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On generalized Siegel domains with exponent $(c_1, c_2, \cdots, c_s)$

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Introduction.

In this paper, extending the notion of generalized Siegel domains in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ with exponent c due to Kaup, Matsushima and Ochiai [5], we introduce the notion of generalized Siegel domains in $\boldsymbol{C}^n\times\boldsymbol{C}^{m_1}\times\boldsymbol{C}^{m_2}\times\cdots\times\boldsymbol{C}^{m_s}$ with exponent $(c_1, c_2, \cdots, c_s)$ and give a generalization of results obtained by Vey [**12**] and Sudo [**11**] for these domains. Many interesting domains are contained in the category of our domains. As a typical example of generalized Siegel domains with exponent $(c_1, c_2, \cdots, c_s)$, we present here the following domain

$$\mathcal{D}=\{(z, w_1, w_2, \cdots, w_s)\in\boldsymbol{C}\times\boldsymbol{C}^s\,|\,\mathrm{Im}.\,z-|w_1|^{1/c_1}-|w_2|^{1/c_2}-\cdots-|w_s|^{1/c_s}>0\}$$

which is a canonical unbounded model of the generalized Thullen domain (cf. [4])

$$D=\{(z, w_1, w_2, \cdots, w_s)\in\boldsymbol{C}^{s+1}\,|\,|z|^2+|w_1|^{1/c_1}+|w_2|^{1/c_2}+\cdots+|w_s|^{1/c_s}<1\}$$

where c_i are all positive real numbers. For examples of our domains, see section 4.

Now, for a domain D in $\boldsymbol{C}^N$ we denote by $\mathrm{Aut}(D)$ the group of all biholomorphic transformations of D onto itself. According to Vey [**12**], we say that D is a sweepable domain if there exist a subgroup Γ of $\mathrm{Aut}(D)$ and a compact subset K of D such that $\Gamma\cdot K=D$. In [**12**], Vey proved the following result: *A sweepable generalized Siegel domain in* $\boldsymbol{C}^n\times\boldsymbol{C}^m$ *with non-zero exponent* c *is a Siegel domain in the sense of Pjateckii-Sapiro* [**10**]. This result was later generalized by Sudo [**11**] to more general domains, called *generalized S-domains*. Recall that, to use our terminology, a generalized S-domain due to Sudo [**11**] is nothing but a generalized Siegel domain with exponent (c_1, c_2) for some non-zero c_i, $i=1, 2$. Extending these results to our domains, we shall prove the following theorem in section 3:

THEOREM. *A sweepable generalized Siegel domain* $\mathcal{D}$ *in* $\boldsymbol{C}^n\times\boldsymbol{C}^{m_1}\times\boldsymbol{C}^{m_2}\times\cdots\times\boldsymbol{C}^{m_s}$ *with non-zero exponents* c_i $(1\leqq i\leqq s)$ *is a Siegel domain in the sense of Pjateckii-Sapiro* [**10**].

Combining this with a result of Vey [**12**], we obtain the following

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COROLLARY. *Let $\mathcal{D}$ be a generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2} \times \cdots \times C^{m_s}$ with non-zero exponents c_i $(1 \leqq i \leqq s)$. Suppose that $\mathcal{D}$ admits a discrete subgroup Γ of* Aut($\mathcal{D}$) *with compact quotient $\mathcal{D}/\Gamma$. Then $\mathcal{D}$ is symmetric.*

The last fact may be interesting when we recall the following outstanding conjecture on bounded domains (cf. [8], p. 128): *If D is a bounded domain in C^N and if there exists a discrete subgroup Γ of* Aut(D) *such that D/Γ is compact, then D is homogeneous.* So far as the author knows, Vey [12] and Sudo [11] seem to be the only known results concerning this conjecture.

§ 1. Preliminaries.

Let $\boldsymbol{R}$ (resp. $\boldsymbol{C}$) denote the field of real (resp. complex) numbers as usual. We fix a coordinate system

$$(z_1, z_2, \cdots, z_n, w_1^1, w_2^1, \cdots, w_{m_1}^1, \cdots, w_1^s, w_2^s, \cdots, w_{m_s}^s)$$

in $C^n \times C^{m_1} \times C^{m_2} \times \cdots \times C^{m_s}$ once and for all. For the sake of simplicity, putting

$$\begin{cases} z=(z_1, z_2, \cdots, z_n); \\ w_\alpha=(w_1^\alpha, w_2^\alpha, \cdots, w_{m_\alpha}^\alpha) \quad \text{for } \alpha=1, 2, \cdots, s, \end{cases}$$

we shall define a generalized Siegel domain with exponent $(c_1, c_2, \cdots, c_s)$ as follows.

DEFINITION 1. A domain $\mathcal{D}$ in $C^n \times C^{m_1} \times C^{m_2} \times \cdots \times C^{m_s}$ is called a *generalized Siegel domain with exponent* $(c_1, c_2, \cdots, c_s)$ if the following conditions are satisfied:

(1) $\mathcal{D}$ is holomorphically equivalent to a bounded domain in $C^{n+m_1+m_2+\cdots+m_s}$ and $\mathcal{D} \cap (C^n \times \{0\} \times \{0\} \times \cdots \times \{0\}) \neq \emptyset$, where $\{0\} \times \{0\} \times \cdots \times \{0\}$ denotes the origin of $C^{m_1+m_2+\cdots+m_s}$.

