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# A theorem on almost analytic equisingularity

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Let $f$ and $g$ denote two germs of real analytic functions on $\boldsymbol{R}^n$ at the origin. We assume that $f(0)=g(0)=0$. If $x(t)$ $(0\leqq t)$ is any analytic path in $\boldsymbol{R}^n$ starting at the origin then we can compare $f$ and $g$ along $x(t)$ by using the orders $O(f(x(t)))$ and $O(g(x(t)))$ respectively. If these orders are the same for all such paths starting at the origin then we say that $f$ and $g$ have the same distribution of Lojasiewicz exponents, or simply the same distribution. (Cf. [**3**]).

Problem: How are the two germs of real analytic functions at the origin related if they have the same distribution?

We show that if two germs have the same distribution then they are represented, after a succession of suitable blow-ups, by functions which differ only by composition with a bianalytic isomorphism.

## A. The theorem.

Let $f$ and $g$ be two germs of real analytic functions on $\boldsymbol{R}^n$ at 0. When it is convenient we will identify $f$ and $g$ with representative functions on some neighbourhood $U$ of 0.

We say that $f$ and $g$ are *almost analytically equivalent* if there exists a neighbourhood $U$ of 0, the composite

$$\pi^*: U^* \longrightarrow U$$

of a finite succession of monoidal transformations with non singular (closed) centres over $U$, and a bianalytic isomorphism

$$\tau^*: V_1^* \longrightarrow V_2^*$$

between two neighbourhoods $V_1^*$ and $V_2^*$ of $(\pi^*)^{-1}(0)$ in $U^*$ such that the diagram

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$$\begin{tikzcd}
V_1^* \arrow[r, hook] \arrow[d, "\tau^*"] & U^* \arrow[r, "\pi^*"] & U \arrow[dr, "g"] & \\
& & & \boldsymbol{R} \\
V_2^* \arrow[r, hook] & U^* \arrow[r, "\pi^*"] & U \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes.

Let $x(t)$, $0 \leqq t$, denote an analytic path starting at the origin. Thus $x(0)=\underset{\sim}{0}$ and the components of $x(t)$ are convergent power series in $t$.

By $\mathcal{D}_f$ we mean the distribution of $f$ at the origin. Thus $\mathcal{D}_f$ is a function which assigns to each analytic path $x(t)$, $0 \leqq t$, starting at the origin, the order $O(f(x(t))$ of the power series $f(x(t))$ in $t$. (Compare [3]).

REMARK. The infimum of the values taken by $\mathcal{D}_f$ yields the Lojasiewicz exponent of $f$. Thus our classification of germs of functions by means of the functions $\mathcal{D}_f$ is finer than that given by the Lojasiewicz exponent. (Here $f$ is positive definite.)

With the notation which we have now introduced we state the main result:

THEOREM. *If $\mathcal{D}_f = \mathcal{D}_g$ then $f$ and $g$ are almost analytically equivalent.*

EXAMPLE. Consider the functions

$$f_a(x, y) = x^4 + y^4 + a x^2 y^2$$

where $a$ is a real number. If $a > -2$ then $\mathcal{D}_{f_a}$ is the constant function with value 4. Thus if $a, b > -2$ then, by the theorem, $f_a$ and $f_b$ are almost analytically equivalent.

When $a \leqq -2$ then the function is no longer positive definite. If $a \neq b$ and $b \leqq -2$ then $\mathcal{D}_{f_a} \neq \mathcal{D}_{f_b}$.

The above notion of almost analytic equivalence, slightly modified, is used in [4] to establish a finite classification theorem for isolated singularities.

### B. Resolution of singularities.

We now state a theorem of Hironaka which is contained in [2].

Let $V_f$ and $V_g$ denote the varieties $f=0$ and $g=0$ respectively. If $\mathcal{D}_f = \mathcal{D}_g$ then for a sufficiently small neighbourhood $U$ of 0,

$$U \cap V_f = U \cap V_g .$$

This follows from the curve selection lemma ([1]).

HIRONAKA'S THEOREM. *Suppose $\mathcal{D}_f = \mathcal{D}_g$. Then there exist a neighbourhood*

*$U$ of 0 and a finite succession of monoidal transformations with non-singular centers*

$$\pi : (\tilde{U}, \Sigma) \longrightarrow (U, V_f \cap U)$$

*where $\Sigma = \pi^{-1}(V_f \cap U) = \pi^{-1}(V_g \cap U)$ with the following property:*

*Near any $p \in \tilde{U}$ there is an (analytic) coordinate system $\{x_1, \cdots, x_n\}$ with respect to which*

$$\left.\begin{aligned} f(\pi(x)) &= \varepsilon_1(x) x_1^{k_1} \cdots x_n^{k_n} \\ g(\pi(x)) &= \varepsilon_2(x) x_1^{k_1} \cdots x_n^{k_n} \end{aligned}\right\} \tag{B1}$$

*where $\varepsilon_1(0) \neq 0 \neq \varepsilon_2(0)$, $k_i \geqq 0$ and $p$ is at the origin.*

COROLLARY 1. *The quotient function $f(\pi(x))/g(\pi(x))$ defined on $\tilde{U} - \Sigma$ has an analytic extension throughout $\tilde{U}$. It is non vanishing at any point.*

The extension is clearly $\varepsilon_1(x)/\varepsilon_2(x)$ near $p$.

