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Weierstrass points on compact Riemann surfaces with nontrivial automorphisms

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1. Introduction.

Let S be a compact Riemann surface of genus g ($\geqq 3$), h be an automorphism of S with fixed points, and T denotes the number of these fixed points. Let $\langle h\rangle$ denote the cyclic group generated by h, whose order is an odd prime number p. Let $S/\langle h\rangle$ be the surface obtained by identifying the equivalent points on S under the elements of $\langle h\rangle$. If $S/\langle h\rangle$ has genus zero, then S can be defined by an equation of the form

$$(1) \qquad y^{p}=\prod_{j=1}^{T}(x-c_{j})^{\delta_{j}},$$

where c_j ($1\leqq j\leqq T$) are complex numbers which are different from each other, $1\leqq\delta_j\leqq p-1$, and $\sum_{j=1}^{T}\delta_j\equiv 0 \pmod p$. Throughout the present paper we consider only these surfaces. We show that they are characterized by non-negative integral solution (under suitable conditions) of the system of linear equations, which are derived from J. Lewittes' method [**4**]. We investigate also the Weierstrass gap sequence at the point Q_j on S corresponding to $(c_j, 0)$.

A matrix representation $R_1(h)$ of $\langle h\rangle$ is obtained by letting it act on the complex g-dimensional space $A_1(S)$ of Abelian differentials of the first kind. Let n_k with $0\leqq k\leqq p-1$ denote the multiplicity of ε^k ($\varepsilon=\exp\{(2\pi i)/p\}$) in the diagonal form of $R_1(h)$. The upper (resp. lower) bound of $\{n_k\}$ is taken over all compact Riemann surfaces of fixed genus g with an automorphism h which satisfy properties mentioned above. This upper (resp. lower) bound we call simply the upper (resp. lower) bound of $\{n_k\}$, and it is denoted by n^* (resp. n_*). Lewittes has given upper and lower bound of $\{n_k\}$ if $T>0$, [**4**, Theorem 4 (c)]. Our bounds given in this paper are ones improved on Lewittes' results except for $T\equiv 0 \pmod p$. In section 4, we consider the condition (A_0), and show that an automorphism h satisfies the condition (A_0) with respect to λ ($1\leqq\lambda\leqq p-1$) if and only if $n_\lambda=n^*$ (resp. $n_{p-\lambda}=n_*$). This condition (A_0) contains the Kato's condition (A), [**3**, p. 398]. Kato has shown that if an automorphism

h satisfies the condition (A), then there exists a Weierstrass point Q on S such that the number $2g-1$ is a gap value at Q. Thus the vector of Riemann constants $K(Q)$ is a half period, [3, p. 400]. In Corollary 1, we show that the converse is also true.

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2. Preliminary.

Throughout this paper, let p and g ($\geqq 3$) always denote an odd prime number and the genus of S respectively. In (1), let an automorphism h be represented such as

$$h(x, y)=(x, \varepsilon y). \tag{2}$$

For each j ($1\leqq j\leqq T$), let z_j be a local parameter at Q_j on S corresponding to $(c_j, 0)$, and let β_j ($1\leqq \beta_j\leqq p-1$) be the solution of $\delta_j\beta_j\equiv -1 \pmod p$. Then we have

$$h^{-1}(z_j)=\varepsilon^{\beta_j}z_j+\cdots. \tag{3}$$

Let $\alpha_j(1)=p-\delta_j$, and let a positive integer $\alpha_j(k)$ ($1\leqq \alpha_j(k)\leqq p-1$) be the solution of

$$\alpha_j(k)\equiv k\alpha_j(1) \pmod p,\quad 2\leqq k\leqq p-1,\quad 1\leqq j\leqq T. \tag{4}$$

Then the multiplicity n_k ($0\leqq k\leqq p-1$) are given by

$$\begin{cases} n_0=0=\text{the genus of } S/\langle h\rangle, \\ n_k=-1+\sum_{j=1}^{T}(1-\alpha_j(k)/p),\quad 1\leqq k\leqq p-1, \text{ [4, p. 743].} \end{cases} \tag{5}$$

Let $\gamma(Q_j)=\{\gamma_1, \gamma_2, \cdots, \gamma_g\}$ denote the Weierstrass gap sequence at Q_j. Then Lewittes [4] has shown that

$$R_1(h)=\text{diagonal } \{\mu^{\gamma_1}, \mu^{\gamma_2}, \cdots, \mu^{\gamma_g}\},\quad \mu=\varepsilon^{\beta_j}. \tag{6}$$

Throughout this paper, for any real number q let $[q]$ denote the integer part of q, and for any integer q let $\bar{q}$ denote the minimum non-negative integer less than p such that $q\equiv\bar{q} \pmod p$.

