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On infinite dimensional unitary representations of certain discrete groups

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§ 0. Introduction.

0.0. For the modular group $SL_2(\boldsymbol{Z})$, M. Saito [7] has constructed certain series of infinite dimensional unitary representations by classifying and decomposing the representations induced from unitary characters of Cartan subgroups of $SL_2(\boldsymbol{Z})$. The purpose of this note is to make a few remarks which either clarify the interconnection or generalize the results of Saito's construction.

0.1. Let G be a group, and $\mathcal{A}$ a family of subgroups of G. The pair $(G, \mathcal{A})$ is said to have Property $(\mathcal{F})$, if the following two requirements are fulfilled.

($\mathcal{F}1$) For $H_1, H_2 \in \mathcal{A}$, and $g \in G$,
$[H_1 : H_1 \cap g^{-1}H_2 g] < \infty \Rightarrow H_1 \subset g^{-1}H_2 g$.

($\mathcal{F}2$) For $H \in \mathcal{A}$, and $g \in G$,
$g^{-1}Hg \subset H \Rightarrow g^{-1}Hg = H$.

Now, suppose moreover that G is a locally compact topological group and any member H_i of $\mathcal{A}$ is an open subgroup of G. Let χ_i be an irreducible unitary representation of H_i and let $U_i = \mathrm{Ind}(\chi_i : H_i \uparrow G)$ denote the representation of G induced by χ_i. The points of [7] can be summarized in the following (I)～(IV).

(I) Assume that χ_i is one dimensional, then the following three conditions are mutually equivalent (Théorème 2 [7]).

(i) U_1 is equivalent to U_2.

(ii) U_1 is not disjoint from U_2.

(iii) There exists $g \in G$ such that $H_2 = g^{-1}H_1 g$ and $\chi_2 = {}^g\chi_1$, where ${}^g\chi_1(x) = \chi_1(gxg^{-1})$ for $x \in H_2$.

(II) If U_1 is not disjoint from U_2 (hence we may assume $H_1 = H_2 = H$ and $\chi_1 = \chi_2 = \chi$, and put $N_\chi = \{g \in N_G(H) \mid {}^g\chi = \chi\}$), then the dimension of the space of all intertwining operators of $U(\chi) = \mathrm{Ind}(\chi : H \uparrow G)$ is given by the group index $[N_\chi : H]$ (Théorème 1 [7]).

(III) If $G = SL_2(\boldsymbol{Z})$ and $\mathcal{A}$ is the set of all Cartan subgroups of G, then the pair $(G, \mathcal{A})$ has Property $(\mathcal{F})$.

(IV) If G is a connected algebraic group defined over an arbitrary field k, and $\mathcal{A}$ is the set of all connected algebraic subgroups of G defined over k, then the pair $(G, \mathcal{A})$ has Property $(\mathcal{F})$.

0.2. The representations of $SL_2(\boldsymbol{Z})$ constructed in [7] are precisely those obtained as the irreducible constituents of U_i's by taking the pair $(G, \mathcal{A})$ of (III), with discrete topology. Since, in this case, each H_i happens to be commutative, any irreducible representation χ_i is one dimensional. Hence, by (I), the classification up to the equivalence of U_i's reduces to the classification up to the conjugacy of Cartan subgroups H_i's and their characters χ_i's.

Furthermore, each Cartan subgroup H has index 2 or 1 in its normalizer, hence the decomposition of U_i is carried out without much difficulty.

0.3. The purpose of this note is to make the following remarks (1)~(3).

(1) Starting with the pair $(G, \mathcal{A})$ which has Property $(\mathcal{F})$, taking a subgroup G' of G and a subfamily $\mathcal{B}$ of $\mathcal{A}$, and setting $B'=\{H\cap G' \mid H\in\mathcal{B}\}$, we can give a simple criterion for the new pair $(G', \mathcal{B}')$ to have Property $(\mathcal{F})$ (Proposition 1.7).

As an application we can associate to the group $G(\boldsymbol{Z})$ of $\boldsymbol{Z}$-valued points of any connected algebraic group G over $\boldsymbol{Q}$, a family $\mathcal{A}$ such that the pair $(G(\boldsymbol{Z}), \mathcal{A})$ has Property $(\mathcal{F})$ (Corollary 1.9). If $G=SL_2$, we show that $\mathcal{A}$ is, up to commensurability, the set of all Cartan subgroups of $SL_2(\boldsymbol{Z})$ (Corollary 2.2). Thus the case (III) and the case (IV), which appear at a glance of a quite different type, can be connected by our criterion.

(2) We prove the statement (I) without assuming χ_i to be one dimensional (but still finite dimensional) (Theorem 3.3). This generalization is indispensable, since in the case of the pair $(G(\boldsymbol{Z}), \mathcal{A})$ for any arbitrary connected algebraic group G, the family $\mathcal{A}$ contains non-commutative subgroups in general.

(3) We can discuss to some extent the decomposition of the induced representation U_i, without any knowledge of the structure of H_i, but only on the basis of Property $(\mathcal{F})$ (Corollary 3.8).

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§1. General remarks on Property $(\mathcal{F})$.

1.0. Let G be a group. Let $\sim$ denote the commensurability relation in G, and for a subgroup H of G, let $Cl(H)$ denote the commensurability class of H, i.e.

(1) $H_1\sim H_2 \Leftrightarrow [H_i : H_1\cap H_2]<\infty$ for $i=1, 2$.

(2) $Cl(H)=\{K \mid K$ is a subgroup of G such that $K\sim H\}$.

For a family $\mathcal{A}$ of subgroups of G, let $\mathcal{A}^*$ (resp. $\bar{\mathcal{A}}$) denote the commensurability (resp. conjugacy) closure of $\mathcal{A}$, i.e.