COROLLARY 2. *In a neighbourhood of $p \in \tilde{U}$, $k_i > 0$ if and only if $\Sigma$ contains the coordinate hyperplane $x_i = 0$.*

*Thus $\Sigma = \bigcup_{k_i > 0} P_i$ where $P_i = \{x \mid x_i = 0\}$.*

## C. A vector field on $\tilde{U} \times I$.

We assume $f$ and $g$ are as in the main theorem. We then choose $U$, $\tilde{U}$, $\Sigma$ and $\pi$ in accordance with Hironaka's theorem.

Let

$$F : \tilde{U} \times I \longrightarrow \boldsymbol{R}$$

be defined by

$$F(x, t) = (1-t) f(\pi(\underset{\sim}{x})) + t g(\pi(\underset{\sim}{x})),$$
$$x \in \tilde{U}, \qquad 0 \leqq t \leqq 1.$$

Choose an analytic Riemannian metric on $\tilde{U}$ so that $\mathrm{grad}_{(x,t)} F$ may be identified with a tangent vector field on $\tilde{U} \times I$. Here we consider $t$ as a parameter; $\mathrm{grad}_{(x,t)} F \equiv \mathrm{grad}\, F$ has the obvious meaning whilst $\mathrm{grad}_x F$ denotes the gradient of $F$ in $\underset{\sim}{x}$-space. Thus if the coordinate system is orthonormal then

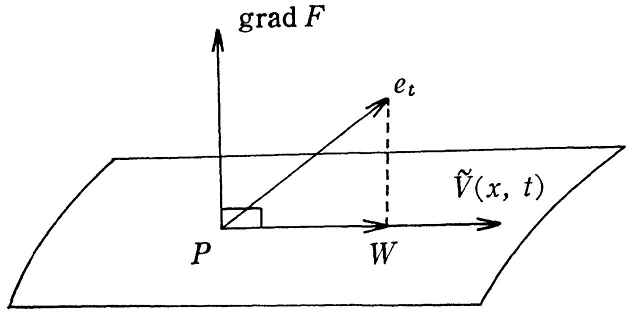
$$\mathrm{grad}_{(x,t)} F = \left( \frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial t} \right)$$

while

$$\mathrm{grad}_x F = \left( \frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}, 0 \right).$$

We now define a vector field $\tilde{V}$ on $\tilde{U} \times I$. At any point $(x, t)$ where $x \notin \Sigma$. This field is to lie in the plane spanned by $\mathrm{grad}\, F$ and the unit vector $e_t$ in the

$t$-direction. (Notice that if $x\notin\Sigma$ then $\mathrm{grad}_{(x,t)}F\neq\underset{\sim}{0}$ by Hironaka's theorem so these two vectors are linearly independent). We also require that this field has component 1 in the direction $e_t$, and is orthogonal to Grad $F$ for $x\notin\Sigma$.




Let $u=(1/|\mathrm{grad}\,F|)\,\mathrm{grad}\,F$ denote the unit vector in the direction of grad $F$. Then the component of $e_t$ in the direction of grad $F$ is the inner product $e_t\cdot u$. Thus $\tilde{V}(x, t)$ has the same direction as the vector

$$w=e_t-(e_t\cdot u)u=e_t-\frac{(e_t\cdot\mathrm{grad}\,F)}{|\mathrm{grad}\,F|^2}\mathrm{grad}\,F\,.$$

Now $\mathrm{grad}\,F=\mathrm{grad}_xF+F_te_t$. Also $\mathrm{grad}_xF$ is orthogonal to $e_t$. Hence

$$\begin{aligned}w&=e_t-\frac{F_t}{|\mathrm{grad}\,F|^2}\mathrm{grad}\,F\\&=\frac{-F_t}{|\mathrm{grad}\,F|^2}\mathrm{grad}_xF+\left[1-\frac{F_t^2}{|\mathrm{grad}\,F|^2}\right]e_t\\&=\frac{-F_t}{|\mathrm{grad}\,F|^2}\mathrm{grad}_xF+\frac{|\mathrm{grad}_xF|^2}{|\mathrm{grad}\,F|^2}e_t\,.\end{aligned}$$

Adjusting the length of $w$ we define, for $x\notin\Sigma$,

$$\tilde{V}(x, t)=\frac{-F_t}{|\mathrm{grad}_xF|^2}\mathrm{grad}_xF+e_t\,.$$

For $x\in\Sigma$ we define

$$\tilde{V}(x, t)=e_t\,.$$

The following proposition is now almost immediate:

PROPOSITION. *Inside $\tilde{U}$ in some neighbourhood of $\Sigma$,*

(1) grad $F\neq0$ *for* $x\notin\Sigma$. *Hence $\tilde{V}$ is analytic off $\Sigma\times I$.*

(2) *Off $\Sigma\times I$ the field $\tilde{V}(x, t)$ is tangent to the level surfaces $F$=constant.*

(3) *$\tilde{V}$ is continuous on $\tilde{U}\times I$ (but not, in general, analytic on $\Sigma\times I$).*

In order to verify (3) it need only be noted that as $x\to\Sigma$,

$$-\frac{F_t}{|\mathrm{grad}_x F|^2}\mathrm{grad}_x F \longrightarrow \underset{\sim}{0}\,.$$

To see this it is sufficient to check that each component $x_i$ of $x$ appears in the vector on the left with exponent at least one in each component.

EXAMPLE. Suppose $n=2$, $\tilde{U}=\boldsymbol{R}^2$ and $F=\varepsilon(t)x^a y^b$ where $a, b>0$. Now

$$\mathrm{grad}_x F=\varepsilon(t)\ (a x^{a-1}y^b \underset{\sim}{i}+b x^a y^{b-1}\underset{\sim}{j})\,.$$

Thus for $(x, y)\notin\Sigma=\{(x, y)\,|\,x=0 \text{ or } y=0\}$ we have

$$\begin{aligned}\frac{-F_t}{|\mathrm{grad}_x F|^2}\mathrm{grad}_x F&=\frac{\varepsilon'(t)}{\varepsilon(t)}\frac{xy}{a^2y^2+b^2x^2}(ay\underset{\sim}{i}+bx\underset{\sim}{j})\\ &\longrightarrow 0 \quad \text{as } x\to 0 \text{ or } y\to 0\,.\end{aligned}$$

Here $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are the unit vectors in the $x$- and $y$-directions.

### D. Stratifications in normal crossing.

Let $\mathcal{A}$ be an analytic manifold of dimension $n$, and

$$\mathcal{S}:\ \mathcal{A}=\bigcup_{s=0}^{n}\Sigma_s$$

a stratification of $\mathcal{A}$. Thus each $\Sigma_s$ is an analytic submanifold of dimension $s$ and $\Sigma_s\cap\Sigma_t=\emptyset$ if $s\neq t$.

We say that $\mathcal{S}$ is in *normal crossing* if the following holds. For any point $p\in\mathcal{A}$ we have $p\in\Sigma_s$ for some $s$. There is to exist an analytic coordinate system $\{x_1, \cdots, x_n\}$ with origin $p$ such that in a sufficiently small neighbourhood of $p$,

(D1) $\Sigma_s$ coincides with the $(x_{n-s+1}, \cdots, x_n)$ coordinate space. (i.e. $\Sigma_s$ is defined by $x_1=\cdots=x_{n-s}=0$).

(D2) $\Sigma_s\cup\cdots\cup\Sigma_{s+k}$ is the union of all $(s+k)$-dimensional coordinate spaces, each of which contains $\Sigma_s$, $1\leqq k\leqq n-s$.

We say that the coordinate system $\{x_1, \cdots, x_n\}$ *displays* $\mathcal{S}$ at $p$. From these we also obtain the property

(D3) If $\Sigma_0=\cdots=\Sigma_{k-1}=\emptyset$, $\Sigma_k\neq\emptyset$ then $\Sigma_k$ is a closed submanifold.

EXAMPLE 1. Let $\mathcal{A}$ denote the $\tilde{U}$ of Hironaka's theorem. Let $\Sigma_s$ consist of all those $p\in\mathcal{A}$ for which the expression (B1) has exactly $n-s$ exponents $k_i$ greater than zero.

Notice that $\Sigma_n$ is open and dense in $\mathcal{A}$ and that

$$\Sigma=\bigcup_{s=0}^{n-1}\Sigma_s\,.$$

This stratification is in normal crossing.

EXAMPLE 2. More generally let $h$ be any analytic function on a manifold $\mathcal{A}$. If $h$ is in normal crossing then a similar stratification can be defined on $\mathcal{A}$. In particular if $\mathcal{A}=\boldsymbol{R}^n$ with coordinates $(x_1, \cdots x_n)$ and $h(x_1, \cdots, x_n)=x_1 \cdots x_n$ then $\Sigma_0=\{0\}$ and for $k>0$, $\Sigma_0 \cup \cdots \cup \Sigma_k$ is the union of all $k$-dimensional coordinate spaces.

## E. Blowing-up the stratification $\mathcal{S}$.

Let

$$\mathcal{S}:\ \mathcal{A}=\bigcup_s \Sigma_s$$

be a stratification in normal crossing. Denote by $k$ the smallest integer for which $\Sigma_k \neq \emptyset$.

Since $\Sigma_k$ is closed, $\mathcal{A}$ can be blown up along $\Sigma_k$. Let

$$\beta:\ B(\mathcal{A}) \longrightarrow \mathcal{A} \tag{E1}$$

denote such a monoidal transformation. If $p \in \Sigma_k$ then

$$\beta^{-1}(p)$$

is an $n-k-1$ dimensional real projective space in $B(\mathcal{A})$.

