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On the existence of harmonic functions in L^p

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Let D be a domain in the n-dimensional Euclidean space R^n ($n\geqq 2$), and let $A_p(D)$ (resp. $H_p(D)$), $1<p<\infty$, be the space of all functions in $L^p(D)$ each of which is holomorphic (resp. harmonic) in D if $n=2$ (resp. $n\geqq 3$). Carleson [2] proved in case $n=2$ that

i) if $p>2$ and $C_q(R^2-D)>0$, $1/p+1/q=1$, then $A_p(D)$ contains a non-constant function;

ii) if $p>2$ and $\Lambda_{2-q}(R^2-D)<\infty$, then $A_p(D)=\{0\}$. Here C_α denotes the Riesz capacity with respect to the kernel $r^{\alpha-n}$, and Λ_α denotes the α-dimensional Hausdorff measure.

To improve this result, it is convenient to use the Bessel capacity; the Bessel capacity of index (α, r), $\alpha>0$, $1<r<\infty$, is denoted by $B_{\alpha,r}$ (cf. Meyers [4]). Further, we say that a class of functions is non-trivial if it contains a non-constant function.

Our main aim is to prove the following theorems.

THEOREM 1. (i) *If* $B_{1,q}(R^2-D)=0$, *then* $A_p(D)=\{0\}$.

(ii) *If* $p\geqq 2$ *and* $B_{1,q}(R^2-D)>0$, *then* $A_p(D)$ *is non-trivial.*

(iii) *If* $p<2$ *and* R^2-D *contains at least two points, then* $A_p(D)$ *is non-trivial.*

THEOREM 2. (i) *If* $B_{2,q}(R^n-D)=0$, *then* $H_p(D)=\{0\}$.

(ii) *If* $2q\leqq n$ *and* $B_{2,q}(R^n-D)>0$, *then* $H_p(D)$ *is non-trivial.*

(iii) *If* $2q>n$, $q\neq n$ *and* R^n-D *contains at least two points, then* $H_p(D)$ *is non-trivial.*

(iv) *If* $q=n$ *and* $R^n-D\supset\{x^0, 0, -x^0\}$, $x^0\neq 0$, *then* $H_p(D)$ *is non-trivial.*

REMARK 1. (i) If $q<n<2q$ and $D=R^n-\{x^{(1)}, x^{(2)}\}$, $x^{(1)}\neq x^{(2)}$, then $H_p(D)=\{cu\,;\, c\in R^1\}$, where

$$u(x)=|x-x^{(1)}|^{2-n}-|x-x^{(2)}|^{2-n}.$$

(ii) If $q>n$ and $D=R^n-\{x^{(1)}, x^{(2)}\}$, $x^{(1)}\neq x^{(2)}$, then $H_p(D)=\left\{\sum_{i=0}^{n} c_i u_i\,;\, c_i\in R^1 \text{ for } i=0, 1, \cdots, n\right\}$, where

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$$u_0(x)=|x-x^{(1)}|^{2-n}-|x-x^{(2)}|^{2-n}$$
$$-\sum_{i=1}^{n}(x_i^{(1)}-x_i^{(2)})\frac{\partial}{\partial x_i}|x-x^{(2)}|^{2-n},$$
$$u_i(x)=\frac{\partial}{\partial x_i}(|x-x^{(1)}|^{2-n}-|x-x^{(2)}|^{2-n}),\qquad i=1,\cdots,n.$$

(iii) If $q\neq n<2q$ and R^n-D consists of one point only, then $H_p(D)=\{0\}$.

(iv) If $q=n$ and R^n-D consists of two points, then $H_p(D)=\{0\}$. If $q=n$ and R^n-D consists of three points $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, then a necessary and sufficient condition for $H_p(D)$ to be non-trivial is that $2x^{(1)}=x^{(2)}+x^{(3)}$, $2x^{(2)}=x^{(3)}+x^{(1)}$ or $2x^{(3)}=x^{(1)}+x^{(2)}$ holds.

REMARK 2. The following follow easily from Theorems 1 and 2.

(1) In case $p\geqq 2$, $B_{1,q}(R^2-D)=0$ if and only if $A_p(D)=\{0\}$.

