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On 3-manifolds admitting orientation-reversing involutions

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1. Statement of main results.

Throughout this paper, spaces and maps will be considered in the piecewise-linear category, unless otherwise specified. The purpose of this paper is to discuss some properties of a pair (M, α), where M is a closed, oriented 3-manifold, and α is an orientation-reversing involution on M (that is, $\alpha^2=$ identity, and $\alpha_*[M]=-[M]$ for the fundamental class $[M]$ of M).

The following is perhaps known, but no reference could be found.

THEOREM I. *Given a pair (M, α), then the torsion subgroup $T_1(M; Z)$ of the homology group $H_1(M; Z)$ is isomorphic to a direct double $A \oplus A$ or a direct sum $A \oplus A \oplus Z_2$ for some A.*

For example, the lens space $L(p, q)$, $p>2$, does not admit any orientation-reversing involution, though the projective 3-space $P^3=L(2, 1)$ admits a unique orientation-reversing involution α, whose fixed point set $\mathrm{Fix}(\alpha, P^3)$ is the topological sum P^0+P^2 of the projective 0-space P^0 (=one point) and the projective 2-space P^2. (Cf. K. W. Kwun [**15**].)

By $\mathfrak{C}$ we denote the class of finitely generated abelian groups with torsion parts of the form $A \oplus A$ or $A \oplus A \oplus Z_2$.

DEFINITION 1.1. For any $G \in \mathfrak{C}$, we define $\sigma(G)$ to be 0 or 1, according to whether the torsion subgroup of G is a direct double or not. By using Theorem I, we define $\sigma(M)=\sigma(H_1(M; Z))$ for any pair (M, α).

The following shows enough that the homological classification of Theorem I is complete, where a 3-manifold is irreducible if any imbedded 2-sphere bounds a 3-ball in it.

THEOREM II. *For any $G \in \mathfrak{C}$ there exists a pair (M, α) with $H_1(M; Z)=G$ so that if $\sigma(G)=0$, then M is connected and irreducible, or if $\sigma(G)=1$, then $M=M_1 \# P^3$ with M_1 connected and irreducible, and α preserves the factors.*

Some G with $\sigma(G)=1$ is probably still realizable by a pair (M, α) with M connected and irreducible, but the following may be noted:

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REMARK TO THEOREM II. Suppose for a pair (M, α) M is connected and $H_1(M; Z)/\text{odd torsion} \approx Z_2$. Then M necessarily splits: $M = M_1 \# P^3$, and α preserves the factors.

This can be derived from the equivariant cohomology theory (cf. G. Bredon [4], W. Y. Hsiang [9]), but we shall give a simple proof by using our Theorem III.

According to Smith theory (cf. Proposition 6.1 and A. Borel [3], III § 4), each component of $\text{Fix}(\alpha, M)$ is a point or a closed surface (unless it is empty), and the Euler characteristic $\chi(\text{Fix}(\alpha, M)) \equiv 0 \pmod 2$, since $\chi(M) = 0$.

DEFINITION 1.2. $\sigma(\alpha, M) = \text{rank}_{Z_2} H_1(\text{Fix}(\alpha, M); Z_2) \pmod 2$.

Clearly, $\sigma(\alpha, M)$ is equal to the Stiefel-Whitney number $w_1^2(F)$ of the 2-dimensional part F of $\text{Fix}(\alpha, M)$. Using $\chi(\text{Fix}(\alpha, M)) \equiv 0 \pmod 2$, we see also that it is equal to the number (mod 2) of the discrete points of $\text{Fix}(\alpha, M)$.

From the following, we see that the number $\sigma(\alpha, M)$ does not depend on a particular involution α on M.

THEOREM III. *For any pair (M, α) the following are equivalent:*

(1) $\sigma(M) = 0$,

(2) $\sigma(\alpha, M) = 0$,

(3) *There exists a compact (possibly, non-orientable) 4-manifold W_1 admitting an involution β_1 such that $\partial W_1 = M$ and $\beta_1 | M = \alpha$,*

(4) *There exists a compact, oriented 4-manifold W_2 admitting an orientation-reversing involution β_2 such that $\partial W_2 = M$ and $\beta_2 | M = \alpha$.*

Let $\Omega_3(Z_2^-)$ be the 3-dimensional, equivariant, oriented bordism group of all pairs (M, α), where M is a closed, oriented 3-manifold, and α is an orientation-reversing involution on M.

The following is direct from Theorem III, since $\sigma(P^3) = 1$.

COROLLARY 1.1. *$\Omega_3(Z_2^-)$ is isomorphic to Z_2 and generated by (P^3, α) where α is any orientation-reversing involution on P^3.*

From an argument concerning Theorem III (cf. Theorems 5.1 and 6.1), we see also the following:

THEOREM IV. *For any pair (M, α) with M a Z_2-homology 3-sphere, the μ-invariant, $\mu(M) = 0$.*

This has been independently obtained by J. S. Birman [2], W. C. Hsiang and P. Pao [8] and D. Galewski and R. Stern [6]. A great difference between their methods ([2], [8], [6]) and our method is that their methods are effective only in the involutional case, but our method is more general. For example, from a direct use of our method, one will see in [14] that the μ-invariant of a Z_2-homology 3-sphere vanishes if it admits an orientation-reversing auto-homeomorphism of finite order.

The invariant $\sigma(M)$ of a given pair (M, α) can be stated in terms of the semi-characteristics of M. For an oriented closed $(2r+1)$-manifold X, the semi-characteristic of X with respect to a field K, denoted by $\sigma(X;K)$ is the sum $\sum_{i=0}^{r}\dim_K H_i(X;K)$ (mod 2). The invariant $\sigma(M)$ is clearly equal to the difference $\sigma(M;Z_2)-\sigma(M;Q)$.

Now we are ready to notice that *there exist analogous structures between a pair* (M, α) *and a closed oriented piecewise-linear* 5-*manifold* X^5. In the first place, the second homology group $H_2(X^5;Z)$ necessarily belongs to the class $\mathfrak{C}$ (cf. Remark 2.1 in §2). Defining $\sigma(X^5)=\sigma(H_2(X^5;Z))$, we see easily that $\sigma(X^5)$ is equal to the difference $\sigma(X;Z_2)-\sigma(X^5;Q)$ (cf. Lemma 6.1), and according to Lusztig, Milnor and Peterson [**16**], this is equal to the Stiefel-Whitney number $w_2w_3[X^5]$. Thus, we see the following known proposition, analogous to our Theorem III:

PROPOSITION 1.1. *The following are equivalent:*

(1) $\sigma(X^5)=0$,

(2) $w_2w_3[X^5]=0$,

(3) X^5 *is the boundary of a compact (possibly, non-orientable)* 6-*manifold,*

(4) X^5 *is the boundary of a compact, oriented* 6-*manifold.*

No proof is given (cf. D. Barden [**1**]).

Let γ be the non-trivial covering transformation of the covering $S^2\to P^2$, which is clearly orientation-reversing. Given a pair (M, α), then we form an orientation-preserving, free involution $\alpha\times\gamma$ on $M\times S^2$ by the identity

$$\alpha\times\gamma(x, y)=(\alpha x, \gamma y)$$

for $(x, y)\in M\times S^2$. Then the orbit space $X(M, \alpha)=M\times S^2/\alpha\times\gamma$ is a closed, oriented 5-manifold.

THEOREM V. $\sigma(M)=\sigma(X(M, \alpha))$ *for all pairs* (M, α) *and the assignment* $(M, \alpha)\to X(M, \alpha)$ *induces a well-defined isomorphism from* $\Omega_3(Z_2^-)$ *onto the* 5-*dimensional oriented cobordism group* Ω_5.

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2. Proof of Theorem I.

Theorem I is a special case of the following:

THEOREM 2.1. *Let* X *be a Poincaré duality space with fundamental class* $[X]$ *of dimension* $2m+1$ *with odd* $m\geqq 1$. *Suppose* X *admits a map* $f: X\to X$ *with* $f_*[X]=-[X]$ *and* $f_*^2=$*identity on* $T_m(X;Z)$ $(=\mathrm{Tor}\, H_m(X;Z))$. *Then* $H_m(X;Z)$ *belongs to the class* $\mathfrak{C}$.

