Compact Lie group actions and fiber homotopy type

By Katsuo KAWAKUBO

1. Introduction.

In [2], Atiyah proved that the stable fiber homotopy type of the tangent sphere bundle of a differentiable manifold M depends only on the homotopy type of M. This theorem not only is interesting in itself, but also plays an important role in constructing and classifying homotopy equivalent manifolds by making use of the Browder-Novikov theory [4], [18], [22], [9], [12].

The purpose of the present paper is to exploit fundamental tools in equivariant topology, and to prove an equivariant analogue of Atiyah's theorem as an immediate consequence.

Throughout the paper, G will denote a compact Lie group.

We shall first develop general theory on equivariant deformation retracts. For this, we give a filtration on a compact G-manifold and introduce a concept of *G-deformation retracts preserving the filtration*. Here we do not assume Yang's results [25], because a gap in the proof was found. Hence we proceed without using [10], [16].

Although Dold's theorem mod k for $k>1$ in the sense of Adams [1] has no counterpart in equivariant theory in general, we shall establish fortunately equivariant Dold's theorem which corresponds to $k=1$. Here we have introduced the notion of a *nice G-deformation retract*, which enables us to carry out the program of the inductive proof of equivariant Dold's theorem one step further.

By combining these results and those of Rubinsztein [20], we shall verify our main theorem.

Needless to say, the theorem can be applied to constructing and classifying G-homotopy equivalent manifolds just as in the ordinary non-equivariant case.

As an application peculiar to the realm of equivariant topology, we shall have, for instance, that the normal representations of the corresponding components of the fixed point sets of G-homotopy equivalent manifolds are stably homotopy equivalent in the sense of tom Dieck [6]. On the other hand, examples will be provided to show that the normal representations of the corre-

sponding components of the fixed point sets of G-homotopy equivalent manifolds are not necessarily isomorphic in general.

Some other applications will also be obtained.

REMARK. Since there is no notion of equivariant S-reducible and equivariant S-coreducible, Atiyah's proof is of no use for our purpose.

2. Statement of results.

The statement of the results concerning G-deformation retracts will be postponed until § 3, because we need to make preparations for the statement.

Let G be a compact Lie group and X be a compact G-space. We now define G-fiber homotopy equivalence. Let ξ and η be orthogonal G-vector bundles over X. Denote by $S(\xi)$ (resp. $S(\eta)$) the sphere bundle associated with ξ (resp. η). Then $S(\xi)$ and $S(\eta)$ are said to be of the same G-fiber homotopy type ($\xi \sim \eta$) if there exist fiber-preserving G-maps:

$$f : S(\xi) \longrightarrow S(\eta), \qquad f' : S(\eta) \longrightarrow S(\xi)$$

and fiber-preserving G-homotopies:

$$h : S(\xi) \times I \longrightarrow S(\xi), \qquad h' : S(\eta) \times I \longrightarrow S(\eta)$$

with $\quad h \,|\, S(\xi) \times 0 = f' \cdot f, \qquad h \,|\, S(\xi) \times 1 = \text{identity}$

$$h' \,|\, S(\eta) \times 0 = f \cdot f', \qquad h' \,|\, S(\eta) \times 1 = \text{identity}.$$

Let W be a compact smooth G-manifold. We shall denote the isotropy group at $x \in W$ by G_x, namely, $G_x = \{g \in G \,|\, g \cdot x = x\}$. Then we have the following equivariant Dold theorem which corresponds to Satz 1 in [7].

THEOREM 2.1. *Let ξ and η be orthogonal G-vector bundles over W. Let $f : S(\xi) \to S(\eta)$ be a fiber-preserving G-map such that the restriction*

$$f_x : S(\xi_x) \longrightarrow S(\eta_x)$$

is a G_x-homotopy equivalence for every $x \in W$. Then f gives a G-fiber homotopy equivalence.

REMARK 2.2. Theorem 2.1. will be valid for G-vector bundles over G-CW-complexes in the sense of Illman [10] and Matumoto [16].

Let $KO_G(X)$ be the Grothendieck-Atiyah-Segal group [3] defined in terms of real G-vector bundles over X. Let $T_G(X)$ be the additive subgroup of $KO_G(X)$ generated by elements of the form $[\xi] - [\eta]$, where ξ and η are othogonal G-vector bundles whose associated sphere bundles are G-fiber homotopy equivalent. We define

$$J_G(X) = KO_G(X)/T_G(X).$$

The natural epimorphism $KO_G(X) \to J_G(X)$ will also be denoted by J_G. Then our main theorem will be:

THEOREM 2.3. *Let M_1, M_2 be closed G-manifolds with tangent bundles TM_1, TM_2 respectively. Let $f: M_1 \to M_2$ be a G-homotopy equivalence. Then we have*

$$J_G(TM_1) = J_G(f^*TM_2).$$

Let $f: M_1 \to M_2$ be a G-homotopy equivalence. Denote by F_1^{μ} each component of the fixed point set of M_i ($i=1, 2$). Then the set $\{F_1^{\mu}\}$ is in one to one correspondence with the set $\{F_2^{\mu}\}$ such that $f(F_1^{\mu}) = F_2^{\mu}$. Let τ_1^{μ} (resp. τ_2^{μ}) be the tangent representation at $x \in F_1^{\mu}$ (resp. $f(x) \in F_2^{\mu}$). Regarding τ_i^{μ} as G-vector bundles over one point, we have

COROLLARY 2.4. $J_G(\tau_1^{\mu}) = J_G(\tau_2^{\mu})$.

Let N_1^{μ} (resp. N_2^{μ}) be the normal bundle of F_1^{μ} (resp. F_2^{μ}) in M_1 (resp. M_2). Then we have

COROLLARY 2.5. $J_G(N_1^{\mu}) = J_G((f|F_1^{\mu})^*N_2^{\mu})$.

Let V_1^{μ} (resp. V_2^{μ}) be the normal representation of F_1^{μ} (resp. F_2^{μ}) in M_1 (resp. M_2). Regarding V_i^{μ} as G-vector bundles over one point, we have in particular

COROLLARY 2.6 $J_G(V_1^{\mu}) = J_G(V_2^{\mu})$.

On the other hand, we have

EXAMPLE 2.7. There are G-homotopy equivalent manifolds M_1, M_2, such that τ_1^{μ} (resp. V_1^{μ}) is not isomorphic to τ_2^{μ} (resp. V_2^{μ}) as G-representation spaces.

Let $G \to EG \to BG$ be the universal principal G-bundle. Our equivariant Stiefel-Whitney class $W_G(\xi)$ of a G-vector bundle $\xi \to X$ is defined by

$$W_G(\xi) = W(EG \underset{G}{\times} \xi)$$

where $W(\ )$ denotes the ordinary total Stiefel-Whitney class. Then we have

COROLLARY 2.8. *Let M_1, M_2 be closed G-manifolds with tangent bundles TM_1, TM_2 respectively. Let $f: M_1 \to M_2$ be a G-homotopy equivalence. Then we have*

$$W_G(TM_1) = W_G(f^*TM_2).$$

REMARK 2.9. Corollary 2.8 follows also from equivariant Wu type formulae [13].

Since $(Z_2)^k$-bordism classes are characterized by equivariant Stiefel-Whitney numbers [5], we have

COROLLARY 2.10. $(Z_2)^k$-*homotopy equivalent manifolds are $(Z_2)^k$-bordant.*

The present paper is arranged as follows. In § 3, we develop general theory of G-deformation retracts. § 4 is devoted to the study of equivariant Dold's theorem. Theorem 2.3 is proved in § 5 and Example 2.7 is given in § 6. Since

Corollaries 2.4, 2.5, 2.6, 2.8 and 2.10 follow immediately from Theorem 2.3, we omit the proofs.

3. G-deformation retracts.

We first give some basic notations. Let G be a compact Lie group. Whenever H is a subgroup of G, (H) denotes the conjugacy class of H in G and $N(H)$ denotes the normalizer of H in G. There is a partial ordering relation among the set of conjugacy classes of subgroups of G, i.e., $(H_1) \leqq (H_2)$ if and only if there exists $g \in G$ such that $gH_1g^{-1} \subset H_2$.

Let X be a compact G-space. We shall denote the isotropy group at $x \in X$ by G_x, namely, $G_x = \{g \in G \mid g \cdot x = x\}$. For a subgroup H of G, we shall put

$$X^H = \{x \in X \mid G_x \supset H\}, \qquad X(H) = \{x \in X \mid G_x \in (H)\}.$$

If a G-vector bundle $\xi \to X$ has a Riemannian metric $\langle , \rangle$, and the G-action is isometric, then ξ is called a Riemannian G-vector bundle. Whenever ξ is a Riemannian G-vector bundle, we put $\|x\| = \langle x, x \rangle^{1/2}$, $\xi(r) = \{x \in \xi \mid \|x\| \leqq r\}$, $S(\xi(r)) = \{x \in \xi \mid \|x\| = r\}$ and $\overset{\circ}{\xi}(r) = \{x \in \xi \mid \|x\| < r\}$ where $r > 0$.

We now recall the definition of an $\langle n \rangle$-manifold due to Jänich [11]. We define a *differentiable manifold with corners* as being locally modelled on the open sets of $\{x = (x_1, \cdots, x_n) \in R^n \mid x_1 \geqq 0, \cdots, x_n \geqq 0\}$. A closed subset Y of an n-dimensional manifold with corners X is called a k-dimensional submanifold of X, if for each $y \in Y$ one can introduce local coordinates in a neighborhood U of y in X, with respect to which $Y \cap U$ becomes an open subset of $\{x \in R^n \mid x_1 \geqq 0, \cdots, x_k \geqq 0, x_{k+1} = \cdots = x_n = 1\}$. Then Y has a canonical structure as a manifold with corners.

If $x \in X$ is represented by $(x_1, \cdots, x_n)$ in a local coordinate system, we denote by $c(x)$ the number of zeros in this n-tuple. $c(x)$ does not depend on the choice of the coordinate system.