(2) $\mathcal{D}$ is invariant by the transformations of $C^{n+m_1+m_2+\cdots+m_s}$ of the following types:

(a) $(z, w_1, w_2, \cdots, w_s) \mapsto (z+a, w_1, w_2, \cdots, w_s)$;

(b) $(z, w_1, \cdots, w_\lambda, \cdots, w_s) \mapsto (z, w_1, \cdots, e^{\sqrt{-1}t} w_\lambda, \cdots, w_s)$ for $\lambda=1, 2, \cdots, s$;

(c) $(z, w_1, \cdots, w_\lambda, \cdots, w_s) \mapsto (e^t z, e^{c_1 t} w_1, \cdots, e^{c_\lambda t} w_\lambda, \cdots, e^{c_s t} w_s)$

for all $a \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}$, where c_λ $(1 \leqq \lambda \leqq s)$ are all real numbers depending only on $\mathcal{D}$. We call $(c_1, c_2, \cdots, c_s)$ the exponent of $\mathcal{D}$.

In the following part of this paper, we denote by $\mathcal{D}$ a generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2} \times \cdots \times C^{m_s}$ with exponent $(c_1, c_2, \cdots, c_s)$. Since $\mathcal{D}$ is holomorphically equivalent to a bounded domain, by a well-known theorem of H. Cartan [2] the group Aut($\mathcal{D}$) has the structure of real Lie group and its Lie algebra is canonically identified with the real Lie algebra $\mathfrak{g}(\mathscr{D})$ consisting of all complete holomorphic vector fields on $\mathscr{D}$.

From the definition, the following holomorphic vector fields on $\mathscr{D}$ are contained in $\mathfrak{g}(\mathscr{D})$:

(a) $\dfrac{\partial}{\partial z_k}$ for $k=1, 2, \cdots, n$;

(b) $\partial^{\lambda}=\sqrt{-1}\sum_{\alpha=1}^{m_\lambda} w_\alpha^\lambda \dfrac{\partial}{\partial w_\alpha^\lambda}$ for $\lambda=1, 2, \cdots, s$;

(c) $\partial=\sum_{k=1}^{n} z_k \dfrac{\partial}{\partial z_k}+\sum_{\lambda=1}^{s} c_\lambda\Big(\sum_{\alpha=1}^{m_\lambda} w_\alpha^\lambda \dfrac{\partial}{\partial w_\alpha^\lambda}\Big)$.

Therefore, by the same way as in [5] we can show the following

THEOREM A. *Every vector field in $\mathfrak{g}(\mathscr{D})$ is a polynomial vector field.*

Thus, every vector field X in $\mathfrak{g}(\mathscr{D})$ is expressed in the following form:

$$X=\sum_{k=1}^{n}\Big(\sum_{\nu\mu} P_{\nu\mu}^{k}\Big)\frac{\partial}{\partial z_k}+\sum_{\lambda=1}^{s}\Big\{\sum_{\alpha=1}^{m_\lambda}\Big(\sum_{\nu\mu} P_{\nu\mu}^{\lambda\alpha}\Big)\frac{\partial}{\partial w_\alpha^\lambda}\Big\}$$

where

$$P_{\nu\mu}^{k}=P_{\nu\mu_1\mu_2\cdots\mu_a\cdots\mu_s}^{k} \quad \text{for } 1\leqq k\leqq n$$

and

$$P_{\nu\mu}^{\lambda\alpha}=P_{\nu\mu_1\mu_2\cdots\mu_a\cdots\mu_s}^{\lambda\alpha} \quad \text{for } 1\leqq \lambda\leqq s$$

are homogeneous polynomials of degree ν in z_l $(1\leqq l\leqq n)$ and μ_a in w_β^a $(1\leqq\beta\leqq m_a)$ for $1\leqq a\leqq s$.

THEOREM B. *We have*

$$P_{\nu\mu_1\mu_2\cdots\mu_s}^{k}=0 \quad \textit{for } \mu_1+\mu_2+\cdots+\mu_s>1$$

and

$$P_{\nu\mu_1\mu_2\cdots\mu_s}^{\lambda\alpha}=0 \quad \textit{for } \mu_1+\mu_2+\cdots+\mu_s>2.$$

PROOF. Taking the vector field $\partial^1+\partial^2+\cdots+\partial^s$ in $\mathfrak{g}(\mathscr{D})$ instead of the vector field ∂' in the proof of Lemma 3.1 in [5], our proof can be done with exactly the same arguments. q.e.d.

For later use, let us fix some notations. Putting $\mu=(\mu_1\mu_2\cdots\mu_s)$ as before, we set

$$\begin{cases} Z_{\nu\mu}=\sum_{k=1}^{n} P_{\nu\mu}^{k}\dfrac{\partial}{\partial z_k}\,; \\ W_{\nu\mu}^{\lambda}=\sum_{\alpha=1}^{m_\lambda} P_{\nu\mu}^{\lambda\alpha}\dfrac{\partial}{\partial w_\alpha^\lambda} \quad \text{for } \lambda=1, 2, \cdots, s. \end{cases}$$

Then, by direct computations we have

$$
(1.1)\qquad \begin{cases} [\partial, Z_{\nu\mu}]=\left(\nu-1+\sum_{a=1}^{s} c_a\mu_a\right)Z_{\nu\mu}\,; \\ [\partial, W^{\lambda}_{\nu\mu}]=\left\{\nu+\sum_{a=1}^{s} c_a(\mu_a-\delta_{a\lambda})\right\}W^{\lambda}_{\nu\mu}\,; \\ [\partial^a, Z_{\nu\mu}]=\sqrt{-1}\mu_a Z_{\nu\mu}\,; \\ [\partial^a, W^{\lambda}_{\nu\mu}]=\sqrt{-1}(\mu_a-\delta_{a\lambda})W^{\lambda}_{\nu\mu} \end{cases}
$$

for $1\leqq a,\ \lambda\leqq s$, where $\delta_{ij}=\begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$

Finally we recall the following basic fact due to H. Cartan [**2**]:

$$
(1.2)\qquad \mathfrak{g}(\mathcal{D})\cap\sqrt{-1}\,\mathfrak{g}(\mathcal{D})=\{0\}.
$$

§ 2. Vector fields which are independent of $z_1, z_2, \cdots, z_n$.