(3) $\mathcal{A}^* = \{K \mid K$ is a subgroup of G such that $K \sim H$ for some $H \in \mathcal{A}\}$.

(4) $\bar{\mathcal{A}} = \{g^{-1}Hg \mid g \in G,\ H \in \mathcal{A}\}$.

1.1. The following lemma can be easily checked.

LEMMA. (i) *If the pair* $(G, \mathcal{A})$ *has Property* $(\mathcal{F})$, *then for any subfamily* $\mathcal{B}$ *of* $\mathcal{A}$, $(G, \mathcal{B})$ *has Property* $(\mathcal{F})$.

(ii) *If* $(G, \mathcal{A})$ *has Property* $(\mathcal{F})$, *then* $(G, \bar{\mathcal{A}})$ *has Property* $(\mathcal{F})$.

(iii) *If* $\mathcal{A}$ *is conjugacy closed, i.e.* $\mathcal{A} = \bar{\mathcal{A}}$, *then the property* $(\mathcal{F}1)$ *of* §0 *for* $(G, \mathcal{A})$ *is equivalent to the following* $(\bar{\mathcal{F}}1)$.

$(\bar{\mathcal{F}}1)$ *For* $H_1, H_2 \in \mathcal{A}$, $[H_1 : H_1 \cap H_2] < \infty \Rightarrow H_1 \subset H_2$.

1.2. As is well known, the commensurability relation $\sim$ is an equivalence relation, and we can consider the quotient set $\mathcal{A}/\sim = \{Cl(H) \mid H \in \mathcal{A}\}$ with the canonical projection p.

(1) $p : \mathcal{A} \to \mathcal{A}/\sim \quad p(H) = Cl(H)$.

Furthermore, for the quotient set $\mathcal{A}/\sim$, we can define a structure of an ordered set by the following inclusion relation.

(2) $Cl(H_1) \subset Cl(H_2) \Leftrightarrow {}^{\exists} H_i' \in Cl(H_i)$, where $i = 1, 2$, such that $H_1' \subset H_2'$.

Indeed the following two facts can be easily checked.

(3) $Cl(H_1) \subset Cl(H_2),\ Cl(H_2) \subset Cl(H_1) \Rightarrow Cl(H_1) = Cl(H_2)$.

(4) $Cl(H_1) \subset Cl(H_2),\ Cl(H_2) \subset Cl(H_3) \Rightarrow Cl(H_1) \subset Cl(H_3)$.

1.3. PROPOSITION. (i) *The following three conditions for* $(G, \mathcal{A})$ *are mutually equivalent.*

(1) $(G, \mathcal{A})$ *has the property* $(\mathcal{F}1)$ *of* §0.

(2) $(G, \bar{\mathcal{A}})$ *has the property* $(\bar{\mathcal{F}}1)$ *of* 1.1.

(3) $\bar{\mathcal{A}}$ *is an inclusion preserving section of the canonical projection* $p : (\bar{\mathcal{A}})^* \to (\bar{\mathcal{A}})^*/\sim$, i.e. *the restriction of* p *to* $\bar{\mathcal{A}}$ *gives an isomorphism of* $\bar{\mathcal{A}}$ *and* $(\bar{\mathcal{A}})^*/\sim$ *as ordered sets with respect to the inclusion.*

(ii) *Suppose* $(G, \mathcal{A})$ *has the property* $(\mathcal{F}1)$, *then the following two conditions are mutually equivalent.*

(4) $(G, \mathcal{A})$ *has the property* $(\mathcal{F}2)$ *of* §0.

(5) $(G, (\bar{\mathcal{A}})^*)$ *has the following property* $(\mathcal{F}^*2)$.

$(\mathcal{F}^*2)$ *For* $K \in (\bar{\mathcal{A}})^*$ *and* $g \in G$, $g^{-1}Kg \subset K \Rightarrow Cl(g^{-1}Kg) = Cl(K)$.

POOOF. (i) (1)$\Leftrightarrow$(2) is clear from 1.1. We show (3)$\Rightarrow$(2). Suppose (3) holds. Take $H_1, H_2 \in \bar{\mathcal{A}}$ such that $[H_1 : H_1 \cap H_2] < \infty$. Then H_1 and $H_1 \cap H_2$ are commensurable. Since $H_1 \cap H_2 \subset H_2$, $Cl(H_1) \subset Cl(H_2)$ by the definition (2) of 1.2. As H_1 (resp. H_2) is the image of $Cl(H_1)$ (resp. $Cl(H_2)$) by the inclusion preserving section of (3), we have $H_1 \subset H_2$. Conversely, suppose (2) holds. Take $H_1, H_2 \in \bar{\mathcal{A}}$. If $H_1 \neq H_2$, then H_1 and H_2 are not commensurable. Hence $Cl(H) \mapsto H$ (for $H \in \bar{\mathcal{A}}$) defines the section of the canonical projection $(\bar{\mathcal{A}})^* \to (\bar{\mathcal{A}})^*/\sim$. We must show that this section preserves the inclusion. Suppose $Cl(H_1) \subset Cl(H_2)$. By the definition (2) of 1.2, there exist $H_i' \in Cl(H_i)$ (for $i = 1, 2$) such that $H_1' \subset H_2'$. By easy index calculation, we see that $[H_1 : H_1 \cap H_2] < \infty$. Hence $H_1 \subset H_2$, because of (2).