For $X \subseteqq \mathcal{A}$ let $\tau(X)$ denote the strict transform of $X$ in $B(\mathcal{A})$. Thus $\tau(X)$ is the closure of $\beta^{-1}(X-\Sigma_k)$ in $B(\mathcal{A})$. In particular $\tau(\Sigma_{k+1})$ is a closed $(k+1)$-dimensional (analytic) submanifold in $B(\mathcal{A})$.

Let

$$B(\Sigma_{k+1})=\tau(\Sigma_{k+1})$$

and for $1 \leqq i \leqq n-k-1$ define

$$B(\Sigma_{k+1+i})=\tau(\Sigma_{k+1+i})-\tau(\Sigma_{k+i})\,.$$

Finally put

$$B(\mathcal{S}):\ B(\mathcal{A})=\bigcup_{i=k+1}^{n} B(\Sigma_i)\,.$$

PROPOSITION. *The stratification $B(\mathcal{S})$ is also in normal crossing. It has no strata of dimension less than $k+1$.*

PROOF. Away from $\Sigma_k$ the monoidal transformation $\beta$ is locally an analytic isomorphism. Thus any analytic coordinate system for $\mathcal{S}$ corresponds to one for $B(\mathcal{S})$ in a neighbourhood of corresponding points.

At a point on $\Sigma_k$ we may choose a coordinate system $\{x_1, \cdots, x_n\}$ which displays $\mathcal{S}$ at $p$. The point $p$ is now at the origin and we refer to it as the origin. The blow-up of a suitably small neighbourhood of the origin may be covered by $n-k$ sets, $U_i$, $k+1 \leqq i \leqq n$.

Fix one such an $i$, say $r$. The set $U_r$ is mapped by $\beta$ into a sector of $\boldsymbol{R}^n$ containing the $x_r$-axis. Now we may coordinatize $U_r$ by coordinates $(x_1, \cdots, x_n)$ and the map $\beta$ when restricted to $U_r$ is given by

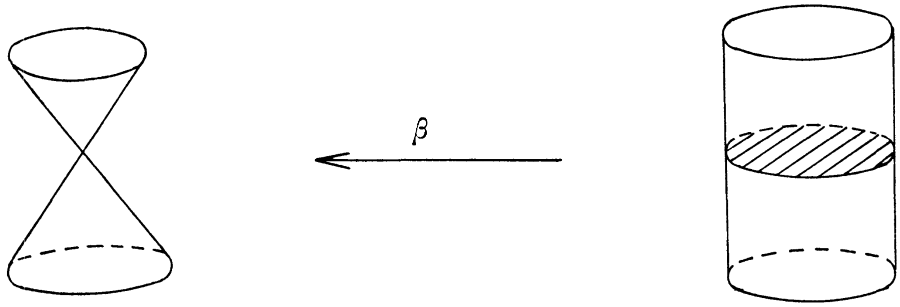
$$((x_1, \cdots, x_n))=(x_1, \cdots, x_k, x_{k+1}x_r, \cdots, x_{r-1}x_r, x_r, x_{r+1}x_r, \cdots, x_n x_r).$$

The blow-up of the point $p \in \Sigma_k$ is thus in the subspace defined by

$$x_1=\cdots=x_k=0\,;\ x_r=0.$$

The required conditions for a normal stratification now are easily seen to hold. This completes the proof.

REMARK. The above situation is illustrated by the diagram




Now we can repeat this construction so that we obtain the successive stratifications:

$$\mathcal{S},\ B(\mathcal{S}),\ B^2(\mathcal{S}),\ \cdots,\ B^{n-k-1}(\mathcal{S}),$$

all of which are in normal crossing.

In addition $B^{n-k-1}(\mathcal{S})$ has no strata of dimension less than $n-1$. Thus $B^{n-k-1}(\mathcal{S})$ consists of only two strata, of dimension $n$ and $n-1$ respectively.

Blowing up along an $(n-1)$-dimensional stratum does not yield anything new. Thus we stop at $B^{n-k-1}(\mathcal{S})$.

NOTATION.

$$\mathcal{A}^*=B^{n-k-1}(\mathcal{A}),\quad \beta^*=\beta\circ\cdots\circ\beta \quad (n-k-1 \text{ times}).$$

$$\mathcal{S}^*=\mathcal{A}^*=\Sigma_n^* \cup \Sigma_{n-1}^*.$$

We call

$$\beta^*: \mathcal{A}^* \longrightarrow \mathcal{A}$$

the *complete blowing-up* of $\mathcal{A}$ with respect to $\mathcal{S}$.

## F. The main lemma.

Consider the stratifications

$$\mathcal{S}_{f\circ\pi} \quad \text{and} \quad \mathcal{S}_{g\circ\pi}$$

of $\tilde{U}$ as in Example 1(D). By (B1) they are the same.