In this section we study holomorphic vector fields in $\mathfrak{g}(\mathcal{D})$ which are independent of $z_1, z_2, \cdots, z_n$.

From now on, the following ranges of indices will be used throughout this paper.

$$
1\leqq i,\ j,\ k, \cdots \leqq n\,,
$$
$$
1\leqq \alpha,\ \beta,\ \gamma, \cdots \leqq s\,.
$$

Denoting by $Z_{\nu\mu_1\mu_2\cdots\mu_s}$ and $W^{\lambda}_{\nu\mu_1\mu_2\cdots\mu_s}$ the polynomial vector fields on $\mathcal{D}$ defined in section 1, we put

$$
Z_0=Z_{00\cdots0}\,,\quad Z_\lambda=Z_{00\cdots\underset{\lambda+1}{\overset{\wedge}{0}1}0\cdots0}\,,
$$
$$
W^{\lambda}_0=W^{\lambda}_{00\cdots0}\,,\quad W^{\lambda}_\alpha=W^{\lambda}_{00\cdots\underset{\alpha+1}{\overset{\wedge}{0}1}0\cdots0}\,,
$$
$$
W^{\lambda}_{2,\alpha}=W^{\lambda}_{00\cdots\underset{\alpha+1}{\overset{\wedge}{0}2}0\cdots0} \quad\text{and}\quad W^{\lambda}_{\alpha\beta}=W^{\lambda}_{00\cdots\underset{\alpha+1}{\overset{\wedge}{0}1}0\cdots\underset{\beta+1}{\overset{\wedge}{0}1}0\cdots0}\quad(\alpha<\beta)
$$

where $\hat{l}$ means that the numbers 1 or 2 appear at the l-th position. Thus, for example $W^{\lambda}_2=W^{\lambda}_{0010\cdots0}$, $W^{\lambda}_{2,1}=W^{\lambda}_{020\cdots0}$, $W^{\lambda}_{23}=W^{\lambda}_{00110\cdots0}$, $\cdots$.

Now, let X be a vector field in $\mathfrak{g}(\mathcal{D})$ which is independent of $z_1, z_2, \cdots, z_n$. Then, by Theorem B X can be written in the form

$$
X=Z_0+\sum_{\lambda} Z_\lambda+\sum_{\lambda}\left(\sum_{\alpha} W^{\lambda}_{2,\alpha}+\sum_{\alpha<\beta} W^{\lambda}_{\alpha\beta}\right)+\sum_{\alpha,\lambda} W^{\lambda}_{\alpha}+\sum_{\lambda} W^{\lambda}_{0}\,.
$$

Using the bracket relation (1.1), we can show by direct computations the following equalities

$$
(2.1)\qquad (\mathrm{ad}(\partial^1+\cdots+\partial^s))^2\cdot X=-\left\{\sum_{\lambda} Z_\lambda+\sum_{\alpha,\lambda} W^{\lambda}_{2,\alpha}+\sum_{\substack{\lambda\\ \alpha<\beta}} W^{\lambda}_{\alpha\beta}+\sum_{\lambda} W^{\lambda}_{0}\right\},
$$

$$\text{(2.2)}\qquad \mathrm{ad}\,\partial\cdot(\mathrm{ad}(\partial^1+\cdots+\partial^s))^2\cdot X$$
$$=\sum_{\lambda}(1-c_\lambda)Z_\lambda+\sum_{\lambda\neq\alpha}c_\lambda W^\lambda_{2,\alpha}-2\sum_{\lambda\neq\alpha}c_\alpha W^\lambda_{2,\alpha}-\sum_{\lambda}c_\lambda W^\lambda_{2,\lambda}$$
$$-\sum_{\substack{\lambda\neq\alpha\\ \alpha<\beta}}c_\alpha W^\lambda_{\alpha\beta}-\sum_{\substack{\lambda\neq\beta\\ \alpha<\beta}}c_\beta W^\lambda_{\alpha\beta}+\sum_{\substack{\lambda\neq\alpha,\beta\\ \alpha<\beta}}c_\lambda W^\lambda_{\alpha\beta}+\sum_{\lambda}c_\lambda W^\lambda_0,$$

$$\text{(2.3)}\qquad (\mathrm{ad}\,\partial^\nu)^2\cdot X=-\{Z_\nu+4\sum_{\lambda\neq\nu}W^\lambda_{2,\nu}+\sum_{\alpha}W^\nu_{2,\alpha}+\sum_{\substack{\alpha,\beta\neq\nu\\ \alpha<\beta}}W^\nu_{\alpha\beta}$$
$$+\sum_{\substack{\lambda\neq\nu\\ \alpha<\nu}}W^\lambda_{\alpha\nu}+\sum_{\substack{\lambda\neq\nu\\ \nu<\alpha}}W^\lambda_{\nu\alpha}+\sum_{\lambda\neq\nu}W^\lambda_\nu+\sum_{\lambda\neq\nu}W^\nu_\lambda+W^\nu_0\}$$

and

$$\text{(2.4)}\qquad (\mathrm{ad}(\partial^1+\cdots+\partial^s))^2\cdot(\mathrm{ad}\,\partial^\nu)^2\cdot X$$
$$=Z_\nu+4\sum_{\lambda\neq\nu}W^\lambda_{2,\nu}+\sum_{\alpha}W^\nu_{2,\alpha}+\sum_{\substack{\alpha,\beta\neq\nu\\ \alpha<\beta}}W^\nu_{\alpha\beta}+\sum_{\substack{\lambda\neq\nu\\ \alpha<\nu}}W^\lambda_{\alpha\nu}+\sum_{\substack{\lambda\neq\nu\\ \nu<\alpha}}W^\lambda_{\nu\alpha}+W^\nu_0.$$

