J. Math. Soc. Japan
Vol. 33, No. 4, 1981

A note on Yoneda product

By L. R. VERMANI

(Received April 24, 1979)
(Revised Dec. 24, 1979)

1. Introduction.

Let G be a group, Z the ring of integers, m a positive integer and Z_m the ring of integers modulo m. It is well known ([**5**], Proposition 5) that Yoneda product in the cohomology ring $\mathrm{Ext}^*_{Z_mG}(Z_m, Z_m)$ is anti-commutative. The aim of the present note is to prove that this anti-commutative property does not hold in the cohomology ring $\mathrm{Ext}^*_{ZG}(Z_m, Z_m)$. Recall that $a, b \in \mathrm{Ext}^*(A, A)$ of degree r, s respectively are said to anti-commute if $ab=(-1)^{r+s}ba$.

2. Preliminaries.

Let G, Z, m and Z_m be as in the introduction. The exact sequence

$$0 \longrightarrow Z \xrightarrow{m} Z \xrightarrow{\alpha} Z_m \longrightarrow 0 \tag{2.1}$$

of trivial G-modules where α is the natural projection determines an element e of $\mathrm{Ext}^1_{ZG}(Z_m, Z)$ ([**4**], pp. 84–85; [**3**], p. 494). For ZG-modules A, B the connecting homomorphisms

$$\delta^r : \mathrm{Ext}^r_{ZG}(A, Z_m) \longrightarrow \mathrm{Ext}^{r+1}_{ZG}(A, Z) \quad \text{and}$$

$$\partial^s : H^s(G, B) \longrightarrow \mathrm{Ext}^{s+1}_{ZG}(Z_m, B)$$

are then given by ([**3**], p. 493)

$$\delta^r(a)=-ea\,, \quad a\in \mathrm{Ext}^r_{ZG}(A, Z_m) \quad \text{and}$$

$$\partial^s(b)=be\,, \quad b\in H^s(G, B)\,.$$

Here the product involved is the Yoneda product and observe that if $x\in \mathrm{Ext}^r_{ZG}(A, B)$, $y\in \mathrm{Ext}^s_{ZG}(B, C)$, then $yx\in \mathrm{Ext}^{r+s}_{ZG}(A, C)$.

For a ZG-module B, let

$$R(B): 0 \longrightarrow B \xrightarrow{\varepsilon_B} R^0(B) \xrightarrow{d^0_B} R^1(B) \xrightarrow{d^1_B} \cdots \longrightarrow R^n(B) \xrightarrow{d^n_B} R^{n+1}(B) \longrightarrow \cdots$$

denote an injective ZG-resolution of B. Then there exists a homomorphism $\phi=\{\phi^k\}:R(Z)\to R(Z_m)$ over the ZG-homomorphism $\alpha: Z\to Z_m$ i.e. $\phi^k: R^k(Z)\to R^k(Z_m)$ are ZG-homomorphisms such that

$$\phi^{k+1}d_Z^k=d_{Z_m}^k\phi^k \quad \text{for all } k\geqq 0 \text{ and} \tag{2.2}$$

$$\phi^0\varepsilon_Z=\varepsilon_{Z_m}\alpha\,.$$

ϕ then determines an element of $H^0(G, Z_m)$ [2] which we again denote by ϕ. The natural projection α also induces homomorphisms

$$\alpha^*: \operatorname{Ext}_{ZG}^r(Z_m, A)\longrightarrow H^r(G, A) \quad \text{and}$$

$$\alpha_*: \operatorname{Ext}_{ZG}^s(A, Z)\longrightarrow \operatorname{Ext}_{ZG}^s(A, Z_m)\,,$$

where A is any ZG-module, which are given by ([3], p. 493)

$$\alpha^*(a)=a\phi\,, \quad a\in\operatorname{Ext}_{ZG}^r(Z_m, A) \quad \text{and}$$

$$\alpha_*(b)=\phi b\,, \quad b\in\operatorname{Ext}_{ZG}^s(A, Z)\,.$$

3. The main result.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{m} & H^{n-1}(G, Z_m) & \xrightarrow{\partial^{n-1}} & \operatorname{Ext}_{ZG}^n(Z_m, Z_m) & \xrightarrow{\alpha^*} & H^n(G, Z_m) & \longrightarrow & \cdots \\
 & & \big\downarrow \delta^{n-1} & & \big\downarrow \delta^n & & \big\downarrow \delta^n & & \\
\cdots & \xrightarrow{m} & H^n(G, Z) & \xrightarrow{\partial^n} & \operatorname{Ext}_{ZG}^{n+1}(Z_m, Z) & \xrightarrow{\alpha^*} & H^{n+1}(G, Z) & \longrightarrow & \cdots
\end{array} \tag{3.1}$$

where the rows are long exact sequences for $\operatorname{Ext}_{ZG}(\ , Z_m)$ and $\operatorname{Ext}_{ZG}(\ , Z)$ corresponding to the extension (2.1) of Z by Z_m. We claim that

(3.2) If the Yoneda product in $\operatorname{Ext}_{ZG}^*(Z_m, Z_m)$ is anti-commutative, then $\alpha^*\delta^r(a)\alpha^*\delta^s(b)=0$ for all $a, b\in\operatorname{Ext}_{ZG}^*(Z_m, Z_m)$ of degree r, s respectively.