Note that x belongs to the closure of at most $c(x)$ different connected components of $\{p \in X \mid c(p) = 1\}$. Now we call X a *manifold with faces*, if each $x \in X$ does belong to the closure of $c(x)$ different components of $\{p \in X \mid c(p) = 1\}$. For a manifold with faces, the closure of a connected component of $\{p \in X \mid c(p) = 1\}$ has in a canonical way the structure of an $(n-1)$-dimensional manifold with corners and is called a *connected face* of X. Any union of pairwise disjoint connected faces is called a *face* of X. Such a face is in fact a manifold with faces itself. Notice that a face may be empty.

Now we are ready to introduce the notion of an $\langle n \rangle$-manifold.

DEFINITION. A manifold with faces X together with an n-tuple $(\partial_0 X, \cdots, \partial_{n-1} X)$ of faces of X is called an $\langle n \rangle$-manifold, if

(i) $\partial_0 X \cup \cdots \cup \partial_{n-1} X = \partial X$,

(ii) $\partial_i X \cap \partial_j X$ is a face of $\partial_i X$ and $\partial_j X$ for $i \neq j$.

Let X be a closed manifold and Y a closed submanifold. If one removes the interior of a small tubular neighborhood of Y from X, one obtains a manifold with boundary. More generally, if X is an $\langle n \rangle$-manifold and Y a submanifold of the underlying manifold with corners, then the removal of an open tubular neighborhood of Y from X creates an $\langle n+1 \rangle$-manifold X'.

We consider this process equivariantly. Let G be a compact Lie group, acting differentiably on an $\langle n \rangle$-manifold X. We call X together with such an action a G-$\langle n \rangle$-manifold, if each $\partial_i X$ is invariant under G. Let Y be a G-invariant submanifold of the G-$\langle n \rangle$-manifold X, and U a small G-invariant open tubular neighborhood of Y in X. Then we have a G-$\langle n+1 \rangle$-manifold $X-U$, which we denote by $X \circledcirc Y$. For the precise definition of $X \circledcirc Y$, see [11].

Let W be a compact G-$\langle n \rangle$-manifold and N be a compact G-invariant submanifold of W in the sense of Jänich [11]. Then the normal bundle ν of N in W is a smooth G-vector bundle and there is an equivariant diffeomorphism $\nu \to U$ where U is an open tubular neighborhood of N in W [15]. We often identify ν with U by this G-diffeomorphism.

Since W is compact, there are only finite G-isotropy types, say

$$\{(G_x) \mid x \in W\} = (H_1) \cup \cdots \cup (H_k).$$

It is possible to arrange (H_i) in such order that $(H_i) \geq (H_j)$ implies $i \leq j$. We shall get a filtration $W = W_1 \supset \cdots \supset W_k$ consisting of compact G-$\langle n+i-1 \rangle$-manifolds W_i such that

$$\{(G_x) \mid x \in W_i\} = (H_i) \cup (H_{i+1}) \cup \cdots \cup (H_k)$$

as follows.

Since (H_1) is a maximal conjugacy class, $W(H_1)$ is a compact G-invariant submanifold. We identify the normal bundle ν_1 of $W(H_1)$ in W with an open tubular neighborhood of $W(H_1)$ in W and impose a Riemannian G-vector bundle structure on ν_1. Set $W_2 = W - \mathring{\nu}_1(2)$. Then W_2 is a compact G-$\langle n+1 \rangle$-manifold and is nothing but $W \circledcirc W(H_1)$, if we employ the notation in [11]. Clearly we have that

$$\{(G_x) \mid x \in W_2\} = (H_2) \cup \cdots \cup (H_k).$$

Suppose that we get a filtration $W = W_1 \supset W_2 \supset \cdots \supset W_i$ such that

$$\{(G_x) \mid x \in W_j\} = (H_j) \cup (H_{j+1}) \cup \cdots \cup (H_k)$$

and W_j is a compact G-$\langle n+j-1 \rangle$-manifold for every $j \leq i$. Since (H_i) is a maximal conjugacy class among the set $\{(G_x) \mid x \in W_i\}$, $W_i(H_i)$ is a compact G-invariant submanifold of W_i. We identify the normal bundle ν_i of $W_i(H_i)$ in

W_i with an open tubular neighborhood of $W_i(H_i)$ in W_i and impose a Riemannian G-vector bundle structure on ν_i. Set $W_{i+1}=W_i-\mathring{\nu}_i(2)$. Then W_{i+1} is a compact G-$\langle n+i\rangle$-manifold and is nothing but $W_i\odot W_i(H_i)$, if we employ the notation in [11]. Clearly we have that

$$\{(G_x) \mid x\in W_{i+1}\} =(H_{i+1})\cup \cdots \cup(H_k).$$

This completes the inductive constructions.

Denote by $\partial_1 W, \cdots, \partial_n W$ the faces of W. Let ΔW be the union $\partial_{i_1}W\cup \cdots \cup\partial_{i_d}W$ of some faces of W. Denote by $(W^{H_i})_\mu$ (resp. $(W^{H_j})_\lambda$) each component of W^{H_i} (resp. W^{H_j}). Then we have

LEMMA 3.1. *Suppose that the inclusion map* $(\Delta W)^{H_i}\to W^{H_i}$ *is a homotopy equivalence for every H_i and that $(H_i)<(H_j)$ implies*

$$\min_\mu\{\dim (W^{H_i})_\mu\} -\max_\lambda\{\dim (W^{H_j})_\lambda\} \geqq\dim G+3.$$

Then there is a G-deformation retraction $D: W\times I\to W$ *such that*

$$D(x, 0)=x, \qquad D(x, 1)\in\Delta W \qquad for \ x\in W,$$

$$D(x, s)=x \qquad\qquad for \ x\in\Delta W, \ s\in I,$$

$$D(S(\nu_j(2))\times I)\subset S(\nu_j(2)) \qquad for \ every \ j.$$

PROOF. Such a G-deformation retraction as in Lemma 3.1 is called briefly a G-deformation retraction preserving $S(\nu_j(2))$. We prove Lemma 3.1 by induction concerning the number of orbit types.

Suppose that W has only one G-isotropy type, say (H). Since $N(H)/H$ acts freely on W^H, we have the following commutative diagram

$$
\begin{array}{ccccc}
N(H)/H & \longrightarrow & (\Delta W)^H & \longrightarrow & (\Delta W)^H/(N(H)/H) \\
\| & & \downarrow & & \downarrow \\
N(H)/H & \longrightarrow & W^H & \longrightarrow & W^H/(N(H)/H)
\end{array}
$$

where the horizontal sequences denote fibrations and the vertical arrows are inclusion maps. Hence we have the following commutative diagram

$$
\begin{array}{ccccccc}
\longrightarrow & \pi_i(N(H)/H) & \longrightarrow & \pi_i((\Delta W)^H) & \longrightarrow & \pi_i((\Delta W)^H/(N(H)/H)) & \longrightarrow \\
& \| & & i_*\downarrow & & \downarrow (i/\sim)_* & \\
\longrightarrow & \pi_i(N(H)/H) & \longrightarrow & \pi_i(W^H) & \longrightarrow & \pi_i(W^H/(N(H)/H)) & \longrightarrow
\end{array}
$$

where i_* is an isomorphism by hypothesis. It follows from the five lemma that $(i/\sim)_*$ is also an isomorphism. Using Whitehead [24], it follows that $(\Delta W)^H/(N(H)/H)$ is a deformation retract of $W^H/(N(H)/H)$. Therefore by the covering homotopy property of principal bundles, $(\Delta W)^H$ is a $N(H)/H$-deforma-

tion retract of W^H. In order to construct the required G-deformation retraction, we have only to remark the natural G-diffeomorphisms

$$\Delta W = (G/H) \underset{N(H)/H}{\times} (\Delta W)^H$$

and

$$W = (G/H) \underset{N(H)/H}{\times} W^H .$$

Let k be an integer greater than one. We now assume that Lemma 3.1 is true for the case where the number of orbit types is smaller than k. Let W be a compact G-$\langle n \rangle$-manifold such that the number of orbit types in W is k and that the assumptions in Lemma 3.1 is satisfied. Following the filtration

$$W = W_1 \supset W_2 \supset \cdots \supset W_k ,$$

we shall construct inductively G-homotopies $D_i : W \times I \to W$ such that

$$D_i(x, 0) = x , \qquad D_i(x, 1) \in W_{i+1} \cup \Delta W \qquad \text{for } x \in W ,$$

$$D_i(x, s) = x \qquad\qquad\qquad \text{for } x \in \Delta W, \, s \in I ,$$

$$D_i(S(\nu_j(2)) \times I) \subset S(\nu_j(2)) \qquad\qquad \text{for every } j .$$

As above we can prove that $(\Delta W)(H_1)$ is a G-deformation retract of $W(H_1)$. In virtue of the equivariant covering homotopy property of G-vector bundles [3], the above G-deformation retraction extends to the normal bundle ν_1 of $W(H_1)$ in W. Denote by

$$d_1 : \nu_1 \times I \longrightarrow \nu_1$$

its extension. Then we have that

$$\{(G_x) \mid x \in S(\nu_1(2))\} \subset (H_2) \cup \cdots \cup (H_k) .$$

It is easy to see that $S(\nu_1(2))$ is a compact G-$\langle n \rangle$-manifold and $S(\nu_1(2) \mid (\Delta W)(H_1))$ is the union of some faces of $S(\nu_1(2))$. Moreover the pair

$$(S(\nu_1(2)), \, S(\nu_1(2) \mid (\Delta W)(H_1)))$$

satisfies the assumptions in Lemma 3.1. The inductive hypothesis guarantees that $S(\nu_1(2) \mid (\Delta W)(H_1))$ is a G-deformation retract of $S(\nu_1(2))$ preserving $S(\nu_1(2)) \cap S(\nu_j(2))$. Here we have employed the filtration.

$$S(\nu_1(2)) \supset S(\nu_1(2)) \cap W_3 \supset \cdots \supset S(\nu_1(2)) \cap W_k .$$