PROOF. Consider the non-singular linking pairing $L: T_m(X;Z)\times T_m(X;Z)\to Q/Z$ defined by Poincaré duality. Since m is odd, L is symmetric. Define a new pairing

$$L_f : T_m(X; Z)\times T_m(X; Z) \longrightarrow Q/Z$$

by the identity

$$L_f(x, y)=L(x, f_*(y))$$

for $x, y\in T_m(X; Z)$. Since $f_*[X]=-[X]$, f_* is an automorphism on $H_*(X; Z)$ by Poincaré duality and the formula $f_*(f^*(u)\cap[X])=u\cap f_*[X]$ $(u\in H^*(X; Z))$, so that L_f is non-singular. Since $L(f_*(x), f_*(y))=-L(x, y)$ and $f_*^2=$identity on $T_m(X; Z)$ we see that $2L_f(x, x)=0$ for all $x\in T_m(X; Z)$. Now the proof is completed by the following lemma:

LEMMA 2.1. *Suppose a finite abelian group T admits a non-singular pairing $L: T\times T\to Q/Z$ such that $2L(x, x)=0$ for all $x\in T$. Then T is of the form $A\oplus A$ or $A\oplus A\oplus Z_2$.*

PROOF OF LEMMA 2.1. First, split T into the primary components which are mutually orthogonal with respect to L. Let T_p be the p-primary component of T. It is known that T_p admits an orthogonal splitting $T_p^1\oplus\cdots\oplus T_p^s$ with respect to L where T_p^i is a direct sum of copies of Z_{p^i}. (See, for example, [**13**], p. 50 for $p=2$.) For either $p=2$ and $i\geqq 2$ or an odd p and $i\geqq 1$, let $\tilde{T}_p^i=T_p^i\otimes Z_p$. Define a non-singular pairing

$$\tilde{L} : \tilde{T}_p^i\times\tilde{T}_p^i \longrightarrow Q/Z$$

by the identity

$$\tilde{L}(x\otimes 1, y\otimes 1)=p^{i-1}L(x, y)$$

for $x, y\in T_p^i$. By translating $1/p$ of Q/Z to 1 of Z_p, $\tilde{L}$ is regarded as a non-singular bilinear form over Z_p. Since $2L(x, x)=0$ for $x\in T_p^i$, we see that $\tilde{L}(a, a)=0$ for $a\in\tilde{T}_p^i$, that is, the form $\tilde{L}$ is symplectic. Thus, $\dim_{Z_p}\tilde{T}_p^i$ is even by taking a symplectic basis. This implies that T is of the form $A\oplus A$ or $A\oplus A\oplus Z_2$. This completes the proof of Lemma 2.1.

REMARK 2.1. Let X be a Poincaré duality space of dimension $2m+1$ with even m. In this case, the non-singular linking pairing $L: T_m(X; Z)\times T_m(X; Z)\to Q/Z$ is skew-symmetric, so that $2L(x, x)=0$ for $x\in T_m(X; Z)$. By Lemma 2.1, $H_m(X; Z)$ belongs to the class $\mathfrak{C}$ (cf. W. Browder [**5**] and D. Barden [**1**], p. 372).

3. Proof of Theorem II.

LEMMA 3.1. *There exists a pair (M, α) such that M is an irreducible Z-homology 3-sphere with $\pi_1(M)$ infinite.*

PROOF. Let $k_i\subset S^3$, $i=1, 2$, be non-trivial knots, invariant under some orientation-reversing involutions α_i such that $\mathrm{Fix}(\alpha_i, S^3)=S_i^0$, 0-spheres, and $S_1^0\subset k_1$ and $S_2^0\cap k_2=\emptyset$. [For example, take as k_1 the composite knot $k\,\sharp\,-k^*$

and, as k_2, the composite knot $k \# k^*$ for a non-trivial knot k, where $-k^*$ and k^* denote the reflected inverse and the reflection of k, respectively.] Let E_i be the manifold obtained from S^3 by removing an α_i-invariant open tubular neighborhood of k_i in S^3. We may assume $S_2^0 \subset \mathrm{Int}\, E_2$. Then one can easily paste the boundaries of E_i, $i=1, 2$, together so that the result M is a Z-homology 3-sphere and admits an orientation-reversing involution defined by $\alpha_i | E_i$, $i=1, 2$. Note that E_i is irreducible (since $E_i \subset S^3$) and the homomorphism $\pi_1(\partial E_i) \to \pi_1(E_i)$ induced from inclusion is injective (since the knot k_i is non-trivial). Then we see that M is irreducible and $\pi_1(M)$ is infinite. This completes the proof.

REMARK 3.1. Another construction of a similar homology 3-sphere has been obtained earlier by W. Jaco and B. Myers. Their construction uses a non-splittable, 2-component link $k_1 \cup k_2 \subset S^3$ such that there is an orientation-reversing involution α of S^3 with $\alpha(k_1)=k_2$. (Such a link exists.) Take a tubular neighborhood $T_1 \cup T_2$ of this link $k_1 \cup k_2$ so that $\alpha(T_1)=T_2$. Replace T_1 and T_2 by two copies of a non-trivial knot exterior E, so that the result M is a Z-homology 3-sphere with an orientation-reversing involution extending $\alpha | S^3 - T$. M is irreducible, since the link is not splittable and E is a non-trivial knot exterior.

Let α be an orientation-reversing involution on a homology 3-sphere M. By Smith theory (cf. Proposition 6.1 and [3], III §4) $\mathrm{Fix}(\alpha, M)=S^0$ or S^2. So, if M is irreducible and not S^3, then we must have $\mathrm{Fix}(\alpha, M)=S^0$. In this case, let M_0 be the 3-manifold obtained from M by removing two open 3-balls B_1, B_2 with $\alpha(B_i)=B_i$. The orbit space $M'=M_0/\alpha$ is a homology $P^2 \times [0, 1]$. Since M is irreducible, it follows that M' is irreducible and any $P^2 \subset \mathrm{Int}\, M'$ is boundary-parallel (that is, the union of this P^2 and one component of $\partial M'$ bounds a manifold homeomorphic to $P^2 \times [0, 1]$). Thus, from Lemma 3.1 we see the following:

COROLLARY 3.1. *There exists an irreducible homology* $P^2 \times [0, 1]$, *not homeomorphic to the product* $P^2 \times [0, 1]$, *such that any* P^2 *in the interior is boundary-parallel.*

PROOF OF THEOREM II. First, suppose G is a direct double $\bigoplus_{i=1}^r Z_{n_i} \oplus Z_{n_i}$ where n_i may be 0. By Lemma 3.1 let S be an irreducible Z-homology 3-sphere with orientation-reversing involution α such that $\pi_1(S)$ is infinite. Let k be a knot in S so that $k \cap \alpha k=\emptyset$ and $[k] \neq 1$ in $\pi_1(S)$. Let k_i, $i=1, \cdots, r$, be mutually disjoint knots isotopic to k in a small tubular neighborhood of k in S such that $\mathrm{Link}_S(k_i, k_j)=0$, $i \neq j$. Let $k'_i=\alpha k_i$. The link $L=\bigcup_{i=1}^r k_i \cup \bigcup_{i=1}^r k'_i$ is clearly α-invariant and any two components of L have the linking number 0 (in S). [Note that $\mathrm{Link}_S(k, \alpha k)=0$, since $\mathrm{Link}_S(k, \alpha k)=-\mathrm{Link}_S(\alpha k, \alpha^2 k)$.] Remove from S a small α-invariant, open tubular neighborhood T of L in S. Then we can easily attach the boundaries of $2r$ copies of a non-trivial knot exterior to the boundary ∂E of $E=S-T$ so that the result M has $H_1(M; Z)=G$ and admits an involution extending $\alpha|E$. M is irreducible, since the map $\pi_1(\partial E)\to\pi_1(E)$ induced by inclusion is injective. Next, to obtain a desired manifold for $G\oplus Z$ with G a direct double as above, we assume by construction of S (by Lemma 3.1) that there is a knot $\bar{k}(\subset \mathrm{Int}\, E)\subset S$ with $[\bar{k}]\neq 1$ in $\pi_1(S)$, $\mathrm{Link}_S(k, \bar{k})=0$, $\alpha(\bar{k})=\bar{k}$ and $\mathrm{Fix}(\alpha, S)\cap\bar{k}=\emptyset$. Remove from E an α-invariant, open tubular neighborhood of $\bar{k}$. The result E' contains $\mathrm{Fix}(\alpha, S)$ $(=S^0)$ in the interior. Let E'' be a knot exterior of $\bar{k}\subset S$ with $\alpha(E'')=E''$ and $\mathrm{Fix}(\alpha, S)\subset \mathrm{Int}\, E''$. Attach $\partial E''$ to $\partial E'$ so that the result $\bar{E}$ admits an involution defined by $\alpha|E'$ and $\alpha|E''$, and the inclusion $E'\subset\bar{E}$ induces an isomorphism $H_1(E'; Z)\approx H_1(\bar{E}; Z)$. Apply the above construction for $\partial\bar{E}=\partial E$ to obtain a desired manifold with $H_1=G\oplus Z$. In case $\sigma(G)=1$, we let $G=G_1\oplus Z_2$. By the above construction, we have a pair (M_1, α_1) such that $H_1(M_1; Z)=G_1$, and M_1 is connected and irreducible. From construction, $\mathrm{Fix}(\alpha_1, M_1)$ contains a discrete point x. Remove from M_1 an α_1-invariant, small 3-ball B containing x, and replace it by a twisted line bundle of P^2. The result is a connected sum $M_1 \# P^3$ and $\alpha_1|M_1-B$ is extendable to an involution on $M_1\# P^3$ preserving the factors. This completes the proof.