DEFINITION. Let t_α ($1\leqq\alpha\leqq p-1$) be the number of fixed points of h at which $\alpha_j(1)=\alpha$ (i.e. $\beta_j\alpha\equiv 1 \pmod p$). In other words, t_α is the number of factors of the form $(x-c_j)^{\delta_j}$ in (1), where

$$\alpha=p-\delta_j \quad \text{for some } j \ (1\leqq j\leqq T). \tag{7}$$

Let $a_{k\alpha}=p-\overline{k\alpha}=p([k\alpha/p]+1)-k\alpha$, and $\Gamma(\beta_j, \alpha_j(k))$ denote the number of gap values at Q_j which are congruent to $\alpha_j(k)$ (mod p).

From the above definition, we have

$$T=\sum_{\alpha=1}^{p-1} t_\alpha \tag{8}$$

and (5) can be written as

$$\sum_{\alpha=1}^{p-1} a_{k\alpha}t_\alpha=p(n_k+1), \quad 1\leqq k\leqq p-1. \tag{9}$$

The condition $\sum_{j=1}^{T}\delta_j\equiv 0$ (mod p) imposed on (1) is contained in the above equation (9). In the case of $p=3$, this idea has been mentioned implicitly by C. Maclachlan [5] using the theory of Fuchsian groups. We generalized his idea to the case of an arbitrary prime number, and represent a compact Riemann surface as an algebraic curve.

LEMMA 1. *The matrix $A=(a_{ij})$ is symmetric and has following properties:*

(i) $\quad a_{ij}=a_{p-i,p-j} \quad (1\leqq i,\ j\leqq (p-1)/2)$,

(ii) $\quad a_{ij}+a_{i,p-j}=p \quad (1\leqq i\leqq p-1,\ 1\leqq j\leqq (p-1)/2)$,

(iii) $\quad \sum_{j=1}^{p-1} a_{ij}=p(p-1)/2 \quad (1\leqq i\leqq p-1)$.

PROOF. (i) follows at once from $(p-i)(p-j)\equiv ij$ (mod p).
(ii) is trivial from $[ij/p]+[i(p-j)/p]+2-i=1$. q.e.d.

LEMMA 2. *Let $m_j(k)p+\alpha_j(k)$ $(1\leqq j\leqq T,\ 1\leqq k\leqq p-1)$ denote the first nongap value at Q_j on S which is congruent to $\alpha_j(k)$* (mod p). *Then*

(i) $\quad m_j(k)=n_k=\Gamma(\beta_j, \alpha_j(k))$,

(ii) $\quad T-2=2g/(p-1)=n_i+n_{p-i} \quad (1\leqq i\leqq (p-1)/2)$.

PROOF. Since the number p is a nongap value at Q_j, natural numbers $ip+\alpha_j(k)$ $(1\leqq k\leqq p-1,\ m_j(k)\leqq i)$ are nongap values at Q_j. This yields $m_j(k)=\Gamma(\beta_j, \alpha_j(k))$. $m_j(k)=n_k$ follows from (4) and (6). Using (ii) and (iii) of Lemma 1, we have $T=n_i+n_{p-i}+2$ from (9). Then the Riemann-Hurwitz relation gives that $g=(n_i+n_{p-i})(p-1)/2$. q.e.d.

We have

$$\Gamma(j, k)=\Gamma(k, j)=n_{\overline{jk}} \quad \text{for } 1\leqq j,\ k\leqq p-1 \tag{10}$$

from (i) of Lemma 2. Thus the number of gap values congruent to k (mod p) at which h^{-1} is locally represented by $z\to\exp\{(2\pi ji)/p\}z$ is equal to the number of gap values congruent to j (mod P) at which h^{-1} is locally represented by

$z \rightarrow \exp\{(2\pi ki)/p\} z$.

LEMMA 3. *Let S be a compact Riemann surface of genus g, Q be a point on S. Let σ_k $(1 \leqq k \leqq p-1)$ denote the first nongap value at Q which is congruent to k (mod p). Suppose that an odd prime number p is the first nongap value at Q.*

(i) *If $g=(p-1)(mp-2)/2$, $m \geqq 2$, and $\sigma_{p-1}=mp-1$, then $\sigma_{p-k}=k\sigma_{p-1}$ $(2 \leqq k \leqq p-1)$.*

(ii) *If $g=(p-1)mp/2$, $m \geqq 1$, and $\sigma_1=mp+1$, then $\sigma_k=k\sigma_1$ $(2 \leqq k \leqq p-1)$.*

PROOF. (i) The numbers $ip+j$ $(0 \leqq i \leqq m-2,\ 1 \leqq j \leqq p-1)$ and $(m-1)p+s$ $(1 \leqq s \leqq p-2)$ are gap values at Q from Jenkins theorem, [2]. Since $mp-1$ is a nongap value at Q, we have $mp+(p-k) \leqq \sigma_{p-k} \leqq k\sigma_{p-1}$ $(2 \leqq k \leqq p-1)$. If there exists some i $(2 \leqq i \leqq p-1)$ such as $\sigma_{p-i} < i\sigma_{p-1}$, then the number of gap values at Q is at most $g-1$. This contradicts the gap theorem. (ii) follows from the same discussion as (i). q. e. d.