We take $\mathcal{A}=\tilde{U}$, $\mathcal{S}=\mathcal{S}_{f\circ\pi}=\mathcal{S}_{g\circ\pi}$ and obtain the complete blow-up

$$\beta^*: U^* \longrightarrow \tilde{U}.$$

LEMMA. *There exists a vector field $V^*$ defined and analytic throughout $U^*\times I$ such that*

$$d\beta^*(V^*(p^*, t))=\tilde{V}(\beta^*(p^*); t) \qquad p^*\in U^*,\ 0\leqq t\leqq 1.$$

The two examples which follow illustrate the proof in the local situation. They can be generalized to deal with the global situation as in the next section.

EXAMPLE 1. Take $n=2$, $\tilde{U}=\boldsymbol{R}^2$, $F=\varepsilon(t)x^a y^b$ where $\varepsilon(t)\neq 0$, $a>0$, $b>0$.

Then

$$\Sigma_1^*=\{(x, y)\,|\,xy=0, \qquad (x, y)\neq(0, 0)\}.$$

$$\Sigma_0^*=\{(0, 0)\}.$$

$$\Sigma_2^*=U-\Sigma_1^*-\Sigma_0^*.$$

If $\beta^*: U^*\to\tilde{U}$ denotes the blow up around the origin then $U^*$ is covered by two pieces, $U_1^*$ and $U_2^*$ say. In $U_i^*$ ($i=1$ or 2) we can choose a coordinate system $(X_i, Y_i)$ and take

$$\beta^*((X_1, Y_1))=(X_1, X_1Y_1)$$

and

$$\beta^*((X_2, Y_2))=(X_2Y_2, Y_2).$$

Thus $\beta^*$ maps $U_1$ onto a sector about the $x$-axis and $U_2$ onto one about the $y$-axis.

From the example in $C$,

$$\begin{aligned}\tilde{V}(x, y, t)&=\frac{\varepsilon'(t)}{\varepsilon(t)}\frac{xy}{a^2y^2+b^2x^2}(ay\underset{\sim}{i}+bx\underset{\sim}{j})+e_t \\ &\qquad\qquad \text{if}\quad x, y\neq 0, \\ &=e_t \qquad\quad \text{if}\quad x=0 \text{ or } y=0.\end{aligned}$$

Now,

$$V^*(X, Y, t)=-\frac{\varepsilon'(t)}{\varepsilon(t)}\frac{X}{a^2+b^2X^2}(a\underset{\sim}{i}+X(bX-a)\underset{\sim}{j})+e_t$$

in the neighbourhood of the $Y$-axis.

In the neighbourhood of the $X$-axis we have

$$V^*(X, Y, t)=-\frac{\varepsilon'(t)}{\varepsilon(t)}\frac{Y}{ay^2+b^2}((aX-b)y\underset{\sim}{i}+b\underset{\sim}{j})+e_t.$$

It is easily seen that

$$d\beta^*(V^*(X, Y, t)) = \tilde{V}(\beta(X, Y), t),$$

in these neighbourhoods. $V^*$ *is* indeed analytic since the denominators in these expressions are *units*.

EXAMPLE 2. Take $n=3$, $\tilde{U}=\boldsymbol{R}^3$ and $F=\varepsilon(t)x^a y^b z^c$.

Now

$$\begin{aligned} \mathrm{grad}_x F &= \frac{aF}{x}\underset{\sim}{i}+\frac{bF}{y}\underset{\sim}{j}+\frac{cF}{z}\underset{\sim}{k} && \text{if } x, y, z\neq 0 \\ &= \underset{\sim}{0} && \text{if } x, y, z=0. \end{aligned}$$

Thus for $(x, y, z)\notin\Sigma$ which is the union of the coordinate planes,

$$\begin{aligned} -\frac{F_t}{|\mathrm{grad}_x F|^2}\mathrm{grad}_x F &= -\frac{\varepsilon'(t)}{\varepsilon(t)}\left(\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}\right)^{-1}\left(\frac{a}{x}\underset{\sim}{i}+\frac{b}{y}\underset{\sim}{j}+\frac{c}{j}\underset{\sim}{k}\right) \\ &= -\frac{\varepsilon'(t)}{\varepsilon(t)}\frac{xyz}{a^2y^2z^2+b^2x^2z^2+c^2x^2y^2}(ayz\underset{\sim}{i}+bxz\underset{\sim}{j}+cyz\underset{\sim}{k}). \end{aligned}$$

Now

$$\tilde{V}(x, y, z, t) = -\frac{\varepsilon'(t)}{\varepsilon(t)}\frac{xyz}{a^2y^2z^2+b^2x^2z^2+c^2x^2y^2}(ayz\underset{\sim}{i}+bxz\underset{\sim}{j}+cyz\underset{\sim}{k})+e_t$$

for $xyz\neq 0$.