(ii) To see (4)$\Rightarrow$(5), take $K\in(\bar{\mathcal{A}})^*$ and $g\in G$ such that $g^{-1}Kg\subset K$. There exists $H\in\bar{\mathcal{A}}$ such that $H\sim K$ by the definition of $(\bar{\mathcal{A}})^*$, and then clearly $g^{-1}Kg\sim g^{-1}Hg$. The image of $Cl(H)=Cl(K)$ (resp. $Cl(g^{-1}Hg)=Cl(g^{-1}Kg)$) by the inclusion preserving section is H (resp. $g^{-1}Hg$). Hence we have $g^{-1}Hg\subset H$ from $Cl(g^{-1}Hg)\subset Cl(H)$. Therefore $g^{-1}Hg=H$ by ($\mathcal{F}$2). Thus we get $Cl(g^{-1}Kg)=Cl(K)$. Conversely, suppose (5) holds. Take $H\in\mathcal{A}$ and $g\in G$ such that $g^{-1}Hg\subset H$. Since $H\in(\bar{\mathcal{A}})^*$, $g^{-1}Hg\in(\bar{\mathcal{A}})^*$ and $g^{-1}Hg\subset H$, we have $Cl(H)=Cl(g^{-1}Hg)$. Thus we have $g^{-1}Hg=H$ as the images by the inclusion preserving section.

1.4. COROLLARY. *Suppose* $\mathcal{A}=\bar{\mathcal{A}}$.

(i) *If* $(G, \mathcal{A})$ *has the property* ($\mathcal{F}$1), *then* $\mathcal{A}$ *is the inclusion preserving section of the canonical projection* $p: \mathcal{A}^*\to\mathcal{A}^*/\sim$.

(ii) *Conversely, if* $\mathcal{B}$ *is any conjugacy closed inclusion preserving section of the canonical projection* $p: \mathcal{A}^*\to\mathcal{A}^*/\sim$, *then the pair* $(G, \mathcal{B})$ *has the property* ($\mathcal{F}$1).

1.5. EXAMPLE. Let k be a field, G an algebraic group defined over k and $\mathcal{A}$ the set of all algebraic subgroups of G defined over k. Let $\mathcal{A}_0$ denote the set of all connected algebraic subgroups of G defined over k. Any two elements of $\mathcal{A}$ are commensurable if and only if they have the same connected component of the identity element. Therefore the pair $(G, \mathcal{A}_0)$ has the property ($\mathcal{F}$1). Since the dimension of any element of $\mathcal{A}_0$ is invariant by the inner automorphisms of G, the pair $(G, \mathcal{A}_0)$ has the property ($\mathcal{F}$2).

Hence the pair $(G, \mathcal{A}_0)$ has Property ($\mathcal{F}$) and obviously $\mathcal{A}_0^*=\mathcal{A}$ and $\bar{\mathcal{A}}_0=\mathcal{A}_0$.

1.6. REMARK. (i) In view of 1.1, we may assume $\mathcal{A}=\bar{\mathcal{A}}$ without important loss of generality for our purpose. However in the statement of 1.4, the assumption, $\mathcal{A}=\bar{\mathcal{A}}$, is essential. For example, if $\mathcal{A}^*/\sim$ has only one point, say $Cl(H)$, then the assumption $\mathcal{A}=\bar{\mathcal{A}}$ reduces the case to the trivial one where H is normal in G.

(ii) $\mathcal{A}$ is not necessarily unique for a given $\mathcal{A}^*$. For example let $\mathcal{A}^*$ be the set of all one dimensional algebraic subgroups of G of Example 1.5. Then any conjugacy closed section of the canonical projection $p: \mathcal{A}^*\to\mathcal{A}^*/\sim$ preserves the inclusion, because there is no non-trivial order relation in $\mathcal{A}^*/\sim$.

1.7. PROPOSITION. *Let* $(G, \mathcal{A})$ *be a pair with Property* ($\mathcal{F}$). *Suppose* G *has the topology such that the left and right translations are closed mappings. For a subgroup* G' *of* G, *put* $\mathcal{A}'=\{H\cap G' \mid H\in\mathcal{A}$ *and* $H\cap G'$ *is dense in* $H\}$. *Then the pair* $(G', \mathcal{A}')$ *has Property* ($\mathcal{F}$).

PROOF. For a subset X of G, let $\bar{X}$ denote its topological closure. If $H_i'\in\mathcal{A}'$, by our definition of $\mathcal{A}'$, H_i' has the form $H_i'=H_i\cap G'$ with $H_i\in\mathcal{A}$ and $\overline{H_i'}=H_i$.

To see ($\mathcal{F}$1), note that:

$[H_1': H_1'\cap x^{-1}H_2'x]<\infty$, where $x\in G'$

$\Leftrightarrow {}^{\exists}g_j \in H_1'$ for $j \leqq N$, $H_1' = \bigcup_{j=1}^{N} g_j(H_1' \cap x^{-1}H_2'x)$,

where N is a suitable natural number

$\Rightarrow H_1 = \overline{H_1'} = \bigcup_{j=1}^{N} \overline{g_j(H_1' \cap x^{-1}H_2'x)}$.

Now, the closedness of translations implies:

$\overline{g_j(H_1' \cap x^{-1}H_2'x)} \subset g_j\overline{(H_1' \cap x^{-1}H_2'x)} \subset g_j(\overline{H_1'} \cap \overline{x^{-1}H_2'x}) \subset g_j(\overline{H_1'} \cap x^{-1}\overline{H_2'}x)$

$= g_j(H_1 \cap x^{-1}H_2x)$.

Namely $[H_1' : H_1' \cap x^{-1}H_2'x] < \infty$ implies $[H_1 : H_1 \cap x^{-1}H_2x] < \infty$, hence $H_1 \subset x^{-1}H_2x$ and $H_1' \subset x^{-1}H_2'x \cap G' = x^{-1}H_2'x$.