Remark that some of them may be equal. We denote its G-deformation retraction by

$${}'d_1 : S(\nu_1(2)) \times I \longrightarrow S(\nu_1(2)) .$$

Then we define the first G-homotopy

$$d_1^1 : \nu_1(1) \times I \longrightarrow \nu_1(1)$$

by putting

$$d_1^1(x, s) = \begin{cases} d_1(x, 2s) & \text{for } 0 \leq s \leq \frac{1}{2}, \\[2mm] d_1(x, 1) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$

We define the second G-homotopy

$$d_1^2 : (\nu_1(2) - \mathring{\nu}_1(1)) \times I \longrightarrow \nu_1(2) - \mathring{\nu}_1(1)$$

by putting

$$d_1^2(x, s) = \begin{cases} d_1(x, 2(2 - \|x\|)s) & \text{for } 0 \leq s \leq \frac{1}{2}, \\[2mm] \dfrac{\|x\|}{2} \, 'd_1\Big(\dfrac{2}{\|x\|} d_1(x, 2 - \|x\|), 2s - 1\Big) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$

The third G-homotopy

$$d_1^3 : (\nu_1(3) - \mathring{\nu}_1(2)) \times I \longrightarrow \nu_1(3) - \mathring{\nu}_1(2)$$

is defined by

$$d_1^3(x, s) = \begin{cases} x & \text{for } 0 \leq s \leq \frac{1}{2}, \\[2mm] \dfrac{\|x\|}{2} \, 'd_1\Big(\dfrac{2}{\|x\|} x, (3 - \|x\|)(2s - 1)\Big) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$

The last trivial G-homotopy

$$d_1^4 : (W - \mathring{\nu}_1(3)) \times I \longrightarrow W - \mathring{\nu}_1(3)$$

is defined by

$$d_1^4(x, s) = x .$$

In this way, we obtain a G-homotopy

$$D_1 : W \times I \longrightarrow W$$

by putting $D_1 = d_1^1 \cup d_1^2 \cup d_1^3 \cup d_1^4$ which satisfies our requirements.

Suppose that we are given G-homotopies

$$D_b : W \times I \longrightarrow W \qquad \text{for all } b < i$$

satisfying the assumptions above. For $b < i$, we set

$${}^b W = W - \bigcup_{c < b} W(H_c) .$$

Then ^bW is an open G-submanifold of W and

$$(^{b+1}W)^{Hi}=(^bW)^{Hi}-W(H_b).$$

Denote by $(W^{Hi})_\mu$ each component of W^{Hi}. Define $((^bW)^{Hi})_\mu$ to be the intersection $^bW\cap(W^{Hi})_\mu$. Then we have that

$$((^{b+1}W)^{Hi})_\mu=((^bW)^{Hi})_\mu-W(H_b)\cap(W^{Hi})_\mu.$$

ASSERTION. $W(H_b)\cap(W^{Hi})_\mu$ is empty or is the disjoint union of some submanifolds of $((^bW)^{Hi})_\mu$ of codimension greater than two.

We have the following commutative diagram:

$$
\begin{array}{ccc}
W(H_b)^{Hi} & \xrightarrow{\ \phi_1\ } & W^{Hi} \\
\Big\downarrow{\phi_2} & & \Big\downarrow{\phi_3} \\
W(H_b) & \xrightarrow{\ \phi_4\ } & W
\end{array}
$$

where ϕ_i are inclusion maps. Since ϕ_2, ϕ_3, ϕ_4 are inclusion maps of submanifolds, $W(H_b)^{Hi}$ is also a submanifolds of W^{Hi}. Since $(W^{Hi})_\mu$ is an open submanifold of W^{Hi}, it follows that $W(H_b)\cap(W^{Hi})_\mu$ is a submanifold of $(W^{Hi})_\mu$. Similarly we are able to prove that $((^bW)^{Hi})_\mu$ is a submanifold of $(W^{Hi})_\mu$. It follows that $W(H_b)\cap(W^{Hi})_\mu$ is a submanifold of $((^bW)^{Hi})_\mu$. Quite a similar argument shows that $W(H_b)\cap(W^{Hi})_\mu$ is a submanifold of $W(H_b)$. Therefore we have that if $W(H_b)\cap(W^{Hi})_\mu$ is non-empty, then

$$\dim\{\text{a component } C \text{ of } W(H_b)\cap(W^{Hi})_\mu\}$$

$$\leqq\dim\{\text{the component of } W(H_b) \text{ including } C\}$$

$$\leqq\dim G+\max_\lambda\{\dim(W^{Hb})_\lambda\}$$

$$\leqq\dim(W^{Hi})_\mu-3.$$

Hence we see by induction that $((^bW)^{Hi})_\mu$ is a connected open submanifold of $(W^{Hi})_\mu$, and hence $\dim(W^{Hi})_\mu=\dim((^bW)^{Hi})_\mu$. This completes the proof of the assertion.

It follows from the assertion that the homomorphism

$$\pi_1(((^{b+1}W)^{Hi})_\mu)\longrightarrow\pi_1(((^bW)^{Hi})_\mu)$$

induced by the inclusion map is an isomorphism. Consequently we obtain that the homomorphism

$$\pi_1(((^iW)^{Hi})_\mu)\longrightarrow\pi_1((W^{Hi})_\mu)$$

is an isomorphism.

Since the inclusion $W_i\rightarrow{}^iW$ is a G-homotopy equivalence, the homomorphism

$$\pi_1(((W_i)^{Hi})_\mu) \longrightarrow \pi_1((W^{Hi})_\mu)$$

is an isomorphism where $((W_i)^{Hi})_\mu$ denotes $W_i \cap (W^{Hi})_\mu$.

Similarly we have that the homomorphism

$$\pi_1(((\Delta W_i)^{Hi})_\mu) \longrightarrow \pi_1(((\Delta W)^{Hi})_\mu)$$

is an isomorphism where $((\Delta W_i)^{Hi})_\mu$ indicates $\Delta W \cap W_i \cap (W^{Hi})_\mu$ and $((\Delta W)^{Hi})_\mu$ indicates $\Delta W \cap (W^{Hi})_\mu$.

On the other hand, we have the following isomorphism by hypothesis

$$\pi_1(((\Delta W)^{Hi})_\mu) \longrightarrow \pi_1((W^{Hi})_\mu) .$$

Combining these, we have that

$$\pi_1(((\Delta W_i)^{Hi})_\mu) \longrightarrow \pi_1(((W_i)^{Hi})_\mu)$$

is an isomorphism.

Given any system $\mathcal{S}$ of local coefficients on W^{Hi}, we have the following commutative diagram:

$$
\begin{array}{ccccccc}
\longrightarrow & H_*((\Delta W_i)^{Hi} : \mathcal{S}) & \longrightarrow & H_*((\Delta W)^{Hi} : \mathcal{S}) & \longrightarrow & H_*((\Delta W)^{Hi}, (\Delta W_i)^{Hi} : \mathcal{S}) & \longrightarrow \\
& \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & \\
\longrightarrow & H_*((W_i)^{Hi} : \mathcal{S}) & \longrightarrow & H_*(W^{Hi} : \mathcal{S}) & \longrightarrow & H_*(W^{Hi}, (W_i)^{Hi} : \mathcal{S}) & \longrightarrow
\end{array}
$$

where ΔW_i indicates $\Delta W \cap W_i$ and we abbreviated the restriction $\mathcal{S}|$ to $\mathcal{S}$. The horizontal sequences are exact and ϕ_2 is an isomorphism by hypothesis. Consider the following commutative diagram:

$$H_*(\bigcup_{b<i} \nu_b(2) \cap (\Delta W)^{Hi}, (\bigcup_{b<i} S(\nu_b(2)) - \bigcup_{b<c<i} S(\nu_b(2)) \cap \mathring{\nu}_c(2)) \cap (\Delta W)^{Hi} : \mathcal{S})$$

$$\downarrow f_1 \qquad \searrow f_2$$

$$H_*((\Delta W)^{Hi}, (\Delta W_i)^{Hi} : \mathcal{S})$$

$$\downarrow \phi_3$$

$$H_*(W^{Hi}, (W_i)^{Hi} : \mathcal{S})$$

$$\nearrow f_3$$

$$H_*(\bigcup_{b<i} \nu_b(2)^{Hi}, \bigcup_{b<i} S(\nu_b(2))^{Hi} - \bigcup_{b<c<i} S(\nu_b(2))^{Hi} \cap \mathring{\nu}_c(2)^{Hi} : \mathcal{S})$$

where f_2 and f_3 are excision isomorphisms. It follows from the inductive hypothesis that f_1 is also an isomorphism. It turns out that ϕ_3 is an isomorphism. Thus we can conclude that ϕ_1 is also an isomorphism by the five lemma. Using Whitehead [24], it follows that the inclusion $(\Delta W_i)^{Hi} \rightarrow W_i^{Hi}$ is a homotopy equi-

valence. Since $N(H_i)/H_i$ acts freely on $W_i^{H_i}$, we see as in the case of one orbit type that $(\Delta W_i)(H_i)$ is a G-deformation retract of $(W_i)(H_i)$. Extending it to the normal bundle ν_i of $W_i(H_i)$ in W_i, we have a G-deformation retraction

$$d_i\colon \nu_i \times I \longrightarrow \nu_i .$$

Then we have that

$$\{(G_x) \mid x \in S(\nu_i(2))\} \subset (H_{i+1}) \cup \cdots \cup (H_k) .$$

It is easy to see that $S(\nu_i(2))$ is a compact G-$\langle n+i-1 \rangle$-manifold and $S(\nu_i(2) | (\Delta W_i)(H_i))$ is the union of some faces of $S(\nu_i(2))$. Moreover the pair

$$(S(\nu_i(2)),\ S(\nu_i(2) \mid (\Delta W_i)(H_i)))$$

satisfies the assumptions in Lemma 3.1. It follows from the inductive hypothesis that $S(\nu_i(2) \mid (\Delta W_i)(H_i))$ is a G-deformation retract of $S(\nu_i(2))$ preserving $S(\nu_i(2)) \cap S(\nu_j(2))$ for $j > i$. Here we employed the filtration

$$S(\nu_i(2)) \supset S(\nu_i(2)) \cap W_{i+2} \supset \cdots \supset S(\nu_i(2)) \cap W_k .$$

