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Regular embeddings of C^*-algebras in monotone complete C^*-algebras

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Introduction.

Let A be a unital C^*-algebra and $A_{s.a.}$ the self-adjoint part of A. If each bounded increasing net (resp. sequence) in $A_{s.a.}$ has a supremum then A is said to be *monotone* (resp. *monotone* σ-) *complete*. [In the literature, e.g., [**10**, **16**, **20**], the adjective "monotone (resp. monotone σ-) closed" is employed as a synonym for "monotone (resp. monotone σ-) complete", but in this paper we will use it in a different sense (cf. Definition 1.2).] As was shown by J. D. M. Wright [**22**], each unital C^*-algebra A possesses a unique *regular σ-completion*, i.e., a monotone σ-complete C^*-algebra $\hat{A}$ which contains A as a C^*-subalgebra and satisfies the following properties:

i) $\hat{A}_{s.a.}$ itself is a unique monotone σ-closed subspace of $\hat{A}_{s.a.}$ which contains $A_{s.a.}$;

ii) each x in $\hat{A}_{s.a.}$ is the supremum in $\hat{A}_{s.a.}$ of $\{a\in A_{s.a.}: a\leqq x\}$; and

iii) whenever a subset $\mathcal{F}$ of $A_{s.a.}$ has a supremum x in $A_{s.a.}$ then x remains the supremum of $\mathcal{F}$ in $\hat{A}_{s.a.}$.

On the other hand the present author proved in [**6**] that each unital C^*-algebra A has a unique *injective envelope*, which will be written as $I(A)$, i.e., a minimal injective C^*-algebra containing A as a C^*-subalgebra. In this paper we give a monotone complete version of the above J. D. M. Wright's result by embedding A in its injective envelope $I(A)$ (Theorem 3.1). Namely it is shown that the monotone closure $\bar{A}$ of A in $I(A)$ is a monotone complete C^*-algebra which satisfies the above properties i), ii) and iii) with $\hat{A}$ replaced by $\bar{A}$ and moreover "monotone σ-" in i) replaced by "monotone". We call $\bar{A}$ the *regular monotone completion* of A. To see that $\bar{A}$ satisfies ii) we consider the family of all unital C^*-algebras which contain A as a C^*-subalgebra and satisfy ii) (called "regular extensions" of A) and we show that, instead of $\bar{A}$, a maximal regular extension of A, written $\tilde{A}$, is realized as a monotone closed C^*-subalgebra of $I(A)$, hence that $\bar{A}\subset\tilde{A}$ satisfies ii). By the construction we have the canonical inclusions $A\subset\hat{A}\subset\bar{A}\subset\tilde{A}\subset I(A)$; however it remains open whether or not the inclusions $\bar{A}\subset\tilde{A}\subset I(A)$ can be proper. In case A is GCR we will see that $\bar{A}=\tilde{A}=I(A)$ (Theorem 6.6).

The contents of the paper are summarized as follows. In section 1 we establish notation and provide preliminary lemmas. In section 2 we give a Banach space-like characterization of regular extensions. Section 3 is devoted to the proof of the existence and uniqueness of $\bar{A}$ and $\tilde{A}$. Section 4 concerns the embedding of a C^*-algebra into another C^*-algebra which preserves suprema and infima, and in section 5 the results of section 4 are applied to examine the regular extensions of the minimal C^*-tensor products of special C^*-algebras. In section 6 we investigate the type I direct summand of the injective envelope $I(A)$. In section 7 we characterize such a C^*-algebra whose regular monotone completion is an AW^*-factor.

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§ 1. Preliminaries and notation.

Throughout the paper C^*-algebras to be considered are always unital, and A and $I(A)$ denote an arbitrary but fixed C^*-algebra and its injective envelope, respectively. The algebra $I(A)$ exists uniquely for any A and is characterized as an injective C^*-algebra, containing A as a C^*-subalgebra, such that the identity map $\mathrm{id}_{I(A)}$ on $I(A)$ is a unique completely positive map of $I(A)$ into itself which fixes A elementwise [6]. The sets of all positive elements, projections, unitary elements of A, the center of A and the state space of A are denoted by A^+, A_p, A_u, Z_A and $S(A)$, respectively.