To see ($\mathcal{F}$2), take $H' \in \mathcal{A}'$ and $x \in G'$ such that $x^{-1}H'x \subset H'$. Since H' has the form $H' = H \cap G'$ with $H \in \mathcal{A}$ and $\overline{H'} = H$, $x^{-1}H'x \subset H'$ implies $H = \overline{H'} \subset \overline{xH'x^{-1}} \subset x\overline{H'}x^{-1} = xHx^{-1}$. Hence $H = xHx^{-1}$ by the property ($\mathcal{F}$2) for $(G, \mathcal{A})$ and $H' = H \cap G' = xHx^{-1} \cap G' = xH'x^{-1}$.

1.8. COROLLARY. *Let k be an infinite perfect field and G an algebraic group defined over k. Let $\mathcal{A}$ be the set of all connected algebraic subgroups of G defined over k and let $G' = G(k)$ the group of k-rational points of G, and $\mathcal{A}' = \{H(k) \mid H \in \mathcal{A}\}$. Then the pair $(G', \mathcal{A}')$ has Property ($\mathcal{F}$).*

PROOF. Combine 1.5 and 1.7, and use the fact that if k is perfect and infinite, then $H(k)$ is Zariski dense in H which is a connected algebraic group defined over k (Rosenlicht [**6**]).

1.9. COROLLARY. *Let k, G and $\mathcal{A}$ be as in 1.8. Let $\mathcal{O}$ be a subring of k with the identity and $G' = G(\mathcal{O})$: the group of $\mathcal{O}$-valued points and put $\mathcal{A}' = \{H(\mathcal{O}) \mid H \in \mathcal{A}$ and $H(\mathcal{O})$ is Zariski-dense in $H\}$. Then the pair $(G', \mathcal{A}')$ has Property ($\mathcal{F}$).*

PROOF. It is immediate from 1.7.

This example will be discussed in more detail in the next section. In particular, it will be seen that the pair $(G, \mathcal{A})$ of (III) in §0 is essentially a special case of our $(G', \mathcal{A}')$.

§2. Remarks on $SL_2(\boldsymbol{Z})$.

2.0. Let G be SL_2 and $\mathcal{A}$ a family of connected algebraic subgroups of G defined over $\boldsymbol{Q}$. Let G' denote $SL_2(\boldsymbol{Z})$ and $\mathcal{B}$ denote the subfamily of $\mathcal{A}$ such that $H \in \mathcal{B}$ if and only if $H \cap G'$ is Zariski dense in H.

2.1. PROPOSITION. (i) *$H \in \mathcal{A}$ belongs to $\mathcal{B}$ if and only if H is equal to one of the following three.*

(1) *$H = G$,*

(2) *$H \cong \boldsymbol{G}_m$ over the algebraic closure $\bar{\boldsymbol{Q}}$ of $\boldsymbol{Q}$ and $|H(\boldsymbol{Z})| = \infty$,*

(3) *$H \cong \boldsymbol{G}_a$ over $\boldsymbol{Q}$.*

(ii) *In the case (2), we have $[N_G(H) : H] = 2$, hence $[N_{G'}(H') : H'] \leqq 2$,* *where* $H'=H(\boldsymbol{Z})$.

(iii) *In the case* (3), *we have* $H\cong\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}$. *Then* $N_G(H)\cong B=\begin{pmatrix}* & * \\ 0 & *\end{pmatrix}$: *a Borel subgroup of* G. *Accordingly,* $H'\cong\begin{pmatrix}1 & \boldsymbol{Z} \\ 0 & 1\end{pmatrix}$ *and* $N_{G'}(H')\cong B(\boldsymbol{Z})=\pm\begin{pmatrix}1 & \boldsymbol{Z} \\ 0 & 1\end{pmatrix}$.

PROOF. To see (i), take $H\in\mathcal{A}$. Since $\dim SL_2=3$, $\dim H\leqq 3$. If $\dim H=3$, then $H=SL_2$ by the connectedness of H and SL_2. Since $SL_2(\boldsymbol{Z})$ is Zariski dense in SL_2 by Borel [2], this is the case (1).

If $\dim H\leqq 2$, then H is solvable over $\bar{\boldsymbol{Q}}$ by Borel [1] (Theorem 11.6). So there exists a Borel subgroup defined over $\bar{\boldsymbol{Q}}$ which contains H.

If $\dim H=2$, then H itself is a Borel subgroup. Since H is defined over $\boldsymbol{Q}$, by the uniqueness of the minimal parabolic subgroup (Borel-Tits [3]) H is a split Borel subgroup, i.e. $H\cong\begin{pmatrix}* & * \\ 0 & *\end{pmatrix}$. But $H(\boldsymbol{Z})\cong\pm\begin{pmatrix}1 & \boldsymbol{Z} \\ 0 & 1\end{pmatrix}$ is not Zariski dense in H.

If $\dim H=1$, then by Borel [1] (Theorem 10.9) H is isomorphic to $\boldsymbol{G}_m$ or $\boldsymbol{G}_a$ over $\bar{\boldsymbol{Q}}$. If H is isomorphic to $\boldsymbol{G}_m$, then clearly $H(\boldsymbol{Z})$ is Zariski dense in $\boldsymbol{H}$ if and only if $|H(\boldsymbol{Z})|=\infty$. This is the case (2). If H is isomorphic to $\boldsymbol{G}_a$, then H is isomorphic to $\boldsymbol{G}_a$ over $\boldsymbol{Q}$ by Borel [1] (remark after Theorem 10.9). Again, by the uniqueness of the minimal $\boldsymbol{Q}$-parabolic subgroup, H is isomorphic to the unipotent radical of a suitable split Borel subgroup. This is the case (3).

(i) In the case (2), H is the maximal torus of SL_2. Therefore $N_G(H)/H$ is isomorphic to the Weyl group of SL_2, which is isomorphic to the symmetric group of degree 2. Thus we get (ii).