3. Gap sequences.

From (i) of Lemma 2, the gap sequence at Q_j is completely determined by $\{n_k\}$. Since the family $\{\beta_j\}$ has only $p-1$ possible values, we have the following result.

PROPOSITION 1. *Assume that $T>4$. The family of gap sequences $\{\gamma(Q_j);\ 1 \leqq j \leqq T\}$ on S constitutes of at most $p-1$ kinds of different gap sequences.*

REMARK 1. The assumption "$T>4$" means that the fixed points of h are all Weierstrass points [4; Theorem 6].

However, there really exists a surface such that $\{\gamma(Q_j)\}$ constitutes of the same gap sequence even if the family $\{\beta_j\}$ has $p-1$ possible values. We give such an example. Consider the condition

$$\text{(I)} \qquad t_\alpha = t_{p-\alpha} \neq 0 \text{ for all } \alpha\ (1 \leqq \alpha \leqq (p-1)/2,\ p \geqq 3) \text{ and } T>4.$$

If an automorphism h satisfies the condition (I), then we have

$$\text{(11)} \qquad n_k = -1 + \sum_{\alpha=1}^{r} t_\alpha = -1 + T/2 \quad \text{for all } k\ (1 \leqq k \leqq p-1)$$

from (ii) of Lemma 1 and (9), where $r=(p-1)/2$. Such a surface S is defined by an equation of the form

$$y^p = \prod_{\alpha=1}^{r} \prod_{j=1}^{t_\alpha} (x-c_j^{(\alpha)})^\alpha (x-c_j'^{(\alpha)})^{p-\alpha}$$

where $c_j^{(\alpha)}$ and $c_j'^{(\alpha)}$ $(1 \leqq j \leqq t_\alpha,\ 1 \leqq \alpha \leqq r)$ are different complex numbers from each other. Since T is an even number in this case, we may set $T=2m$ $(m \geqq 3)$. Then $\gamma(Q_j)=\{ip+k\,;\ 0 \leqq i \leqq m-2,\ 1 \leqq k \leqq p-1\}$ for all j $(1 \leqq j \leqq T)$. Farkas has given such a surface in the case of $p=3$ and $T=6$ [1; p. 135]. In the last section we give compact Riemann surfaces which have Weierstrass points with exactly $p-1$ kinds of different gap sequences.

4. The upper and the lower bounds of $\{n_k\}$.

In this section, we give explicitly the upper bound n^* (resp. the lower bound n_*) of $\{n_k\}$ in the case of $T>4$. This has been given by Theorem 4(c) of Lewittes [**4**]. Our bounds given in this section are ones improved on Lewittes' results except for $T\equiv 0 \pmod{p}$. From (i) of Lemma 2 and the fact that a gap value does not exceed $2g-1$, we have

$$(12) \qquad n^*=\begin{cases} T-T/p-1 & \text{if } T\equiv 0 \pmod{p}, \\ T-[T/p]-2 & \text{if } T\not\equiv 0 \pmod{p}, \text{ where } T>4. \end{cases}$$

We get $n_*=T-n^*-2$ by (ii) of Lemma 2. This yields

$$(13) \qquad n_*=\begin{cases} T/p-1 & \text{if } T\equiv 0 \pmod{p}, \\ [T/p] & \text{if } T\not\equiv 0 \pmod{p}, \text{ where } T>4. \end{cases}$$

PROPOSITION 2. *Assume that $T>p$ for $p\geqq 5$, and that $T>4$ for $p=3$. Then the number p is the first nongap value at every fixed point of h, i.e. $n_k\neq 0$ for every k $(1\leqq k\leqq p-1)$.*

PROOF. Every fixed point of h is a Weierstrass point, for $T>4$. Suppose that the number p is not the first nongap value at a fixed point of h. Then there exists a certain number k $(1\leqq k\leqq p-1)$ such that $n_k=0$. This yields $n_{p-k}=T-2$ from (ii) of Lemma 2. This contradicts equation (12). q.e.d.

We define the number J as follows:

$$(14) \qquad J=\begin{cases} 1 & \text{if } \xi=0, \\ p-1 & \text{if } \xi=1, \\ p-\xi+1 & \text{if } 2\leqq \xi<p-1, \end{cases}$$

where $T=mp+\xi>4$ and $0\leqq \xi<p$.

