{}^2-b_{12}{}^2+c_2{}^2\}-\sum_{j=3}^{l} b_{2j}{}^2, \tag{3.14'}$$

where we denote $a_{ij}=a_{\mu_i+\mu_j}$, $b_{ij}=a_{\mu_i-\mu_j}$ $(1\leqq i<j\leqq l)$, and $c_i=a_{2\mu_i}$ $(1\leqq i\leqq l)$. Therefore we may give a_α $(\alpha\in\Delta^+)$ satisfying that both (3.14) and (3.14′) are zero. Thus such $\{a_\alpha;\ \alpha\in\Delta^+\}$ give a hermitian metric on this complex manifold $(K/T_0, J)$ of complex dimension l^2+1, which satisfies $\boldsymbol{\Delta}'=\boldsymbol{\Delta}''=(1/2)\,\boldsymbol{\Delta}$.

Summing up the above results, we have the following theorem.

THEOREM 3.1. *For classical groups $K=SU(l+1)$, $SO(2l)$, $SO(2l+1)$, or $Sp(l)$ $(l\geqq3)$, there exist a $(l-2)$-dimensional toral subgroup T_0 of K, a K-invariant complex structure J and a K-invariant hermitian metric g with respect to J on the coset space K/T_0 satisfying that*

$$\Delta'=\Delta''=\frac{1}{2}\Delta .$$

REMARK. The complex manifolds $(K/T_0, J)$ in Theorem 3.1 have no Kaehler metric since $\mathrm{rank}(K)=l>l-2=\mathrm{rank}(T_0)$ (cf. [**3**]).

§ 4. Compact complex parallelisable manifolds.

In this section, we assume that a compact complex manifold M of complex dimension n is parallelisable, that is, there exist n holomorphic vector fields $\{X_1, \cdots, X_n\}$ on M which are linearly independent everywhere (cf. [**16**]). Let $\{\omega_1, \cdots, \omega_n\}$ be n holomorphic 1-forms given by $\omega_\alpha(X_\beta)=\delta_{\alpha\beta}$ $(1\leqq\alpha, \beta\leqq n)$. Then the complex symmetric form $\sum_{\alpha=1}^{n}\omega_\alpha\cdot\bar{\omega}_\alpha$ gives a hermitian metric g on M, where $\bar{\omega}_\alpha$ denotes the complex conjugate of ω_α $(\alpha=1, \cdots, n)$. The complex Laplacians Δ', Δ'' of (M, g) can be calculated as follows:

$$\Delta'=-\sum_{\alpha=1}^{n}\bar{X}_\alpha X_\alpha , \qquad \Delta''=-\sum_{\alpha=1}^{n}X_\alpha\bar{X}_\alpha ,$$

where $\bar{X}$ is the complex conjugate of a vector field X. Since each vector field X_α $(\alpha=1, \cdots, n)$ is holomorphic, we have $[X_\alpha, \bar{X}_\alpha]=0$. Thus we obtain the following proposition.

PROPOSITION 4.1. *Each compact complex parallelisable manifold admits a hermitian metric satisfying* $\Delta'=\Delta''=(1/2)\Delta$.

REMARK 1. A compact complex parallelisable manifold which is not a complex torus, admits no Kaehler metric (cf. [**16**]).

REMARK 2. Recently, K. Tsukada [**17**] shows that for a compact complex hermitian manifold (M, g), a condition $\Delta''=(1/2)\Delta$ is equivalent to that (M, g) is semi-Kaehler, that is, the Kaehler form is coclosed (cf. [**18**]). Thus, due to his result, Theorem 3.1 and Proposition 4.1, we obtain examples of semi-Kaehler compact complex manifolds which admit no Kaehler metric.

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