

Some remarks on lagrangian imbeddings

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§ 1. Introduction.

In this note we shall investigate some properties of lagrangian imbeddings of compact smooth n -manifolds into the complex n -space \mathbf{C}^n from the view point of differential topology.

As for lagrangian immersions, considerably many facts are known. One of the most interesting results already obtained would be the theorem due to Gromov [3], Lees [5], Weinstein [11] which says that a smooth n -manifold M admits a lagrangian immersion into \mathbf{C}^n if and only if the complexification $\tau(M) \otimes_{\mathbf{R}} \mathbf{C}$ of the tangent bundle $\tau(M)$ of M is a trivial complex vector bundle. On the other hand, Lees [5] obtained the homotopy theoretic classification theorem of lagrangian immersions of a smooth n -manifold L into a smooth symplectic $2n$ -manifold M .

On the contrary, as for lagrangian imbeddings very few are known. First we consider what kind of compact manifolds admit lagrangian imbeddings into \mathbf{C}^n . A familiar example of a compact manifold admitting lagrangian imbeddings into \mathbf{C}^n is the n -torus T^n whose lagrangian imbeddings are defined by n functions in involution (see [2]). However we can prove the following theorem which shows that there are many other examples of compact orientable manifolds than T^n which admit lagrangian imbeddings into \mathbf{C}^n if $n \geq 3$. (For the case $n=2$, it is easily seen that T^2 is the only compact orientable surface that admits a lagrangian imbedding into \mathbf{C}^2 .)

THEOREM 1. *Let M be a compact orientable smooth n -manifold which admits an immersion into the euclidean $(n+1)$ -space \mathbf{R}^{n+1} . Then $M \times S^1$ admits a lagrangian imbedding into \mathbf{C}^{n+1} .*

Since an orientable smooth n -manifold M admits an immersion into \mathbf{R}^{n+1} if and only if it is s -parallelizable, Theorem 1 can be restated as follows.

COROLLARY 1. *Let M be a compact smooth s -parallelizable n -manifold. Then $M \times S^1$ admits a lagrangian imbedding into \mathbf{C}^{n+1} .*

Recalling that every compact n -manifold M with $n \leq 3$ is immersible in \mathbf{R}^{n+1} (for $n=3$, see [4]), we have:

COROLLARY 2. *Let M be a compact orientable smooth n -manifold with $n \leq 3$.*

Then $M \times S^1$ admits a lagrangian imbedding into C^{n+1} .

Secondly, we consider if the standard n -sphere S^n admits a lagrangian imbedding into C^n . Since S^n is s -parallelizable, the complexification of its tangent bundle is trivial, and therefore, S^n admits lagrangian immersions into C^n . Weinstein [11] has given an explicit form of a lagrangian immersion of S^n into C^n with one point of self-intersection.

We shall prove the following.

THEOREM 2. For $n \neq 1, 3$ S^n does not admit a lagrangian imbedding into C^n .

For the case $n=1$, every imbedding of S^1 into C^1 is trivially lagrangian. For the case $n=3$, at present we could not decide whether S^3 admits a lagrangian imbedding into C^3 or not.

We also give some examples of non-orientable compact n -manifolds which admit lagrangian imbeddings into C^n at the end of §2 and §3.

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§2. Definitions and basic properties.

First we give some definitions.

A smooth manifold M is called a *symplectic manifold* (with symplectic structure ω), if there is given a non-degenerate closed 2-form ω on M .

Let $f: L \rightarrow M$ be an immersion [or an imbedding] of a smooth n -manifold L into a symplectic $2n$ -manifold M with symplectic structure ω . f is called a *lagrangian immersion* [or a *lagrangian imbedding*], if $f^*\omega=0$. If $f: L \rightarrow M$ is a lagrangian imbedding, then $f(L)$ (or L) is called a *lagrangian submanifold* of M .

A real vector space V is called a *symplectic vector space* (with symplectic structure ω), if there is given a non-degenerate skew-symmetric bilinear form ω on it. As an example, we take the complex vector space C^n of n -tuples of complex numbers (z^1, \dots, z^n) . If we set

$$z^i = x^i + \sqrt{-1}y^i, \quad x^i, y^i \in \mathbf{R}, \quad i=1, \dots, n,$$

then C^n can be identified with the real vector space \mathbf{R}^{2n} of $2n$ -tuples of real numbers by means of the correspondence $(z^1, \dots, z^n) \rightarrow (x^1, \dots, x^n, y^1, \dots, y^n)$. This space has the standard symplectic structure ω_0 given by

$$\omega_0(z, w) = \text{Im}(z, w) \quad \text{for } z, w \in C^n,$$

using the standard hermitian inner product

$$(z, w) = \sum_{i=1}^n z^i \bar{w}^i.$$

Let λ be an n -dimensional subspace of a $2n$ -dimensional symplectic vector space V with symplectic structure ω . λ is called a lagrangian subspace, if $\omega(u, v)=0$ for all $u, v \in \lambda$.

A real vector bundle ξ is called a *symplectic vector bundle* (with symplectic structure ω), if, for each point x of the base space, the fibre ξ_x of ξ over x is a symplectic vector space with symplectic structure ω_x which varies continuously with x .

Let η be an n -dimensional sub-bundle of a $2n$ -dimensional symplectic vector bundle ξ with symplectic structure ω . η is called a *lagrangian sub-bundle*, if ω vanishes when restricted to η .

If M is a symplectic manifold with symplectic structure ω , then the tangent bundle $\tau(M)$ of M is clearly a symplectic vector bundle with symplectic structure ω . If $f: L \rightarrow M$ is an immersion, then f is lagrangian if and only if $\tau(L)$ is a lagrangian sub-bundle of the induced symplectic vector bundle $f^*\tau(M)$ with symplectic structure $f^*\omega$.