Clearly as $x\to 0$ or $y\to 0$ or $z\to 0$ then $\tilde{V}(x, y, z, t)\to\underset{\sim}{0}$. If $x$, $y$ or $z$ is zero then $\tilde{V}(x, y, z, t)=\underset{\sim}{0}$. A simple computation shows that $(d\beta^*)^{-1}(\tilde{V})$ has a unique analytic extension in $U^*$.

## G. Analysis of $\tilde{V}$.

We prove the main lemma by a careful analysis of the field $\tilde{V}$. Consider the general case first. Let the manifold $\mathcal{A}$ with stratification

$$\mathcal{S}:\ \Sigma=\bigcup_{s=0}^{n-1}\Sigma_s$$

be given. For a point $p\in\Sigma_{n-s}$ let $\{x_1, \cdots, x_n\}$ be a coordinate system displaying $\mathcal{S}$ at $p$ (as in $(D)$).

Write

$$X^{(s)}=\prod_{i=1}^{s}x_i$$

$$\xi_i^{(s)}=\frac{1}{x_i}X^{(s)} \qquad 1\leqq i\leqq s.$$

$$\underset{\sim}{\xi}^{(s)}=(\xi_1^{(s)}, \cdots, \xi_s^{(s)}).$$

(In the case $s=1$ notice that $X^{(1)}=x_1$, $\xi_1^{(1)}=1$).
These notations can only be used when $p$, $\Sigma_{n-s}$ and the coordinate system are given.

Let $V$ be a vector field defined and analytic on $(\mathcal{A}-\Sigma)\times I$. We say that $V$ is *modifiable* along $\Sigma$ if for any $p\in\Sigma$, say $p\in\Sigma_{n-s}$, there is a coordinate system $\{x_1, \cdots, x_n\}$ displaying $\mathcal{S}$ at $p$ for which $V$ has the form

$$V(x, t)=\sum_{i=1}^{n}\frac{X^{(s)}L_i(\xi^{(s)}; x, t)}{P(\xi^{(s)}; x, t)}\frac{\partial}{\partial x_i}+\frac{\partial}{\partial t} \tag{G1}$$

where the $L_i$ $(1\leqq i\leqq n)$ are linear and $P$ is a quadratic form in the $s$-variables with coefficients which are analytic in $(x, t)$. Moreover, we require that $P$ is positive definite near $0\times I$ (that is $P(y_1, \cdots, y_s; x, t)>0$ if $\underset{\sim}{y}\neq\underset{\sim}{0}$). By $\partial/\partial x_i$ we mean the unit vector in the $x_i$-direction. Similarly $\partial/\partial t$, for consistency, replaces $e_t$. Note that $V$ has an analytic extension on $\Sigma_{n-1}$.

Example. Let $\mathcal{A}=\tilde{U}$, $\mathcal{S}=\mathcal{S}_{f\circ\pi}=\mathcal{S}_{g\circ\pi}$ and $V=\tilde{V}$ as given in C. We show that $\tilde{V}$ is modifiable.

For $p\in\Sigma_{n-s}$, choose $\{x_1, \cdots, x_n\}$ displaying $\mathcal{S}$ so that

$$\left.\begin{aligned} f(\pi(x))&=\varepsilon_1(x)x_1^{k_1}\cdots x_s^{k_s}\\ g(\pi(x))&=\varepsilon_2(x)x_1^{k_1}\cdots x_s^{k_s}\end{aligned}\right\}\ k_i>0\,.$$

Then

$$F(x, t)=\varepsilon(x, t)x_1^{k_1}\cdots x_s^{k_s}$$

where $\varepsilon(x, t)=(1-t)\varepsilon_1+t\varepsilon_2$.

$$\begin{aligned}\frac{\partial F}{\partial x_i}&=e_i(x, t)x_1^{k_1}\cdots x_i^{k_i-1}\cdots x_s^{k_s}\\ &=x_1^{k_1-1}\cdots x_s^{k_s-1}e_i\xi_i^{(s)}\end{aligned}$$

where $e_i(0, t)\neq 0$, $1\leqq i\leqq s$.

$$\frac{\partial F}{\partial x_{s+i}}=x_1^{k_1-1}\cdots x_s^{k_s-1}X^{(s)}q_{s+i}$$

where

$$q_{s+i}=\frac{\partial\varepsilon}{\partial x_{s+i}}\qquad 1\leqq i\leqq n-s\,.$$

In terms of the Riemannian metric, which has been chosen on $\tilde{U}$, the components of $\operatorname{grad} F$ are expressed as certain linear combinations of the above partial derivatives. Hence each has the form

$$x_1^{k_1-1}\cdots x_s^{k_s-1}\sum_{i=1}^{s}a_i\xi_i^{(s)} \tag{G2}$$

where the $a_i$ are functions of $\underset{\sim}{x}$ and $t$. Moreover

$$\frac{\partial F}{\partial t}=(\varepsilon_2-\varepsilon_1)x_1^{k-1}\cdots x_s^{k_s-1}X^{(s)}. \tag{G3}$$