(iii) In the case (3), clearly $N_{G'}(H')=G'\cap N_G(H)=B(\boldsymbol{Z})$.

2.2. COROLLARY. Let $\mathcal{B}'=\{H\cap G'\,|\,H\in\mathcal{B},\ H\neq G\}$ *and* $\mathcal{C}$ *be the set of all Cartan subgroups of* G'. *Then* $(\mathcal{B}')^*=\mathcal{C}^*$. Here, the definition of a Cartan subgroup C is in the sense of Chevalley characterized by the following.

(1) C is a maximal nilpotent subgroup, and

(2) every subgroup of finite index in C has finite index in its normalizer in G' (cf. Borel [1] p. 290).

PROOF. Let $C\in\mathcal{C}$ and $\bar{C}$ (resp. $\bar{C}^0$) be the Zariski closure of C in SL_2 (resp. the connected component of the identity of $\bar{C}$). $[\bar{C}:\bar{C}^0]<\infty$ and $\bar{C}$ normalizes $\bar{C}^0$. Since $\bar{C}^0$ is nilpotent and connected, $\bar{C}^0$ is isomorphic to $\boldsymbol{G}_a$ or $\boldsymbol{G}_m$ over $\bar{Q}$. By the nilpotency of $\bar{C}$ and the maximality of C, it follows that $C=\bar{C}\cap SL_2(\boldsymbol{Z})$. Moreover $|C|=\infty$ by the definition of a Cartan subgroup. Hence $|\bar{C}^0(\boldsymbol{Z})|=|\bar{C}^0\cap SL_2(\boldsymbol{Z})|=\infty$. Therefore by a proper H of the type of (2) or (3) in $\mathcal{B}$, we have $N_{G'}(H')\supset C\supset H'$. In the case (2), $N_{G'}(H')$ induces the action of the Weyl group on H', i.e. $nhn^{-1}=h^{-1}$ for $h\in H'$ and $n\in N_{G'}(H')$, $n\notin H'$, hence $N_{G'}(H')$ is not nilpotent. Thus C must be equal to H'.

In the case (3), by (3) of 2.1, $N=N_{G'}(H')$ is nilpotent. Since C is a maximal nilpotent subgroup, C must be equal to N.

Therefore we see that for any $H\in \mathcal{B}'$ (resp. $C\in \mathcal{C}$) there exists a suitable $C\in \mathcal{C}$ (resp. $H\in \mathcal{B}'$) such that $H\sim C$. Hence we have $(\mathcal{B}')^*=\mathcal{C}^*$.

In particular, if we denote by $\mathcal{D}$ the set $\{N_{G'}(C)\mid C\in \mathcal{C}\}$, then we have $\mathcal{D}^*=\mathcal{C}^*$, because $[N_{G'}(C): C]<\infty$ by the definition of a Cartan subgroup.

2.3. REMARK. (i) In the view points of the construction of the representations of $SL_2(\boldsymbol{Z})$ induced from the characters of a subgroup of $SL_2(\boldsymbol{Z})$ as will be seen in § 3, the choice of an inclusion preserving section of the canonical projection $\mathcal{C}^*=(\mathcal{B}')^*\to \mathcal{C}^*/\sim$ does not yield any essential difference.

(ii) Given an algebraic group defined over a field k, the problem of the classification of $\mathcal{A}'$ in 1.9 can be very complicated. However there are some cases where such classifications are essentially known. For example, let $G=SL_2$ and $G'=\Gamma_0(N)$. Then the classification is implicitly done in efforts to give an explicit formula for the traces of Hecke operators (cf. Hijikata [4]).

§ 3. Representations.

3.0. In this section, let G be a separable locally compact group, and $\mathcal{A}$ be a conjugacy closed family of open subgroups of G. Suppose that the pair $(G, \mathcal{A})$ has Property $(\mathcal{F})$.

Let $K\in \mathcal{A}^*$ and $\rho: K\to GL(V)$ be a finite dimensional unitary representation, where V denotes a finite dimensional vector space over the complex number field $\boldsymbol{C}$ with the scalar product $(\ ,\ )$.

Let $U(\rho)$ denote the representation of G induced from ρ. By the definition of $\mathcal{A}^*$, there exists some $H\in \mathcal{A}$ such that $H\sim K$. Since such H is unique by 1.3, let us denote this H by $\boldsymbol{H}(K)$. Let K' be another member of $\mathcal{A}^*$ in the same commensurability class as K, $\boldsymbol{H}(K)=\boldsymbol{H}(K')$. If $K\supset K'$, let ρ' be the restriction of ρ to K'. Then every irreducible constituent of $U(\rho')$ is contained in $U(\rho)$. Hence in the view points of the construction of the representations we may restrict our attention to only large enough K in $Cl(H)$.

For example we may assume $K\supset \boldsymbol{H}(K)$ without any important loss of generality.

3.1. LEMMA. *Assume* $K\supset \boldsymbol{H}(K)=H$, *and put* $\chi=\rho\mid_H$ *and* $N_\chi=\{g\in N_G(H)\mid \chi\sim {}^g\chi\}$. *Then* K *is a subgroup of* N_χ.

PROOF. If $k\in K$, then $[H: H\cap k^{-1}Hk]\leqq [K: k^{-1}Hk]=[K: H]<\infty$. Hence $H\subset k^{-1}Hk$ by the property $(\mathcal{F}1)$ and then $H=k^{-1}Hk$ by the property $(\mathcal{F}2)$. Since $\chi(k^{-1}hk)=\rho(k^{-1}hk)=\rho(k)^{-1}\chi(h)\rho(k)$ for any $k\in K$ and any $h\in H$, we have $k\in N_\chi$.