4. The Arf invariant of a Z_2-homology handle and a cobordism theory.

A closed (possibly, non-orientable) 3-manifold M is a *Z_2-homology handle*, if $H_*(M; Z_2)\approx H_*(S^1\times S^2; Z_2)$ and $H_1(M; Z)$ is infinite. Note that $H_1(M; Z)$ is always infinite, if M is non-orientable. Throughout this section we denote by M a Z_2-homology handle. We shall define an invariant of an integer (mod 2) for M, which is analogous to an invariant of Robertello [**18**] for classical knots. Let $f: M\to S^1$ be a map such that $f_*: H_1(M; Z)\to H_1(S^1; Z)$ is onto. Using $H_1(M; Q)=Q$ and $H_1(M; Z_2)=Z_2$, we can assume that for a point $p\in S^1$, $f^{-1}(p)=F$ is a closed, connected, orientable surface (cf. [**11**], Lemma 2.5). Define a Z_2-linking pairing $L: H_1(F; Z_2)\times H_1(F; Z_2)\to Z_2$ by the identity

$$L(x, y)=Z_2\text{-linking number}_M(x', i_*(y'))$$

for Z_2-cycles x', y' in F with $x=\{x'\}$, $y=\{y'\}$, where $i_*(y')$ denotes a cycle in M obtained from y' by translating in the positive normal direction (associated with f and S^1). [Note that the Z_2-linking number is well-defined, since x' and $i_*(y')$ are Z_2-null-homologous in M (cf. [**12**], 2.19).] Clearly,

$$L(x, y)+L(y, x)=x\cdot y$$

where $x\cdot y$ denotes the usual intersection number (mod 2) on F. Define a map $q: H_1(F; Z_2)\to Z_2$ by the identity

$$q(x)=L(x, x)$$

for $x\in H_1(F; Z_2)$. It follows that

$$q(x+y)=q(x)+q(y)+x\cdot y .$$

So q is a quadratic form (mod 2), and the Arf invariant of q is defined.

DEFINITION 4.1. $\varepsilon(M)$ is the Arf invariant of q.

To show that $\varepsilon(M)$ is an invariant of the topological type of M, we make use of the Z_2-Alexander polynomial $A(t)$ ($\in Z_2\langle t\rangle$) of M associated with an epimorphism $\pi_1(M)\to\langle t\rangle$, $\langle t\rangle$ being the infinite cyclic group with a fixed generator t. This is defined to be any generator of the order ideal of the $Z_2\langle t\rangle$-module $H_1(\tilde{M}; Z_2)$, which is a finitely generated torsion $Z_2\langle t\rangle$-module, where $\tilde{M}$ is the infinite cyclic cover of M associated with an epimorphism $\pi_1(M)\to\langle t\rangle$. As an analogy of [**11**], $A(t)\in Z_2\langle t\rangle$ has that $A(t)=t^m A(t^{-1})$ for some m and $A(1)=1$. Therefore, $A(t)$ (up to multiples of t) is an invariant of the topological type of M. Using $A(1)=1$, we see that $A(t)$ is of the form $a(t)(t^4+1)+t^b$ or $a(t)(t^4+1)+t^b(t^2+t+1)$. [These forms do not occur at the same time.]

LEMMA 4.1. *$\varepsilon(M)=0$ if $A(t)=a(t)(t^4+1)+t^b$, and*

$$\varepsilon(M)=1 \text{ if } A(t)=a(t)(t^4+1)+t^b(t^2+t+1).$$

COROLLARY TO LEMMA 4.1. *$\varepsilon(M)$ is an invariant of the topological type of M.*

PROOF OF LEMMA 4.1. By choosing a symplectic basis for $H_1(F; Z_2)$, the linking pairing $L: H_1(F; Z_2)\times H_1(F; Z_2)\to Z_2$ is represented by a matrix $V=(v_{ij})$ ($v_{ij}\in Z_2$) so that

$$V+V'=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\oplus\cdots\oplus\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{(block sum)}$$

since $L(x, y)+L(y, x)=x\cdot y$, where V' is the transpose of V. Note that $A(t)=\det(tV+V')$ and $\varepsilon(M)=\sum_{i=1}^{g} v_{2i\,2i}\cdot v_{2i-1\,2i-1}$, where g is the genus of F. Now we can apply the same calculation as R. Robertello [**18**], pp. 551-553 and obtain our desired result.

LEMMA 4.2. *If M' is a connected double cover of a Z_2-homology handle M, then M' is also a Z_2-homology handle, and the Z_2-Alexander polynomial of M' is equal to the Z_2-Alexander polynomial of M. In particular, $\varepsilon(M')=\varepsilon(M)$.*

PROOF. Let $\tilde{M}$ be an infinite cyclic cover of M associated with an epimorphism $\pi_1(M)\to\langle t\rangle$. Note that M' is identical with the orbit space $\tilde{M}/\langle t^2\rangle$. Consider the following part of Wang exact sequence

$$H_1(\tilde{M}; Z_2)\xrightarrow{t-1} H_1(\tilde{M}; Z_2)\longrightarrow H_1(M; Z_2)\longrightarrow \dot{H}_0(\tilde{M}; Z_2)\longrightarrow 0 .$$

Similarly,

$$H_1(\tilde{M}; Z_2)\xrightarrow{t^2-1} H_1(\tilde{M}; Z_2)\longrightarrow H_1(M'; Z_2)\longrightarrow H_0(\tilde{M}; Z_2)\longrightarrow 0 .$$

Since $H_1(M; Z_2) \approx H_0(\tilde{M}; Z_2)$ $(=Z_2)$, $t-1: H_1(\tilde{M}; Z_2) \to H_1(\tilde{M}; Z_2)$ is onto. Hence $(t-1)^2 = t^2-1: H_1(\tilde{M}; Z_2) \to H_1(\tilde{M}; Z_2)$ is so, which implies that $H_1(M'; Z_2) \approx H_0(\tilde{M}; Z_2) \approx Z_2$. Using an epimorphism $\pi_1(M') \to \langle t^2 \rangle$, we see that M' is a Z_2-homology handle. Let $A(t)$, $A'(t)$ be the Z_2-Alexander polynomials of M, M', respectively. Note that $A(t)$ is the characteristic polynomial of $t: H_1(\tilde{M}; Z_2) \to H_1(\tilde{M}; Z_2)$ and $A'(t^2)$ is the characteristic polynomial of $t^2: H_1(\tilde{M}; Z_2) \to H_1(\tilde{M}; Z_2)$. So using the field Z_2,

$$A'(t^2) = A(t)A(-t) = A(t)^2 = A(t^2)$$

(cf. [**12**], Lemma 3.11), which implies $A'(t) = A(t)$. By Lemma 4.1, $\varepsilon(M) = \varepsilon(M')$. This completes the proof.

We shall show the following:

THEOREM 4.1. *$\varepsilon(M)=0$ if and only if there exists a compact connected 4-manifold W with $\partial W = M$ such that*

(1) *the inclusion $M \subset W$ induces an isomorphism*

$$H_1(M; Z)/\text{odd torsion} \approx H_1(W; Z)/\text{odd torsion} \ (\approx Z),$$

(2) *the Z_2-intersection number, $x \cdot x = 0$ for all $x \in H_2(W; Z_2)$.*

PROOF. Suppose $\varepsilon(M)=0$. Then there is a symplectic basis $a_1, \cdots, a_g$, $b_1, \cdots, b_g$ ($a_i \cdot a_j = b_i \cdot b_j = 0$ for all i, j, and $a_i \cdot b_j = \delta_{ij}$) of $H_1(F; Z_2)$ such that $L(a_i, a_i) = 0$, $i=1, 2, \cdots, g$, where F is a closed, connected, orientable surface of genus g, transversal to a circle representing a generator of $H_1(M; Z)$/odd torsion $(\approx Z)$. We proceed to the proof by assuming the following lemma:

LEMMA 4.3. *A symplectic basis $a_1, \cdots, a_g$, $b_1, \cdots, b_g$ of $H_1(F; Z_2)$ is represented by simple closed curves $a_1^0, \cdots, a_g^0$, $b_1^0, \cdots, b_g^0$ on F such that $a_i^0 \cap a_j^0 = b_i^0 \cap b_j^0 = a_i^0 \cap b_j^0 = \emptyset$, $i \neq j$, and $a_i^0 \cap b_i^0 =$ one point.*