Let a natural number λ $(1\leqq \lambda\leqq p-1)$ be given, and let $\beta(k)$ $(1\leqq k\leqq J$, $1\leqq \beta(k)\leqq p-1)$ be the solution of

$$(15) \qquad k\beta(k)\equiv \lambda \pmod{p}.$$

We define the number $\alpha(k)$ $(1\leqq \alpha(k)\leqq p-1,\ 1\leqq k\leqq J)$ to be the solution of

$$(16) \qquad \alpha(k)\beta(k)\equiv 1 \pmod{p}.$$

We consider the following condition:

$$(\mathrm{A}_0)\qquad \begin{cases} T=\sum_{k=1}^{J} t_{\alpha(k)}>4, \text{ and} \\ J-1=\sum_{k=2}^{J}(k-1)t_{\alpha(k)} \quad \text{if } T\not\equiv 1 \pmod p, \\ p-1=\sum_{k=2}^{J}(k-1)t_{\alpha(k)} \quad \text{if } T\equiv 1 \pmod p. \end{cases}$$

REMARK 2. These $\alpha(k)$ are different from $\alpha_j(k)$ defined by (4).

THEOREM 1. *Let a natural number* λ $(1\leqq\lambda\leqq p-1)$ *be given. An automorphism* h *satisfies the condition* (A_0) *with respect to* λ *if and only if* $n_\lambda=n^*$ *(resp.* $n_{p-\lambda}=n_*$*).*

PROOF We, at first, prove only if part of the theorem. Assume that $n_\lambda=n^*$. We may complete the proof by considering four cases:

Case 1. $T=mp$, where $m>1$ for $p=3$ and $m\geqq 1$ for $p\geqq 5$. Then $J=1$ from (14). Suppose there exists a fixed point Q_j of h at which h^{-1} is locally represented by $z_j\to\exp\{(2\pi\beta i)/p\}z_j$, where β $(1\leqq\beta\leqq p-1)$ satisfies $\beta k\equiv\lambda$ (mod p) for a certain k $(1\leqq k\leqq p-1)$. We get $n_\lambda=m(p-1)-1$ by the assumption. Then t_α is not equal to zero, where α $(1\leqq\alpha\leqq p-1)$ is the solution of $\alpha\beta\equiv 1$ (mod p). If $k\neq 1$, then we have $\Gamma(\beta, k)=n_\lambda$ from (10). This yields that there is a gap value greater than $2g-1$ at Q_j, for $2g-1=p\{m(p-1)-2\}+1$. This contradicts the gap theorem. Thus $k=1$, and $\beta=\lambda$. This means $a_{\beta\alpha}=a_{\lambda\alpha}=(p-1)$, because $\beta=\lambda\equiv\lambda\alpha\beta$ (mod p). Since $a_{\beta\mu}=a_{\lambda\mu}\leqq(p-2)$ for every μ $(\neq\alpha)$ and $T=mp=\sum_{\mu=1}^{p-1}t_\mu$, we have $p(n_\lambda+1)=p\{m(p-1)\}=a_{\lambda\alpha}t_\alpha+\sum_{\mu\neq\alpha}a_{\lambda\mu}t_\mu\leqq(p-1)t_\alpha+\sum_{\mu\neq\alpha}(p-2)t_\mu=t_\alpha+(p-2)mp$ from (9). Therefore $T=t_\alpha=mp$. Thus h satisfies the condition (A_0).

Case 2. $T=mp+\xi>4$, where $2\leqq\xi\leqq p-1$. Then $J=p-\xi+1$ from (14). Suppose there exists a fixed point Q_j of h at which h^{-1} is locally represented by $z_j\to\exp\{(2\pi\beta i)/p\}z_j$, where β $(1\leqq\beta\leqq p-1)$ satisfies $\beta k\equiv\lambda$ (mod p) for a certain k $(1\leqq k\leqq p-1)$. And for this λ, we have $n_\lambda=m(p-1)+\xi-2$ by the assumption. Then t_α is not equal to zero, where α $(1\leqq\alpha\leqq p-1)$ is the solution of $\alpha\beta\equiv 1$ (mod p) from the definition of t_α. If the number k satisfies the inequality $J<k\leqq p-1$, then we get $\Gamma(\beta, k)=n_\lambda$ by (10). There is a gap value greater than $2g-1$ at Q_j, because $(2g-1)=p\{m(p-1)+\xi-3\}+J$. This is the contradiction. Thus the number k satisfy the inequality $1\leqq k\leqq J$. We conclude that t_μ is equal to zero for all μ, where μ satisfies $\mu\eta\equiv\lambda$ (mod p) for every η satisfying $J<\eta\leqq p-1$. From the above discussion we see that there exists at most J pairs of positive integers $\alpha(k)$ and $\beta(k)$ which satisfy (15) and (16). Since $k\beta(k)\equiv\lambda\equiv\lambda\alpha(k)\beta(k)$ (mod p), we have $\beta(k)\cdot(\lambda\alpha(k)-k)\equiv 0$ (mod p). This yields that $\lambda\alpha(k)\equiv k$ (mod p) for each k, because $1\leqq\beta(k)\leqq p-1$. Hence we get $a_{\lambda\alpha(k)}=p-k$ from the definition, and (9) can be written as $p(n_\lambda+1)=$

$p\{m(p-1)+\xi-1\}=\sum_{k=1}^{J} a_{\lambda\alpha(k)}t_{\alpha(k)}=\sum_{k=1}^{J}(p-k)t_{\alpha(k)}=pT-\sum_{k=1}^{J} kt_{\alpha(k)}$. This reduces to $p(m+1)=\sum_{k=1}^{J} kt_{\alpha(k)}=T+\sum_{k=2}^{J}(k-1)t_{\alpha(k)}$. Then we have the condition (A_0).