Next we list some basic facts about symplectic vector bundles. For further details, see for example [11].

First let us consider the symplectic vector space \mathbf{C}^n and its lagrangian subspace λ (cf. [1]). Note that \mathbf{C}^n has the standard euclidean inner product

$$\langle u, v \rangle = \operatorname{Re}(u, v) \quad \text{for } u, v \in \mathbf{C}^n.$$

Hence λ has a euclidean structure induced from that of \mathbf{C}^n . On the other hand it follows from the definition that

$$\omega_0(u, v) = \operatorname{Im}(u, v) = 0 \quad \text{for all } u, v \in \lambda,$$

hence we have

$$(u, v) = \langle u, v \rangle \quad \text{for all } u, v \in \lambda.$$

This shows that \mathbf{C}^n considered a hermitian vector space has a structure of the complexification of the euclidean vector space λ . In other words if we define a linear map $\varphi: \lambda \otimes_{\mathbf{R}} \mathbf{C} \rightarrow \mathbf{C}^n$ by

$$\varphi(u \otimes (a + \sqrt{-1}b)) = au + b\sqrt{-1}u \quad \text{for } u \in \lambda, a, b \in \mathbf{R},$$

then φ is an isomorphism between hermitian vector spaces. (Here $\lambda \otimes_{\mathbf{R}} \mathbf{C}$ is considered a hermitian vector space in the usual way.) In particular, $\sqrt{-1}\lambda$ is orthogonal to λ .

The following proposition enables us to generalize the above facts to symplectic vector bundles.

PROPOSITION 1. *Let ξ be a symplectic vector bundle with symplectic structure ω . Then there exists a complex structure J on ξ and a hermitian inner product h with respect to J on ξ such that*

$$(*) \quad \omega_x(u, v) = \operatorname{Im} h_x(u, v)$$

for all $u, v \in \xi_x$ and all points x of the base space B . Such a complex structure J and a hermitian inner product h are unique up to homotopy.

Such a complex structure J is said to be compatible with ω .

PROOF. Making use of a similar argument as in the case of a euclidean vector bundle or a hermitian vector bundle, we can show that there exists a set of local coordinate systems $\{(U_i, \varphi_i)\}_{i \in I}$, $\bigcup_{i \in I} U_i = B$, such that $(\varphi_i)_x: \mathbf{C}^n \rightarrow \xi_x$ is an isomorphism between symplectic vector spaces for each $x \in U_i$, $i \in I$. Hence we can reduce the structure group of ξ to the real symplectic group $Sp(2n, R)$. Since the unitary group $U(n)$ is the maximal compact subgroup of $Sp(2n, R)$, we can further reduce the structure group of ξ to $U(n)$, and here the reduction is unique up to homotopy. Let $\{(U_i, \varphi_i)\}_{i \in I'}$ be a set of local coordinate systems such that

$$\bigcup_{i \in I'} U_i = B, \quad (\varphi_i)_x^{-1} \circ (\varphi_j)_x \in U(n) \quad \text{for } x \in U_i \cap U_j.$$

Then we can define a complex structure J on ξ and a hermitian inner product h with respect to J by the formulas

$$J_x = (\varphi_i)_x \circ J_0 \circ (\varphi_i)_x^{-1}, \quad (\varphi_i)_x^* h = (\cdot, \cdot),$$

where J_0 is the canonical complex structure on \mathbf{C}^n . Now it is clear that h satisfies the condition (*). Conversely, any compatible complex structure is obtained in this way. This completes the proof of Proposition 1.

Thus every symplectic vector bundle has a structure of a hermitian vector bundle determined by its symplectic structure uniquely up to homotopy. As an example, let M be a Kähler manifold with complex structure J and fundamental two form ω . Then we can regard M as a symplectic manifold with symplectic structure ω . In this case, J is a compatible complex structure on the symplectic vector bundle $\tau(M)$.

In view of the above facts, we assume henceforth that every symplectic vector bundle is equipped with a complex structure J compatible with its symplectic structure, and hence is given a structure of a hermitian vector bundle.

Now the next lemma is an immediate consequence of the above arguments.

LEMMA 1. *Let ξ be a symplectic vector bundle with symplectic structure ω . If ξ has a lagrangian sub-bundle η , then ξ considered a hermitian vector bundle has a structure of the complexification of η .*

Applying Lemma 1 to lagrangian immersions, we have:

LEMMA 2. *Let $f: L \rightarrow M$ be a lagrangian immersion. Then the induced hermitian vector bundle $f^*\tau(M)$ has a structure of the complexification of $\tau(L)$. In particular, we have an isomorphism*

$$\nu(f) = (-1)^{n(n-1)/2} \tau(L)$$

between oriented vector bundles, where $\nu(f)$ is the normal bundle of f with orientation defined in the usual way.

REMARK 1. Suppose that a smooth n -manifold L admits a lagrangian immersion into C^n . Then it follows from Lemma 2 that $\tau(L) \otimes_R C$ is trivial. (According to Weinstein [11], the converse is also true.) Hence the total Pontrjagin class $p(L)$ and the total Stiefel-Whitney class $w(L)$ of $\tau(L)$ satisfy the conditions

$$p(L) = 1, \quad w(L)^2 = 1.$$

Now let us specialize to the case L is an orientable compact lagrangian submanifold.