In Lemma 4.3 let $a_i^0 \times [0, 1]$ be a small neighborhood of a_i^0 in F. Further, thicken $a_i^0 \times [0, 1]$ by a collar of F in M. From the results $a_i^0 \times [0, 1] \times [0, 1]$, we construct an adjunction space W_1 as follows:

$$W_1 = M \times [0, 1] \cup D_1 \times [0, 1] \times [0, 1] \cup \cdots \cup D_g \times [0, 1] \times [0, 1]$$

where D_i is a disk, and $(\partial D_i) \times [0, 1] \times [0, 1]$ is identified with $a_i^0 \times [0, 1] \times [0, 1] \times 1$ so that $(x, t, u) \equiv (h_i(x), t, u, 1)$ for a homeomorphism $h_i: \partial D_i \to a_i^0$. Since $L(a_i, a_i) = 0$, $i = 1, 2, \cdots, g$, it follows that $x \cdot x = 0$ for all $x \in H_2(W_1; Z_2)$. Note that $\partial W_1 - M \times 0$ is homeomorphic to $S^1 \times S^2 \# N$ or $S^1 \times_\tau S^2 \# N$ for some closed connected orientable 3-manifold N according to whether M is orientable or non-orientable. ($S^1 \times_\tau S^2$ is the non-orientable handle, that is, the twisted S^2 bundle over S^1.) By attaching $S^1 \times B^3$ or $S^1 \times_\tau B^3$ (=the twisted B^3 bundle over S^1) to the factor $S^1 \times S^2$ or $S^1 \times_\tau S^2$ of $\partial W_1 - M \times 0$, we obtain a manifold W_2 with $\partial W_2 - M \times 0 = N$. Note that the canonical homomorphism $H_1(M \times 0; Z)$/odd torsion$\to H_1(W_2; Z)$/odd torsion is an isomorphism. In particular, the boundary homomorphism $\partial: H_2(W_2, N; Z_2)\to H_1(N; Z_2)$ is onto. If $H_1(N; Z_2)\neq 0$, let $x\in H_1(N; Z_2)$ be a non-zero element represented by a simple closed curve S. Let c be a 2-chain (mod 2) in W_2 with $\partial c=S$. Let S' be a simple closed curve on a tubular neighborhood T of S in N, homotopic to S in T and bounding a 2-chain c' (mod 2) in W so that c' is Z_2-homologous to c in W_2 mod N, and the Z_2-intersection number, $c\cdot c'=0$. Let W_2' be a 4-manifold obtained from W_2 by attaching a 2-handle $D^2\times D^2$ along T with framing determined by S'. Note that $H_2(W_2'; Z_2)$ has a basis $x_1, \cdots x_g, y$, where $\{x_1, \cdots, x_g\}$ is the image of a basis of $H_2(W_2; Z_2)$ by the canonical map $H_2(W_2; Z_2)\to H_2(W_2'; Z_2)$, and y is the homology class represented by $c\cup D^2\times 0$. ($D^2\times 0$ is a core of the 2-handle $D^2\times D^2$.) From construction, we have $x_i\cdot x_i=y\cdot y=0$, $i=1, 2, \cdots, g$. Let $\partial W_2'-M\times 0=N'$. Since $H_1(N\text{-Int}\,T; Z_2)\to H_1(N; Z_2)$ is an isomorphism (cf. Lemma 4.4), we see that $\dim_{Z_2}H_1(N'; Z_2)=\dim_{Z_2}H_1(N; Z_2)-1$. By induction on $\dim_{Z_2}H_1(N; Z_2)$, we may assume that $H_1(N; Z_2)=0$. Then let V be a 1-connected 4-manifold V with $\partial V=N$ such that $x\cdot x=0$ for all $x\in H_2(V; Z_2)$ (cf. J. W. Milnor [**17**], S. J. Kaplan [**10**]). The manifold $W=W_2\cup V$, then, satisfies (1) and (2). Conversely, assume that M bounds a manifold W satisfying (1) and (2). Let $F\subset M$ be a closed, connected, orientable surface of genus g, transversal to a circle of a generator of $H_1(M; Z)$/odd torsion. By (1) F is the boundary of a compact 3-manifold $U\subset W$, transversal to a circle of a generator of $H_1(W; Z)$/odd torsion$\approx Z$. We proceed to the proof by assuming the following lemma:

LEMMA 4.4. *There exists a symplectic basis* $a_1, \cdots, a_g, b_1, \cdots, b_g$ *of* $H_1(F; Z_2)$ *such that* $a_1, \cdots, a_g$ *generate the kernel of the canonical homomorphism* $i_*: H_1(F; Z_2)\to H_1(U; Z_2)$.

Suppose $L(a_1, a_1)=1$. Let c_1 be a representative of a_1. Let c_1' be a slight translation of c_1 in a positive normal direction in M. Since $L(a_1, a_1)=1$, c_1 bounds a 2-chain c_2 (mod 2) in M such that $c_2\cdot c_1'=1$. c_1 is Z_2-null-homologous in U. So it bounds a 2-chain $\tilde{c}_2$ (mod 2) in U. Let $\bar{c}_2$ be a slight translation of the cycle $c_2\cup\tilde{c}_2$ into Int W by using a boundary collar of W. Since U admits a collar in W, we may consider that the cycle c_1' is in the boundary of a slight translation U' of U, and bounds a 2-chain $\tilde{c}_2'$ (mod 2) in U', Z_2-homologous to $\tilde{c}_2$ in W (mod M). Since $c_2\cdot c_1'=1$, we see easily that $\bar{c}_2\cdot\tilde{c}_2'=1$. Let c_2' be a 2-chain (mod 2) in M with $\partial c_2'=c_1'$. Then $\bar{c}_2\cdot(c_2'\cup\tilde{c}_2')=1$, which contradicts (2), since $c_2'\cup\tilde{c}_2'$ is Z_2-homologous to $\bar{c}_2$ in W by the canonical isomorphism $H_2(W; Z_2)\approx H_2(W, M; Z_2)$. Hence $L(a_1, a_1)=0$. Similarly, $L(a_i, a_i)=0$, $i=2, \cdots, g$. Therefore, $\varepsilon(M)=0$. This completes the proof except for the proofs of Lemmas 4.3 and 4.4.

PROOF OF LEMMA 4.3. Regard F as a connected sum $F_1\#\cdots\# F_g$ of g copies of a torus of genus 1. Then $a_1\in H_1(F; Z_2)$ is written as a sum $c_1+c_2+\cdots+c_g$, $c_i\in H_1(F_i; Z_2)$. Similarly, $b_1\in H_1(F; Z_2)$ as a sum $d_1+d_2+\cdots+d_g$,

$d_i \in H_1(F_i; Z_2)$. (Some c_i or d_i may be 0.) Since $a_1 \cdot b_1 = 1$, we can assume that for some *odd* $m \geqq 1$, $c_1 \cdot d_1 = \cdots = c_m \cdot d_m = 1$ and $c_{m+1} \cdot d_{m+1} = \cdots = c_g \cdot d_g = 0$. Then c_i and d_i are represented by simple closed curves c_i^0 and d_i^0 on F_i such that $c_i^0 \cap d_i^0 =$ one point (if $i \leqq m$) or $\emptyset$ (if $i \geqq m+1$). When $m \geqq 3$, $c_1 + c_2$ and $d_1 + d_2$ can be represented by mutually disjoint, simple closed curves on $F_1 \# F_2$. By induction on m, $c_1 + \cdots + c_{m-1}$ and $d_1 + \cdots + d_{m-1}$ are represented by mutually disjoint, simple closed curves c^0 and d^0 on $F_1 \# \cdots \# F_{m-1}$. Then suitable connected sums $a_1^0 = c^0 \# c_m^0 \# c_{m+1}^0 \# \cdots \# c_g^0$ and $b_1^0 = d^0 \# d_m^0 \# d_{m+1}^0 \# \cdots \# d_g^0$ are simple closed curves on F representing a_1 and b_1 such that $a_1^0 \cap b_1^0 =$ one point. Taking a regular neighborhood of $a_1^0 \cup b_1^0$, we obtain a new connected sum $F = F_1' \# F''$ where a_1^0, b_1^0 represent a basis for $H_1(F_1'; Z_2)$. Then $a_2, \cdots, a_g, b_2, \cdots, b_g$ form a symplectic basis for $H_1(F''; Z_2)$. By induction, we complete the proof.