Remark that some of them may be equal. We denote its G-deformation retraction by

$$'d_i\colon S(\nu_i(2)) \times I \longrightarrow S(\nu_i(2)) .$$

As in the case of (H_1), we obtain G-homotopies d_i^1, d_i^2, d_i^3, d_i^4 and we set

$$'D_i = d_i^1 \cup d_i^2 \cup d_i^3 \cup d_i^4\colon W_i \times I \longrightarrow W_i .$$

Extend $'D_i$ to $''D_i\colon (W_i \cup \Delta W) \times I \to W_i \cup \Delta W$ by putting $''D_i(x, s) = x$ for $x \in \Delta W$, $s \in I$. Then the i-th G-homotopy

$$D_i\colon W \times I \longrightarrow W$$

is defined by

$$D_i(x, s) = \begin{cases} D_{i-1}(X, 2s) & \text{for } 0 \le s \le \dfrac{1}{2}, \\[2mm] ''D_i(D_{i-1}(x, 1), 2s-1) & \text{for } \dfrac{1}{2} \le s \le 1. \end{cases}$$

It follows from the construction that D_i is a G-homotopy preserving $S(\nu_j(2))$ for every j such that

$$D_i(x, 0) = x, \qquad D_i(x, 1) \in W_{i+1} \cup \Delta W \qquad \text{for } x \in W,$$

$$D_i(x, s) = x \qquad\qquad\qquad\qquad \text{for } x \in \Delta W,\ s \in I .$$

In this way, we obtain G-homotopies $D_1, \cdots, D_k$. The last G-homotopy D_k gives our required G-deformation retraction. Hence Lemma 3.1 is true for the case where the number of orbit types is k. Consequently Lemma 3.1 is valid

by induction.

This makes the proof of Lemma 3.1 complete.

Let M_1, M_2 be closed G-manifolds and $f: M_1 \rightarrow M_2$ be a G-homotopy equivalence. It is well-known that there exist orthogonal G-representation spaces V_a and G-embeddings $e_a: M_a \rightarrow V_a$ $(a=1, 2)$ [19], [23]. We may assume that $e_a(M_a) \subset V_a - \{0\}$. Set $V = V_1 \oplus V_2$. For a positive integer m, we denote by V^m the direct sum of m-copies of V. Denote by N_2 the normal disk bundle of the embedding defined by the composition:

$$M_2 \xrightarrow{e_2} V_2 \xrightarrow{j} V = V_1 \oplus V_2 \xrightarrow{k} V^m$$

where j is the natural inclusion and k is the inclusion to the first factor. We identify N_2 with an invariant tubular neighborhood. Then we may assume without loss of generalities that $N_2 \subset V^m - \{0\}$. For any positive number ε, the composition

$$M_1 \xrightarrow{\varepsilon e_1 \oplus e_2 f} V = V_1 \oplus V_2 \xrightarrow{k} V^m$$

is a G-embedding and we can choose $\varepsilon > 0$ such that

$$k \cdot (\varepsilon e_1 \oplus e_2 f)(M_1) \subset \mathrm{Int}\ N_2.$$

Let N_1 be the normal disk bundle of M_1 in V^m by the above G-embedding. We identify N_1 with an invariant tubular neighborhood in such a way that

$$N_1 \subset \mathrm{Int}\ N_2.$$

Set $W = N_2 - \mathrm{Int}\ N_1$. Then we have

LEMMA 3.2. *If $m \geq \dim G + 3$, then the total spaces $S(N_a)$ of the associated sphere bundles are G-deformation retracts of W $(a=1, 2)$.*

PROOF. Let $\{(H_i) \mid i=1, \cdots, k\}$ be the set of the conjugacy classes of the isotropy groups which appear in W. As before we arrange (H_i) in such order that $(H_i) \geq (H_j)$ implies $i \leq j$. Remark that the set $\{(H_i)\}$ does not necessarily coincide with the set of the conjugacy classes of the isotropy groups which appear in M_a. But it is easily seen that $M_a^{H_i} \neq \emptyset$ for any $i \in \{1, \cdots, k\}$ and any $a \in \{1, 2\}$. The normal disk bundle of $M_a^{H_i}$ in $(V^m)^{H_i}$ is nothing but $N_a^{H_i}$ and $W^{H_i} = N_2^{H_i} - (\mathrm{Int}\ N_1)^{H_i}$. Since there is a one to one correspondence by f between the set of components of $M_1^{H_i}$ and that of $M_2^{H_i}$, we denote by $M_{a\mu}^{H_i}$ each component of $M_a^{H_i}$ such that $f(M_{1\mu}^{H_i}) = M_{2\mu}^{H_i}$. Denote by $N_{a\mu}^{H_i}$, the restriction $N_a^{H_i} \mid M_{a\mu}^{H_i}$ and by $(W^{H_i})_\mu$, $N_{2\mu}^{H_i} - \mathrm{Int}\ N_{1\mu}^{H_i}$. It is easy to see that $\dim M_{1\mu}^{H_i} = \dim M_{2\mu}^{H_i}$ and $\dim N_{a\mu}^{H_i} = \dim (W^{H_i})_\mu = \dim (V^m)^{H_i}$. Then one verifies that

$$\dim (V^m)^{H_i} - \dim M_{a\mu}^{H_i} \geq m \geq 3 \quad \text{for every } \mu.$$

It follows by the argument of the proof of Lemma 2' in [17] that $S(N_a^{H_i})$ are

deformation retracts of W^{H_i} for every H_i.

For reader's convenience, we repeat the proof. First notice that the inclusion $N_1 \subset N_2$ is a G-homotopy equivalence, hence the inclusion $N_1^{H_i} \subset N_2^{H_i}$ is a homotopy equivalence for any H_i. By a dimensional argument, any map of a 2-dimensional complex into $N_2^{H_i}$ can be deformed off $k \cdot (\varepsilon e_1 \oplus e_2 f)(M_1^{H_i})$, and hence can be pushed into W^{H_i}.

This implies that

$$\pi_1(W^{H_i}) \xrightarrow{\ \cong\ } \pi_1(N_2^{H_i})$$

and hence that

$$\pi_1(S(N_a^{H_i})) \xrightarrow{\ \cong\ } \pi_1(W^{H_i}) \qquad \text{for } a=1, 2,$$

for any common base point.

Given any system S of local coefficients on $N_2^{H_i}$ we have

$$H_*(W^{H_i}, S(N_1^{H_i}); S) \xrightarrow{\ \cong\ } H_*(N_1^{H_i}, N_1^{H_i}; S)$$

by excision. But the inclusion $N_1^{H_i} \subset N_2^{H_i}$ is a homotopy equivalence, hence these groups are zero. Using Whitehead [24, Theorem 3] it follows that $S(N_1^{H_i})$ is a deformation retract of W^{H_i}.

The group $H_p(W^{H_i}, S(N_2^{H_i}); S)$ is isomorphic by Poincaré duality to $H^{n-p}(W^{H_i}, S(N_1^{H_i}); S)$ where $n = \dim W^{H_i}$, and therefore is zero. This implies that $S(N_2^{H_i})$ is a deformation retract of W^{H_i}.

Let (H_i), (H_j) be conjugacy classes such that $(H_i) < (H_j)$. Then we have that

$$\dim (V^m)^{H_i} - \dim (V^m)^{H_j} = m(\dim V^{H_i} - \dim V^{H_j}) \geq m.$$

It follows that

$$\min_\mu \{\dim (W^{H_i})_\mu\} - \max_\lambda \{\dim (W^{H_j})_\lambda\} \geq m \geq \dim G + 3.$$

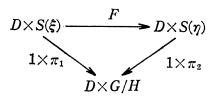We are now in a position to apply Lemma 3.1.

This makes the proof of Lemma 3.2 complete.

4. Equivariant Dold's theorem.

The purpose of this section is to establish an equivariant version of Dold's theorem. The following lemma corresponds to Hilfssatz 2 of Dold [7].

LEMMA 4.1. *Let D be a disk of dimension greater than or equal to one with the trivial G-action. Let $\pi_1: \xi \to G/H$ and $\pi_2: \eta \to G/H$ be G-vector bundles over G/H where H is a closed subgroup of G. We shall suppose that we are given:*
   α) a fiberwise G-map

$$D \times S(\xi) \xrightarrow{\quad F \quad} D \times S(\eta)$$

$$1 \times \pi_1 \searrow \qquad \swarrow 1 \times \pi_2$$

$$D \times G/H$$

such that the restriction

$$F| : d \times S(\xi_H) \longrightarrow d \times S(\eta_H)$$

is an H-homotopy equivalence for every $d \in D$ *where* $S(\xi_H)$, $S(\eta_H)$ *denote the fibers over the coset H,*

β) *a fiberwise G-map*

$$f' : S \times S(\eta) \longrightarrow S \times S(\xi),$$

where S denotes the boundary of D,

γ) *a fiberwise G-homotopy*

$$\phi : S \times S(\xi) \times I \longrightarrow S \times S(\xi)$$

such that $\phi(x, y, 0) = f' \cdot F(x, y)$, $\phi(x, y, 1) = (x, y)$, *for* $x \in S$, $y \in S(\xi)$.
    Then there exist a fiberwise G-map

$$F' : D \times S(\eta) \longrightarrow D \times S(\xi)$$

and a fiberwise G-homotopy

$$\Phi : D \times S(\xi) \times I \longrightarrow D \times S(\xi)$$

such that

$$F'(x, y') = f'(x, y'), \qquad \Phi(x, y, t) = \phi(x, y, t)$$

$$\Phi(d, y, 0) = F' \circ F(d, y), \qquad \Phi(d, y, 1) = (d, y)$$

for $x \in S$, $d \in D$, $y \in S(\xi)$, $y' \in S(\eta)$, $t \in I$.