Case 3. $T=mp+1$ and $m\geqq 1$. We have $(2g-1)=p\{m(p-1)-1\}$.

According to the similar discussion as above, we get the condition (A_0).

Case 4. $p\geqq 7$ and $5\leqq T\leqq p-1$. We have $(2g-1)=p(T-3)+(p-T+1)$ and $J=p-T+1$. Therefore $n_\lambda=T-2$ from the assumption. Then there exists at most J pairs of positive integers $\alpha(k)$ and $\beta(k)$ which satisfy (15) and (16). This yields $p(T-1)=\sum_{k=1}^{J}(p-k)t_{\alpha(k)}$ by (9). Thus the condition (A_0) holds.

Conversely, we assume that an automorphism h satisfies the condition (A_0). If $T=mp>4$, then $J=1$ by (14). Thus we have $k=1$ and $t_{\alpha(1)}=mp$ i.e. $\beta(1)=\lambda$ and $\alpha(1)\beta(1)=\alpha(1)\lambda\equiv 1 \pmod p$ from (15) and (16). This yields $p\{n_\lambda+1\}=a_{\lambda\alpha(1)}t_{\alpha(1)}=(p-1)mp$, which reduces to $n_\lambda=m(p-1)-1=n^*$. In the case of $T=mp+\xi>4$ $(\xi\neq 0)$, if we discuss the preceding argument conversely, then we have (12). q.e.d.

The case of $J=1$ (i.e. $T\equiv 0 \pmod p$, so that $g\equiv 1 \pmod p$) in (A_0), we have

(II) $$T=mp=t_{\alpha(1)}>4.$$

If an automorphism h satisfies the condition (II), then all β_j (defined by (3)) have the same common value, whence all gap sequences $\gamma(Q_j)$ are the same.

Suppose that $3\leqq J\leqq p-1$. If $t_{\alpha(q)}\neq 0$ for a certain q $([(J+1)/2]+1\leqq q\leqq J)$, then $t_{\alpha(\zeta)}=0$ for ζ $(\zeta\neq q,\ [(J+1)/2]+1\leqq\zeta\leqq J)$ and $t_{\alpha(q)}=1$. In this case the condition (A_0) can be written as

(A_q) $$\begin{cases} T=1+\sum_{k=1}^{B} t_{\alpha(k)}\not\equiv 1 \pmod p, \\ J-q=\sum_{k=2}^{B}(k-1)t_{\alpha(k)}, \text{ and } t_{\alpha(q)}=1 \\ \text{for a certain } q\ (B+1\leqq q\leqq J,\ 3\leqq J\leqq p-1), \text{ where } B=[(J+1)/2]. \end{cases}$$

THEOREM 2. *Assume that* $T>4$, $T\not\equiv 0$, *and* $T\not\equiv 1 \pmod p$. *The number* $2g-(J-q+1)$, $B+1\leqq q\leqq J$, *is a gap value at a fixed point of an automorphism* h *if and only if* h *satisfies the condition* (A_q), *where* $B=[(J+1)/2]$.

PROOF. Every fixed point of h is a Weierstrass point, for $T>4$. We set $T=mp+\xi>4$, $2\leqq\xi<p$, and $m\geqq 0$. Assume that h satisfies the condition (A_q) for a certain q $(B+1\leqq q\leqq J)$. Since $t_{\alpha(q)}=1$, there exists a point Q_j on S at which h^{-1} is locally represented by $z_j\rightarrow\exp\{(2\pi\beta(q)i)/p\}z_j$, where natural numbers $\alpha(q)$ and $\beta(q)$ satisfy $q\beta(q)\equiv\lambda \pmod p$ and $\alpha(q)\beta(q)\equiv 1 \pmod p$. We have $\Gamma(\beta(q),q)=n_\lambda=T-m-2=n^*$ from (10) and Theorem 1. Let $\sigma_q=m_qp+q$ denote the first nongap value at Q_j which is congruent to $q \pmod p$. Then $m_q=n_\lambda=$

$[(2g-J+q-1)/p]+1$ from (i) of Lemma 2, because $2g-(J-q+1)=p(T-m-3)+q$ $=p(n_\lambda-1)+q$. This shows that the number $2g-(J-q+1)$ is a gap value at Q_j.