3.2. We assume the quotient $K\backslash G$ is denumerable for any $K\in \mathcal{A}^*$. Then recall that $U(\rho)$ is realized on the Hilbert space

$$\mathcal{V}=\{f:\Theta\to V\mid \|f\|^2=\sum_{x\in\Theta}|f(x)|^2<\infty\}$$

by the action of $g\in G$ as follows,

$$(U(\rho)(g)f)(x)=\rho(\eta(xg))f(\theta(xg)) \quad \text{for } f\in\mathcal{V} \text{ and } x\in\Theta.$$

Here Θ denotes a system of representatives of the quotient $K\backslash G$, $|f(x)|^2=(f(x), f(x))$, and θ is the section $K\backslash G\to\Theta$, and $\eta(g)=g\theta(Kg)^{-1}$ is a mapping from G into K.

This action of $g\in G$ is essentially independent of the choice of the system Θ. For, if Θ' denotes another system of representatives of the quotient $K\backslash G$, and $\mathcal{V}'$ denotes another space with respect to Θ', then we can define a unitary operator $I:\mathcal{V}\to\mathcal{V}'$ such that $I\circ U(\rho)(g)=U(\rho)(g)\circ I$ for $g\in G$ as follows.

$$(I(f))(x')=\rho(\eta(x')f(\theta(x')) \quad \text{for } f\in\mathcal{V} \text{ and } x'\in\Theta'.$$

In particular, we may assume that the system Θ contains the identity element of G.

3.3. THEOREM. *Let G be a separable locally compact group and $\mathcal{A}$ be a family of open snbgroups of G such that the pair $(G, \mathcal{A})$ has Property $(\mathcal{F})$. Let $\mathcal{A}^*$ be a commensurability closure of $\mathcal{A}$. Let $K_i\in\mathcal{A}^*$ and let $H_i=\boldsymbol{H}(K_i)$ and assume $K_i\supset H_i$, where $i=1, 2$. Let ρ_i be a unitary representation of K_i acting on a finite dimensional vector space V_i over $\boldsymbol{C}$, and $\chi_i=\rho_i|_{H_i}$ the restriction of ρ_i to H_i.*

If χ_i's are irreducible, then

(i) *$U(\rho_1)$ and $U(\rho_2)$ are disjoint from each other unless there exists $g\in G$ such that $H_2=g^{-1}H_1g$ and $\chi_2={}^g\chi_1$.*

(ii) *If $H_1=H_2=H$ and $\chi_1=\chi_2=\chi$, then the dimension of the space of all intertwining operators from $U(\rho_2)$ to $U(\rho_1)$ is not greater than the group index $[N_\chi: K_1]$.*

(iii) *In particular, if $K_1=K_2=N_\chi$, then $U(\rho_1)$ and $U(\rho_2)$ are equivalent to each other if and only if ρ_1 and ρ_2 are equivalent to each other.*

PROOF. We use the notations in 3.2 attaching the index i as $\mathcal{V}_i$, θ_i, η_i, etc. for $i=1, 2$.

Suppose $\dim V_i=n_i$ and let $\{v_t\mid t=1, 2, \cdots, n_1\}$ (resp. $\{u_j\mid j=1, 2, \cdots, n_2\}$) be a basis of V_1 (resp. V_2). We may assume these bases are orthonormal.

Then we can set, for any $k_i\in K_i$,

$$\rho_2(k_2)u_j=\sum_{s=1}^{n_2} a_{j,s}(k_2)u_s \quad a_{j,s}(k_2)\in\boldsymbol{C}$$

and

$$\rho_1(k_1)v_t=\sum_{r=1}^{n_1} b_{t,r}(k_1)v_r \quad b_{t,r}(k_1)\in\boldsymbol{C}.$$

Let φ_x (resp. ψ_y) denote the characteristic function on Θ_2 (resp. Θ_1) of x (resp. y).

Under these notations, we have

$$U(\rho_2)(g)(u_j\varphi_x)=\sum_{s=1}^{n_2} a_{j,s}(\eta_2(\theta_2(xg^{-1})g))u_s\varphi_{\theta_2(xg^{-1})}$$

for any $g\in G$, where $x\in\Theta_2$ and $u_j\psi_x$ denotes the assignment $x'\mapsto\varphi_x(x')u_j$ for $x'\in\Theta_2$.

3.4. LEMMA. *Let $\mathcal{E}(U(\rho_2), U(\rho_1))$ be the space of all intertwining operators from $U(\rho_2)$ to $(U\rho_1)$. If there exists a non trivial $M\in\mathcal{E}(U(\rho_2), U(\rho_1))$, then $H_2\subset x^{-1}H_1x$ for any $x\in\bigcup_{j=1}^{n_2}$ Supp$\|M(u_j\varphi_e)\|\subset\Theta_1$.*

PROOF. Since we have

$$U(\rho_1)(k)M(u_j\varphi_e)=\sum_{s=1}^{n_2} a_{j,s}(k)M(u_s\varphi_e) \quad \text{for any } k\in K_1,$$

it holds that

$$\sum_{j=1}^{n_2} |U(\rho_1)(k)M(u_j\varphi_e)(x)|^2=\sum_{j=1}^{n_2} |M(u_j\varphi_e)(x)|^2$$

for each $x\in\Theta_1$. On the other hand we have

$$\sum_{j=1}^{n_2} |U(\rho_1)(k)M(u_j\varphi_e)(x)|^2=\sum_{j=1}^{n_2} |M(u_j\varphi_e)(\theta_1(xk))|^2$$

by the definition of $U(\rho)$. Hence we get

$$(1) \qquad \sum_{j=1}^{n_2} |M(u_j\varphi_e)(x)|^2=\sum_{j=1}^{n_2} |M(u_j\varphi_e)(\theta_1(xk))|^2$$

for each $x\in\Theta_1$ and each $k\in K_2$.