PROOF. Since the proof is almost parallel to that of Hilfssatz 2 in [7], we only point out the differences. We denote by $F(S(\xi_H), S(\eta_H))$ the space of all continuous mappings from $S(\xi_H)$ to $S(\eta_H)$ with the compact-open topology. An H-action

$$\rho : H \times F(S(\xi_H), S(\eta_H)) \longrightarrow F(S(\xi_H), S(\eta_H))$$

is given by $\rho(h, f) = hfh^{-1}$ for $h \in H$, $f \in F(S(\xi_H), S(\eta_H))$. Similarly we have a space $F(S(\eta_H), S(\xi_H))$ with an H-action. The maps F, f', ϕ yield the following continuous H-maps,

$$\tilde{F} : D \longrightarrow F(S(\xi_H), S(\eta_H))$$

$$\tilde{f}' : S \longrightarrow F(S(\eta_H), S(\xi_H))$$

$$\tilde{\phi}: S \times I \longrightarrow F(S(\xi_H), S(\xi_H))$$

by putting $\tilde{F}(d)(y)=F(d, y)$, $\tilde{f}'(x)(y')=f'(x, y')$ and $\tilde{\phi}(x, t)(y)=\phi(x, y, t)$ for $x \in S$, $d \in D$, $y \in S(\xi_H)$, $y' \in S(\eta_H)$, $t \in I$.

As in the manner of the proof of Hilfssatz 2 in [7], we obtain the following H-maps by using H-spaces $F(S(\xi_H), S(\eta_H))$, $F(S(\eta_H), S(\xi_H))$ and $F(S(\xi_H), S(\xi_H))$,

$$\tilde{F}': D \longrightarrow F(S(\eta_H), S(\xi_H))$$

$$\tilde{\Phi}: D \times I \longrightarrow F(S(\xi_H), S(\xi_H))$$

such that

$$\tilde{F}'|S=\tilde{f}', \qquad \tilde{\Phi}(d, 0)=\tilde{F}'(d)\circ\tilde{F}(d),$$

$$\tilde{\Phi}(d, 1)=\text{identity}, \qquad \tilde{\Phi}|S \times I=\tilde{\phi} \quad \text{for } d \in D.$$

Then we define

$$F': D \times S(\eta) \longrightarrow D \times S(\xi)$$

$$\Phi: D \times S(\xi) \times I \longrightarrow D \times S(\xi)$$

by putting

$$F'(d, y')=(d, g'\tilde{F}'(d)(g'^{-1}y'))$$

$$\Phi(d, y, t)=(d, g\tilde{\Phi}(d, t)(g^{-1}y))$$

where $g', g \in G$ are chosen as $\pi_2(y')=g'H$, $\pi_1(y)=gH$. It is easily seen that these maps are a well-defined fiberwise G-map and a well-defined fiberwise G-homotopy, and satisfy the required properties.

This completes the proof of Lemma 4.1.

Let X be a compact G-space and Y be a closed G-invariant subspace of X. A G-homotopy $d: X \times I \to X$ satisfying $d(x, 0)=x$, $d(x, 1) \in Y$, $d(y, t)=y$, $d(d(x, t), 1)=d(x, 1)$ for $x \in X$, $y \in Y$, $t \in I$ is called a *nice G-deformation retraction* of X to Y. When such a map d exists, we say also that Y is a *nice G-deformation retract* of X. Then we have

LEMMA 4.2. *Let (X, Y) be a compact G-space pair such that Y is a nice G-deformation retract of X. Let $\pi_1: \xi \to X$, $\pi_2: \eta \to X$ be G-vector bundles over X and $f: S(\xi) \to S(\eta)$ be a fiberwise G-map such that the restriction $f|: S(\xi|Y) \to S(\eta|Y)$ gives a G-fiber homotopy equivalence. Then f itself gives a G-fiber homotopy equivalence.*

PROOF. By assumption, there exist a fiberwise G-map $\tilde{f}': S(\eta|Y) \to S(\xi|Y)$ and fiberwise G-homotopies $\phi: S(\xi|Y) \times I \to S(\xi|Y)$ and $\psi: S(\eta|Y) \times I \to S(\eta|Y)$ with

$$\phi| S(\xi|Y) \times 0=\tilde{f}' \cdot f|, \qquad \phi| S(\xi|Y) \times 1=\text{identity}$$

$$\psi| S(\eta|Y) \times 0=f| \cdot \tilde{f}', \qquad \psi| S(\eta|Y) \times 1=\text{identity}.$$

Let $d: X \times I \to X$ be the nice G-deformation retraction and $r: X \to X$ be the G-retraction given by $r(x)=d(x, 1)$. It follows from the G-homotopy property of G-vector bundles that there exist isomorphisms

$$\alpha: \xi \longrightarrow r^*(\xi \mid Y)=X \underset{Y}{\times}(\xi \mid Y) \subset X \times (\xi \mid Y)$$

$$\beta: \eta \longrightarrow r^*(\eta \mid Y)=X \underset{Y}{\times}(\eta \mid Y) \subset X \times (\eta \mid Y)$$

of G-vector bundles such that

$$\alpha \mid (\xi \mid Y)=\text{identity}, \qquad \beta \mid (\eta \mid Y)=\text{identity},$$

where $X \underset{Y}{\times}(\xi \mid Y)$ denotes the subset of $X \times (\xi \mid Y)$ consisting of (x, z) with $r(x)$ $=\pi_1(z)$ and similar for $X \underset{Y}{\times}(\eta \mid Y)$.

Define $\bar{f}: X \underset{Y}{\times} S(\xi \mid Y) \to X \underset{Y}{\times} S(\eta \mid Y)$ to be $\beta f \alpha^{-1}$ where we abbreviated the restrictions $\alpha \mid$, $\beta \mid$ to the associated sphere bundles to α, β respectively. Then we have that

$$\bar{f} \mid S(\xi \mid Y)=f \mid S(\xi \mid Y).$$

Let $f_2: X \underset{Y}{\times} S(\xi \mid Y) \to S(\eta \mid Y)$ be the G-map defined by

$$\bar{f}(x, z)=(x, f_2(x, z))$$

for $x \in X$, $z \in S(\xi \mid Y)$ with $r(x)=\pi_1(z)$. We now construct a fiberwise G-homotopy

$$E: X \underset{Y}{\times} S(\xi \mid Y) \times I \longrightarrow X \underset{Y}{\times} S(\eta \mid Y)$$

by setting

$$E(x, z, t)=(x, f_2(d(x, t), z))$$

for $x \in X$, $z \in S(\xi \mid Y)$, $t \in I$ with $r(x)=\pi_1(z)$. Since d is a nice G-deformation retraction, E is a well-defined fiberwise G-homotopy with

$$E \mid X \underset{Y}{\times} S(\xi \mid Y) \times 0=\bar{f},$$

$$E \mid X \underset{Y}{\times} S(\xi \mid Y) \times 1=1 \times (f \mid S(\xi \mid Y)).$$

We are now ready to construct the desired fiberwise G-map $f': S(\eta) \to S(\xi)$ and fiberwise G-homotopies $\Phi: S(\xi) \times I \to S(\xi)$ and $\Psi: S(\eta) \times I \to S(\eta)$ as follows. Put

$$f'=\alpha^{-1}(1 \times \tilde{f}')\beta,$$

$$\Phi(z, t)=\begin{cases} \alpha^{-1}(1 \times \tilde{f}')E(\alpha(z), 2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \alpha^{-1}(1 \times \phi)(\alpha(z), 2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

and

$$\Psi(z', t) = \begin{cases} \beta^{-1} E((1 \times \tilde{f}') \beta(z'), 2t) & \text{for } 0 \le t \le \dfrac{1}{2} \\ \\ \beta^{-1}(1 \times \phi)(\beta(z'), 2t-1) & \text{for } \dfrac{1}{2} \le t \le 1 \end{cases}$$

where $z \in S(\xi)$, $z' \in S(\eta)$. Then it is easy to see that Φ and Ψ are well-defined fiberwise G-homotopies with

$$\Phi| \ S(\xi) \times 0 = f' \cdot f, \qquad \Phi| \ S(\xi) \times 1 = \text{identity}$$

$$\Psi| \ S(\eta) \times 0 = f \cdot f', \qquad \Psi| \ S(\eta) \times 1 = \text{identity}.$$

This completes the proof of Lemma 4.2.

PROOF OF THEOREM 2.1. We shall use freely the notions and notations in §3. For examples, $\{(H_i)\}$, ν_i, W_i will be used as in §3. We shall construct a G-fiber homotopy inverse by induction concerning the i of $\{(H_i)\}$. We denote by f_i the restriction $f| \ S(\xi| \bigcup_{j<i} \nu_j(2) \cup W(H_i))$ and by $\bar{f}_i$ the restriction $f| S(\xi| \bigcup_{j \le i} \nu_j(2))$.

We shall first show that there exist a fiberwise G-map

$$f_1' : \ S(\eta| W(H_1)) \longrightarrow S(\xi| W(H_1))$$

and a fiberwise G-homotopy

$$h_1 : \ S(\xi| W(H_1)) \times I \longrightarrow S(\xi| W(H_1))$$

with

$$h_1| \ S(\xi| W(H_1)) \times 0 = f_1' \cdot f_1, \qquad h_1| \ S(\xi| W(H_1)) \times 1 = \text{identity}.$$

Consider the fiber bundle

$$G/H \longrightarrow W(H_1) \overset{p}{\longrightarrow} W(H_1)/G.$$

Since $W(H_1)/G$ is a smooth manifold, we can triangulate it and we shall suppose this done. Denote by $(W(H_1)/G)^n$, the n-skeleton and by $W(H_1)^n$, $p^{-1}(W(H_1)/G)^n$. Set $f_{1,n} = f| \ S(\xi| W(H_1)^n)$. Then $W(H_1)^0$ is the disjoint union $\bigcup_\mu (G/H_1)_\mu$ of some copies of G/H_1. The restriction of $f_{1,0}$ to $S(\xi|(G/H_1)_\mu)$ is denoted by $f_{1,0}^\mu$. Let H_1^μ be the coset H_1 in $(G/H_1)_\mu$. It follows from the assumption that there exist an H_1-map

$$\tilde{f}_{1,0}'^{\mu} : \ S(\eta_{H_1^\mu}) \longrightarrow S(\xi_{H_1^\mu})$$

and an H_1-homotopy

$$\tilde{h}_{1,0}^\mu : \ S(\xi_{H_1^\mu}) \times I \longrightarrow S(\xi_{H_1^\mu})$$

with

$$\tilde{h}_{1,0}^\mu| S(\xi_{H_1^\mu}) \times 0 = \tilde{f}_{1,0}'^\mu \cdot f_{1,0}^\mu|, \qquad \tilde{h}_{1,0}^\mu| S(\xi_{H_1^\mu}) \times 1 = \text{identity}.$$