By an *extension* of A we mean a pair (B, κ) of a C^*-algebra B and a unital *-monomorphism κ of A into B. In what follows we sometimes identify A with $\kappa(A)\subset B$ and abbreviate (B, κ) to B. In the family of all extensions of A we define a relation $\prec$ (resp. equivalence relation $\sim$) by $(B, \kappa)\prec(C, \lambda)$ [resp. $(B, \kappa)\sim(C, \lambda)$] if there exists a unital *-monomorphism (resp. *-isomorphism) ι of B into (resp. onto) C with $\iota\circ\kappa=\lambda$.

DEFINITION 1.1. An extension (B, κ) of A is *regular* if each x in $B_{s.a.}$ is the supremum in $B_{s.a.}$ of $\{a\in\kappa(A)_{s.a.}: a\leqq x\}$ (written $(-\infty, x]_{\kappa(A)}$ for short). In this situation $\kappa(A)$ is said to be *order dense* in B. A *maximal regular extension* of A, written $\tilde{A}$, is a regular extension of A such that $\tilde{A}\prec(B, \kappa)$ with (B, κ) a regular extension of A implies $\tilde{A}\sim(B, \kappa)$.

As will be seen below (Lemma 2.5) if (B, κ) is a regular extension of A and (C, λ) is a regular extension of B then the extension $(C, \lambda\circ\kappa)$ of A is regular.

Hence $\tilde{A}$ is a regular extension of A which has no proper regular extension of itself, so that $(\tilde{A})^{\sim}=\tilde{A}$.

DEFINITION 1.2. Let B be an extension of A. A subset S of $B_{s.a.}$ is *monotone closed* in $B_{s.a.}$ if it is closed with respect to taking suprema (resp. infima) of bounded increasing (resp. decreasing) nets, i.e., whenever a bounded increasing (resp. decreasing) net $\mathcal{F}$ in S has a supremum (resp. infimum) in $B_{s.a.}$, written $\sup_B\mathcal{F}$ (resp. $\inf_B\mathcal{F}$), then $\sup_B\mathcal{F}$ (resp. $\inf_B\mathcal{F}$) is in S. Similarly *monotone σ-closedness* of S in $B_{s.a.}$ is defined with "nets" replaced by "sequences". The *monotone* (resp. *monotone σ-*) *closure* of $A_{s.a.}$ in $B_{s.a.}$, written $m\text{-cl}_B A_{s.a.}$ (resp. $\sigma\text{-cl}_B A_{s.a.}$), is the smallest monotone (resp. monotone σ-) closed subset of $B_{s.a.}$ containing $A_{s.a.}$, and that of A in B is the set

$$m\text{-cl}_B A = m\text{-cl}_B A_{s.a.} + i\, m\text{-cl}_B A_{s.a.}$$

$$(\text{resp. } \sigma\text{-cl}_B A = \sigma\text{-cl}_B A_{s.a.} + i\, \sigma\text{-cl}_B A_{s.a.})\,.$$

The algebra A is *monotone* (resp. *monotone σ-*) *closed* in B if $m\text{-cl}_B A$ (resp. $\sigma\text{-cl}_B A)=A$, and it is *monotone* (resp. *monotone σ-*) *dense* in B if $m\text{-cl}_B A$ (resp. $\sigma\text{-cl}_B A)=B$.

DEFINITION 1.3. A *monotone completion* of A is an extension B of A such that B is monotone complete and $m\text{-cl}_B A=B$. We write $\bar{A}$ for the regular monotone completion of A.

Let B be an extension of A. In case B is a W^*-algebra the arguments by R. V. Kadison [**9**; pp. 316-318] and G. K. Pedersen [**16**; the proof of Theorem 1] show that $m\text{-cl}_B A$ (resp. $\sigma\text{-cl}_B A$) is a monotone (resp. monotone σ-) closed C^*-subalgebra of B. Note also that $m\text{-cl}_B A$ is the weak closure of A in B (R. V. Kadison [8; Lemma 1]). But the same argument can be applicable for a not necessarily W^*, C^*-algebra B since, as is readily seen, $m\text{-cl}_B A_{s.a.}$ (resp. $\sigma\text{-cl}_B A_{s.a.}$) is a real linear subspace of $B_{s.a.}$ and the existence of $\sup_B\mathcal{F}$ with $\mathcal{F}$ a bounded increasing net in $B_{s.a.}$ implies $\sup_B b^*\mathcal{F}b=b^*(\sup_B\mathcal{F})b$ for every b in B [**10**; the proof of Lemma 2.1]. Hence we obtain:

LEMMA 1.4. *If B is an extension of A then the monotone* (*resp. monotone σ-*) *closure $m\text{-cl}_B A$* (*resp. $\sigma\text{-cl}_B A$*) *of A in B is a monotone* (*resp. monotone σ-*) *closed C^*-subalgebra of B.*

From now on we use the following notation: With B an extension of A, $x\in B_{s.a.}$ and $\mathcal{F}\subset B_{s.a.}$ we write

$$(-\infty,\ x]_A=\{a\in A_{s.a.}:\ a\leqq x\},\quad [x,\ +\infty)_A=\{a\in A_{s.a.}:\ x\leqq a\}.$$

The symbol "$\sup_B\mathcal{F}=x$" means that $\sup_B\mathcal{F}$ (the supremum of $\mathcal{F}$ in $B_{s.a.}$) exists and equals x, and "$\mathcal{F}\leqq x$" means that $y\leqq x$ for all y in $\mathcal{F}$. Moreover similar notations should be naturally understood.

DEFINITION 1.5. Let B be an extension of A and put

$$s\text{-cl}_B A_{s.a.} = \{x \in B_{s.a.} : x = \sup_B(-\infty, x]_A\},$$

$$s\text{-cl}_B A = s\text{-cl}_B A_{s.a.} + i\, s\text{-cl}_B A_{s.a.}\,.$$

We call $s\text{-cl}_B A$ the *sup-closure* of A in B. If $s\text{-cl}_B A = A$ then A is said to be *sup-closed* in B.

Clearly B is a regular extension of A if and only if $s\text{-cl}_B A = B$, and it is also immediate to see that $s\text{-cl}_B A_{s.a.}$ is the smallest subset of $B_{s.a.}$ which contains $A_{s.a.}$ and is closed with respect to taking suprema which exist in $B_{s.a.}$. In contrast to $m\text{-cl}_B A$ or $\sigma\text{-cl}_B A$, $s\text{-cl}_B A$ is generally not a C^*-subalgebra of B; whereas it is the case under an additional hypothesis:

LEMMA 1.6. *With notation as above suppose that* $s\text{-cl}_B A_{s.a.}$ *is a real linear subspace of* $B_{s.a.}$. *Then* $s\text{-cl}_B A$ *is a monotone closed* C^**-subalgebra of* B (*hence it is a regular extension of* A).

PROOF. For simplicity we write $C = s\text{-cl}_B A$; hence $C_{s.a.} = s\text{-cl}_B A_{s.a.}$. The monotone closedness of C in B is immediate since by hypothesis $C_{s.a.}$ is closed with respect to taking both suprema and infima. Moreover $C_{s.a.}$ is norm closed in B. For if $x_n \to x$ in norm with $\{x_n\}$ a sequence in $C_{s.a.}$ and $x \in B_{s.a.}$ then

$$\begin{aligned} x &= \sup_B\{x_n - \|x - x_n\| : n = 1, 2, \cdots\} \\ &= \sup_B\{\sup_B(-\infty, x_n - \|x - x_n\|]_A : n = 1, 2, \cdots\} \\ &= \sup_B \bigcup_{n=1}^{\infty} (-\infty, x_n - \|x - x_n\|]_A \\ &= \sup_B(-\infty, x]_A \in C_{s.a.} \end{aligned}$$

since $x \geqq x_n - \|x - x_n\| \to x$ in norm and $x_n - \|x - x_n\| \in C_{s.a.}$ (note that $A_{s.a.}$, hence $C_{s.a.}$ contains the unit of B).

We follow a reasoning analogous to those of R. V. Kadison and G. K. Pedersen cited above. By the linearity of $C_{s.a.}$ we see that $x \in C_{s.a.}$ if and only if $x = \inf_B[x, +\infty)_A$, if and only if $x = \sup_B \mathcal{F}$ (resp. $x = \inf_B \mathcal{G}$) for some $\mathcal{F}$ (resp. $\mathcal{G}$) $\subset A_{s.a.}$.