PROOF OF LEMMA 4.4. Consider the following exact sequence

$$H_2(U, F; Z_2) \xrightarrow{\partial} H_1(F; Z_2) \xrightarrow{i_*} H_1(U; Z_2).$$

From this, we see that $\operatorname{Im} \partial$ is a self-orthogonal complement with respect to the non-singular intersection pairing $H_1(F; Z_2) \times H_1(F; Z_2) \to Z_2$ (since $(\partial x) \cdot y = x \cdot i_*(y)$). In particular, $2 \dim_{Z_2} \operatorname{Im} \partial = \dim_{Z_2} H_1(F; Z_2) = 2g$. Let $a_1, \cdots, a_g$ be a basis for $\operatorname{Im} \partial$. Since $x \cdot x = 0$ for $x \in H_1(F; Z_2)$, we can find $b_1, \cdots, b_g$ such that $a_1, \cdots, a_g$, $b_1, \cdots, b_g$ give a symplectic basis for $H_1(F; Z_2)$. This completes the proof.

REMARK 4.1. Let F be a closed, connected (possibly, non-orientable) surface. From an idea of the proof of Lemma 4.3, we can see that any element of $H_1(F; Z_2)$ is represented by a simple closed curve on F. By the proof of Lemma 4.4, we see also that if F is the boundary of a compact 3-manifold U, then the kernel of $i_*: H_1(F; Z_2) \to H_1(U; Z_2)$ has the half-dimension of $H_1(F; Z_2)$.

LEMMA 4.5. (1) *If W is a finite cover of a compact 4-manifold W' with $x \cdot x = 0$ for all $x \in H_2(W'; Z_2)$, then we have $x \cdot x = 0$ for all $x \in H_2(W; Z_2)$.*

(2) *A compact 4-manifold W is spin ($w_1(W) = w_2(W) = 0$) if and only if W is orientable and $x \cdot x = 0$ for all $x \in H_2(W; Z_2)$.*

PROOF. By Wu formula,

$$w_2(W') = w_1(W') \cup w_1(W') + v_2(W'),$$

where $v_2(W') \in H^2(W'; Z_2)$ is defined by the identity $x \cup v_2(W') = x \cup x$ for all $x \in H^2(W', \partial W'; Z_2)$. Since $x \cdot x = 0$ for all $x \in H_2(W'; Z_2)$, we see that $x \cup x = 0$ for all $x \in H^2(W', \partial W'; Z_2)$. Hence by Poincaré duality, $v_2(W') = 0$. So $w_2(W') = w_1(W') \cup w_1(W')$. Applying the covering projection $p: W \to W'$ to this identity, we obtain that

$$w_2(W) = p^*(w_2(W')) = p^*(w_1(W')) \cup p^*(w_1(W')) = w_1(W) \cup w_1(W).$$

Hence by Wu formula, $v_2(W) = w_1(W) \cup w_1(W) + w_2(W) = 0$. That is, $x \cup x = 0$ for all $x \in H^2(W, \partial W; Z_2)$. This implies that $x \cdot x = 0$ for all $x \in H_2(W; Z_2)$, showing (1). (2) follows easily from Wu formula, since $w_1(W)=0$ if and only if W is orientable. This completes the proof.

COROLLARY 4.1. *Assume that M admits a free involution α such that the orbit space M/α is a Z_2-homology handle. Then $\varepsilon(M)=0$ if and only if there exists a compact connected 4-manifold W with $\partial W = M$ such that*

(1) *W admits a free involution β extending α,*

(2) *the inclusion $M \subset W$ induces an isomorphism $H_1(M; Z)/$ odd torsion $\approx H_1(W; Z)/$odd torsion,*

(3) *the Z_2-intersection number, $x \cdot x = 0$ for all $x \in H_2(W; Z_2)$.*

PROOF. Let $M_1 = M/\alpha$. By Lemma 4.2 $\varepsilon(M_1) = \varepsilon(M)$. By Theorem 4.1, $\varepsilon(M_1)=0$ if and only if M_1 bounds a 4-manifold W_1 satisfying (1) and (2) of Theorem 4.1. By (1) there is a double cover W of W_1 extending the covering $M \to M_1$. By Wang exact sequence, $H_1(W; Z)$/odd torsion$\approx Z$ (cf. the proof of Lemma 4.2). It follows that the inclusion $M \subset W$ induces an isomorphism $H_1(M; Z)$/odd torsion$\approx H_1(W; Z)$/odd torsion. By Lemma 4.5 (1), we complete the proof.

REMARK 4.2. In Theorem 4.1, if M is orientable, then W is spin. [In fact, by (1) $w_1(M)=0$ if and only if $w_1(W)=0$. Hence, if $w_1(M)=0$, then by Lemma 4.5 (2), (2) implies W is spin.] In Corollary 4.1, M is necessarily orientable and W is necessarily spin, and α is orientation-preserving if and only if β is orientation-preserving. [To see this, it suffices to check that M is orientable. In general, for a connected manifold X_1 with $H^1(X_1; Z_2) \approx Z_2$, any 2-fold connected cover X of X_1 is orientable. In fact, if $w_1(X_1) \neq 0$, X is the cover of X_1 associated with $w_1(X_1) \in H^1(X_1; Z_2)$ $(\approx Z_2)$, that is, the orientation cover of X_1.]

5. Proof of Theorem IV.

We shall show the following:

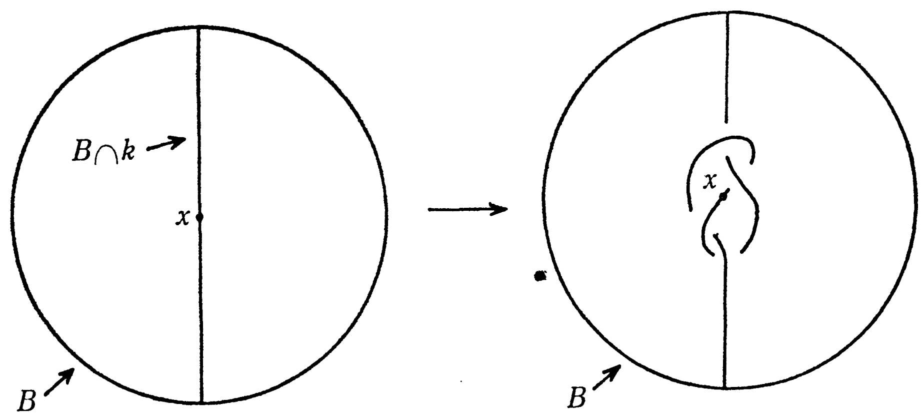
THEOREM 5.1. *Given a pair (M, α) where M is a Z_2-homology 3-sphere and α is an orientation-reversing involution on M, then there exists a compact, connected, oriented, spin 4-manifold W with an orientation-reversing involution β such that $\partial W = M$, $\beta | M = \alpha$ and $H_1(W; Z_2)=0$.*

PROOF OF THEOREM IV. $\mu(M)=\mathrm{sign}(W)/16=0$ for W in Theorem 5.1 which clearly has $\mathrm{sign}(W)=0$.

PROOF OF THEOREM 5.1. By Smith theory (cf. Proposition 6.1 and [3], III §4), $\mathrm{Fix}(\alpha, M)=S^0$ or S^2. If $\mathrm{Fix}(\alpha, M)=S^2$, then M splits: $M=M_1 \# M_2$, and α interchanges the factors. (M, α) is equivalent to $(M_1 \# -M_1, \alpha_1)$ where α_1 is a reflection of the factors. Then we obtain easily a desired pair (W, β) with $\partial W = M$, $\beta | M = \alpha$ and $\widetilde{H}_*(W; Z_2)=0$. Now we assume $\mathrm{Fix}(\alpha, M)=S^0$. Let k be a knot in M such that $\alpha(k)=k$ and $S^0\subset k$. [Such a knot k is easily obtained by considering the canonical projection $M\to M/\alpha$.] Let T be an α-invariant, tubular neighborhood of k in M. Let $E=M-\text{Int}\, T$. Since $\alpha|E$ acts freely on E, the orbit space $E'=E/\alpha$ is a non-orientable manifold with $\partial E'$ a Klein bottle. We proceed to the proof by assuming the following lemma:

LEMMA 5.1. *$H_1(E'; Z)$/odd torsion$\approx Z$.*

By this lemma let F be a proper surface transversal to a circle of a generator of $H_1(E'; Z)$/odd torsion $(\approx Z)$. F is properly imbedded in E so that $\alpha F\cap F=\emptyset$. Since $H_1(E', \partial E'; Z_2)=0$ by Remark 4.1, we can assume that ∂F has some odd components of circles all of which are isotopic in $\partial E'$ and hence in ∂E. Let a be an α-invariant meridian on T, which represents clearly a generator of $H_1(E; Z_2)$. Let b be any component of ∂F, which generates the kernel of $H_1(\partial E; Z_2)\to H_1(E; Z_2)$ and satisfies $\alpha b\cap b=\emptyset$. We may have $a\cap b=$ one point. We construct a 4-manifold $W_1=M\times[0, 1]\cup D^2\times D^2$ by identifying $T\times 1$ with $(\partial D^2)\times D^2$ so that $a\times 1=\text{pt}\times\partial D^2$ and $b\times 1=(\partial D^2)\times\text{pt}$. Then α is extendable to an orientation-reversing involution β_1 on W_1 so that $\beta_1|M_1=\alpha_1$ is an orientation-reversing free involution on M_1 where $M_1=\partial W_1-M\times 0$. From construction, $H_1(M_1; Z_2)=H_1(M_1/\alpha_1; Z_2)=Z_2$. Since M_1/α_1 is non-orientable, $H_1(M_1/\alpha_1; Z)$ is infinite. Hence M_1/α_1 and M_1 are Z_2-homology handles. In case $\varepsilon(M_1)=0$, then by Corollary 4.1 and Remark 4.2, M_1 bounds a compact, connected, spin 4-manifold W_2 admitting an orientation-reversing involution β_2 extending α_1 such that the canonical homomorphism $H_1(M_1; Z_2)\to H_1(W_2; Z_2)$ is an isomorphism. Let $W=W_1\cup W_2$. Note that the canonical homomorphism $H_2(W_2; Z_2)\to H_2(W; Z_2)$ is an isomorphism and $H_1(W; Z_2)=0$. The pair (W, β), where β is defined by $\beta|W_i=\beta_i$, is a desired pair. In case $\varepsilon(M_1)=1$, we make a restart by using, instead of k, a new knot k' constructed from k as follows: Let $x\in M$ be a fixed point of α, and B be an α-invariant, regular neighborhood



Figure

of x in M. We may consider that $B\cap k$ is an unknotted arc in B. Replace this arc by an arc of the figure eight knot, illustrated in the Figure, where B and $\alpha|B$ are regarded as a unit ball in R^3 and the antipodal map sending $(x, y, z)\rightarrow(-x, -y, -z)$. Then the resulting knot k' still satisfies $\alpha(k')=k'$ and $\mathrm{Fix}(\alpha, M)=S^0\subset k'$.

Let M_1' be a Z_2-homology handle resulting from k'. Let $A(t)$, $A'(t)$ be the Z_2-Alexander polynomials of M_1, M_1', respectively. Since k' is a knot sum of k and the figure eight knot, it follows that $A'(t)=A(t)\cdot f(t^r)$, where $f(t)=t^2-3t+1$ $=t^2+t+1$ is the Z_2-Alexander polynomial of the figure eight knot, and r is the number of the components of ∂F which is *odd*. [To see this, notice that $H_1(\tilde{M}_1'; Z_2)$ is $Z_2\langle t\rangle$-isomorphic to $H_1(\tilde{M}_1; Z_2)\oplus Z_2\langle t\rangle/(f(t^r))$.] Since $\varepsilon(M_1)=1$, by Lemma 4.1 $A(t)=a(t)(t^4+1)+t^b(t^2+t+1)$ for some $a(t)$ and b. Using that r is odd, $f(t^r)=c(t)(t^4+1)+t^d(t^2+t+1)$ for some $c(t)$ and d. Hence $A'(t)=a'(t)(t^4+1)+t^e$ for some $a'(t)$ and e. By Lemma 4.1 this implies that $\varepsilon(M_1')=0$. Now we reduced the case $\varepsilon(M_1)=1$ to the case $\varepsilon(M_1)=0$. This completes the proof of Theorem 5.1 except for the proof of Lemma 5.1.

PROOF OF LEMMA 5.1. From the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\pi_1(E) & \longrightarrow & \pi_1(E') & \longrightarrow & Z_2 & \longrightarrow & 1\\
\uparrow & & \uparrow & & \| & & \\
\pi_1(\partial E) & \longrightarrow & \pi_1(\partial E') & \longrightarrow & Z_2 & \longrightarrow & 1,
\end{array}$$

we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H_1(E; Z_2) & \longrightarrow & H_1(E'; Z_2) & \longrightarrow & Z_2 & \longrightarrow & 0\\
\uparrow & & \uparrow & & \| & & \\
H_1(\partial E; Z_2) & \longrightarrow & H_1(\partial E'; Z_2) & \longrightarrow & Z_2 & \longrightarrow & 0.
\end{array}$$

Since $H_1(\partial E; Z_2)\rightarrow H_1(E; Z_2)$ is onto, $H_1(\partial E'; Z_2)\rightarrow H_1(E'; Z_2)$ is so. By Remark 4.1, $H_1(E'; Z_2)=Z_2$. The double $D(E')$ of E' is closed and non-orientable, so that $H_1(D(E'); Q)\neq 0$. By Mayer-Vietoris sequence we have $H_1(E'; Q)\neq 0$. Hence $H_1(E'; Z)/\text{odd torsion}\approx Z$. This completes the proof.

6. Proof of Theorem III.

DEFINITION 6.1. A pair (M, α) is *spin bordant* to a pair (M', α'), if $M+(-M')$ bounds a compact spin 4-manifold with an orientation-reversing involution extending $\alpha+\alpha'$. If $M'=\emptyset$, then we say that (M, α) is a *spin boundary*.

THEOREM 6.1. *Assume that $\sigma(\alpha, M)=0$. Then there exists a sequence of pairs $(M, \alpha)=(M_0, \alpha_0), (M_1, \alpha_1), \cdots, (M_r, \alpha_r)$ such that for each i, $0\leqq i\leqq r-1$,*

(M_i, α_i) *is spin bordant to* (M_{i+1}, α_{i+1}), *and* (M_r, α_r) *is a spin boundary.*

PROOF. (1) *Any pair* (M, α) *is spin bordant to a pair* (M', α') *where* M' *is connected.* The proof of (1) is not difficult. [If M contains two components $M^{(0)}$, $M^{(1)}$ such that $\alpha(M^{(0)})=M^{(1)}$, then choose 3-balls $B^{(i)}\subset M^{(i)}$ so that $\alpha(B^{(0)})=B^{(1)}$ and construct a 4-manifold $W=M\times[0, 1]\cup D^3\times[0, 1]$ by identifying $B^{(i)}\times 1$ with $D^3\times i$, $i=0, 1$. For any two components $M^{(0)}$, $M^{(1)}$ with $\alpha(M^{(i)})=M^{(i)}$, choose 3-balls $B^{(i)}\subset M^{(i)}$ such that $\alpha(B^{(i)})\cap B^{(i)}=\emptyset$, and then construct a 4-manifold $W=M\times[0, 1]\cup D^3\times[0, 1]_1\cup D^3\times[0, 1]_2$ identifying $B^{(i)}\times 1$ with $D^3\times i_1$, and $\alpha(B^{(i)})\times 1$ with $D^3\times i_2$, $i=0, 1$.]

(2) *Given a pair* (M, α) *where* M *is connected and* $\sigma(\alpha, M)=0$, *then* (M, α) *is spin bordant to a pair* (M', α') *such that* M' *is connected and* $\mathrm{Fix}(\alpha', M')$ *is* $\emptyset$ *or a closed connected surface.* The proof is not difficult. [Since $\sigma(\alpha, M)=0$, the number of discrete points of $\mathrm{Fix}(\alpha, M)$ is even. For any two discrete points $x^{(0)}$, $x^{(1)}$ of $\mathrm{Fix}(\alpha, M)$, choose 3-balls $B^{(i)}\subset M$ which are α-invariant neighborhoods of $x^{(i)}$, and then form a 4-manifold $W=M\times[0, 1]\cup D^3\times[0, 1]$ identifying $B^{(i)}\times 1$ with $D^3\times i$, $i=0, 1$. For any two surfaces $F^{(0)}$, $F^{(1)}$ in $\mathrm{Fix}(\alpha, M)$, choose proper ball pairs $B^{2(i)}\subset B^{3(i)}$ such that $B^{2(i)}\subset F^{(i)}$, $B^{3(i)}\subset M$ and $\alpha(B^{3(i)})=B^{3(i)}$, and then construct a 4-manifold $W=M\times[0, 1]\cup D^3\times[0, 1]$ identifying $B^{3(i)}\times 1$ with $D^3\times i$, $i=0, 1$.]