PROPOSITION 2. Let M be a symplectic $2n$ -manifold with symplectic structure ω and let $f: L \rightarrow M$ be a lagrangian imbedding of an oriented compact smooth n -manifold L . If a homology class $\mathbf{a} \in H_n(M, \mathbf{Z})$ is represented by L , then we have

- (i) $\mathbf{a} \cdot \mathbf{a} = (-1)^{n(n-1)/2} \chi(L),$
(ii) $\langle c_{n/2}, \mathbf{a} \rangle = 0$ if $n \equiv 2$ modulo 4,

where $\mathbf{a} \cdot \mathbf{a}$ denotes the self-intersection number of \mathbf{a} , $\chi(L)$ the Euler number of L , $c_{n/2}$ the $n/2$ -th Chern class of M and \langle, \rangle the Kronecker index.

Using the Poincaré isomorphism $D: H_n(M, \mathbf{Z}) \rightarrow H_{\text{comp}}^n(M, \mathbf{Z})$, the self-intersection number $\mathbf{a} \cdot \mathbf{a}$ of \mathbf{a} is defined by $\mathbf{a} \cdot \mathbf{a} = \langle Da, \mathbf{a} \rangle$.

PROOF. (i) By Lemma 2 we have $f^*Da = e(\nu(L)) = (-1)^{n(n-1)/2} e(\tau(L))$. Hence we have $\mathbf{a} \cdot \mathbf{a} = \langle Da, \mathbf{a} \rangle = \langle Da, f_*[L] \rangle = \langle f^*Da, [L] \rangle = (-1)^{n(n-1)/2} \langle e(\tau(L)), [L] \rangle = (-1)^{n(n-1)/2} \chi(L)$. Here $e(\xi)$ is the Euler class of ξ , and $[L]$ is the fundamental class of L .

(ii) From Lemma 2 it follows that f^*c_{2k+1} is of order two. Hence we have $\langle c_{2k+1}, \mathbf{a} \rangle = \langle c_{2k+1}, f_*[L] \rangle = \langle f^*c_{2k+1}, [L] \rangle = 0$. This completes the proof.

COROLLARY 3. Let L be an orientable compact smooth n -manifold. If L admits a lagrangian imbedding into C^n , then we have

$$\chi(L) = 0.$$

PROOF. This follows immediately from Proposition 2 (i).

Finally we remark on the existence of a non-orientable compact lagrangian submanifold of C^n . (See also the remark at the end of §3.)

Let $S^{2n-1} \subset C^n$ be the standard unit sphere in C^n , and let $\pi: S^{2n-1} \rightarrow CP(n-1)$ be the natural projection to the complex projective space. Then we have the next proposition which is also stated implicitly in [11]. For the proof, see [11, Lecture 3].

PROPOSITION 3. *Let L be an $(n-1)$ -dimensional submanifold of $CP(n-1)$. Then L is a lagrangian submanifold of $CP(n-1)$ if and only if $\pi^{-1}(L)$ is a lagrangian submanifold of C^n .*

Since $RP(n-1)$ naturally imbedded in $CP(n-1)$ is a lagrangian submanifold of $CP(n-1)$, it follows from Proposition 3 that $\tilde{L}_n = \pi^{-1}(RP(n-1))$ is a compact lagrangian submanifold of C^n . \tilde{L}_n is the total space of a non-trivial principal S^1 -bundle over $RP(n-1)$. Since $RP(n-1)$ is non-orientable for n odd, \tilde{L}_n is non-orientable for n odd as well. (If n is even, then $\tilde{L}_n = S^{n-1} \times S^1$.)

§ 3. Proof of Theorem 1.

Suppose now that a compact orientable smooth n -manifold M admits an immersion $f: M \rightarrow \mathbf{R}^{n+1}$. Without loss of generality we may assume that the immersion f is in general position (cf. [9, Theorem 4.6]). Let $N \subset M \times \mathbf{R}^{n+1}$ be defined by

$$N = \{(x, v); x \in M, v \text{ perpendicular to } f_*\tau(M)_x\}$$

and let $E: N \rightarrow \mathbf{R}^{n+1}$ be the map defined by

$$E(x, v) = f(x) + v \quad \text{for } x \in M, v \in \mathbf{R}^{n+1}.$$

N is the total space of the normal bundle of f and E is the exponential map of f . Now we choose a smooth field of normal vectors $e: M \rightarrow N$ and put $e(x) = (x, e_x)$. Then we can identify N with $M \times \mathbf{R}$ by the map sending (x, te_x) to (x, t) and E can be considered a map from $M \times \mathbf{R}$ into \mathbf{R}^{n+1} written in the form

$$E(x, t) = f(x) + te_x \quad \text{for } x \in M, t \in \mathbf{R}.$$

Considering \mathbf{R}^{n+1} the real plane of C^{n+1} , we next define the map $E^c: (M \times \mathbf{R}) \times \mathbf{R}^{n+1} \rightarrow C^{n+1}$ to be a natural extension of E :

$$E^c(x, t, y) = E(x, t) + \sqrt{-1}y \quad \text{for } x \in M, t \in \mathbf{R}, y \in \mathbf{R}^{n+1}.$$

Here and for the rest of § 3, the symbol $\sqrt{-1}$ denotes the imaginary unit, and $\sqrt{-1}$ also denotes the canonical complex structure on C^{n+1} .