Assume that the number $2g-(J-q+1)$ is a gap value at a fixed point Q_j of h, where h^{-1} is locally represented by $z_j \to \exp\{(2\pi\beta i)/p\} z_j$ at Q_j. The first nongap value which is congruent to $q \pmod p$ can be written as $p\{(2g-J+q-1)/p+1\}+q=p\{T-(m+2)\}+q$. This means that $\Gamma(\beta, q)=n_\lambda=T-(m+2)$ for a certain λ $(1 \leqq \lambda \leqq p-1)$. Here the number λ is the solution of $q\beta \equiv \lambda \pmod p$ and $n_\lambda=n^*$. Thus an automorphism h must satisfy the condition (A_0). Clearly $t_{\alpha(q)} \neq 0$, where $\alpha(q)$ $(1 \leqq \alpha(q) \leqq p-1)$ is the solution of $\beta(q)\alpha(q)=\beta\alpha(q) \equiv 1 \pmod p$. Thus an automorphism h satisfies the condition (A_q). q. e. d.

Now we consider the case of $q=J$ $(2 \leqq J \leqq p-1)$ and $T \not\equiv 1 \pmod p$ in (A_q). Then we get

$$(A_J) \qquad t_{\alpha(J)}=1 \quad \text{and} \quad t_{\alpha(1)}=T-1,$$

where natural numbers $\alpha(1)$ and $\alpha(J)$ satisfy $\alpha(J)\beta(J) \equiv 1$, $J\beta(J) \equiv \beta(1)$, and $\alpha(1)\beta(1) \equiv 1 \pmod p$ for an arbitrary given natural number $\beta(1)$ $(1 \leqq \beta(1) \leqq p-1)$.

An automorphism h satisfies the condition (II) or (A_J) is equivalent that h satisfies the Kato's condition (A) [**3**]. Kato has shown that if h satisfies the condition (A), then $2g-1$ is a gap value at a fixed point of h [**3**; p. 400]. We show that the converse is also true.

COROLLARY 1. *Assume that* $T>4$ *and* $T \not\equiv 1 \pmod p$. *The number* $2g-1$ *is a gap value at a certain fixed point of* h *if and only if* h *satisfies the Kato's condition* (A).

PROOF. Every fixed point of h is a Weierstrass point. Assume that the number $2g-1$ is a gap value at a certain fixed point of h. It is sufficient to prove in the case of $T=mp>4$, that is the condition (II). Since $J=1$, the condition (A_0) is equivalent to the condition (II). The fact that the number $2g-1$ is a gap value at a certain fixed point of h implies the condition (A_0).

q. e. d.

REMARK 3. (i) Let Q be a fixed point of h corresponding to $t_{\alpha(J)}$ in the condition (A_J). Then the vector of Riemann constants $K(Q)$ is a half period, ([**3**], [**6**]).

(ii) Suppose that $T>4$ and that $T \equiv 0$ or $T \equiv (p-1) \pmod p$. Then an automorphism h satisfies the condition (A_0) if and only if h satisfies the Kato's condition (A).

Consider the case of $q=J-1$ $(3 \leqq J \leqq p-1)$ in (A_q). Then we have

$$(A_{J-1}) \qquad t_{\alpha(J-1)}=t_{\alpha(2)}=1 \quad \text{and} \quad t_{\alpha(1)}=T-2,$$

where $\alpha(k)$ $(k=1, 2,$ and $J-1;$ $1 \leqq \alpha(k) \leqq p-1)$ are natural numbers which satisfy $\alpha(k)\beta(k) \equiv 1$, $k\beta(k) \equiv \beta(1) \pmod p$ for an arbitrary given natural number $\beta(1)$ $(1 \leqq \beta(1) \leqq p-1)$.

COROLLARY 2. *Assume that* $T>4$, *and that* $T\not\equiv j$ $(j=0, 1,$ *and* $p-1)$. *An automorphism* h *satisfies the condition* (A_{J-1}) *if and only if the number* $2g-2$ *is a gap value at a certain fixed point of* h.

PROOF. This is an immediate consequence of Theorem 2. Moreover, if h satisfies the condition (A_{J-1}), then $2g-2$ is a gap value at the fixed point of h corresponding to $t_{\alpha(J-1)}$. q.e.d.

When $q=J-2$ $(4\leqq J\leqq p-1)$, the condition (A_{J-2}) can be written as $t_{\alpha(J-2)}=1$, $t_{\alpha(2)}+2t_{\alpha(3)}=2$, and $T=1+\sum_{k=1}^{3} t_{\alpha(k)}$. We therefore have following solutions:

$(\mathrm{A}_{J-2})_2$ $\qquad t_{\alpha(J-2)}=1,\ t_{\alpha(2)}=2,$ and $t_{\alpha(1)}=T-3,$

$(\mathrm{A}_{J-2})_3$ $\qquad t_{\alpha(J-2)}=t_{\alpha(3)}=1,$ and $t_{\alpha(1)}=T-2.$

In each case, the number $2g-3$ is a gap value at the fixed point of h corresponding to $t_{\alpha(J-2)}$.