(3) *Given a pair* (M, α) *where* M *is connected and* $\mathrm{Fix}(\alpha, M)$ *is a closed connected surface, then there is a sequence of pairs* $(M, \alpha)=(M_0, \alpha_0)$, (M_1, α_1), $\cdots$, (M_r, α_r) *such that* (M_i, α_i) *is spin bordant to* (M_{i+1}, α_{i+1}), *and* (M_r, α_r) *satisfies either that* M_r *is connected and* $\mathrm{Fix}(\alpha_r, M_r)=\emptyset$ *or that* M_r *has just two components* $M_r^{(1)}$, $M_r^{(2)}$ *and* $\alpha(M_r^{(1)})=M_r^{(2)}$. To prove (3), let $F=\mathrm{Fix}(\alpha, M)$, and first assume $F\neq S^2$, $P^2\,\sharp\,P^2$. Note that the orbit space $M/\alpha=M_F$ is a compact connected manifold with boundary F. By Remark 4.1 and the following canonical commutative triangle

$$\begin{array}{ccc} H_1(F; Z_2) & \longrightarrow & H_1(M; Z_2) \\ & \searrow & \downarrow \\ & & H_1(M_F; Z_2), \end{array}$$

we find a two-sided simple closed curve S on F which represents a non-zero element in $H_1(M_F; Z_2)$ and hence in $H_1(M; Z_2)$. Let T be an α-invariant, tubular neighborhood of S in M, so that $T\cap F$ is a proper annulus in T. Construct a 4-manifold $W=M\times[0, 1]\cup D^2\times D^2$ identifying $T\times 1$ (with framing determined by the annulus $T\cap F$) with $(\partial D^2)\times D^2$. Since $H_1(T; Z_2)\to H_1(M; Z_2)$ is injective, $H_2(M\times 0; Z_2)\to H_2(W; Z_2)$ is onto, which implies that W is spin. From construction, α is extendable to an involution β on W such that $M_1=$

$\partial W - M\times 0$ is connected and $\mathrm{Fix}(\beta|M_1, M_1)$ is connected with Euler characteristic $\chi(F)+2$. Since $\chi(F)$ is even, we may assume by induction that $F=S^2$ or $P^2 \# P^2$. In case $F=S^2$, construct a 4-manifold $W=M\times[0,1]\cup D^3\times[0,1]$, where $(\partial D^3)\times[0,1]$ is identified with an α-invariant neighborhood of $F\times 1$ in $M\times 1$. Then W gives a spin bordism from (M,α) to a pair (M',α') satisfying either that M' is connected and $\mathrm{Fix}(\alpha', M')=\emptyset$ or that M' has just two components $M'^{(1)}$, $M'^{(2)}$ and $\alpha'(M'^{(1)})=M'^{(2)}$. Now assume $F=P^2 \# P^2$. Let S be a circle defining the connected sum $P^2 \# P^2$. Let T be an α-invariant, tubular neighborhood of S in M, so that $F\cap T$ is a proper annulus in T. Construct a 4-manifold $W=M\times[0,1]\cup D^2\times D^2$ identifying $T\times 0$ (with framing specified by a proper annulus $F\cap T$) with $(\partial D^2)\times D^2$. $M'=\partial W - M\times 0$ is connected, and α is extendable to an involution β on W such that $\mathrm{Fix}(\alpha', M')$ $(\alpha'=\beta|M')$ consists of two copies of P^2. By considering an α'-invariant, regular neighborhood of $\mathrm{Fix}(\alpha', M')$, we see that $M'=M'' \# P^3 \# P^3$ for some M'', and α' preserves the factors. W is spin since $H_2(\partial W; Z_2)\to H_2(W; Z_2)$ is onto. Thus, (M,α) is spin bordant to $(M'' \# P^3 \# P^3, \alpha')$. The latter is easily spin bordant to a pair (M''', α''') with M''' connected and $\mathrm{Fix}(\alpha''', M''')=\emptyset$ by considering the product $(P^3-\mathrm{Int}\,\Delta^3)\times[0,1]$ with a standard involution whose fixed point set is $P^2\times[0,1]$. This shows (3).

(4) *Suppose, for a pair (M,α), M has just two components $M^{(1)}$, $M^{(2)}$, and $\alpha(M^{(1)})=M^{(2)}$. Then (M,α) is a spin boundary.* The proof is easy. [Note that (M,α) is equivalent to a pair $(M^{(1)}+(-M^{(1)}), \alpha_0)$ where $\alpha_0|M^{(1)}\to -M^{(1)}$ is defined by the identity on the underlying set.]

(5) *Suppose for a pair (M,α), M is connected and* $\mathrm{Fix}(\alpha, M)=\emptyset$. *Then there is a finite sequence of pairs $(M,\alpha)=(M_0,\alpha_0), (M_1,\alpha_1), \cdots, (M_r,\alpha_r)$ such that (M_i,α_i) is spin bordant to (M_{i+1},α_{i+1}), and M_r is a Z_2-homology* 3*-sphere.* To see this, consider the orbit space $M'=M/\alpha$, which is non-orientable. Clearly, $H_1(M'; Z_2)\neq 0$. If $\dim_{Z_2} H_1(M'; Z_2)\geqq 2$, then choose a simple closed curve S which represents a non-zero element of $H_1(M'; Z_2)$ and whose tubular neighborhood T is a solid torus. [Use the map $w_1: H_1(M'; Z_2)\to Z_2$ giving the first Stiefel-Whitney class.] Construct a 4-manifold $W'=M'\times[0,1]\cup D^2\times D^2$ identifying T (with any framing) with $(\partial D^2)\times D^2$. Since $H_1(T; Z_2)\to H_1(M'; Z_2)$ is injective, $H_2(\partial W'; Z_2)\to H_2(W'; Z_2)$ is onto, so that $x\cdot x=0$ for $x\in H_2(W'; Z_2)$. Let $M_1'=\partial W' - M'\times 0$. We have $\dim_{Z_2} H_1(M_1'; Z_2)=\dim_{Z_2} H_1(M'; Z_2)-1$, since $H_1(M'-T; Z_2)\to H_1(M'; Z_2)$ is an isomorphism. By Lemma 4.5(1), the orientation cover W of W' gives a spin bordism from (M,α) to a pair (M_1,α_1) with $\mathrm{Fix}(\alpha_1, M_1)=\emptyset$ and $\dim_{Z_2} H_1(M/\alpha; Z_2)-1=\dim_{Z_2} H_1(M_1/\alpha_1; Z_2)$. By induction, there is a sequence $(M,\alpha)=(M_0,\alpha_0), \cdots, (M_s,\alpha_s)$ such that (M_i,α_i) is spin bordant to (M_{i+1},α_{i+1}) with α_{i+1} a free involution, and $H_1(M_s/\alpha_s; Z_2)=Z_2\cdot M_s/\alpha_s$ is a Z_2-homology handle, since it is non-orientable. By Lemma 4.2 M_s is a Z_2-homology handle. Let T be an α_s-invariant, solid torus in M_s whose core

represents a generator of $H_1(M_s, Z_2)$. Let β be a standard, orientation-reversing involution on $D^2\times D^2$ such that $\beta((\partial D^2)\times D^2)=(\partial D^2)\times D^2$ and $\beta|(\partial D^2)\times D^2$ is free and $\beta|D^2\times(\partial D^2)$ has the fixed point set S^0. Construct $W=M_s\times[0, 1]\cup D^2\times D^2$ by identifying $(T\times 1, \alpha_s)$ with $((\partial D^2)\times D^2, \beta|(\partial D^2)\times D^2)$. W gives a spin bordism from (M_s, α_s) to a pair (M_{s+1}, α_{s+1}) with $H_1(M_{s+1}; Z_2)=0$. This shows (5).

To conclude the proof of Theorem 6.1, it suffices to prove that (M, α) with M a Z_2-homology 3-sphere is a spin boundary. But this was proved in Theorem 5.1. This completes the proof of Theorem 6.1.

Let X be a Poincaré duality space with fundamental class $[X]$ of dimension $4m-1$ $(m\geqq 1)$. Assume X admits a map $f: X\to X$ with $f_*[X]=-[X]$ and $f_*^2=$identity on $T_{2m-1}(X; Z)$. By Theorem 2.1 $H_{2m-1}(X; Z)$ belongs to the class $\mathfrak{C}$. Let $\sigma(X)=\sigma(H_{2m-1}(X; Z))$.

THEOREM 6.2. *Suppose there is a Poincaré duality space Y of dimension $4m$ with boundary X such that*

(1) *Y admits a map $g: Y\to Y$ extending f with $g_*^2=$identity on $H_{2m}(Y; Q)$,*

(2) *the Z_2-intersection number, $y\cdot y=0$ for all $y\in H_{2m}(Y; Z_2)$. Then we have $\sigma(X)=0$.*

Let q be an integer >0. Let (A, B) be a topological pair such that $H_i(A, B; Z)$ is finitely generated for $i\leqq q$. For any field K, define $\sigma^{(q)}(A, B; K)$ to be the sum $\Sigma_{i=0}^{q}\dim_K H_i(A, B; K)$ (mod 2).