Now we denote by D_ε^{n+2} the ε -neighborhood of $0 \in \mathbf{R} \times \mathbf{R}^{n+1}$. We choose $\varepsilon_0 > 0$ sufficiently small so that the restriction of E^c to $M \times D_{\varepsilon_0}^{n+2}$ gives rise to an immersion. From now on, we use the notation E_0^c for $E^c|_{M \times D_{\varepsilon_0}^{n+2}}$. Since E_0^c is locally a diffeomorphism, we can define a symplectic structure ω_1 on $M \times D_{\varepsilon_0}^{n+2}$ by $\omega_1 = (E_0^c)^*\omega_0$, where ω_0 is the standard symplectic structure on C^{n+1} . Thus we can make $M \times D_{\varepsilon_0}^{n+2}$ into a symplectic manifold so that E_0^c becomes locally a symplectic diffeomorphism. Henceforth we regard $M \times D_{\varepsilon_0}^{n+2}$ as a symplectic manifold with symplectic structure ω_1 .

Next we define an imbedding $g: M \times C \rightarrow (M \times \mathbf{R}) \times \mathbf{R}^{n+1}$ to be the map de-

finned by

$$g(x, a + \sqrt{-1}b) = (x, a, be_x),$$

or equivalently by

$$(\pi_M \circ g)(x, a + \sqrt{-1}b) = x,$$

$$(E^c \circ g)(x, a + \sqrt{-1}b) = f(x) + (a + \sqrt{-1}b)e_x,$$

where π_M denotes the natural projection from $M \times \mathbf{R} \times \mathbf{R}^{n+1}$ to M . Let $S_\varepsilon^1 \subset \mathbf{C}$ be the ε -circle around 0 and let

$$g' : M \times S_\varepsilon^1 \longrightarrow M \times \mathbf{R} \times \mathbf{R}^{n+1}$$

be the restriction of g to $M \times S_\varepsilon^1$. Clearly we have

$$(3.1) \quad g'(x, \varepsilon e^{i\theta}) = (x, \varepsilon \cos \theta, \varepsilon(\sin \theta)e_x),$$

$$(3.2) \quad (E^c \circ g')(x, \varepsilon e^{i\theta}) = f(x) + e^{i\theta}e_x.$$

Here $e^{i\theta}$ denotes $\cos \theta + \sqrt{-1} \sin \theta$. For $\varepsilon < \varepsilon_0$, $g'(M \times S_\varepsilon^1)$ is contained in $M \times D_{\varepsilon_0}^{n+2}$, and hence $E_0^c \circ g'$ gives rise to an immersion of $M \times S^1$ into \mathbf{C}^{n+1} . From now on we fix ε so as to satisfy $\varepsilon < \varepsilon_0$. Now we have the following.

LEMMA 3. *The immersion $E_0^c \circ g' : M \times S_\varepsilon^1 \rightarrow \mathbf{C}^{n+1}$ is a lagrangian immersion.*

PROOF. First we define two functions $\lambda, \mu : M \times D_{\varepsilon_0}^{n+2} \rightarrow \mathbf{R}$ by the identity

$$(3.3) \quad p = (x(p), \lambda(p), \mu(p)e_{x(p)} + v(p)),$$

where $v(p) \in f_*\tau(M)_{x(p)}$. Above formula implies that

$$E_0^c(p) = f(x(p)) + \bar{v}(p) + \{\lambda(p) + \sqrt{-1}\mu(p)\}e_{x(p)},$$

where $x(p) = \pi_M(p)$ and $\bar{v}(p) \in \sqrt{-1}f_*\tau(M)_{x(p)}$. In terms of these functions λ, μ we define a function $h : M \times D_{\varepsilon_0}^{n+2} \rightarrow \mathbf{R}$ by

$$h = \frac{1}{2}(\lambda^2 + \mu^2),$$

and consider the Hamiltonian vector field X_h associated with the Hamiltonian h .

We shall prove that each $g'(\{x\} \times S_\varepsilon^1)$ is a trajectory of X_h . For any point $p_0 \in g'(M \times S_\varepsilon^1)$ we can choose a neighborhood U of p_0 in such a way that the restriction of E_0^c to U

$$E_0^c|_U : U \longrightarrow \mathbf{C}^{n+1}$$

gives rise to a diffeomorphism (hence a symplectic diffeomorphism) onto its image $\bar{U} \subset \mathbf{C}^{n+1}$. Taking a smaller U , if necessary, we can choose local coordinates $(x^1, \dots, x^n, \lambda, y^1, \dots, y^n, \mu)$ on U which satisfy the following conditions:

- (a) (x^1, \dots, x^n) are local coordinates of M at $x_0 = \pi_M(p_0)$,

$$(b) \quad \begin{aligned} \bar{p} = & \left(x(\bar{p}), \lambda(\bar{p}), \sum_{i=1}^n y^i(\bar{p}) f_* \left(\frac{\partial}{\partial x^i} \right)_{x(\bar{p})} + \mu(\bar{p}) e_{x(\bar{p})} \right), \\ (\text{or } E_{\bar{U}}^{\varepsilon}(\bar{p}) = & f(x(\bar{p})) + \sum_{i=1}^n y^i(\bar{p}) \sqrt{-1} f_* \left(\frac{\partial}{\partial x^i} \right)_{x(\bar{p})} + \{ \lambda(\bar{p}) + \sqrt{-1} \mu(\bar{p}) \} e_x, \\ & x(\bar{p}) = \pi_M(\bar{p})). \end{aligned}$$

To simplify the argument, we consider the vector field $(E_{\bar{U}}^{\varepsilon})_* X_{\bar{h}}$ instead of $X_{\bar{h}}$. Clearly we have a local coordinates $(\bar{x}^1, \dots, \bar{x}^n, \bar{\lambda}, \bar{y}^1, \dots, \bar{y}^n, \bar{\mu})$ on \bar{U} defined by

$$\bar{x}^i = x^i \circ (E_{\bar{U}}^{\varepsilon})^{-1}, \quad \bar{\lambda} = \lambda \circ (E_{\bar{U}}^{\varepsilon})^{-1}, \quad \bar{y}^i = y^i \circ (E_{\bar{U}}^{\varepsilon})^{-1}, \quad \bar{\mu} = \mu \circ (E_{\bar{U}}^{\varepsilon})^{-1}.$$