COROLLARY 3. *Assume that* $T>4$ *and that* $T\not\equiv j$ $(j=0, 1, p-1,$ *and* $p-2)$ (mod p). *An automorphism* h *satisfies the condition* (A_{J-2}) *if and only if the number* $2g-3$ *is a gap value at a fixed point of* h.

5. Examples.

Throughout this section, let $\{c_j^{(\delta_\alpha)};\ 1\leqq j\leqq T\}$ always denote complex numbers which are different from each other. Then equation (1) can be written as

$$(1)' \qquad y^p=\prod_{\alpha=1}^{p-1}\prod_{j=1}^{t_\alpha}(x-c_j^{(\delta_\alpha)})^{\delta_\alpha}.$$

In (1)′, let an automorphism h be represented such as (2). For a given t_α $(1\leqq\alpha\leqq p-1)$, δ_α is determined by

$$(7)' \qquad \delta_\alpha=p-\alpha.$$

We, at first, show two examples related to Proposition 1.

EXAMPLE 1. Let S be defined by

$$y^p=\prod_{j=1}^{l_1}(x-c_j^{(p-1)})^{p-1}\cdot\prod_{j=1}^{l_1(mp+1)}(x-c_j^{(1)})\cdot\prod_{\delta=2}^{r}\prod_{j=1}^{l_\delta}(x-c_j^{(\delta)})^{\delta}(x-c_j^{(p-\delta)})^{p-\delta},$$

where $p\geqq5$, $r=(p-1)/2$, and l_δ, m $(1\leqq\delta\leqq r)$ are natural numbers. We have $t_1=l_1$, $t_{p-1}=l_1(mp+1)$, and $t_\delta=t_{p-\delta}=l_\delta$ $(2\leqq\delta\leqq r)$. This yields $n_k=kl_1m-1+\sum_{\delta=1}^{r} l_\delta$ $(1\leqq k\leqq p-1)$ from (9), and $g=r(mpl_1+2\sum_{\delta=1}^{r} l_\delta-2)$. Let Q_k be the point corresponding to $(c_1^{(k)}, 0)$ for each k $(1\leqq k\leqq p-1)$. The gap sequences $\gamma(Q_k)$ $(k=1, 2, \cdots, p-1)$ are different from each other, because $n_k\neq n_{k'}$ for $k\neq k'$.

EXAMPLE 2. Let S be defined by

$$y^p=\prod_{j=1}^{2(l+2-p)}(x-c_j^{(p-1)})^{p-1}\cdot\prod_{j=1}^{p+r-l}(x-c_j^{(p-2)})^{p-2}\cdot\prod_{j=3}^{p-1}(x-c_j^{(p-j)})^{p-j},$$

where $p\geqq 7$, $r=(p-1)/2$, l is a natural number such that $2r\leqq l\leqq 3r$. Then $t_1=2(l+2-p)$, $t_2=p+r-l$, and $t_\alpha=1$ $(3\leqq\alpha\leqq p-1)$. We have $n_k=l-k$, $n_{p-k}=r+k-1$ for $k=1, 2, \cdots, r$ and $g=r(l+r-1)$. Let Q_k be the point corresponding to $(c_1^{(k)}, 0)$ for each k $(1\leqq k\leqq p-1)$. The gap sequences $\gamma(Q_k)$ $(1\leqq k\leqq p-1)$ are different from each other.

The next example has respect to the condition (II).

EXAMPLE 3. Let S be defined by

$$y^p=\prod_{j=1}^{mp}(x-c_j)^{p-\alpha}\quad\text{for any } \alpha\ (1\leqq\alpha\leqq p-1),$$

where $m>1$ for $p=3$, and $m\geqq 1$ for $p\geqq 5$. We have $\gamma(Q_j)=\{ip+(p-k);\ 0\leqq i\leqq mk-2,\ 1\leqq k\leqq p-1,\ m\geqq 2\}$ or $\gamma(Q_j)=\{ip+(p-k);\ 0\leqq i\leqq k-2,\ 2\leqq k\leqq p-1,\ m=1\}$ for all j $(1\leqq j\leqq T)$ from (i) of Lemma 3.

EXAMPLE 4. We consider the condition (A_J). Without loss of generality, we take $\beta(1)=p-1$. Then $\alpha(1)=p-1$ and $\alpha(J)=p-J$. Therefore (A_J) can be written as

$$t_{p-J}=1 \text{ and } t_{p-1}=T-1 \text{ for a certain } J\ (2\leqq J\leqq p-1).$$

The number δ can be determined from (7)′. The compact Riemann surface with an automorphism h which satisfies the condition (A_J) is defined by an equation

$$y^p=(x-c_1)^J\prod_{j=1}^{T-1}(x-c_{j+1})\quad\text{for a certain } J\ (2\leqq J\leqq p-1).$$

This has been given by Kato [**3**, p. 406 (71)].