Therefore we have

$$\text{(2.5)}\qquad Y:=\mathrm{ad}\,\partial\cdot(\mathrm{ad}(\partial^1+\cdots+\partial^s))^2\cdot X-\sum_{\nu}c_\nu(\mathrm{ad}(\partial^1+\cdots+\partial^s))^2\cdot(\mathrm{ad}\,\partial^\nu)^2\cdot X$$
$$=\sum_{\lambda}(1-2c_\lambda)Z_\lambda-2\sum_{\lambda}c_\lambda W^\lambda_{2,\lambda}-6\sum_{\lambda\neq\alpha}c_\alpha W^\lambda_{2,\alpha}$$
$$-2\sum_{\substack{\lambda\neq\alpha,\beta\\ \alpha<\beta}}(c_\alpha+c_\beta)W^\lambda_{\alpha\beta}-2\sum_{\alpha<\beta}c_\alpha W^\beta_{\alpha\beta}-2\sum_{\alpha<\beta}c_\beta W^\alpha_{\alpha\beta}.$$

It follows then by a routine calculation that

$$\text{(2.6)}\qquad \mathrm{ad}(\partial^1+\cdots+\partial^s)\cdot Y=\sqrt{-1}\,Y.$$

Recalling that $\mathfrak{g}(\mathcal{D})\cap\sqrt{-1}\,\mathfrak{g}(\mathcal{D})=\{0\}$ by (1.2), this implies

$$\text{(2.7)}\qquad Y=0.$$

We are now in a position to prove the following lemma, which is essential to the proof of our theorem.

LEMMA 2.1. *Let X be a holomorphic vector field in $\mathfrak{g}(\mathcal{D})$ which is independent of $z_1, z_2, \cdots, z_n$. Suppose that $\mathcal{D}$ satisfies the following two conditions:*

(*) $c\neq 1/2$.

(**) *The exponents c_α $(1\leqq\alpha\leqq s)$ are all different from each other and $c_\alpha\neq 0$ $(1\leqq\alpha\leqq s)$.*

Then X can be written in the form

$$X=Z_0+\sum_{1<\lambda}Z_\lambda+\sum_{\substack{1<\alpha<\beta\\ \lambda\neq\alpha,\beta\\ 1<\lambda}}W^\lambda_{\alpha\beta}+W^1_1+\sum_{1<\alpha,\lambda}W^\lambda_\alpha+\sum_{1<\lambda}W^\lambda_0.$$

PROOF. By (2.7) and our assumptions (*), (**), we conclude that

$$\begin{cases} Z_1,\ W^{\lambda}_{2,\alpha}\ (1\leqq\alpha,\ \lambda\leqq s), \\ W^{\beta}_{\alpha\beta}\ \text{and}\ W^{\alpha}_{\alpha\beta}\ (1\leqq\alpha<\beta\leqq s) \end{cases}$$

are all equal to zero. So that the vector field X is of the form

$$X=Z_0+\sum_{1<\lambda}Z_\lambda+\sum_{\substack{\alpha<\beta\\ \lambda\neq\alpha,\beta}}W^{\lambda}_{\alpha\beta}+\sum_{\lambda,\alpha}W^{\lambda}_{\alpha}+\sum_{\lambda}W^{\lambda}_{0}. \tag{2.8}$$

Direct computations now give the following equalities

$$(\mathrm{ad}(\partial^1+\cdots+\partial^s))^2\cdot(\mathrm{ad}\,\partial^\nu)^2\cdot X=Z_\nu+\sum_{\substack{\alpha<\beta\\ \alpha,\beta\neq\nu}}W^{\nu}_{\alpha\beta}+\sum_{\substack{\lambda\neq\nu\\ \alpha<\nu}}W^{\lambda}_{\alpha\nu}+\sum_{\substack{\lambda\neq\nu\\ \nu<\alpha}}W^{\lambda}_{\nu\alpha}+W^{\nu}_{0}, \tag{2.9}$$

$$\mathrm{ad}\,\partial^\nu\cdot\mathrm{ad}(\partial^1+\cdots+\partial^s)\cdot(\mathrm{ad}\,\partial^\nu)^2\cdot X=Z_\nu-\sum_{\substack{\alpha<\beta\\ \alpha,\beta\neq\nu}}W^{\nu}_{\alpha\beta}+\sum_{\substack{\lambda\neq\nu\\ \alpha<\nu}}W^{\lambda}_{\alpha\nu}+\sum_{\substack{\lambda\neq\nu\\ \nu<\alpha}}W^{\lambda}_{\nu\alpha}+W^{\nu}_{0} \tag{2.10}$$

and hence

$$\begin{cases} \sum_{\substack{\alpha<\beta\\ \alpha,\beta\neq\nu}}W^{\nu}_{\alpha\beta}\in\mathfrak{g}(\mathcal{D}); \\ Z_\nu+\sum_{\substack{\lambda\neq\nu\\ \alpha<\nu}}W^{\lambda}_{\alpha\nu}+\sum_{\substack{\lambda\neq\nu\\ \nu<\alpha}}W^{\lambda}_{\nu\alpha}+W^{\nu}_{0}\in\mathfrak{g}(\mathcal{D}). \end{cases} \tag{2.11}$$