Let us define \bar{h} by $\bar{h} = h \circ (E_{\bar{U}}^{\varepsilon})^{-1}$. Since $E_{\bar{U}}^{\varepsilon}$ is a symplectic diffeomorphism, we have

$$(3.4) \quad (E_{\bar{U}}^{\varepsilon})_* X_{\bar{h}} = X_{\bar{h}},$$

where $X_{\bar{h}}$ is the Hamiltonian vector field with Hamiltonian \bar{h} on \bar{U} . Here we recall that the Hamiltonian vector field $X_{\bar{h}}$ is related to the gradient vector field of \bar{h} $grad \bar{h}$ by the formula

$$(3.5) \quad X_{\bar{h}} = \sqrt{-1} grad \bar{h},$$

while $grad \bar{h}$ is defined by the condition

$$\langle grad \bar{h}, X \rangle = X(\bar{h}) \quad \text{for every vector field } X \text{ on } \bar{U},$$

where \langle, \rangle is the standard Riemannian metric on \mathbf{C}^{n+1} . Using the definition of $grad \bar{h}$, we have

$$(3.6) \quad \left\{ \begin{aligned} \left\langle (grad \bar{h})_{\bar{p}}, \left(\frac{\partial}{\partial \bar{x}^i} \right)_{\bar{p}} \right\rangle &= \left(\frac{\partial \bar{h}}{\partial \bar{x}^i} \right)_{\bar{p}} = 0, \\ \left\langle (grad \bar{h})_{\bar{p}}, \left(\frac{\partial}{\partial \bar{y}^i} \right)_{\bar{p}} \right\rangle &= \left(\frac{\partial \bar{h}}{\partial \bar{y}^i} \right)_{\bar{p}} = 0, \\ \left\langle (grad \bar{h})_{\bar{p}}, e_{x(\bar{p})} \right\rangle &= \left\langle (grad \bar{h})_{\bar{p}}, \left(\frac{\partial}{\partial \bar{\lambda}} \right)_{\bar{p}} \right\rangle = \left(\frac{\partial \bar{h}}{\partial \bar{\lambda}} \right)_{\bar{p}} = \bar{\lambda}(\bar{p}), \\ \left\langle (grad \bar{h})_{\bar{p}}, \sqrt{-1} e_{x(\bar{p})} \right\rangle &= \left\langle (grad \bar{h})_{\bar{p}}, \left(\frac{\partial}{\partial \bar{\mu}} \right)_{\bar{p}} \right\rangle = \left(\frac{\partial \bar{h}}{\partial \bar{\mu}} \right)_{\bar{p}} = \bar{\mu}(\bar{p}), \end{aligned} \right.$$

where $\bar{p} = E_{\bar{U}}^{\varepsilon}(p)$. These formulas imply that $(grad \bar{h})_{\bar{p}}$ is orthogonal to the subspace spanned by the vectors

$$\left(\frac{\partial}{\partial \bar{x}^1} \right)_{\bar{p}}, \dots, \left(\frac{\partial}{\partial \bar{x}^n} \right)_{\bar{p}}, \left(\frac{\partial}{\partial \bar{y}^1} \right)_{\bar{p}}, \dots, \left(\frac{\partial}{\partial \bar{y}^n} \right)_{\bar{p}}.$$

Next we assume that the point $\bar{p} \in \bar{U}$ satisfies the condition $\bar{y}^1(\bar{p}) = \dots = \bar{y}^n(\bar{p}) = 0$, that is, \bar{p} is in the image of $E_{\bar{U}}^{\varepsilon} \circ g$. Then it is easy to see that the

subspace spanned by the vectors $(\partial/\partial\bar{x}^1)_{\bar{p}}, \dots, (\partial/\partial\bar{x}^n)_{\bar{p}}, (\partial/\partial\bar{y}^1)_{\bar{p}}, \dots, (\partial/\partial\bar{y}^n)_{\bar{p}}$ is orthogonal to the vectors $e_{x(p)}$ and $\sqrt{-1}e_{x(p)}$. Therefore by (3.6) we have

$$(\text{grad } \bar{h})_{\bar{p}} = \bar{\lambda}(\bar{p})e_{x(p)} + \bar{\mu}(\bar{p})\sqrt{-1}e_{x(p)},$$

hence by (3.5)

$$(3.7) \quad X_{\bar{h}}(\bar{p}) = -\bar{\mu}(\bar{p})e_{x(p)} + \bar{\lambda}(\bar{p})\sqrt{-1}e_{x(p)}.$$

Let $p = g'(x, \varepsilon e^{it})$ be any point of $g'(M \times S^1_\varepsilon)$. Then by (3.1), (3.3), (3.4) and (3.7) we have

$$\begin{aligned} (E^c_0)_* X_h(p) &= X_{\bar{h}}(\bar{p}) = \{-\bar{\mu}(\bar{p}) + \sqrt{-1}\bar{\lambda}(\bar{p})\} e_{x(p)} = \{-\mu(p) + \sqrt{-1}\lambda(p)\} e_{x(p)} \\ &= \varepsilon(-\sin t + \sqrt{-1}\cos t)e_{x(p)}. \end{aligned}$$

Comparing this formula with (3.2), we see that

$$(3.8) \quad H_t(g'(x, \varepsilon e^{i\theta})) = g'(x, \varepsilon e^{i(\theta+t)}),$$

where H_t denotes the time t integral of X_h .