Therefore if $x\in$ Supp$\|M(u_j\varphi_e)\|$ for some j, then the orbit of the action of K_2 on Θ_1 containing x must be a finite set, because we have (1) and

$$\sum_{x\in\Theta_1}\sum_{j=1}^{n_2} |M(u_j\varphi_e)(x)|^2=\sum_{j=1}^{n_2} \|M(u_j\varphi_e)\|<\infty.$$

In other words $[K_2: K_2\cap x^{-1}K_1x]<\infty$. This implies $[H_2: H_2\cap x^{-1}H_1x]<\infty$ and hence $H_2\subset x^{-1}H_1x$ by the property ($\mathcal{F}$1). This completes the proof of the lemma.

3.5. PROOF OF 3.3 (i). Suppose $U(\rho_2)$ and $U(\rho_1)$ are not disjoint. That is to say that there exist non trivial members $M\in\mathcal{E}(U(\rho_2), U(\rho_1))$ and $N\in\mathcal{E}(U(\rho_1), U(\rho_2))$. Accordingly we have $\bigcup_j$ Supp$\|M(u_j\varphi_e)\|\neq\emptyset$ and $\bigcup_t$ Supp$\|N(v_t\psi_e)\|\neq\emptyset$. By the lemma of 3.4, there exists $x\in\Theta_1$ and $y\in\Theta_2$ such that $H_2\subset x^{-1}H_1x$ and $H_1\subset y^{-1}H_2y$. Thus we get $H_2\subset x^{-1}H_1x\subset x^{-1}y^{-1}H_2yx$, hence $yx\in N_G(H_2)$ by the property ($\mathcal{I}2$) and $H_2=x^{-1}H_1x$. This shows the first part of (i).

To see the second part of (i), we may assume $H_1=H_2=H$ by the first part of (i). Then it is clear that $\bigcup_j \mathrm{Supp}\|M(u_j\varphi_e)\| \subset N_G(H)\cap\Theta_1$. So we get, for each j,

$$M(u_j\varphi_e)=\sum_{\substack{1\leq t\leq n_1\\ x\in N_G(H)\cap\Theta_1}} \alpha_{j,t}(x)(v_t\phi_x), \quad \alpha_{j,t}(x)\in \boldsymbol{C}.$$

This is symbolically

$$(1) \qquad {}^t(\cdots, M(u_j\varphi_e), \cdots)=\sum_{x\in N_G(H)\cap\Theta_1}(\alpha_{j,t}(x))^t(\cdots, v_t\phi_x, \cdots)$$

where t on the left shoulder denotes the transposing symbol and $(\alpha_{j,t}(x))$ is an $n_2\times n_1$ matrix.

Applying $U(\rho_1)(h)$ $(h\in H)$ to the both sides of (1), we have

$$\begin{aligned}\text{the left side} &= {}^t(\cdots, MU(\rho_2)(h)(u_j\varphi_e), \cdots)\\ &= {}^t(\cdots, M(\sum_s a_{j,s}(h)(u_s\varphi_e)), \cdots)\\ &= (a_{j,s}(h))^t(\cdots, M(u_s\varphi_e), \cdots)\end{aligned}$$

$$(2) \qquad = \sum_{x\in N_G(H)\cap\Theta_1}(a_{j,s}(h))(\alpha_{j,t}(x))^t(\cdots, v_t\phi_x, \cdots)$$

and

$$\text{the right side}=\sum_{x\in N_G(H)\cap\Theta_1}(\alpha_{j,t}(x))^t(\cdots, U(\rho_1)(h)(v_t\phi_x), \cdots)$$

$$(3) \qquad = \sum_{x\in N_G(H)\cap\Theta_1}(\alpha_{j,t}(x))(b_{t,r}(xhx^{-1}))^t(\cdots, v_r\phi_x, \cdots).$$

Since $\{v_t\phi_x | j, x\}$ is a linearly independent subset of V_1, comparing (2) and (3) we get

$$(a_{j,s}(h))(\alpha_{j,t}(x))=(\alpha_{j,t}(x))(b_{t,r}(xhx^{-1})).$$

Since M is non trivial, there exists some $x\in N_G(H)\cap\Theta_1$ such that $(\alpha_{j,t}(x))\neq 0$. Hence the irreducibility of χ_i's shows that $n_1=n_2$ and $(\alpha_{j,t}(x))$ is invertible by the Schur's lemma. This implies $\chi_2\sim{}^x\chi_1$, because $\chi_2(h)=(a_{j,s}(h))$, and ${}^x\chi_1(h)=(b_{t,r}(xhx^{-1}))$ by the definition.

3.6. PROOF OF 3.3 (ii). We may assume $\chi_2=\chi_1=\chi$ by (i). Since $\chi\sim{}^x\chi$ for any x such that $(\alpha_{j,t}(x))\neq 0$, it follows that if $M\in\mathcal{E}(U(\rho_2), U(\rho_1))$, then $M(u_j\varphi_e)$ appears in

$$\langle v_t\phi_x | t=1, \cdots, n, x\in N_\chi\cap\Theta_1\rangle_C$$

for each j, where $n=\dim\chi$ and $\langle S\rangle_C$ denotes the vector subspance spanned by the subset S of $\mathcal{V}$ over $\boldsymbol{C}$. Since the action of G on Θ_2 is transitive, $\{u_j\varphi_e | j\}$ generates the space $\mathcal{V}_2$ as a G-space. Therefore, to define a member M in

$\mathcal{E}(U(\rho_2), U(\rho_1))$, we must define a suitable linear mapping:

$$\langle u_j\varphi_e \mid j\rangle_C \to \langle v_j\psi_x \mid j,\ x\in N_\chi\cap\Theta_1\rangle_C .$$