In particular, putting $\nu=1$ in (2.11), we see that the vector fields

$$X_1:=\sum_{1<\alpha<\beta}W^{1}_{\alpha\beta} \tag{2.12}$$

and

$$X_2:=\sum_{1<\alpha,\lambda}W^{\lambda}_{1\alpha}+W^{1}_{0} \tag{2.13}$$

are both contained in $\mathfrak{g}(\mathcal{D})$. Here we assert that

$$X_1=0\quad\text{and}\quad X_2=0. \tag{2.14}$$

First of all, we show the first assertion $X_1=0$. Putting

$$A_\alpha=\sum_{\nu=\alpha+1}^{s}W^{1}_{\alpha\nu}\quad\text{for}\ \alpha=2,3,\cdots,s-1, \tag{2.15}$$

we have $X_1=\sum_{\alpha=2}^{s-1}A_\alpha$. Using induction on α, we shall prove that any $A_\alpha=0$, which assures $X_1=0$. Now, since $(\mathrm{ad}\,\partial^2)^2\cdot X_1=-A_2$, we get

$$\begin{cases} \mathrm{ad}\,\partial^\nu\cdot(\mathrm{ad}\,\partial^2)^2\cdot X_1=-\sqrt{-1}\,W^{1}_{2\nu}; \\ (\mathrm{ad}\,\partial^\nu)^2\cdot(\mathrm{ad}\,\partial^2)^2\cdot X_1=W^{1}_{2\nu} \end{cases} \tag{2.16}$$

for $3\leqq\nu\leqq s$, which implies from (1.2) that $W^{1}_{2\nu}=0$ for $3\leqq\nu\leqq s$ and so $A_2=0$. Suppose that $A_\alpha=0$ for $2\leqq\alpha\leqq\mu$, $2\leqq\mu<s-1$. Then we have

$$(\mathrm{ad}\,\partial^{\mu+1})^2\cdot X_1=-A_{\mu+1} \tag{2.17}$$

and hence

$$(2.18)\qquad \begin{cases} \operatorname{ad}\partial^{\nu}\cdot(\operatorname{ad}\partial^{\mu+1})^2\cdot X_1=-\sqrt{-1}\,W^1_{(\mu+1)\nu}\,; \\ (\operatorname{ad}\partial^{\nu})^2\cdot(\operatorname{ad}\partial^{\mu+1})^2\cdot X_1=W^1_{(\mu+1)\nu} \end{cases}$$

for $\mu+2\leqq\nu\leqq s$, which implies as above that $A_{\mu+1}=0$. Therefore we have shown that $X_1=0$. Next, we shall prove that $X_2=0$. Direct computations now give the following

$$(2.19)\qquad \begin{cases} (\operatorname{ad}\partial^2)^2\cdot X_2=-\left\{\sum_{\alpha=3}^{s} W^{\alpha}_{12}+\sum_{\alpha=3}^{s} W^2_{1\alpha}\right\}; \\ \operatorname{ad}\partial^2\cdot\operatorname{ad}(\partial^1+\cdots+\partial^s)\cdot(\operatorname{ad}\partial^1)^2\cdot X=\sum_{\alpha=3}^{s} W^{\alpha}_{12}-\sum_{\alpha=3}^{s} W^2_{1\alpha}\,. \end{cases}$$

From this we have

$$(2.20)\qquad \sum_{\alpha=3}^{s} W^2_{1\alpha},\ \sum_{\alpha=3}^{s} W^{\alpha}_{12}\in\mathfrak{g}(\mathcal{D})\,.$$

Since

$$\begin{cases} \operatorname{ad}\partial^{\alpha}\cdot\left\{\sum_{\alpha=3}^{s} W^2_{1\alpha}\right\}=\sqrt{-1}\,W^2_{1\alpha}\,; \\ (\operatorname{ad}\partial^{\alpha})^2\left\{\sum_{\alpha=3}^{s} W^2_{1\alpha}\right\}=-W^2_{1\alpha} \end{cases}$$

for $3\leqq\alpha\leqq s$, (2.20) says that

$$(2.21)\qquad W^2_{1\alpha}=0\quad \text{for } 3\leqq\alpha\leqq s,$$

because $\mathfrak{g}(\mathcal{D})\cap\sqrt{-1}\,\mathfrak{g}(\mathcal{D})=\{0\}$ by (1.2). Analogously we obtain the following

$$(2.22)\qquad W^{\alpha}_{12}=0\quad \text{for } 3\leqq\alpha\leqq s.$$

Then it is an easy matter to verify that

$$(2.23)\qquad \begin{cases} (\operatorname{ad}\partial^3)^2\cdot X_2=-\left\{\sum_{\alpha=4}^{s} W^{\alpha}_{13}+\sum_{\alpha=4}^{s} W^3_{1\alpha}\right\}; \\ \operatorname{ad}\partial^3\cdot\operatorname{ad}(\partial^1+\cdots+\partial^s)\cdot(\operatorname{ad}\partial^1)^2\cdot X=\sum_{\alpha=4}^{s} W^{\alpha}_{13}-\sum_{\alpha=4}^{s} W^3_{1\alpha}\,. \end{cases}$$

From this we have

$$(2.24)\qquad \sum_{\alpha=4}^{s} W^3_{1\alpha},\ \sum_{\alpha=4}^{s} W^{\alpha}_{13}\in\mathfrak{g}(\mathcal{D})$$

and hence

$$(2.25)\qquad W^3_{1\alpha}=0\quad\text{and}\quad W^{\alpha}_{13}=0\quad \text{for } 4\leqq\alpha\leqq s$$

by a similar argument as above. By continuing these calculations, we finally arrive at the following result

$$W^{\lambda}_{1\alpha}=0 \quad \text{for } 1<\alpha,\ \lambda. \tag{2.26}$$

It remains to show that $W^1_0=0$, but this is now obvious, since W^1_0 and $\sqrt{-1}\,W^1_0 = -\mathrm{ad}\,\partial^1\cdot W^1_0$ are both contained in $\mathfrak{g}(\mathcal{D})$ and $\mathfrak{g}(\mathcal{D})\cap\sqrt{-1}\,\mathfrak{g}(\mathcal{D})=\{0\}$ by (1.2). Therefore we have shown the second assertion $X_2=0$, and thus we obtain (2.14).