Finally we show that the subspace $(E^c_0)_*\tau(g'(M \times S^1_\varepsilon))_p$ of \mathbf{C}^{n+1} is lagrangian for every point $p \in g'(M \times S^1_\varepsilon)$. First we consider the case $p = g'(x, \varepsilon)$. Then $(E^c_0)_*\tau(g'(M \times S^1_\varepsilon))_p$ is spanned by the vectors in $f_*\tau(M)_x$ and $\sqrt{-1}e_x$. Clearly $f_*\tau(M)_x$ and $\sqrt{-1}e_x$ are complex orthogonal. Hence $(E^c_0)_*\tau(g'(M \times S^1_\varepsilon))_p$ is lagrangian. We next consider the case $p = g'(x, \varepsilon e^{it})$ is an arbitrary point of $g'(M \times S^1_\varepsilon)$. From (3.8) it follows that $p = H_t(g'(x, \varepsilon))$ and that H_t leaves $g'(M \times S^1_\varepsilon)$ invariant. Hence we have $(E^c_0)_*\tau(g'(M \times S^1_\varepsilon))_p = (E^c_0)_*(H_t)_*\tau(g'(M \times S^1_\varepsilon))_{g'(x, \varepsilon)}$. On the other hand H_t is known to preserve the symplectic structure. Hence $(E^c_0)_*\tau(g'(M \times S^1_\varepsilon))_p$ is lagrangian. This completes the proof of Lemma 3.

Next we shall construct a lagrangian imbedding of $M \times S^1$ into \mathbf{C}^{n+1} which is close to the lagrangian immersion $E^c_0 \circ g'$ just obtained. Though we can not remove the self-intersections of a given lagrangian immersion in general (cf. [10], [11] and Theorem 2), this can be done easily in our case. In fact, replacing g' by $\tilde{g}' = F \circ g'$ with an appropriate symplectic diffeomorphism F of $M \times D^{n+2}_\varepsilon$, we obtain a lagrangian imbedding $E^c_0 \circ \tilde{g}'$ of $M \times S^1$ into \mathbf{C}^{n+1} as follows.

First consider the Hamiltonian vector field X_λ with Hamiltonian λ on the symplectic manifold $M \times D^{n+2}_\varepsilon$. Then, if $t_0 > 0$ is small enough, we can choose an open neighborhood $W \subset M \times D^{n+2}_\varepsilon$ of $g(M \times D^2_\varepsilon)$ in such a way that the time t integral $F_t : W \rightarrow M \times D^{n+2}_\varepsilon$ of X_λ gives rise to a symplectic diffeomorphism onto an open set for $|t| < t_0$. Since f is in general position, we have $e_x \neq e_y$ if $x \neq y$ and $f(x) = f(y)$. This implies the injectivity of $(E^c_0 \circ F_t)|_M : M \rightarrow \mathbf{C}^{n+1}$, since $(E^c_0 \circ F_t)|_M$ is explicitly given by $E^c_0 \circ F_t(x, 0, 0) = f(x) + t\sqrt{-1}e_x$. Replacing ε

by a smaller one, if necessary, $(E_0^i \circ F_i) |_{g'(M \times S^1)}$ is also injective; hence $(E_0^i \circ F_i \circ g') : M \times S^1 \rightarrow \mathbb{C}^{n+1}$ is a lagrangian imbedding. This completes the proof.

REMARK 2. Let M be a non-orientable compact n -manifold which admits an immersion $f : M \rightarrow \mathbb{R}^{n+1}$ in general position. Then we can construct a lagrangian immersion of the total space of a certain principal S^1 -bundle over M in the same way as in the proof of Theorem 1. If there exists an orientable neighborhood $U \subset M$ of the inverse image of the self-intersections of f , then our method of removing the self-intersections applies to this case also, and hence we obtain a non-orientable compact lagrangian submanifold of \mathbb{C}^{n+1} . For example, it can be shown that the product of the Klein's bottle and S^1 admits a lagrangian imbedding into \mathbb{C}^3 .

§ 4. Proof of Theorem 2.

To begin with, we shall introduce the notion of a weakly lagrangian immersion.

Suppose given a vector bundle ξ over a topological space M and two sub-bundles η_0, η_1 of ξ . η_0 and η_1 are said to be homotopic, if there exists a sub-bundle $\tilde{\eta}$ of $\xi \times I$ such that $\tilde{\eta}|_{M \times 0} = \eta_0$ and $\tilde{\eta}|_{M \times 1} = \eta_1$.

DEFINITION 1. Let ξ be a symplectic vector bundle with symplectic structure ω and let η be a sub-bundle of ξ . We call η a *weakly lagrangian sub-bundle*, if η is homotopic to a lagrangian sub-bundle of ξ .

Let $f : L \rightarrow M$ be an immersion of a smooth manifold L into a symplectic manifold M with symplectic structure ω . We call f a *weakly lagrangian immersion*, if $\tau(L)$ is a weakly lagrangian sub-bundle of the symplectic vector bundle $f^*\tau(M)$ with symplectic structure $f^*\omega$.

Now we are getting down to the proof of Theorem 2. To prove Theorem 2 we shall determine which S^n admits a weakly lagrangian imbedding into \mathbb{C}^n .