PROOF OF CLAIM. From the definitions of the maps α^*, δ^k and the assumed anti-commutativity of Yoneda product it follows that

$$\begin{aligned}\alpha^*\delta^r(a)\alpha^*\delta^s(b)&=(-ea\phi)(-eb\phi)=e(a(\phi e))(b\phi)\\ &=(-1)^{r+1}e((\phi e)a)(b\phi)=(-1)^{r+1}(e\phi)(eab\phi)\,.\end{aligned}$$

Set $(-1)^{r+1}eab\phi=c$ which is an element of $H^{r+s+1}(G, Z)$. Then $\alpha^*\delta^r(a)\alpha^*\delta^s(b)=e\phi c=-(\delta^{r+s+1}\alpha_*)(c)$ which is zero because the sequence $H^k(G, Z)\xrightarrow{\alpha_*}H^k(G, Z_m)\xrightarrow{\delta^k}H^{k+1}(G, Z)$ is exact.

Now suppose that G is a cyclic group of order m. For n odd, $H^n(G, Z)=0$ ([1], p. 251) and, therefore, in (3.1) $\alpha^*: \mathrm{Ext}_{ZG}^{n+1}(Z_m, Z)\to H^{n+1}(G, Z)$ is an isomorphism. Also $\delta^n: \mathrm{Ext}_{ZG}^n(Z_m, Z_m)\to \mathrm{Ext}_{ZG}^{n+1}(Z_m, Z)$ is an epimorphism. Hence for any non-zero element $\lambda\in H^{n+1}(G, Z)$ we can choose a non-zero $a\in \mathrm{Ext}_{ZG}^n(Z_m, Z_m)$ such that $\lambda=\alpha^*\delta^n(a)$. Since the integral cohomology ring $H^*(G, Z)$ is non-trivial ([1], p. 252) and the cup product coincides with Yoneda product in this case ([5], Proposition 5), we can find λ, $\mu\in H^*(G, Z)$ both non-zero such that $\lambda\mu\neq 0$. But then there exist a, $b\in \mathrm{Ext}_{ZG}^*(Z_m, Z_m)$ such that $\alpha^*\delta(a)\alpha^*\delta(b)=\lambda\mu\neq 0$. Observation (3.2), in view of this example, then shows that Yoneda product in $\mathrm{Ext}_{ZG}^*(Z_m, Z_m)$ is not anti-commutative.

(3.3) REMARKS (i) Since the cup product in $\mathrm{Ext}_{ZG}^*(Z_m, Z_m)$ is anti-commutative ([1], p. 212), cup product in $\mathrm{Ext}_{ZG}^*(Z_m, Z_m)$ does not coincide with the Yoneda product.

(ii)*) An explicit counter example for the anti-commutativity of Yoneda product in $\mathrm{Ext}_{ZG}^*(Z_m, Z_m)$ can be constructed as follows.

Let G be a finite cyclic group of order m generated by σ (say) and let $\bar{r}$ denote the residue class $\alpha(r)$ of r mod mZ. Let a be the element of $\mathrm{Ext}_{ZG}^1(Z_m, Z_m)$ determined by the exact sequence

$$0 \longrightarrow Z_m \longrightarrow Z_{m^2} \longrightarrow Z_m \longrightarrow 0$$

which is defined by the maps: $\bar{r}\to mr \pmod{m^2}$ and $s \pmod{m^2}\to \bar{s}$, G acting trivially on Z_{m^2}; and $b\in \mathrm{Ext}_{ZG}^1(Z_m, Z_m)$ determined by the exact sequence

$$0 \longrightarrow Z_m \longrightarrow Z_m\times Z_m \longrightarrow Z_m \longrightarrow 0$$

defined by the maps $\bar{r}\to(\bar{r}, 0)$, $(\bar{r}, \bar{s})\to\bar{s}$ and G acting on $Z_m\times Z_m$ by $\sigma(\bar{r}, \bar{s})=(\bar{r}+\bar{s}, \bar{s})$. It can then be proved (using arguments of sections 2 and 3) that $ab=0$ while $ba\neq 0$.

References

[1] H. Cartan and S. Eilenberg, Homological Algebra, Princeton, 1956.

[2] P. Cartier, The groups $\mathrm{Ext}^s(A, B)$, Séminaire A. Grothendieck (Algebre Homologique), 1956/57.

[3] P. J. Hilton and D. Rees, Natural maps of extension functors and a theorem of R.G. Swan, Proc. Cambridge Philos. Soc., **57** (1961), 489-502.

[4] S. MacLane, Homology, Springer Verlag, 1963.

[5] N. Yoneda, Notes on products in Ext, Proc. Amer. Math. Soc., **9** (1958), 873-875.

*) The author is thankful to the referee for suggesting the present explicit counter-example.

L. R. VERMANI
Department of Mathematics
Kurukshetra University
Kurukshetra, India
and
Department of Mathematics
University of Manitoba
Winnipeg, Canada