Then we define

$$f'^{\mu}_{1,0} : S(\eta | (G/H_1)_{\mu}) \longrightarrow S(\xi | (G/H_1)_{\mu})$$

and

$$h^{\mu}_{1,0} : S(\xi | (G/H_1)_{\mu}) \times I \longrightarrow S(\xi | (G/H_1)_{\mu})$$

by putting

$$f'^{\mu}_{1,0}(z') = g' \tilde{f}'^{\mu}_{1,0}(g'^{-1}z'), \qquad h^{\mu}_{1,0}(z, t) = g \tilde{h}^{\mu}_{1,0}(g^{-1}z, t)$$

where g', $g \in G$ are chosen as $\pi_2(z') = g'H^{\mu}_1$, $\pi_1(z) = gH^{\mu}_1$. We now set

$$f'_{1,0} = \bigcup_{\mu} f'^{\mu}_{1,0}, \qquad h_{1,0} = \bigcup_{\mu} h^{\mu}_{1,0}.$$

Suppose that we are given for $n \geqq 1$ a fiberwise G-map

$$f'_{1,n-1} : S(\eta | W(H_1)^{n-1}) \longrightarrow S(\xi | W(H_1)^{n-1})$$

and a fiberwise G-homotopy

$$h_{1,n-1} : S(\xi | W(H_1)^{n-1}) \times I \longrightarrow S(\xi | W(H_1)^{n-1})$$

with

$$h_{1,n-1} | S(\xi | W(H_1)^{n-1}) \times 0 = f'_{1,n-1} \cdot f_{1,n-1},$$

$$h_{1,n-1} | S(\xi | W(H_1)^{n-1}) \times 1 = \text{identity}.$$

Let D be an n-simplex in $(W(H_1)/G)^n$. Then $p^{-1}(D)$ is G-homeomorphic to $D \times (G/H_1)$ where G acts trivially on D. We are now in a position to apply our Lemma 4.1 and obtain f'_1 and h_1 by induction.

Since $W(H_1)$ is a nice G-deformation retract of $\nu_1(2)$, we can apply Lemma 4.2 and extend f'_1 to a fiberwise G-map

$$\bar{f}'_1 : S(\eta | \nu_1(2)) \longrightarrow S(\xi | \nu_1(2))$$

and extend h_1 to a fiberwise G-homotopy

$$\bar{h}_1 : S(\xi | \nu_1(2)) \times I \longrightarrow S(\xi | \nu_1(2))$$

with

$$\bar{h}_1 | S(\xi | \nu_1(2)) \times 0 = \bar{f}'_1 \cdot \bar{f}_1,$$

$$\bar{h}_1 | S(\xi | \nu_1(2)) \times 1 = \text{identity}.$$

Suppose that we are given a fiberwise G-map

$$\bar{f}'_{i-1} : S(\eta | \bigcup_{j<i} \nu_j(2)) \longrightarrow S(\xi | \bigcup_{j<i} \nu_j(2))$$

and a fiberwise G-homotopy

$$\bar{h}_{i-1} : S(\xi | \bigcup_{j<i} \nu_j(2)) \times I \longrightarrow S(\xi | \bigcup_{j<i} \nu_j(2))$$

with

$$\bar{h}_{i-1} | S(\xi | \bigcup_{j<i} \nu_j(2)) \times 0 = \bar{f}'_{i-1} \cdot \bar{f}_{i-1},$$

$$\bar{h}_{i-1}|\ S(\xi|\bigcup_{j<i}\nu_j(2))\times 1=\text{identity}.$$

Consider the fiber bundle

$$G/H_i \longrightarrow W_i(H_i) \longrightarrow W_i(H_i)/G .$$

Since $\bigcup_{j<i}\nu_j(2)\cap W_i(H_i)$ is the union of some faces of $W_i(H_i)$ in the sense of Jänich [11], we can triangulate $W_i(H_i)/G$ so that $\{\bigcup_{j<i}\nu_j(2)\cap W_i(H_i)\}/G$ is a sub-complex. Then, applying Lemma 4.1 inductively as before to each simplex which is not included in $\{\bigcup_{j<i}\nu_j(2)\cap W_i(H_i)\}/G$, we get a fiberwise G-map

$$f_i': S(\eta|\bigcup_{j<i}\nu_j(2)\cup W_i(H_i)) \longrightarrow S(\xi|\bigcup_{j<i}\nu_j(2)\cup W_i(H_i))$$

and a fiberwise G-homotopy

$$h_i: S(\xi|\bigcup_{j<i}\nu_j(2)\cup W_i(H_i))\times I \longrightarrow S(\xi|\bigcup_{j<i}\nu_j(2)\cup W_i(H_i))$$

with

$$h_i|\ S(\xi|\bigcup_{j<i}\nu_j(2)\cup W_i(H_i))\times 0=f_i'\cdot f_i$$

$$h_i|\ S(\xi|\bigcup_{j<i}\nu_j(2)\cup W_i(H_i))\times 1=\text{identity}.$$

Let

$$c: \{\bigcup_{j<i}\nu_j(2)\cap W_i(H_i)\}\times I \longrightarrow W_i(H_i)$$

be an invariant collar such that

$$c|\ \{\bigcup_{j<i}\nu_j(2)\cap W_i(H_i)\}\times 0=\text{identity}.$$

It follows from the equivariant covering homotopy theorem that there exists an isomorphism

$$\gamma: [\nu_i(2)|\{\bigcup_{j<i}\nu_j(2)\cap W_i(H_i)\}]\times I \longrightarrow \nu_i(2)|\text{Image } c$$

of G-disk bundles covering c with

$$\gamma|\ [\nu_i(2)|\{\bigcup_{j<i}\nu_j(2)\cap W_i(H_i)\}]\times 0=\text{identity}.$$

Then we define a G-homotopy

$$d: \nu_i(2)\times I \longrightarrow \nu_i(2) \qquad \text{by}$$

$$d(\gamma(uy, s), t)=\begin{cases} \gamma((u-2st)y, (1-t)s) & \text{for } u\geq 2s \\ \gamma\left((1-t)uy, s+\dfrac{ut(s-1)}{2-u}\right) & \text{for } u\leq 2s \end{cases}$$

where $y\in S(\nu_i(2)|\{\bigcup_{j<i}\nu_j(2)\cap W_i(H_i)\})$, $0\leq s\leq 1$, $0\leq t\leq 1$ and $0\leq u\leq 1$, and by

$$d(x, t)=(1-t)x$$

where $0 \leq t \leq 1$ and $x \in \nu_i(2) \mid \{W_i(H_i)-\text{Image } c\}$. It is easy to see that d is a well-defined nice G-deformation retraction of $\nu_i(2)$ to

$$[\nu_i(2) \mid \{ \bigcup_{j<i} \nu_j(2) \cap W_i(H_i)\}] \cup W_i(H_i).$$

We can now apply Lemma 4.2 and extend f'_i to a fiberwise G-map

$$\bar{f}'_i \colon S(\eta \mid \bigcup_{j \leq i} \nu_j(2)) \longrightarrow S(\xi \mid \bigcup_{j \leq i} \nu_j(2))$$

and extend h_i to a fiberwise G-homotopy

$$\bar{h}_i \colon S(\xi \mid \bigcup_{j \leq i} \nu_j(2)) \times I \longrightarrow S(\xi \mid \bigcup_{j \leq i} \nu_j(2))$$

with $\bar{h}_i \mid S(\xi \mid \bigcup_{j \leq i} \nu_j(2)) \times 0 = \bar{f}'_i \cdot \bar{f}_i$,

$$\bar{h}_i \mid S(\xi \mid \bigcup_{j \leq i} \nu_j(2)) \times 1 = \text{identity}.$$

Thus we have by induction the required fiberwise G-map

$$f'=f'_k \colon S(\eta) \longrightarrow S(\xi)$$

and the required fiberwise G-homotopy

$$h=h_k \colon S(\xi) \times I \longrightarrow S(\xi)$$

with $h \mid S(\xi) \times 0 = f' \cdot f$, $h \mid S(\xi) \times 1 = \text{identity}$.

Next consider the restrictions $f_x = f \mid S(\xi_x) \colon S(\xi_x) \to S(\eta_x)$ and $f'_x = f' \mid S(\eta_x) \colon S(\eta_x) \to S(\xi_x)$ for $x \in W$. Since $f'_x \cdot f_x$ is G_x-homotopic to the identity and since f_x is a G_x-homotopy equivalence, it follows by routine arguments that f'_x gives also a G_x-homotopy equivalence for every $x \in W$. Hence quite similar arguments prove that there exist a fiberwise G-map $f'' \colon S(\xi) \to S(\eta)$ and a fiberwise G-homotopy $h' \colon S(\eta) \times I \to S(\eta)$ with $h' \mid S(\eta) \times 0 = f'' \cdot f'$, $h' \mid S(\eta) \times 1 = \text{identity}$. Therefore we have that

$$f'' \sim f'' \cdot (f' \cdot f) = (f'' \cdot f') \cdot f \sim f$$

and hence $f \cdot f' \sim \text{identity}$ where $g \sim g'$ means that g is fiberwise G-homotopic to g'.

This makes the proof of Theorem 2.1 complete.