The following is proved easily by using the universal coefficient theorem:

LEMMA 6.1. $\sigma^{(q)}(A, B; Z_p)-\sigma^{(q)}(A, B; Q)\equiv\dim_{Z_p}T_q(A, B; Z)\otimes Z_p$ (mod 2) *for all prime $p\geqq 2$.* $(T_*(A, B; Z)=\mathrm{Tor}\, H_*(A, B; Z).)$

PROOF OF THEOREM 6.2. By dropping the components of Y not intersecting X into the discard, we can assume $H_0(Y, X; Z)=0$. Then the boundary operator $\partial: H_{4m}(Y, X; Z)\to H_{4m-1}(X; Z)$ is injective and, by definition, sends the fundamental class $[Y]$ of Y to $[X]$. Hence $g_*[Y]=-[Y]$ since $f_*[X]=-[X]$. Let $S_g: H_{2m}(Y; Q)\times H_{2m}(Y; Q)\to Q$ be the pairing defined by

$$S_g(x, y)=x\cdot g_*y$$

for $x, y\in H_{2m}(Y; Q)$, where $\cdot$ denotes the Q-intersection pairing, defined by using Q-Poincaré duality. Since $g_*[Y]=-[Y]$ and $g_*^2=$identity on $H_{2m}(Y; Q)$, we see that $S_g(x, y)=-S_g(y, x)$, so that $S_g(y, y)=0$ for all y. For a field K, let $j_*^K: H_{2m}(Y, K)\to H_{2m}(Y, X; K)$ be the canonical homomorphism. Noting that g_* is an automorphism on $H_{2m}(Y; Q)$, we obtain a non-singular, symplectic bilinear Q-form $\mathrm{Im}\, j_*^Q\times\mathrm{Im}\, j_*^Q\to Q$ induced by S_g. It follows that

$$\mathrm{Im}\, j_*^Q\equiv 0 \pmod 2.$$

By Z_2-Poincaré duality and (2), we obtain a non-singular symplectic bilinear Z_2-form $\mathrm{Im}\, j_*^{Z_2}\times\mathrm{Im}\, j_*^{Z_2}\to Z_2$ induced by the Z_2-intersection pairing of Y. Hence

$$\mathrm{Im}\ j_*^{Z_2} \equiv 0 \pmod 2.$$

From the exact sequence of the pair (Y, X), we see that

$$\dim_K \mathrm{Im}\ j_*^K \equiv \sigma^{(2m)}(Y, X; K) + \sigma^{(2m-1)}(X; K) + \sigma^{(2m-1)}(Y; K) \pmod 2$$

for any field K. Then by using Lemma 6.1,

$$\begin{aligned} 0 &\equiv \dim_{Z_2} \mathrm{Im}\ j_*^{Z_2} - \dim_Q \mathrm{Im}\ j_*^Q \pmod 2 \\ &\equiv \dim_{Z_2} T_{2m}(Y, X; Z) \otimes Z_2 + \sigma(X) + \dim_{Z_2} T_{2m-1}(Y; Z) \otimes Z_2 \pmod 2 \\ &\equiv \sigma(X) \pmod 2, \end{aligned}$$

since $T_{2m}(Y, X; Z)$ and $T_{2m-1}(Y; Z)$ are isomorphic by Poincaré duality. This completes the proof.

DEFINITION 6.2. A finite polyhedron X is a *Z_2-homology n-manifold* if

$$H_*(X, X-x; Z_2) \approx H_*(R^n, R^n-0; Z_2),$$

or equivalently (by excision)

$$H_*(\mathrm{Link}(x); Z_2) \approx H_*(S^{n-1}; Z_2)$$

for any $x \in X$ and any triangulation of X with x as a vertex.

The following is known (cf. A. Borel [3], p. 76, p. 79).

PROPOSITION 6.1. *If X is a closed piecewise-linear n-manifold, and α is a piecewise-linear involution on X, then each non-empty component C of* Fix(α, X) *is a Z_2-homology manifold. Further, if X is oriented, then C is a Z_2-homology m-manifold with $n-m$ even or odd, according to whether α is orientation-preserving or -reversing.* [Note that Fix(α, X) is a subpolyhedron of X.]

PROOF OF THEOREM III. By Theorem 6.1, (2)$\Rightarrow$(4). (4)$\Rightarrow$(3) is obvious. To prove (3)$\Rightarrow$(2), let W_1 be a compact 4-manifold with an involution β_1 such that $\partial W_1 = M$ and $\beta_1 | M \times [0, 1] = \alpha \times$identity for a boundary collar $M \times [0, 1]$ in W_1. The double $D(W_1)$ of W_1 admits an involution $\bar{\beta}_1$ defined by β_1. By Proposition 6.1, each component of Fix$(\bar{\beta}_1, D(W_1))$ is a Z_2-homology manifold. This implies that the set of discrete points of Fix(α, M) is the boundary of a compact 1-manifold. Hence $\sigma(\alpha, M)=0$ and (3)$\Rightarrow$(2). (2)$\Rightarrow$(1) follows from Theorems 6.1 and 6.2. To prove that (1)$\Rightarrow$(2), assume $\sigma(\alpha, M)=1$. Then $\sigma(\alpha+\alpha_0, M+P^3)=0$ for (P^3, α_0), since Fix$(\alpha_0, P^3)=P^0+P^2$. By Theorems 6.1 and 6.2, $\sigma(M+P^3)=0$, and $\sigma(M)=\sigma(P^3)=1$, proving (1)$\Rightarrow$(2). This completes the proof.

7. Proof of the Remark to Theorem II.

Since $H_1(M; Z)$/odd torsion$\approx Z_2$, $Sq^1: H^1(M; Z_2) \approx H^2(M; Z_2)$. The cohomology algebra $H^*(M; Z_2)$ is isomorphic to $Z_2[a]/a^4$, since $Sq^1(x) = x \cup x$ for

$x \in H^1(M; Z_2)$. By Thom-Gysin sequence, the connected double cover M' of M is a Z_2-homology 3-sphere. Since $\sigma(M)=1$, $\mathrm{Fix}(\alpha, M)$ contains a closed connected surface F with $\chi(F)$ odd by Theorem III. Let $p: M' \to M$ be the projection. Let F' be a component of $p^{-1}(F)$. By the unique-lifting property of a covering, α lifts to an involution α' on M' such that $\alpha' | F' = \text{identity}$. Since $\mathrm{Fix}(\alpha', M') = S^0$ or S^2 by Smith theory, we have $F'=S^2$, so that $F=P^2$ (and $F'=p^{-1}(F)=S^2$). Let N be an α-invariant, regular neighborhood of P^2 in M. Since $\partial N = S^2$, the union $(M - \mathrm{Int}\, N) \cup N$ gives a connected sum $M'' \# P^3$ for a Z_2-homology 3-sphere M''. Clearly, α preserves the factors. This completes the proof.

8. Proof of Theorem V.

It suffices to check that $\sigma(P^3)=\sigma(X)$ where $X=X(P^3, \alpha)$ (cf. Theorem III, Corollary 1.1 and Proposition 1.1). Let $\mathrm{Fix}(\alpha, P^3)=P^0+P^2$. Let $P^3=N \cup B^3$, where N is an α-invariant, regular neighborhood of P^2 in P^3, and B^3 is an α-invariant 3-ball containing P^0. Then $X=X(P^3, \alpha)=X_1 \cup X_2$ where $X_1 = N \times S^2/\alpha \times \gamma \simeq P^2 \times P^2$ and $X_2 = B^3 \times S^2/\alpha \times \gamma \simeq P^2$ and $\partial X_1 = \partial X_2 = S^2 \times S^2/\alpha \times \gamma = S^2 \times S^2/\gamma \times \gamma$. From the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(N \times S^2) & \longrightarrow & \pi_1(X_1) & \longrightarrow & Z_2 & \longrightarrow & 1 \\ & & \downarrow \approx & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(P^3 \times S^2) & \longrightarrow & \pi_1(X) & \longrightarrow & Z_2 & \longrightarrow & 1, \end{array}$$

we see that $\pi_1(X_1) \approx \pi_1(X) \approx Z_2 + Z_2$. By Mayer-Vietoris sequence, $H_2(X_1; Z) \to H_2(X; Z)$ is onto, so that $H_2(X; Z)=0$ or Z_2. But $H_2(X; Z) \to H_2(\pi_1(X); Z)$ $(=Z_2)$ is always onto by H. Hopf [7]. Thus, $H_2(X; Z)=Z_2$ and $\sigma(X)=1=\sigma(P^3)$. This completes the proof.

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