The vector field X is now of the form

$$X=Z_0+\sum_{1<\lambda}Z_\lambda+\sum_{\substack{\lambda\neq\alpha,\beta\\1<\alpha<\beta\\1<\lambda}}W^{\lambda}_{\alpha\beta}+\sum_{\alpha,\lambda}W^{\lambda}_{\alpha}+\sum_{1<\lambda}W^{\lambda}_{0}. \tag{2.27}$$

Then, by routine calculations we have

$$\begin{cases}\mathrm{ad}\,\partial\cdot(\mathrm{ad}\,\partial^1)^2\cdot X=\sum_{1<\alpha}(c_1-c_\alpha)W^1_\alpha-\sum_{1<\alpha}(c_1-c_\alpha)W^\alpha_1;\\ \mathrm{ad}\,\partial^\alpha\cdot\mathrm{ad}\,\partial^1\cdot X=W^1_\alpha+W^\alpha_1 \quad \text{for } \alpha=2,3,\cdots,s,\end{cases} \tag{2.28}$$

which yields that the vector field

$$\mathrm{ad}\,\partial\cdot(\mathrm{ad}\,\partial^1)^2\cdot X+\sum_{1<\alpha}(c_1-c_\alpha)\mathrm{ad}\,\partial^\alpha\cdot\mathrm{ad}\,\partial^1\cdot X=2\sum_{1<\alpha}(c_1-c_\alpha)W^1_\alpha \tag{2.29}$$

is contained in $\mathfrak{g}(\mathcal{D})$. Since

$$\begin{cases}\mathrm{ad}\,\partial^\alpha\cdot\{\sum_{1<\alpha}(c_1-c_\alpha)W^1_\alpha\}=(c_1-c_\alpha)\sqrt{-1}\,W^1_\alpha;\\ (\mathrm{ad}\,\partial^\alpha)^2\cdot\{\sum_{1<\alpha}(c_1-c_\alpha)W^1_\alpha\}=-(c_1-c_\alpha)W^1_\alpha\end{cases}$$

and $c_1\neq c_\alpha$ $(1<\alpha)$ by our assumption (**), we find that

$$W^1_\alpha=0 \quad \text{for } 1<\alpha\leqq s, \tag{2.30}$$

because $\mathfrak{g}(\mathcal{D})\cap\sqrt{-1}\,\mathfrak{g}(\mathcal{D})=\{0\}$ by (1.2). A similar reasoning yields also the following

$$W^\alpha_1=0 \quad \text{for } 1<\alpha\leqq s. \tag{2.31}$$

Finally, by (2.14), (2.27), (2.30) and (2.31) we conclude that X has the desired form as in our lemma. q. e. d.

By means of Lemma 2.1, the following lemma can be proved in the same way as in Lemmas 3.2 and 3.3 in Vey [**12**], and hence is omitted.

LEMMA 2.2. *Under the same conditions* (*) *and* (**) *as in Lemma* 2.1, *we have the followings:*

(a) *The vector field* ∂^1 *belongs to the center* $\mathfrak{z}$ *of* $\mathfrak{g}(\mathcal{D})$.

(b) *Let* V *be the set of common zeros of vector fields belonging to* $\mathfrak{z}$. *Then*

$$\mathcal{D}\supset\mathcal{D}\cap(\boldsymbol{C}^n\times\{0\}\times\boldsymbol{C}^{m_2}\times\cdots\times\boldsymbol{C}^{m_s})$$
$$\supset V\supset\mathcal{D}\cap(\boldsymbol{C}^n\times\{0\}\times\{0\}\times\cdots\times\{0\}).$$

The proofs of the following two lemmas are almost identical to those of

Lemmas 2.1 and 2.2 above. So we will omit the proofs.

LEMMA 2.3. *Let X be a holomorphic vector field in $\mathfrak{g}(\mathcal{D})$ which is independent of $z_1, z_2, \cdots, z_n$. Suppose that $\mathcal{D}$ satisfies the following two conditions:*

(*)′ $c_1=1/2$.

(**) *The exponents c_α $(1\leqq\alpha\leqq s)$ are all different from each other and $c_\alpha\neq 0$ $(1\leqq\alpha\leqq s)$.*

Then X can be written in the form

$$X=Z_0+Z_1+\sum_{2<\alpha<\beta} W^1_{\alpha\beta}+\sum_{\substack{2<\alpha<\beta\\ \lambda\neq\alpha,\beta\\ 2<\lambda}} W^\lambda_{\alpha\beta}+\sum_{\substack{2<\alpha,\lambda\\ \lambda\neq\alpha}} W^\lambda_{1\alpha}$$

$$+W^2_2+\sum_{\alpha,\lambda\neq 2} W^\lambda_\alpha+\sum_{\lambda\neq 2} W^\lambda_0\,.$$

LEMMA 2.4. *Under the same conditions* (*)′ *and* (**) *as in Lemma 2.3, we have the followings:*

(a)′ *The vector field ∂^2 belongs to the center $\mathfrak{z}$ of $\mathfrak{g}(\mathcal{D})$.*

(b)′ *Let V be the set of common zeros of vector fields belonging to $\mathfrak{z}$. Then*

$$\mathcal{D}\supset\mathcal{D}\cap(\boldsymbol{C}^n\times\boldsymbol{C}^{m_1}\times\{0\}\times\boldsymbol{C}^{m_3}\times\cdots\times\boldsymbol{C}^{m_s})$$

$$\supset V\supset\mathcal{D}\cap(\boldsymbol{C}^n\times\{0\}\times\cdots\times\{0\})\,.$$

§ 3. Proof of Theorem.

In order to prove the theorem, we need some preparations.