5. G-homotopy type invariance of G-fiber homotopy type of stable tangent sphere bundles.

In this section, we shall prove Theorem 2.3. Let us first recall Lemma 3.2, that is, $S(N_a)$ are G-deformation retracts of W $(a=1, 2)$. Let $d(a) \colon W \times I \to W$

be such G-deformation retractions. We set

$$r = d(2) \mid S(N_1) \times 1 : S(N_1) \longrightarrow S(N_2).$$

Then r is a G-homotopy equivalence. Since the following diagram:

$$
\begin{array}{ccc}
S(N_1) & \xrightarrow{\ \ r\ \ } & S(N_2) \\
{\scriptstyle p_1}\Big\downarrow & & \Big\downarrow{\scriptstyle p_2} \\
M_1 & \xrightarrow[\ \ f\ \ ]{} & M_2
\end{array}
$$

is G-homotopy commutative, the equivariant covering homotopy theorem implies that r is G-homotopic to a fiberwise G-map $\bar{f} : S(N_1) \rightarrow S(N_2)$ covering f.

Let $f' : M_2 \rightarrow M_1$ be a G-homotopy inverse of f. Then in a similar way we get a fiberwise G-map $\bar{f}' : S(N_2') \rightarrow S(N_1')$ covering f'. If we use the same G-embeddings $e_a : M_a \rightarrow V_a$ that we used in § 3, then we see by the G-homotopy property of G-vector bundles that N_1', N_2' are G-isomorphic to N_1, N_2 respectively. Thus we have a fiberwise G-map $\bar{f}' : S(N_2) \rightarrow S(N_1)$ covering f'.

Let x be a point of M and H be a closed subgroup of G_x. Denote by $M_{1\mu}^H$ (resp. $M_{2\mu}^H$) the connected component of M_1^H (resp. M_2^H) including x (resp. $f(x)$). Then we have the following commutative diagram

$$
\begin{array}{ccccc}
S(N_{1x})^H & \longrightarrow & S(N_1 \mid M_{1\mu}^H)^H & \longrightarrow & M_{1\mu}^H \\
{\scriptstyle \bar{f}\mid}\Big\downarrow & & {\scriptstyle \bar{f}\mid}\Big\downarrow & & \Big\downarrow{\scriptstyle f\mid} \\
S(N_{2f(x)})^H & \longrightarrow & S(N_2 \mid M_{2\mu}^H)^H & \longrightarrow & M_{2\mu}^H
\end{array}
$$

consisting of sphere bundles and a bundle map. Since $f\mid : M_{1\mu}^H \rightarrow M_{2\mu}^H$ and $\bar{f}\mid : S(N_1 \mid M_{1\mu}^H)^H \rightarrow S(N_2 \mid M_{2\mu}^H)^H$ are homotopy equivalences, it follows by the five lemma that the degree of the map $\bar{f}\mid : S(N_{1x})^H \rightarrow S(N_{2f(x)})^H$ is ± 1. Similarly we have that the degree of the map $\bar{f}'\mid : S(N_{2f(x)})^H \rightarrow S(N_{1f' \cdot f(x)})^H$ is ± 1.

Since $f' \cdot f$ is G-homotopic to the identity, there exists an isomorphism $\theta : N_{1f' \cdot f(x)} \rightarrow N_{1x}$ of G_x-representation spaces. Then the degree of the map

$$(\theta \cdot \bar{f}' \cdot \bar{f} \cdot \theta \cdot \bar{f}')\bar{f} \mid S(N_{1x})^H : S(N_{1x})^H \longrightarrow S(N_{1x})^H$$

is $+1$. Similarly the degree of the map

$$\bar{f} \cdot (\theta \cdot \bar{f}' \cdot \bar{f} \cdot \theta \cdot \bar{f}') \mid S(N_{2f(x)})^H : S(N_{2f(x)})^H \longrightarrow S(N_{2f(x)})^H$$

is $+1$.

For a subgroup H of G_x, we have $S(N_{ax})(H)$ $(a=1, 2)$ (see § 3). We now show

LEMMA 5.1. *If $S(N_{ax})$ (H) is non-empty, then $S(N_{ax})$ $(H)/G_x$ is connected $(a=1, 2)$.*

PROOF. We prove Lemma 5.1 for $a=1$. We begin with some general remarks.

For a G-space X and for a subgroup H of G, we set

$$X^{[H]} = \{x \in X \mid G_x = H\}.$$

If X is a smooth G-manifold, then $X^{[H]}$ is a smooth submanifold of X and we have an equivariant diffeomorphism:

$$X(H) = G/H \underset{N(H)/H}{\times} X^{[H]}.$$

Consequently we have a diffeomorphism:

$$X(H)/G = X^{[H]}/(N(H)/H).$$

We may assume without loss of generality that V includes a direct summand with trivial G-action. Then in view of the inclusions $M_1 \subset V \subset V^m$, we see that

$$\dim S(N_{1x})^{G_x} \geq m-1 \geq 2.$$

Hence $\dim S(N_{1x})^H \geq 2$ for any subgroup H of G_x. If $S(N_{1x})^{[H]} = S(N_{1x})^H$, then we have

$$S(N_{1x})(H)/G_x = S(N_{1x})^H/(N_{G_x}(H)/H),$$

where $N_{G_x}(H)$ is the normalizer of H in G_x, and hence $S(N_{1x})(H)/G_x$ is connected. Namely Lemma 5.1 holds in this case.

Next we consider the case where

$$S(N_{1x})^{[H]} \neq S(N_{1x})^H.$$

Denote by $T_x(M_1)$ the tangent space at x, which is a G_x-representation space. Notice that M_1 is equivariantly embedded in V. Denote by ν the equivariant normal bundle of M_1 in V. Then the fiber ν_x over x is a G_x-representation space and we have isomorphisms of G_x-representation spaces:

$$V \cong T_x(M_1) \oplus \nu_x$$

and

$$N_{1x} \cong V^{m-1} \oplus \nu_x$$

where V is regarded as a G_x-representation space.

Let H_i and H_j be closed subgroups of G_x such that $H_i \subsetneqq H_j$. Then we shall show that $N_{1x}^{H_i} \supsetneqq N_{1x}^{H_j}$ implies $V^{H_i} \supsetneqq V^{H_j}$. If $V^{H_i} = V^{H_j}$, then

$$T_x(M_1)^{H_i} \oplus \nu_x^{H_i} \cong V^{H_i} = V^{H_j} \cong T_x(M_1)^{H_j} \oplus \nu_x^{H_j}.$$

Generally we have

$$T_x(M_1)^{H_i} \supseteq T_x(M_1)^{H_j} \quad \text{and} \quad \nu_x^{H_i} \supseteq \nu_x^{H_j}.$$

It follows that

$$T_x(M_1)^{H_i} = T_x(M_1)^{H_j} \quad \text{and} \quad \nu_x^{H_i} = \nu_x^{H_j}.$$

Hence we have

$$N_{1x}^{H_i} \cong (V^{H_i})^{m-1} \oplus \nu_x^{H_i} = (V^{H_j})^{m-1} \oplus \nu_x^{H_j} \cong N_{1x}^{H_j}$$

which is a contradiction. Accordingly if $N_{1x}^{H_i} \supsetneq N_{1x}^{H_j}$, then we have

$$\dim N_{1x}^{H_i} - \dim N_{1x}^{H_j} = (m-1)(\dim V^{H_i} - \dim V^{H_j}) + \dim \nu_x^{H_i} - \dim \nu_x^{H_j}$$

$$\geqq m-1 \geqq \dim G + 2 \geqq \dim G_x + 2.$$

Thus we have

$$\dim S(N_{1x})^{H_i} - \dim S(N_{1x})^{H_j} \geqq \dim G_x + 2.$$

Let $\{(H_i) \mid i = 1, \cdots, k\}$ be the set of the conjugacy classes of the isotropy groups which appear in the G_x-manifold $S(N_{1x})$. As before we arrange (H_i) in such order that $(H_i) \leqq (H_j)$ implies $i \geqq j$.

Fix a conjugacy classes (H_i) and its representative H_i. Then for any (H_j) with $(H_i) \leqq (H_j)$, we choose representatives H_j of (H_j) such that $H_i \subset H_j$.

For $b < i$, we set

$$^b S(N_{1x}) = S(N_{1x}) - \bigcup_{c < b} S(N_{1x})(H_c).$$

Then $^b S(N_{1x})$ is an open G_x-submanifold of $S(N_{1x})$ and

$$(^{b+1} S(N_{1x}))^{H_i} = (^b S(N_{1x}))^{H_i} - (S(N_{1x})(H_b))^{H_i}.$$

ASSERTION. $(S(N_{1x})(H_b))^{H_i}$ *is empty or is the disjoint union of some submanifolds of* $(^b S(N_{1x}))^{H_i}$ *of codimension greater than one.*

PROOF OF ASSERTION. We have the following commutative diagram:

$$
\begin{array}{ccc}
(S(N_{1x})(H_b))^{H_i} & \xrightarrow{\ \phi_1\ } & S(N_{1x})^{H_i} \\
\Big\downarrow{\scriptstyle \phi_2} & & \Big\downarrow{\scriptstyle \phi_3} \\
S(N_{1x})(H_b) & \xrightarrow{\ \phi_4\ } & S(N_{1x})
\end{array}
$$

where ϕ_i are inclusion maps. Since ϕ_2, ϕ_3, ϕ_4 are inclusion maps of submanifolds, $(S(N_{1x})(H_b))^{H_i}$ is also a submanifold of $S(N_{1x})^{H_i}$. Since $^b S(N_{1x})$ is an open G_x-submanifold of $S(N_{1x})$, $(^b S(N_{1x}))^{H_i}$ is a submanifold of $S(N_{1x})^{H_i}$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
(S(N_{1x})(H_b))^{H_i} & \xrightarrow{\ \phi_1\ } & S(N_{1x})^{H_i} \\
\Big\downarrow{\scriptstyle \phi_6} & \nearrow{\scriptstyle \phi_5} & \\
(^b S(N_{1x}))^{H_i} & &
\end{array}
$$

where ϕ_i are inclusion maps. Since ϕ_1 and ϕ_5 are inclusion maps of submanifolds, $(S(N_{1x})(H_b))^{H_i}$ is also a submanifold of $(^b S(N_{1x}))^{H_i}$. Since $S(N_{1x})(H_b)$ is closed in $^b S(N_{1x})$, $(S(N_{1x})(H_b))^{H_i}$ is closed in $(^b S(N_{1x}))^{H_i}$. If $(S(N_{1x})(H_b))^{H_i}$ is non-empty, then by the choice of representatives we have $H_b \supsetneqq H_i$. It follows that $N_{1x}^{H_i} \supsetneqq N_{1x}^{H_b}$. Hence we have

$$\dim \{\text{a component } C \text{ of } S(N_{1x})(H_b)^{H_i}\}$$

$$\leqq \dim \{\text{the component of } S(N_{1x})(H_b) \text{ including } C\}$$

$$\leqq \dim G_x + \dim S(N_{1x})^{H_b}$$

$$\leqq \dim S(N_{1x})^{H_i} - 2 .$$

It follows by induction that $(^b S(N_{1x}))^{H_i}$ is a connected open submanifold of $S(N_{1x})^{H_i}$, and hence $\dim S(N_{1x})^{H_i} = \dim (^b S(N_{1x}))^{H_i}$.