(1) If $x \in C_{s.a.}$ is positive and invertible in B then $x^{-1} \in C_{s.a.}$.

From above $x = \inf_B[x, +\infty)_A$. Then we have $x^{-1} = \sup_B[x, +\infty)_A^{-1} \in C_{s.a.}$, where $[x, +\infty)_A^{-1} = \{a^{-1} : a \in [x, +\infty)_A\}$. In fact, since $x \leqq [x, +\infty)_A$ and each element in $[x, +\infty)_A$ is invertible, we have $x^{-1} \geqq [x, +\infty)_A^{-1}$; moreover $B_{s.a.} \ni y \geqq [x, +\infty)_A^{-1}$ implies $y^{-1} \leqq [x, +\infty)_A$, so that $y^{-1} \leqq \inf_B[x, +\infty)_A = x$ and $y \geqq x^{-1}$.

(2) If $x \in C_{s.a.}$ then $x^n \in C_{s.a.}$ for $n = 1, 2, \cdots$.

To make an induction on n we assume that $x, x^2, \cdots, x^n \in C_{s.a.}$. We may also assume that $\|x\| \leqq 1/2$. Then for each $0 < \alpha \leqq 1$, $C_{s.a.} \ni 1 + \alpha x \geqq 1 - \|x\| \geqq 1/2$; hence $(1 + \alpha x)^{-1} \in C_{s.a.}$ by (1). Thus

$$x^{n+1}(1 + \alpha x)^{-1} = (-\alpha)^{-(n+1)}[(1 + \alpha x)^{-1} - \{1 + (-\alpha x) + \cdots + (-\alpha x)^n\}] \in C_{s.a.}$$

and so $x^{n+1}\in C_{s.a.}$ since $x^{n+1}(1+\alpha x)^{-1}\to x$ in norm as $\alpha\downarrow 0$ and $C_{s.a.}$ is norm closed.

(3) If $x, y\in C_{s.a.}$ then $yxy\in C_{s.a.}$.

We have

$$yxy\pm xyx=(x\pm y)^3-(x^2\pm y)^2-(x+y^2)^2+x^4-x^3+x^2+y^4\mp y^3+y^2\in C_{s.a.}$$

by (2), so that we add these equalities to obtain $yxy\in C_{s.a.}$.

(4) If $x\in C_{s.a.}$ and $y\in A_{s.a.}$ then $[x, y]=i(xy-yx)\in C_{s.a.}$.

Since $[x, \alpha y]=\alpha[x, y]$, we may assume by replacing y by αy with α a suitable scalar that $1+iy$ is invertible. Then the map $B_{s.a.}\ni z\mapsto(1+iy)^*z(1+iy)\in B_{s.a.}$ is one-to-one, onto and bipositive, so that

$$(1+iy)^*x(1+iy)=(1+iy)^*\sup_B(-\infty, x]_A(1+iy)$$
$$=\sup_B(1+iy)^*(-\infty, x]_A(1+iy)\in C_{s.a.}.$$

Hence $[x, y]=(1+iy)^*x(1+iy)-x-yxy\in C_{s.a.}$ by (3).

(5) If $x, y\in C_{s.a.}$ then $[x, y]\in C_{s.a.}$.

For $a\in A_{s.a.}$ and $y\in C_{s.a.}$ we have

$$(1+iy)^*a(1+iy)=-[y, a]+a+aya\in C_{s.a.}$$

by (3), (4). Hence for $x, y\in C_{s.a.}$,

$$(1+iy)^*x(1+iy)=(1+iy)^*\sup_B(-\infty, x]_A(1+iy)$$
$$=\sup_B(1+iy)^*(-\infty, x]_A(1+iy)\in C_{s.a.}$$

by the reasoning as in (4) and the sup-closedness of $C_{s.a.}$ in $B_{s.a.}$, so that $[x, y]=(1+iy)^*x(1+iy)-x-yxy\in C_{s.a.}$ by (3).

(6) If $x, y\in C_{s.a.}$ then $(x+iy)^*(x+iy)=x^2+y^2+[x, y]\in C_{s.a.}$ by (2), (5). This and the polarization identity imply $xy\in C_{s.a.}+iC_{s.a.}=C$ for all $x, y\in C_{s.a.}$. Hence C is a C^*-subalgebra of B. q.e.d.