Clearly $\langle v_j\psi_x \mid j,\ x\rangle_C = \bigoplus_x \langle v_j\psi_x \mid j\rangle_C$ (direct sum of vector spaces). We can easily check that $\langle u_j\varphi_e \mid j\rangle_C$ (resp. $\langle v_j\psi_x \mid j\rangle_C$) is closed under the action of H by $U(\rho_2)(H)$ (resp. $U(\rho_1)(H)$) and is isomorphic to V_2 (resp. V_1) as an H-space. We note that V_1 and V_2 are isomorphic to each other as H-spaces by our assumption. Thus, since V_i's are irreducible H-spaces, we have

$$\dim \mathrm{Hom}_H(\langle u_j\varphi_e \mid j\rangle_C,\ \langle v_j\psi_x \mid j\rangle_C)=1 .$$

Therefore $\dim \mathcal{E}(U(\rho_2), U(\rho_1))$ is not greater than the cardinality $|N_\chi\cap\Theta_1|$. Since $|N_\chi\cap\Theta_1|=[N_\chi : K_1]$, we get (ii).

3.7. PROOF OF 3.3 (iii). Since $K_1=K_2=N_\chi\subset N_G(H)$, we may assume that $\Theta_1=\Theta_2$, $v_j=u_j$ for each j, and $\varphi_e=\psi_e$. Then we have, for each j,

$$M(u_j\varphi_e)=\sum_{t=1}^{n}\alpha_{j,t}(u_t\varphi_e) \quad \alpha_{j,t}\in \boldsymbol{C} .$$

(Note that $N_G(H)\cap\Theta_1=\{e\}$.) This is symbolically

$$(1) \qquad {}^t(\cdots,\ M(u_j\varphi_e),\ \cdots)=(\alpha_{j,t})\,{}^t(\cdots,\ u_t\varphi_e,\ \cdots) .$$

Applying $U(\rho_1)(k)$ $(k\in K_1=K_2=N_\chi)$ to the both side of (1), we get

$$(a_{j,t}(k))(\alpha_{j,t})\,{}^t(\cdots,\ u_t\varphi_e,\ \cdots)=(\alpha_{j,t})(b_{j,t}(k))\,{}^t(\cdots,\ u_t\varphi_e,\ \cdots) .$$

Since $\{u_j\varphi_e \mid j\}$ is a linearly independent subset of $\mathcal{V}$, we have $(a_{j,t}(k))(\alpha_{j,t})=(\alpha_{j,t})(b_{j,t}(k))$. If $U(\rho_1)$ and $U(\rho_2)$ are not disjoint, then there exists non trivial member $M\in\mathcal{E}(U(\rho_2), U(\rho_1))$, and then $(\alpha_{j,t})\neq 0$. Since ρ_i's are irreducible, $(\alpha_{j,t})$ is invertible. That is to say $\rho_1\sim\rho_2$. This completes the proof.

3.8. Under the same notations as in 3.3, let $W(\rho_2, \rho_1)$ be the set of all $x\in N_\chi\cap\Theta_1$ which satisfy the following two condition.

(1) x is fixed by K_2, i.e. $x=\theta_1(xk)$ for $\forall k\in K_2$.

(2) $\rho_2={}^x\rho_1$ on $x^{-1}K_1x\cap K_2$.

COROLLARY. *If* $\dim\chi=1$ *in* 3.3 (ii), *then*

$$|W(\rho_2, \rho_1)|\leqq \dim\mathcal{E}(U(\rho_2), U(\rho_1))\leqq |K_1\backslash N_\chi/K_2| .$$

PROOF. From Mackey [5] Theorem 3′, we have

$$\dim\mathcal{E}(U(\rho_2), U(\rho_1))=\sum_{D\in\mathcal{D}_f}\dim\mathcal{E}(\rho_2, \rho_1 : D) ,$$

where $\mathcal{D}_f$ denote the set of all double cosets, namely $D=K_1xK_2$ $(x\in G)$, such that K_2 and $x^{-1}K_1x$ are commensurable, and $\mathcal{E}(\rho_2, \rho_1 : D)$ denotes the space of all intertwining operators between the restrictions of ρ_2 and ${}^x\rho_1$ to $x^{-1}K_1x\cap K_2$.

The dimension of $\mathcal{E}(\rho_2, \rho_1: D)$ is independent of the choice of the representative x of $D=K_1xK_2$.

If $D=K_1xK_2\in\mathcal{D}_f$, then the commensurability of $x^{-1}K_1x$ and K_2 shows $x\in N_G(H)$ by Property $(\mathcal{T})$ for $(G, \mathcal{A})$. Moreover $\dim\mathcal{E}(\rho_2, \rho_1: D)=1$ or 0, because $\dim\rho_2=\dim\rho_1=1$. If this value is equal to 1, then $x\in N_\chi$. Thus we have

$$\dim\mathcal{E}(U(\rho_2), U(\rho_1))\leqq|K_1\backslash N_\chi/K_2|.$$

On the other hand, if $x\in W(\rho_2, \rho_1)$, then $K_2\subset x^{-1}K_1x$ from $x=\theta_1(xk)$ for any $k\in K_2$, and then we can define a member of $\mathcal{E}(U(\rho_2), U(\rho_1))$ by setting $\varphi_e\mapsto\psi_x$ from $\rho_2={}^x\rho_1$. So we get $|W(\rho_2, \rho_1)|\leqq\dim\mathcal{E}(U(\rho_2), U(\rho_1))$.

3.9. REMARK. In 3.8, if we take $K_2=K_1=H_2=H_1=H$ and $\rho_1=\rho_2=\chi_1=\chi_2=\chi$, then it holds that $\dim\mathcal{E}(U(\chi), U(\chi))=|N_\chi/H|$. This is the result of Théorème 1 of Saito [7].

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