Now $|\mathrm{grad}\, F|^2=\tilde{P}(\partial F/\partial x_1, \cdots, \partial F/\partial x_n)$ where $\tilde{P}(z_1, \cdots, z_n)$ is a quadratic form in $z_1, \cdots, z_n$ depending on the metric. Thus

$$\begin{aligned}|\mathrm{grad}\, F|^2&=\tilde{P}\Big(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_s}, \cdots, \frac{\partial F}{\partial x_n}\Big)\\ &=(x_1^{k_1-1}\cdots x_s^{k_s-1})^2\tilde{P}(e_1\xi_1^{(s)}, \cdots, e_s\xi_s^{(s)}, X^{(s)}q_{s+1}, \cdots, X^{(s)}q_n)\\ &=(x_1^{k_1-1}\cdots x_s^{k_s-1})^2 P(\xi_1^{(s)}, \cdots, \xi_s^{(s)}; \underset{\sim}{x}, t)\end{aligned}$$

where $P(\xi_1^{(s)}, \cdots, \xi_s^{(s)}; \underset{\sim}{x}, t)=\tilde{P}(e_1\xi_1^{(s)}, \cdots, e_s\xi_s^{(s)}, X^{(s)}q_{s+1}, \cdots, X^{(s)}q_n)$ is positive definite for $\underset{\sim}{x}$ near $\underset{\sim}{0}$ and $0\leqq t\leqq 1$.

That is, for $0\leqq t\leqq 1$,

$$P(y_1, \cdots, y_s; \underset{\sim}{x}, t)>0.$$

(Notice that $P$ is not uniquely determined as a form). Thus

$$|\mathrm{grad}\, F|^2=(x_1^{k_1-1}\cdots x_s^{k_s-1})^2 P(\xi^{(s)}; x, t). \tag{G4}$$

Now G1 follows from G2, G3 and G4.

In the general case let

$$\beta: B(\mathcal{A})\longrightarrow \mathcal{A}$$

be the blow-up of $\mathcal{S}$ in E1.

LEMMA. *Let $V$ denote a modifiable vector field on $\mathcal{A}$. Then there exists a modifiable vector field $B(V)$ on $B(\mathcal{A})$ such that where $(d\beta)^{-1}$ is defined,*

$$B(V)=(d\beta)^{-1}(V).$$

PROOF. Let $\Sigma_k$ denote the lowest dimensional stratum of $\mathcal{S}$. Let $s=n-k$ and choose $p\in\Sigma_k=\Sigma_{n-s}$.

Since $V$ is modifiable we can choose a coordinate system $\{x_1, \cdots x_n\}$ at $p$ such that $V(x, t)$ is expressed as in G1. This coordinate chart about $p$ is blown up by $\beta$ into $n$ coordinate charts, a typical one of which has coordinate system $\{y_1, \cdots, y_n\}$ related to the system $\{x_1, \cdots, x_n\}$ by

$$\left.\begin{aligned}x_i&=y_iy_s & 1\leqq i\leqq s-1\\ x_s&=y_s & \\ x_{s+j}&=y_{s+j} & 1\leqq j\leqq k.\end{aligned}\right\} \tag{G5}$$

(This amounts to the blow up of a cone around the $x_s$-axis in $\{x_1, \cdots, x_s\}$ space, a fact which we have emphasised by not condensing the middle equation into the last one. The other charts are obtained by blowing up about the other axes of course). Observe that $\{y_1, \cdots, y_n\}$ displays $B(\mathcal{S})$.

By the chain rule we have

$$\left.\begin{aligned}\frac{\partial}{\partial x_i}&=\frac{1}{y_s}\frac{\partial}{\partial y_i} \qquad 1\leqq i\leqq s-1\\ \frac{\partial}{\partial x_s}&=-\sum_{i=1}^{s-1}\frac{y_i}{y_s}\frac{\partial}{\partial y_i}+\frac{\partial}{\partial y_s}\\ \frac{\partial}{\partial x_{s+j}}&=\frac{\partial}{\partial y_{s+j}} \qquad 1\leqq j\leqq k\,.\end{aligned}\right\} \tag{G6}$$

Define

$$Y^{(s-1)}=y_1\cdots y_{s-1}$$

$$\eta_i^{(s-1)}=\frac{1}{y_i}Y^{(s-1)} \qquad 1\leqq i\leqq s-1$$

$$\eta^{(s-1)}=(\eta_1^{(s-1)},\ \cdots,\ \eta_{s-1}^{(s-1)})\,.$$

Then (a) $X^{(s)}=Y^{(s-1)}y_s^s$,

(b) if $L$ is a linear form in $s$ variables with coefficients which are functions of $(x, t)$ then there exists a linear form $L'$ in $s-1$ variables with coefficients which are functions of $(y, t)$ such that

$$L(\xi^{(s)};\ x,\ t)=y_s^{s-1}L'(\eta^{(s-1)},\ y,\ t)$$

and (c) if $P(\xi^{(s)};\ x,\ t)$ is a quadratic form in $\xi^{(s)}$ with coefficients which are functions of $(x, t)$ then there exists a quadratic form $P'(\eta^{(s-1)};\ y,\ t)$ in $\eta^{(s-1)}$ whose coefficients are functions of $(y, t)$ and which satisfies

$$P(\xi^{(s)};\ x,\ t)=y_s^{2(s-1)}P'(\eta^{(s-1)};\ y,\ t)\,.$$

Furthermore if $P$ is positive definite for $x$ near 0, $0\leqq t\leqq 1$ then $P'$ is positive definite for $y$ near 0, $0\leqq t\leqq 1$.