REMARK 1.7. Under the same hypothesis as in Lemma 1.6 we see that for each $x\in s\text{-cl}_B A_{s.a.}$ there exists a *bounded* subset $\mathcal{F}$ of $A_{s.a.}$ such that $x=\sup_B\mathcal{F}$. In fact, if $x\in s\text{-cl}_B A_{s.a.}$ is positive and invertible then $x^{-1}=\inf_B[x^{-1}, +\infty)_A$ and so $x=\sup_B[x^{-1}, +\infty)_A^{-1}=\sup_B[0, x]_A$, where $[0, x]_A=\{a\in A_{s.a.}: 0\leqq a\leqq x\}$ (see (1) above). Hence for each $x\in s\text{-cl}_B A_{s.a.}$,

$$x=x+\|x\|+1-(\|x\|+1)=\sup_B\{[0, x+\|x\|+1]_A-(\|x\|+1)\}.$$

We close this section with a remark on the regular monotone completion of a C^*-algebra, whose existence and uniqueness will be proved in section 3. First we need the following definition and lemmas.

DEFINITION 1.8. A subset S of a partially ordered vector space V is *order* *dense* in V if for each $v\in V$ we have $v=\sup_V\{w\in S: w\leqq v\}=\inf_V\{w\in S: w\geqq v\}$.

LEMMA 1.9. *Let B be a C^*-algebra and $a\in B^+$. If $\mathcal{F}$ is a bounded subset of $B_{s.a.}$ such that $\sup_B\mathcal{F}$ exists then $\sup_B a\mathcal{F}a=a(\sup_B\mathcal{F})a$.*

PROOF. Put $\sup_B\mathcal{F}=x_0$. By hypothesis there exists an $\alpha>0$ such that $\|x\|\leqq\alpha$ for all $x\in\mathcal{F}$. If $B_{s.a.}\ni y\geqq axa$ for all $x\in\mathcal{F}$ then for each $\varepsilon>0$ we have

$$(a+\varepsilon)x(a+\varepsilon)=axa+\varepsilon(ax+xa)+\varepsilon^2x\leqq y+\alpha\varepsilon(2\|a\|+\varepsilon),$$
$$x\leqq(a+\varepsilon)^{-1}\{y+\alpha\varepsilon(2\|a\|+\varepsilon)\}(a+\varepsilon)^{-1};$$

hence

$$x_0\leqq(a+\varepsilon)^{-1}\{y+\alpha\varepsilon(2\|a\|+\varepsilon)\}(a+\varepsilon)^{-1},$$
$$(a+\varepsilon)x_0(a+\varepsilon)\leqq y+\alpha\varepsilon(2\|a\|+\varepsilon).$$

Therefore $ax_0a\leqq y$ and so $ax_0a=\sup_B a\mathcal{F}a$. q. e. d.

LEMMA 1.10. *Let A be a C^*-algebra and B a regular extension of A. Then for each $a\in B^+$, $(aAa)_{s.a.}$ is order dense in $(aBa)_{s.a.}$.*

PROOF. The regularity of B is equivalent to $s\text{-cl}_BA=B$, so that for each $x\in B_{s.a.}$ there exists a bounded subset $\mathcal{F}$ of $A_{s.a.}$ with $\sup_B\mathcal{F}=x$ (Remark 1.7). Hence Lemma 1.9 completes the proof. q. e. d.

PROPOSITION 1.11. *Let A be a C^*-algebra, $\bar{A}$ its regular monotone completion and e a projection of A. Then $e\bar{A}e$ is a regular monotone completion of eAe, i.e., $e\bar{A}e=(eAe)^{-}$.*

PROOF. By Lemma 1.10 $e\bar{A}e$ is a regular extension of eAe, and it is monotone closed in $\bar{A}$, hence monotone complete. In fact, if $\mathcal{F}$ is a bounded increasing net in $(e\bar{A}e)_{s.a.}$ then $\sup_{\bar{A}}\mathcal{F}$ exists, so that $\sup_{\bar{A}}\mathcal{F}=\sup_{\bar{A}}e\mathcal{F}e=e(\sup_{\bar{A}}\mathcal{F})e\in(e\bar{A}e)_{s.a.}$ (Lemma 1.9). Hence $m\text{-cl}_{\bar{A}}eAe=m\text{-cl}_{e\bar{A}e}eAe\subset e\bar{A}e$ is a regular monotone completion of eAe. Similarly $m\text{-cl}_{\bar{A}}(1-e)A(1-e)\subset(1-e)\bar{A}(1-e)$. Put

$$V=m\text{-cl}_{\bar{A}}eAe+e\bar{A}(1-e)+(1-e)\bar{A}e+m\text{-cl}_{\bar{A}}(1-e)A(1-e).$$