This completes the proof of the assertion.

It follows from the assertion that $(^i S(N_{1x}))^{H_i}$ is connected. Notice that

$$S(N_{1x})^{[H_i]} = (^i S(N_{1x}))^{H_i}$$

and that

$$S(N_{1x})(H_i)/G_x = S(N_{1x})^{[H_i]}/(N_{G_x}(H_i)/H_i) .$$

Consequently we can conclude that $S(N_{1x})(H_i)/G_x$ is connected. Note that if $S(N_{1x})(H)$ is non-empty, then $(H) = (H_i)$ for some H_i.

Quite a similar argument works for $a = 2$.

This makes the proof of Lemma 5.1 complete.

Now we are ready to apply [20] to the G_x-maps $(\theta \cdot \bar{f}' \cdot \bar{f} \cdot \theta \cdot \bar{f}') \cdot \bar{f} \mid S(N_{1x})$ and $\bar{f} \cdot (\theta \cdot \bar{f}' \cdot \bar{f} \cdot \theta \cdot \bar{f}') \mid S(N_{2f(x)})$. As a consequence, both maps are G_x-homotopic to the identity. Namely $\bar{f} \mid : S(N_{1x}) \to S(N_{2f(x)})$ gives a G_x-homotopy equivalence. Let $\bar{f}: S(N_1) \to S(f^*N_2)$ be the G-map induced naturally from $\bar{f}$. Then $\bar{\bar{f}}_x : S(N_{1x}) \to S((f^*N_2)_x)$ gives a G_x-homotopy equivalence. We are now in a position to apply Theorem 2.1 and have that $\bar{\bar{f}}: S(N_1) \to S(f^*N_2)$ is a G-fiber homotopy equivalence.

This makes the proof of Theorem 2.3 complete.

6. Normal representations.

In this section, we shall prove Example 2.7. Let V, V' be two orthogonal G-representation spaces whose restrictions $S(V)$, $S(V')$ to the unit spheres have the same G-homotopy type. Then we first remark that V and V' appear as the tangent representations at the corresponding fixed points of G-homotopy equivalent manifolds. For this purpose, we consider G-manifolds $S(V \oplus R)$, $S(V' \oplus R)$ where R denotes the set of real numbers with the trivial G-action.

Naturally $S(V \oplus R)$ is G-homotopy equivalent to $S(V' \oplus R)$ so that the fixed point $a=(0, 1) \in S(V \oplus R)$ corresponds to the fixed point $b=(0, 1) \in S(V' \oplus R)$. The tangent representation at a (resp. b) is isomorphic to V (resp. V').

Next we show that there are many examples of representations, V, V' such that V is not isomorphic to V' as G-representation spaces, but $S(V)$ is G-homotopy equivalent to $S(V')$. Let k be an integer greater than one and $q_1, \cdots, q_n$ be integers relatively prime to k. Set $\zeta = \exp 2\pi\sqrt{-1}/k$. Regarding Z_k as $\{\zeta^i \mid i=0, \cdots, k-1\}$, we define a Z_k-action on the n-dimensional complex vector space C^n by

$$\zeta \cdot (z_1, \cdots, z_n) = (\zeta^{q_1} z_1, \cdots, \zeta^{q_n} z_n).$$

This Z_k-representation space is denoted by $V(k; q_1, \cdots, q_n)$ and its restriction to the unit sphere is denoted by $S(V(k; q_1, \cdots, q_n))$. The orbit space $S(V(k; q_1, \cdots, q_n))/Z_k$ is the so called lens space $L(k; q_1, \cdots, q_n)$.

According to [14], $L(k; q_1, \cdots, q_n)$ and $L(k; q_1', \cdots, q_n')$ have the same homotopy type if and only if there exists an integer a with

$$q_1 \cdots q_n = \pm a^n q_1' \cdots q_n' \pmod{k}.$$

On the other hand, it follows by the uniqueness of the universal regular covering space that $L(k; q_1, \cdots, q_n)$ and $L(k; q_1', \cdots, q_n')$ have the same homotopy type if and only if $S(V(k; q_1, \cdots, q_n))$ is Z_k-homotopy equivalent to $S(V(k; bq_1', \cdots, bq_n'))$ for some b with $(b, k)=1$.

Let $\phi(\ )$ be the Euler function, that is, $\phi(k)$ is the cardinal number of the set $\{j \mid 1 \leqq j \leqq k, (j, k)=1\}$. As is well-known,

$$q = q^{\phi(k)+1} \pmod{k}$$

for any q with $(q, k)=1$. Hence taking $a=q$ and $n=\phi(k)+1$, we have that

$$L(k; q, 1, \cdots, 1) \simeq L(k; 1, \cdots, 1),$$

that is, $S(V(k; q, 1, \cdots, 1))$ is Z_k-homotopy equivalent to $S(V(k; b, \cdots, b))$ for some b with $(b, k)=1$. However $V(k; q, 1, \cdots, 1)$ is not isomorphic to $V(k; b, \cdots, b)$ as Z_k-representation spaces if $q \neq \pm 1 \pmod{k}$.

Added in July, 1979.

(1) According to M. Cohen "A course in Simple-Homotopy Theory, Springer", Two Z_k-spheres $S(V(k; q_1, \cdots, q_n))$ and $S(V(k; q_1', \cdots, q_n'))$ in § 6 are Z_k-homotopy equivalent if and only if

$$\Pi q_i \equiv \pm \Pi q_i' \mod k.$$

(2) Although we did not mention it explicitly, we proved actually that a

smooth G-manifold has a G-CW-complex structure without assuming Yang's result [25]. In the case of finite group actions, S. Illman proved that a smooth G-manifold has a triangulation such that the action is simplicial in "Smooth equivariant triangulations of G-manifolds for G a finite group, Math. Ann., **233** (1978), 199-220".

References

[1] J. F. Adams, On the group $J(X)-I$, Topology, **2** (1963), 181-195.

[2] M. F. Atiyah, Thom complexes, Proc. London Math. Soc., (3) **11** (1961), 291-310.

[3] M. F. Atiyah and G. B. Segal, Equivariant K-theory, (mimeographed note), University of Warwick.

[4] W. Browder, Homotopy type of differentiable manifolds, Proceedings of the Aarhus Symposium on Algebraic Topology, Aarhus, (1962), 42-46.

[5] T. tom Dieck, Characteristic numbers of G-manifolds, I, Invent. Math., **13** (1971), 213-224.

[6] T. tom Dieck, Oberwolfach talk, 1976, September.

[7] A. Dold, Über fasernweise Homotopieäquivalenz von Faserräumen, Math. Z., **62** (1955), 111-136.

[8] M. W. Hirsch, On the fiber homotopy type of normal bundles, Michigan Math. J., **12** (1965), 225-229.

[9] W. C. Hsiang, A note on free differentiable actions of S^1 and S^3 on homotopy spheres, Ann. of Math., **83** (1966), 266-272.

[10] S. Illman, Equivariant singular homology and cohomology for actions of compact Lie groups, Proc. of the Second Conf. on Compact Transformation Groups, Amherst, Springer Lecture Note, **298** (1971), 403-415.

[11] K. Jänich, On the classification of $O(n)$ manifolds, Math. Ann., **176** (1968), 53-76.

[12] K. Kawakubo, Free and semi-free differentiable actions on homotopy spheres, Proc. Japan Acad., **45** (1969), 651-655.

[13] K. Kawakubo, Equivariant Riemann-Roch theorems, localization and formal group law, (mimeographed note), Universität Bonn, 1976.

[14] M. A. Kervaire, S. Maumary and G. de Rham, Torsion et type simple d'homotopie, Springer Lecture Note, **48** (1967).

[15] J. L. Koszul, Sur certain groupes de transformation de Lie, Colloque de Geometrie Differentiable, Strasbourg, 1953.

[16] T. Matumoto, On G-CW complexes and a theorem of J. H. C. Whitehead, J. Fac. Sci. Univ. Tokyo Sect. IA, **18** (1971), 363-374.

[17] J. Milnor, Two complexes which are homeomorphic but combinatorially distinct, Ann. of Math., **74** (1961), 575-590.

[18] S. P. Novikov, Homotopy equivalent smooth manifolds, I, Izv. Acad. Nauk SSSR Ser. Mat., **28** (1964), 365-474, A. M. S. Translations, **48** (1965), 271-396.

[19] R. Palais, Imbeddings of compact, differentiable transformation groups in orthogonal representations, J. Math. Mech., **6** (1957), 673-678.

[20] R. L. Rubinsztein, On the equivariant homotopy of spheres, Dissertationes Mathematicae, Warszawa, 1976.

[21] G. Segal, Equivariant stable homotopy theory, Actes, Congrés internat. math. (Nice, 1970), T.2, 59-63, Gauthier-Villars, Paris, 1971.

[22] C. T. C. Wall, Surgery on compact manifolds, Academic Press, 1970.

[23] A. G. Wasserman, Equivariant differential topology, Topology, **8** (1969), 127-150.
[24] J. H. C. Whitehead, Combinatorial homotopy, I, Bull. Amer. Math. Soc., **55** (1949), 213-245.
[25] C. T. Yang, The triangulability of the orbit space of a differentiable transformation group, Bull. Amer. Math. Soc., **69** (1963), 405-408.

Katsuo KAWAKUBO
Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560
Japan