Now off the hyperplane $y_s=0$, $d\beta$ is not singular and substituting G5 and G6 into G1 we find that $(d\beta)^{-1}$ transforms $V(x, t)$ into a vector field of the form

$$\sum_{i=1}^{s-1}\frac{Y^{(s-1)}(L_i'-y_iL_0')}{P'}\frac{\partial}{\partial y_i}+y_s\sum_{j=s}^{n}\frac{Y^{(s-1)}L_j'}{P'}\frac{\partial}{\partial y_j}+\frac{\partial}{\partial t}\,. \tag{G7}$$

We are now in a position to define the vector field $B(V)$.

Let $\bar{p}$ be a point represented by $(a_1,\ \cdots,\ a_n)$ in the above coordinate system, such that $\beta(\bar{p})=p$. Of course, $a_s=a_{s+1}=\cdots=a_n=0$. There are two cases to consider:

(i) Each $a_i\neq 0$, $1\leqq i\leqq s-1$. This implies $\bar{p}\notin B(\Sigma)$ and hence $P'$ is a unit; G7 is thus an analytic expression and we take it as the definition of $B(V)$ at $\bar{p}$. Hence $B(V)$ is defined and analytic off $B(\Sigma)$.

(ii) Some $a_i=0$. By permuting the indices $(1,\ \cdots,\ s-1)$, we may assume

$$\bar{p}=(0, \cdots, 0, a_{l+1}, \cdots, a_{s-1}, 0, \cdots, 0), \quad a_i \neq 0, \quad 0 \leqq l \leqq s-2.$$

An important observation is that $\bar{p} \in B(\Sigma_{n-l})$. This follows from the definition of blowing-up.

We must show that $B(V)$ has the form G1 near $\bar{p}$ (with $l$ replacing $s$). Let us translate the origin to $\bar{p}$ by setting

$$\begin{aligned} z_i &= y_i - a_i & \quad l+1 \leqq i \leqq s-1 \\ z_i &= y_i & \quad \text{otherwise}. \end{aligned}$$

Let

$$Z^{(l)} = \prod_{i=1}^{l} z_i$$

$$\zeta_i^{(l)} = \frac{1}{Z_i} z^{(l)}$$

and

$$\zeta^{(l)} = (\zeta_1^{(l)}, \cdots, \zeta_l^{(l)}).$$

Then $Y^{(s-1)} = Z^{(l)} \delta(z, t)$ where $\delta(z, t)$ is a unit;

$$L'(\eta^{(s)}; y, t) = L''(\zeta^{(l)}; z, t);$$

and

$$P'(\eta^{(s)}; y, t) = P''(\zeta^{(l)}; z, t).$$

Here $L''$ and $P''$ are respectively linear and quadratic forms in $l$ variables with properties analogous to those of $L'$ and $P'$.

G7 then is of the form G1 near $\bar{p}$. This shows that $B(V)$ is modifiable.

## H. Proof of the theorem.

We now use the notation of $A$ and assume that the germs $f$ and $g$ satisfy $\mathcal{D}_f = \mathcal{D}_g$. Let $U$ and $\tilde{U}$ be as defined by Hironaka's Theorem (B). Let $\mathcal{S}$ denote the corresponding strata of $\tilde{U}$ and, as in $E$, form the complete blow up $U^*$ of the stratification $\mathcal{S}$.

By the main lemma there exists a vector field $V^*$ which is defined and analytic throughout $U^* \times I$ such that

$$d\beta^*(V^*(p^*; t)) = \tilde{V}(\beta^*(p^*); t).$$

Now $\tilde{V}$ is tangent to the level surfaces $F$=constant off $\Sigma \times I$. Thus $V^*$ is tangent to the level surfaces of the composite function

$$F^* = F \circ (\beta^* \times 1)$$

where $1: I \to I$ is the identity map. Since $V^*$ has $t$-component one the integral lines of $V^*$ (which are contained in level surfaces $F^*$=constant) give an analytic map

$$\tau^*: V_1^* \times \{1\} \longrightarrow V_2^* \times \{0\}$$

where $V_1^*$ and $V_2^*$ are suitable neighborhoods of $(\pi \circ \beta^*)^{-1}(0)$ in $U^*$. Now as

$$F_{|V_1^* \times \{1\}} = g \circ \pi \circ \beta^* \quad \text{and} \quad F_{|V_2^* \times \{0\}} = f \circ \pi \circ \beta^*$$

we see that $f$ and $g$ are almost analytically equivalent. The theorem is proved.

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