Then $V\supset A$ is monotone closed in $\bar{A}$. For if $\mathcal{F}$ is a bounded increasing net in $V_{s.a.}$ then $\sup_{\bar{A}}\mathcal{F}$ exists, and since $e\mathcal{F}e\subset m\text{-cl}_{\bar{A}}eAe$, $e(\sup_{\bar{A}}\mathcal{F})e=\sup_{\bar{A}}e\mathcal{F}e\in m\text{-cl}_{\bar{A}}eAe$ (Lemma 1.9). Similarly $(1-e)(\sup_{\bar{A}}\mathcal{F})(1-e)\in m\text{-cl}_{\bar{A}}(1-e)A(1-e)$. Hence

$$\sup_{\bar{A}}\mathcal{F}=e(\sup_{\bar{A}}\mathcal{F})e+e(\sup_{\bar{A}}\mathcal{F})(1-e)+(1-e)(\sup_{\bar{A}}\mathcal{F})e$$
$$+(1-e)(\sup_{\bar{A}}\mathcal{F})(1-e)\in V$$

and so $V=\bar{A}$. Thus $e\bar{A}e=eVe=m\text{-cl}_{\bar{A}}eAe$ is a regular monotone completion of eAe. q. e. d.

§ 2. A Characterization of regular extensions.

The self-adjoint part of a C^*-algebra is regarded as a *function system*, i.e., an Archimedean partially ordered vector space having the unit of the C^*-algebra as the order unit (cf. [**2**; pp. 588-589]). And the definition of regularity for extensions of C^*-algebras depends only on the order structure, as function systems, of the self-adjoint parts of the C^*-algebras. Therefore it will be convenient to generalize the notion of regularity to function systems and characterize it for function systems.

In the following a function system V with a distinguished order unit 1 will be viewed as a normed linear space with the order-unit norm: $\|v\|=\inf\{\lambda>0: -\lambda 1\leqq v\leqq\lambda 1\}$. Note that a unital linear map between function systems is contractive (resp. isometric) if and only if it is positive (resp. bipositive). The *state space* of V is the set $S(V)=\{f\in V^*: \|f\|=f(1)=1\}$.

Definition 2.1. An *extension* of a function system V is a pair (W, α) of a function system W and a unital order injection α of V into W (i.e., $\alpha(1)=1$ and α is an order isomorphism of V onto $\alpha(V)$). The extension (W, α) is a *regular extension* of V if $w=\sup_W(-\infty, w]_{\alpha(V)}$ for all $w\in W$, where as before "$\sup_W$" means the supremum taken in W and $(-\infty, w]_{\alpha(V)}=\{w'\in\alpha(V): w'\leqq w\}$. The extension (W, α) is a *bound extension* of V if the norm on W is a unique seminorm p on W such that $p(w)\leqq\|w\|$ and $p(\alpha(v))=\|v\|$ for all $w\in W$ and $v\in V$, and it is an *essential extension* of V if given any unital positive linear map β of W into a function system Z, β is bipositive whenever $\beta\circ\alpha$ is (cf. [**14**], [**15**; pp. 38-39]).

The equivalence of boundness and essentiality is known in the context of the extensions of normed linear spaces [**15**; p. 89, Corollary to Lemma 2], and we will see that the equivalence of regularity and essentiality (Proposition 2.6). Then it will result that these three notions coincide. We need another definition.

Definition 2.2. A positive linear map α of a function system V into another W is *sup-preserving* (resp. *normal*) if $\sup_V\mathscr{F}=v$ with $\mathscr{F}$ a subset (resp. a bounded increasing net) of V and $v\in V$ implies $\sup_W\alpha(\mathscr{F})=\alpha(v)$. A positive linear map ϕ of a C^*-algebra A into another B is *sup-preserving* (resp. *normal*) if its restriction $\phi|_{A_{s.a.}}: A_{s.a.}\to B_{s.a.}$ is so.