(2) In case $2q\leqq n$, $B_{2,q}(R^n-D)=0$ if and only if $H_p(D)=\{0\}$.

The assertion (1) for the case $p>2$ is also an easy consequence of [**3**; Theorem 5.1]; the assertion (2) for the case $2q<n$ follows also from [**3**; Lemma 5.3].

We give only a proof of Theorem 2, because Theorem 1 can be proved similarly.

PROOF OF THEOREM 2. The statement (i) is an easy consequence of [**1**; Theorem B] and the fact that $H_p(R^n)=\{0\}$, which follows from the mean-value property for harmonic functions.

Assume that the assumptions of (ii) are satisfied. Then we can find mutually disjoint compact subsets K_1, K_2 of R^n-D such that $B_{2,q}(K_i)>0$ for $i=1$, 2. By [**4**; Theorem 16] there exist non-negative measures μ_1, μ_2 such that the support of μ_i is included in K_i, $\mu_i(K_i)=1$ and $g_2*\mu_i\in L^p(R^n)$ for each i, where g_2 denotes the Bessel kernel of order 2. Set

$$u(x)=\int|x-y|^{2-n}d\mu_1(y)-\int|x-y|^{2-n}d\mu_2(y),\qquad x\in R^n.$$

Then $u\in L^p_{\mathrm{loc}}(R^n)$ and $u=O(|x|^{1-n})$ as $|x|\to\infty$, so that $u\in H_p(D)$.

Assume that $2q>n$ and $R^n-D\supset\{x^{(1)}, x^{(2)}\}$, $x^{(1)}\neq x^{(2)}$. If $q<n$, then the function u in Remark 1 (i) belongs to $H_p(D)$. If $q>n$, then the functions u_0, u_1, $\cdots$, u_n in Remark 1 (ii) belong to $H_p(D)$.

Finally assume that $q=n$ and $R^n-D\supset\{x^0, 0, -x^0\}$, $x^0\neq 0$. Then the function

$$v(x)=|x+x^0|^{2-n}-2|x|^{2-n}+|x-x^0|^{2-n}$$

belongs to $H_p(D)$.

To prove Remark 1, it suffices to use the following result.

LEMMA. *Let u be a tempered distribution in R^n such that*

$$\Delta u=0\qquad\text{on } R^n-\{x^{(1)},\cdots,x^{(k)}\}.$$

Then u is of the form

$$u(x)=\sum_{i,\lambda} c_{i,\lambda} D^{\lambda}(|x-x^{(i)}|^{2-n})+P(x),$$

where $c_{i,\lambda}\in R^1$, $D^{\lambda}=(\partial/\partial x_1)^{\lambda_1}\cdots(\partial/\partial x_n)^{\lambda_n}$ for a multi-index $\lambda=(\lambda_1, \cdots, \lambda_n)$ and P is a harmonic polynomial.

As an application of Theorem 2, we give a partial answer to Problem 2 in [**6**]. Assume hereafter $p<2<q$. In the three cases listed below, $H_p(D)$ is non-trivial and $H_q(D)=\{0\}$, so that the dual of $H_p(D)$ is not equal to $H_q(D)$.

(1) Let $2p\leqq n$ and $2q\leqq n$. Find a compact set $K\subset R^n$ such that $B_{2,p}(K)=0$ but $B_{2,q}(K)>0$, and let $D=R^n-K$, which is a domain on account of [**5**; Theorem 3].

(2) Let $2p\leqq n$, $2q>n$, $q\neq n$ and $D=R^n-\{x^{(1)}, x^{(2)}\}$, $x^{(1)}\neq x^{(2)}$.

(3) Let $2p\leqq n$, $q=n$ and $D=R^n-\{x^0, 0, -x^0\}$, $x^0\neq 0$.

Finally we note that if $p<n<2p$, $q<n<2q$ and $D=R^n-\{x^{(1)}, x^{(2)}\}$, $x^{(1)}\neq x^{(2)}$, then both $H_p(D)$ and $H_q(D)$ are one-dimensional, so that the dual of $H_p(D)$ is equal to $H_q(D)$.

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