As we can see easily, the condition that an immersion f is weakly lagrangian depends only on the regular homotopy class of f . Furthermore we have $\tau(L) = (-1)^{n(n-1)/2} \nu(f)$ for a weakly lagrangian immersion f of L , just as in the case of a lagrangian immersion. On the other hand, we recall that every imbedding of S^n into \mathbb{C}^n is regularly homotopic to the standard one (cf. [7]). Thus we conclude that in case there exists a weakly lagrangian imbedding of S^n into \mathbb{C}^n , the standard imbedding ι is weakly lagrangian, and hence satisfies the condition $\tau(S^n) = (-1)^{n(n-1)/2} \nu(\iota)$. Clearly the normal bundle of the standard imbedding of S^n into \mathbb{C}^n is trivial. Thus in order that there exists a weakly lagrangian imbedding of S^n into \mathbb{C}^n , $\tau(S^n)$ should be trivial and eventually $n \neq 1, 3, 7$ is not in the case. Here making use of the homotopy theory, we reach the following conclusion.

PROPOSITION 4. S^n admits a weakly lagrangian imbedding into C^n if and only if $n=1, 3$.

Let $S^n = \{(x^1, \dots, x^{n+1}) \in R^{n+1}; (x^1)^2 + \dots + (x^{n+1})^2 = 1\}$. Then the standard imbedding $\iota: S^n \rightarrow C^n$ is given by $\iota(x^1, \dots, x^{n+1}) = (x^1, \dots, x^{n+1}, 0, \dots, 0)$. Let $f_n: S^n \rightarrow G_{2n, n}$ denote the Gauss map of ι which carries $x \in S^n$ to the tangent plane of $\iota(S^n)$ at $\iota(x)$, where $G_{2n, n}$ is the Grassmann manifold consisting of all n -planes in C^n ; and let $[f_n] \in \pi_n(G_{2n, n})$ denote the homotopy class represented by f_n . According to Arnold [1], we use the notation $A(n)$ for the lagrangian Grassmann manifold consisting of all lagrangian subspaces of C^n . Let $i_n: A(n) \rightarrow G_{2n, n}$ be the natural inclusion and let $(i_n)_*: \pi_n(A(n)) \rightarrow \pi_n(G_{2n, n})$ be the homomorphism induced by i_n . Now the proof of Proposition 4 is based on the following lemma.

LEMMA 4. S^n admits a weakly lagrangian imbedding into C^n if and only if $[f_n] \in \text{Image } (i_n)_*$.

PROOF. Using the fact that S^n admits a weakly lagrangian imbedding if and only if ι is weakly lagrangian, the proof is immediate from Definition 1.

Now our problem is reduced to determining the homomorphism $(i_n)_*: \pi_n(A(n)) \rightarrow \pi_n(G_{2n, n})$.

Here we note the following. The lagrangian Grassmann manifold $A(n)$ is identified with the coset space $U(n)/O(n)$ by means of the correspondence $U\lambda_R \rightarrow [U]$, where λ_R is the real plane of C^n and $[U]$ is the coset of $U \in U(n)$ (cf. [1]). On the other hand, the Grassmann manifold $G_{2n, n}$ is identified with $O(2n)/O(n) \times O(n)$ by means of the correspondence $O\lambda_R \rightarrow [O]$, where $O \in O(2n)$. Under these identifications $i_n: A(n) \rightarrow G_{2n, n}$ takes the form

$$i_n([U]) = [j_n(U)] = \left[\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \right],$$

where $j_n: U(n) \rightarrow O(2n)$ is the natural inclusion and

$$A = (U + \bar{U})/2, \quad B = (U - \bar{U})/2\sqrt{-1}.$$

PROOF OF PROPOSITION 4. Now the proof will be divided into four steps.

Let $p_n: U(n) \rightarrow A(n)$ be the natural projection.

STEP 1. $(p_n)_*: \pi_n(U(n)) \rightarrow \pi_n(A(n))$ is surjective, if $n=3, 7$.

$$\pi_3(A(3)) \cong Z_4, \quad \pi_7(A(7)) \cong Z_2.$$

PROOF. Consider the diagram

$$\begin{array}{ccccccc}
 \pi_n(O(n)) & \longrightarrow & \pi_n(U(n)) & \xrightarrow{(p_n)_*} & \pi_n(A(n)) & \longrightarrow & \pi_{n-1}(O(n)) \\
 \downarrow & & & & & & \\
 \pi_n(O(n+2)) & \longrightarrow & \pi_n(U(n+2)) & \longrightarrow & \pi_n(A(n+2)) & \longrightarrow & \pi_{n-1}(O(n+2)) \\
 \downarrow & & & & & & \\
 \pi_n(V_{n+2,2}) & & & & & & \\
 \downarrow & & & & & & \\
 \pi_{n-1}(O(n)) & & & & & &
 \end{array}$$

where the horizontal lines and the vertical line are parts of the homotopy exact sequences of the natural fibrations. Let A and O be the direct limits as $n \rightarrow \infty$ of $A(n)$ and $O(n)$, with the direct limit topology respectively. Then it is easily checked that $\pi_i(A(n)) \cong \pi_i(A)$ for $n \geq i+1$. Furthermore we have $\pi_i(A) \cong \pi_{i+6}(O)$ according to [6, §24]. Then the facts $\pi_{n-1}(O(n))=0$ for $n=3, 7$ imply the surjectivity of $(p_n)_*$, and the facts $\pi_n(O(n)) \cong \mathbf{Z}$ for $n=3, 7$, $\pi_n(V_{n+k,k}) \cong \mathbf{Z}_2$ for n odd, $k \geq 2$ and the informations on $\pi_i(O)$ yield the latter part of Step 1.

STEP 2. $\pi_3(G_{6,3}) \cong \pi_3(V_{6,3}) \cong \mathbf{Z}_2$, $\pi_7(G_{14,7}) \cong \pi_7(V_{14,7}) \cong \mathbf{Z}_2$.