Let $\mathcal{D}$ be a generalized Siegel domain in $\boldsymbol{C}^n\times\boldsymbol{C}^{m_1}\times\boldsymbol{C}^{m_2}\times\cdots\times\boldsymbol{C}^{m_s}$ with exponent $(c_1, c_2, \cdots, c_s)$. Supposing that at least some two exponents c_α and c_β are not identical, there exists a partition $I_1, I_2, \cdots, I_k$ $(2\leqq k\leqq s)$ of $\{1, 2, \cdots, s\}$ so that

(3.1) $\quad I_q\neq\emptyset$ for all $q=1, 2, \cdots, k$;

(3.2) if $I_q=\{n_{q1}, n_{q2}, \cdots, n_{qs_q}\}$, then there is a constant $\tilde{c}_q$ satisfying $c_{n_{q1}}=c_{n_{q2}}=\cdots=c_{n_{qs_q}}=\tilde{c}_q$ and such that $\tilde{c}_p\neq\tilde{c}_q$ if $p\neq q$. Putting

$$\tilde{w}_q=(w_{n_{q1}}, w_{n_{q2}}, \cdots, w_{n_{qs_q}}) \quad \text{for } q=1, 2, \cdots, k,$$

we now define a non-singular linear mapping

$$\mathcal{L}:\ \boldsymbol{C}^{n+m_1+m_2+\cdots+m_s}\to\boldsymbol{C}^{n+m_1+m_2+\cdots+m_s}$$

by

$$\mathcal{L}(z, w_1, w_2, \cdots, w_s)=(z, \tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_k)\,.$$

Then, it is easily seen from the definition that the image domain

(3.3) $\tilde{\mathcal{D}}=\mathcal{L}(\mathcal{D})$ is a generalized Siegel domain in $\boldsymbol{C}^n\times\boldsymbol{C}^{\tilde{m}_1}\times\boldsymbol{C}^{\tilde{m}_2}\times\cdots\times\boldsymbol{C}^{\tilde{m}_k}$ with exponent $(\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_k)$ such that $\tilde{c}_p \neq \tilde{c}_q$ if $p \neq q$, where $\tilde{m}_q = m_{n_{q1}} + m_{n_{q2}} + \cdots + m_{q s_q}$, $q = 1, 2, \cdots, k$.

In the proof below, we shall use the following notation. Let $\mathfrak{h}$ be a vector subspace of $\mathfrak{g}(\mathscr{D})$ and p a point of $\mathscr{D}$. Then we put

$$\mathfrak{h}(p) = \{X(p) \mid X \in \mathfrak{h}\} \subset T_p^c(\mathscr{D})$$

and

$$\mathfrak{h}^c(p) = \mathfrak{h}(p) + \sqrt{-1}\, \mathfrak{h}(p),$$

where we denote by $X(p)$ the value of the vector field X at p and $T_p^c(\mathscr{D})$ the complex tangent space of $\mathscr{D}$ at p.

PROOF OF THEOREM. Let $\mathscr{D}$ be a sweepable generalized Siegel domain with exponent $(c_1, c_2, \cdots, c_s)$. By a result of Vey [12], we have only to show that all the exponents c_α are identical. Supposing that this is not true, we shall obtain a contradiction. First of all, there exists a subgroup Γ of $\mathrm{Aut}(\mathscr{D})$ such that $\mathscr{D}/\Gamma$ is compact, since $\mathscr{D}$ is a sweepable domain. We denote by $\mathfrak{z}$ the center of $\mathfrak{g}(\mathscr{D})$ and V the common zeros of vector fields belonging to $\mathfrak{z}$. Since $\mathfrak{z}$ is stable under the adjoint action of Γ, it follows from Proposition 2.3 in Vey [12] that $\dim \mathfrak{z}^c(p) = k$, where k is a constant independent of $p \in \mathscr{D}$. Now, we suppose that $c_\alpha \neq c_\beta$ for some two α and β. Consider the domain $\tilde{\mathscr{D}}$ defined in (3.3). Since $\tilde{\mathscr{D}}$ is holomorphically equivalent to $\mathscr{D}$, $\tilde{\mathscr{D}}$ is then a sweepable generalized Siegel domain with exponent $(\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_k)$ such that $\tilde{c}_p \neq \tilde{c}_q$ if $p \neq q$. Therefore, by considering the domain $\tilde{\mathscr{D}}$ instead of $\mathscr{D}$ if necessary, we may assume without loss of generality that the exponents c_α of $\mathscr{D}$ are all different from each other from the beginning. We must now point out a contradiction. We have two cases to consider. Consider first the case where $c_1 \neq 1/2$. Then, by Lemma 2.2, we have

$$V \neq \emptyset \tag{3.4}$$

and

$$\begin{cases} \dim \mathfrak{z}^c(p) = 0 & \text{for } p \in V; \\ \dim \mathfrak{z}^c(p) \neq 0 & \text{for } p \in \mathscr{D} - \{\mathscr{D} \cap (\boldsymbol{C}^n \times \{0\} \times \boldsymbol{C}^{m_2} \times \cdots \times \boldsymbol{C}^{m_s})\}. \end{cases} \tag{3.5}$$

Since, as is stated above, the dimension of $\mathfrak{z}^c(p)$ is independent of the point $p \in \mathscr{D}$ this is obviously a contradiction. Consider next the case where $c_1 = 1/2$. Then, by using Lemma 2.4, instead of Lemma 2.2 in the above arguments, we can prove an analogous contradiction. Therefore, in any cases we have a contradiction. This completes the proof.

§ 4. Examples.

We conclude this paper by a few examples of generalized Siegel domains with exponent $(c_1, c_2, \cdots, c_s)$.

EXAMPLE 1. We recall *generalized Siegel domains with exponent* c due to Kaup, Matsushima and Ochiai [5]. A domain $\mathcal{D}$ in $C^n \times C^m$ is called a *generalized Siegel domain with exponent* c if the following conditions are satisfied:

(1) $\mathcal{D}$ is holomorphically equivalent to a bounded domain in C^{n+m} and $\mathcal{D} \cap (C^n \times \{0\}) \neq \emptyset$, where 0 denotes the origin of C^m.