We consider the case of $g=mp(p-1)/2$ $(m\geqq 1)$ and represent the gap sequence explicitly. Such a surface is defined by

$$y^p=(x-c_1)^{p-\alpha}\prod_{j=1}^{mp+1}(x-c_{j+1})^\alpha\quad\text{for any } \alpha\ (1\leqq\alpha\leqq(p-1)/2).$$

Then $t_\alpha=1$ and $t_{p-\alpha}=mp+1$. Let Q_β and $Q_{p-\beta}$ denote the fixed points of h corresponding to $(c_1, 0)$ and $(c_2, 0)$ respectively, where $\alpha\beta\equiv 1 \pmod p$. We have $n_k=ma_{k,p-\alpha}$ $(1\leqq k\leqq p-1)$ from (9) and (ii) of Lemma 1. Since $\beta(p-\alpha)\equiv(p-1)$ (mod p), we get $\Gamma(\beta, 1)=n_\beta=ma_{\beta,p-\alpha}=m$ from (10) and the definition of $a_{\beta,p-\alpha}$. Let σ_k $(1\leqq k\leqq p-1)$ denote the first nongap value at Q_β which is congruent to k (mod p). Then $\sigma_1=mp+1$. This yields $\sigma_k=k\sigma_1$ $(2\leqq k\leqq p-1)$ from (ii) of Lemma 3. Thus $\Gamma(\beta, k)=km$ $(1\leqq k\leqq p-1)$. From (i) of Lemma 2, we have

$\gamma(Q_\beta)=\{jp+k\,;\ 1\leqq k\leqq p-1,\ 0\leqq j\leqq km-1\}$. Moreover we have $\Gamma(p-\beta,\ p-k)=\Gamma(\beta,\ k)=mk$, which shows that $\gamma(Q_{p-\beta})=\{(j+1)p-k\,;\ 1\leqq k\leqq p-1,\ 0\leqq j\leqq mk-1\}$. Therefore $2g-1=\{m(p-1)-1\}p+(p-1)$ is a gap value at Q_β.

EXAMPLE 5. We consider the condition (A_{J-1}) in the case of $T\not\equiv j$ ($j=0$, 1, and $p-1$) (mod p) and $T>4$. Without loss of generality, we take $\beta(1)=1$. Then $\alpha(j)=j$ for each j ($j=1$, 2, and $J-1$). In this case (A_{J-1}) can be written as follows:

(A_{J-1}) $\qquad t_{J-1}=t_2=1$ and $t_1=T-2$ for a certain J $(3\leqq J\leqq p-1)$.

The compact Riemann surface S of genus g with an automorphism h which satisfies the condition (A_{J-1}) is defined by an equation

$$y^p=(x-c_1)^{p-J+1}(x-c_2)^{p-2}\cdot\prod_{j=1}^{T-2}(x-c_{j+2})^{p-1}\quad\text{for a certain } J$$

$(3\leqq J\leqq p-1)$, where $p\geqq 7$. Then the number $2g-2$ is a gap value at the fixed point corresponding to $(c_1,\ 0)$.

The next two examples are related to the condition (A_{J-2}).

EXAMPLE 6. Let S be defined by

$$y^p=(x-c_1)^{p-J+2}(x-c_2)^{p-3}\prod_{j=1}^{T-2}(x-c_{j+2})^{p-1}\quad\text{for a certain } J$$

$(4\leqq J\leqq p-1)$, where $p\geqq 11$, $T>4$, and $T\not\equiv j$ ($j=0$, 1, $p-1$, and $p-2$) (mod p). Then $t_1=T-2$, and $t_3=t_{J-2}=1$. Thus an automorphism h satisfies the condition $(A_{J-2})_3$. The number $2g-3$ is a gap value at the point on S corresponding to $(c_1,\ 0)$.

EXAMPLE 7. Let S be defined by

$$y^p=(x-c_1)^{p-J+2}\prod_{j=1}^{2}(x-c_{j+1})^{p-2}\prod_{j=1}^{T-3}(x-c_{j+3})^{p-1}$$

for a certain J $(4\leqq J\leqq p-1)$, where $p\geqq 11$, $T>4$, and $T\not\equiv j$ ($j=0$, 1, $p-1$, and $p-2$) (mod p). Then $t_1=T-3$, $t_2=2$, and $t_{J-2}=1$. Thus an automorphism h satisfies the condition $(A_{J-2})_2$. The number $2g-3$ is a gap value at the point on S corresponding to $(c_1,\ 0)$.

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