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & \pi_n(O(n)) & & & & \\
 & & \downarrow j_* & & & & \\
 & & \pi_n(O(2n)) & & & & \\
 h_* \nearrow & & \downarrow (p'_n)_* & & & & \\
 \pi_n(O(n)) & \longrightarrow & \pi_n(V_{2n,n}) & \longrightarrow & \pi_n(G_{2n,n}) & \longrightarrow & 0,
 \end{array}$$

where the horizontal line and the vertical line are parts of the homotopy exact sequences of the natural fibrations. The homomorphisms $h, j: O(n) \rightarrow O(2n)$ are defined by $h(A) = \begin{pmatrix} A & O \\ O & I \end{pmatrix}$, $j(A) = \begin{pmatrix} I & O \\ O & A \end{pmatrix}$ respectively. Since h and j are conjugate in $O(2n)$, they are homotopic to each other and hence $h_* = j_*$. Now using the fact that $\pi_n(V_{n+k,k}) \cong \mathbf{Z}_2$ ($k \geq 2, n$ odd) we obtain the required results.

STEP 3. $(i_3)_*: \pi_3(A(3)) \rightarrow \pi_3(G_{6,3})$ is surjective.

$(i_7)_*: \pi_7(A(7)) \rightarrow \pi_7(G_{14,7})$ is zero.

PROOF. Consider the diagram

$$\begin{array}{ccccccc}
 \pi_n(U(n)) & \xrightarrow{(j_n)_*} & \pi_n(O(2n)) & \longrightarrow & \pi_n(O(2n)/U(n)) & \longrightarrow & 0 \\
 \downarrow & & \downarrow (p'_n)_* & & & & \\
 & & (p_n)_* & \pi_n(V_{2n,n}) & & & \\
 \downarrow & & (i_n)_* & \downarrow (p''_n)_* & & & \\
 \pi_n(A(n)) & \longrightarrow & \pi_n(G_{2n,n}) & & & & \\
 \downarrow & & & & & & \\
 0 & & & & & &
 \end{array}$$

where the upper horizontal line is a part of the homotopy exact sequence of the natural fibration. According to Step 2, $(p''_n)_*$ is bijective, and Step 1 and the fact that $\pi_{n-1}(O(n))=0$ for $n=3, 7$ imply that $(p_n)_*$ and $(p'_n)_*$ are surjective. Hence the generators of $\pi_n(U(n))$ and $\pi_n(O(n))$ determine those of $\pi_n(A(n))$ and $\pi_n(G_{2n,n})$ respectively. Since we know that $\pi_3(O(6)/U(3))=0$, $\pi_7(O(14)/U(7))\cong \mathbf{Z}_2$, using the knowledge of stable homotopy groups (cf. [6]), we can conclude that $(j_3)_*$ is an isomorphism and $(j_7)_*$ sends a generator of $\pi_7(U(7))$ into twice a generator or $\pi_7(O(14))$. This implies the assertion of Step 3.

STEP 4. $[f_n]$ is the generator of $\pi_n(G_{2n,n})$ for $n=3, 7$.

PROOF. Let $\iota': S^n \rightarrow \mathbf{R}^{n+k}$ be the standard imbedding. If n is odd and $k \geq 2$, the generator of $\pi_n(V_{n+k,k}) \cong \mathbf{Z}_2$ is represented by a map $\tilde{g}: S^n \rightarrow V_{n+k,k}$ which assigns to a point $x \in S^n$ the orthogonal k -frame consisting of the vector $\iota'(x)$ followed by the i -th basis vectors e_i with $n+2 \leq i \leq n+k$ (cf. [8, §25]). Here applying Step 2, the generator of $\pi_n(G_{2n,n})$ is represented by a map $g_n: S^n \rightarrow G_{2n,n}$ which assigns to a point $x \in S^n$ the normal plane of $\iota(S^n)$ at $\iota(x)$. Then the duality of Grassmann manifolds $G_{n+k,n} \cong G_{n+k,k}$ shows that f_n also represents the generator of $\pi_n(G_{2n,n})$.

Proposition 4 clearly follows from Steps 1, 2, 3 and 4.

Now it would be clear that Theorem 2 is a direct consequence of Proposition 4.

References

- [1] V.I. Arnold, Characteristic class entering in quantization conditions (translation), *Functional Anal. Appl.*, 1 (1967), 1-13.
- [2] V.I. Arnold and A. Avez, *Problemes ergodiques de la mecanique classique*, Gauthier-Villars, Paris, 1967.
- [3] M.L. Gromov, A topological technique for the construction of solutions of differential equations and inequalities, *Actes, Congres Intern. Math.*, 1970, vol. 2, 221-225.
- [4] M. Hirsch, Immersions of manifolds, *Trans. Amer. Math. Soc.*, 93 (1959), 242-276.
- [5] J.A. Lees, On the classification of lagrange immersions, *Duke Math. J.*, 43 (1976), 217-224.
- [6] J. Milnor, *Morse Theory*, Ann. of Math. Studies, No. 51, Princeton Univ. Press, Princeton.
- [7] S. Smale, The classification of immersions of spheres in euclidean spaces, *Ann. of Math.*, 69 (1959), 327-344.
- [8] N.E. Steenrod, *The Topology of Fibre Bundles*, Princeton Univ. Press, Princeton, 1951.
- [9] S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.
- [10] A. Weinstein, Lagrangian submanifolds and Hamiltonian systems, *Ann. of Math.*, 98 (1973), 377-410.

- [11] A. Weinstein, Lectures on symplectic manifolds, Regional Conference Series in Mathematics, vol. 29, Amer. Math. Soc., Providence, R.I., 1977.

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