(2) $\mathcal{D}$ is invariant by the transformations of C^{n+m} of the following types:

(a) $(z, w) \mapsto (z+a, w)$ for all $a \in R^n$;

(b) $(z, w) \mapsto (z, e^{\sqrt{-1}t} w)$ for all $t \in R$;

(c) $(z, w) \mapsto (e^t z, e^{ct} w)$ for all $t \in R$,

where c is a fixed real number depending only on $\mathcal{D}$. We call c the exponent of $\mathcal{D}$. Comparing this definition with our definition 1 in section 1, we see that the notion of generalized Siegel domains with exponent $(c_1, c_2, \cdots, c_s)$ may be considered as a natural generalization of the notion of generalized Siegel domains with exponent c by Kaup, Matsushima and Ochiai [5].

EXAMPLE 2. For $\alpha=1, 2, \cdots, r$, let $\Phi_\alpha : C^{n_\alpha} \to R$ be a non-negative continuous function defined on C^{n_α} such that

$$D_\alpha = \{w \in C^{n_\alpha} \mid \Phi_\alpha(w) < 1\} \text{ is a bounded domain;} \tag{4.1}$$

$$\Phi_\alpha(\lambda w) = |\lambda|^{1/c_\alpha} \Phi_\alpha(w) \tag{4.2}$$

for all $w \in C^{n_\alpha}$ and all $\lambda \in C$, where c_α is a fixed non-zero real number. Taking $(1/2, c_1, c_2, \cdots, c_r)$ for the exponent in the definition 1 in section 1, it is easily seen that the domain

$$\mathcal{D} = \Big\{ (z, u, w_1, w_2, \cdots, w_r) \in C \times C^{m-1} \times C^{n_1} \times C^{n_2} \times \cdots \times C^{n_r} \Big| \; \operatorname{Im} z - \sum_{i=1}^{m-1} |u_i|^2 - \sum_{\alpha=1}^{r} \Phi_\alpha(w_\alpha) > 0 \Big\}$$

satisfies the condition (2) and $(\sqrt{-1}\, l, 0) \in \mathcal{D}$ for every positive real number l, where $u = (u_1, u_2, \cdots, u_{m-1}) \in C^{m-1}$. Moreover, it can be shown that the domain $\mathcal{D}$ is holomorphically equivalent to the following bounded domain

$$D = \Big\{ (\tilde{z}, \tilde{w}) \in C^{m+|n|} \Big| \sum_{i=1}^{m} |\tilde{z}_i|^2 + \sum_{\alpha=1}^{r} \Phi_\alpha(\tilde{w}_\alpha) < 1 \Big\}$$

where $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_m)$, $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_r)$ and $|n| = n_1 + n_2 + \cdots + n_r$. Indeed, the following mapping $\Phi : C^{m+|n|} \to C^{m+|n|}$ gives rise to a biholomorphic isomorphism of $\mathcal{D}$ onto D:

$$
\text{(4.3)} \qquad \Phi : \begin{cases} \tilde{z}_1 = \dfrac{z-\sqrt{-1}}{z+\sqrt{-1}} & \\ \tilde{z}_i = \dfrac{2u_{i-1}}{z+\sqrt{-1}} & \text{for } i=2, 3, \cdots, m; \\ \tilde{w}_\alpha = \dfrac{4^{c_\alpha} w_\alpha}{(z+\sqrt{-1})^{2c_\alpha}} & \text{for } \alpha=1, 2, \cdots, r. \end{cases}
$$

Therefore we have seen that the domain $\mathcal{D}$ is a generalized Siegel domain in $C\times C^{m-1}\times C^{n_1}\times C^{n_2}\times\cdots\times C^{n_r}$ with exponent $(1/2, c_1, c_2, \cdots, c_r)$. This Example 2 yields the following interesting

Example 3. Consider the following special case in the Example 2: We put

$$
\Phi_\alpha(w)=\Big(\sum_{\beta=1}^{n_\alpha}|w_\beta|^2\Big)^{1/p_\alpha}
$$

for all $w=(w_1, w_2, \cdots, w_{n_\alpha})\in C^{n_\alpha}$, where p_α is a natural number such that $p_\alpha>1$. Then it follows from the Example 2 that the domain

$$
\mathcal{D}_{m,n,p}=\Big\{(z, u, w_1, w_2, \cdots, w_r)\in C\times C^{m-1}\times C^{n_1}\times C^{n_2}\times\cdots\times C^{n_r}\,\Big|\, \operatorname{Im}. z-\sum_{j=1}^{m-1}|u_j|^2-\sum_{\alpha=1}^{r}\Big(\sum_{\beta=1}^{n_\alpha}|w_{\beta\alpha}|^2\Big)^{1/p_\alpha}>0\Big\}
$$

is a generalized Siegel domain with exponent $(1/2, p_1/2, p_2/2, \cdots, p_r/2)$, where $u=(u_1, u_2, \cdots, u_{m-1})\in C^{m-1}$ and $w_\alpha=(w_{1\alpha}, w_{2\alpha}, \cdots, w_{n_\alpha\alpha})\in C^{n_\alpha}$. Moreover, $\mathcal{D}_{m,n,p}$ is holomorphically equivalent to the following bounded Reinhardt domain

$$
D_{m,n,p}=\Big\{(z_i, w_{jk})\in C^{m+|n|}\,\Big|\, \sum_{i=1}^{m}|z_i|^2+\sum_{k=1}^{r}\Big(\sum_{j=1}^{n_k}|w_{jk}|^2\Big)^{1/p_k}<1\Big\}.
$$

In connection with the structure theory of the group $\operatorname{Aut}(D)$ for a bounded domain D, this domain $D_{m,n,p}$ is studied by Ise [3]. To use his terminology, this domain $D_{m,n,p}$ is nothing but the *Naruki domain of type* (m, n, p).

Therefore, our generalized Siegel domain $\mathcal{D}_{m,n,p}$ is a natural unbounded model of the Naruki domain $D_{m,n,p}$.

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