Given a C^*-algebra A and its extension B it is obvious that if A is monotone complete and the inclusion map $A\hookrightarrow B$ is normal then A is monotone closed in B, and that if B is monotone complete and A is monotone closed in B then the map $A\hookrightarrow B$ is normal.

For a while V denotes a fixed function system. The *Dedekind completion* of V is a regular extension $(\hat{V}, j_V)$ of V such that $\hat{V}$ is a boundedly complete vector lattice (cf. [**21**]). Such a $\hat{V}$ is unique and is the self-adjoint part of a commutative AW^*-algebra or an injective real Banach space in the sense of H.B. Cohen [4]; moreover we have:

Lemma 2.3. *$(\hat{V}, j_V)$ is the injective envelope of V in the sense of H.B. Cohen* [**4**], *i.e., $\hat{V}$ is an injective Banach space and is the only injective subspace* *of itself which contains* $j_V(V)$.

PROOF. To see this it suffices to show that $\phi\circ j_V=j_V$ with ϕ a contractive linear map of $\hat{V}$ into itself implies $\phi=\mathrm{id}_{\hat{V}}$ (cf. [7]). But ϕ is then positive, and the regularity of $(\hat{V}, j_V)$ implies that for each $w\in\hat{V}$, $w=\sup_{\hat{V}}(-\infty, w]_{j_V(V)}$ and $\phi(w)\geqq\phi((-\infty, w]_{j_V(V)})=(-\infty, w]_{j_V(V)}$, hence that $\phi(w)\geqq w$. Similarly $\phi(-w)\geqq -w$ and so $\phi(w)=w$ for all $w\in\hat{V}$. q.e.d.

The following fact is stated without proof in [**22**; p. 303, ll.18-20]:

LEMMA 2.4. *If* (W, α) *is a regular extension of* V *then* α *is sup-preserving.*

PROOF. Suppose that $\sup_V\mathcal{F}=v$ for some $\mathcal{F}\subset V$ and $v\in V$. If $W\ni w\geqq\alpha(\mathcal{F})$ then $[w, +\infty)_{\alpha(V)}\geqq\alpha(\mathcal{F})$, $\alpha^{-1}([w, +\infty)_{\alpha(V)})\geqq\mathcal{F}$ and so $\alpha^{-1}([w, +\infty)_{\alpha(V)})\geqq v$, $[w, +\infty)_{\alpha(V)}\geqq\alpha(v)$. Moreover by regularity $w=\inf_W[w, +\infty)_{\alpha(V)}$; hence $w\geqq\alpha(v)$, so that $\sup_W\alpha(\mathcal{F})=\alpha(v)$. q.e.d.

LEMMA 2.5. *If* (W, α) *is a regular extension of* V *and* (Z, β) *is a regular extension of* W *then* $(Z, \beta\circ\alpha)$ *is a regular extension of* V.

PROOF. If $w\in W$ then $w=\sup_W(-\infty, w]_{\alpha(V)}$ and $\beta(w)=\sup_Z\beta((-\infty, w]_{\alpha(V)})=\sup_Z(-\infty, \beta(w)]_{\beta\circ\alpha(V)}$ by Lemma 2.4. Hence for $z\in Z$,

$$z=\sup_Z(-\infty, z]_{\beta(W)}=\sup_Z\{\beta(w):\ \beta(w)\leqq z,\ w\in W\}$$

$$=\sup_Z\{\sup_Z(-\infty, \beta(w)]_{\beta\circ\alpha(V)}:\ \beta(w)\leqq z,\ w\in W\}$$

$$=\sup_Z(-\infty, z]_{\beta\circ\alpha(Z)}.$$

q.e.d.

PROPOSITION 2.6. *Let* V *be a function system and* (W, α) *its extension. Then* (W, α) *is regular if and only if it is essential.*

PROOF. Suppose (W, α) is regular and take the Dedekind completion $(\hat{W}, j_W)$ of W. Then the extension $(\hat{W}, j_W\circ\alpha)$ of V is the Dedekind completion of V since it is regular by Lemma 2.5, so that it is the injective envelope of V by Lemma 2.3. Since the injective envelope is a maximal essential extension, $(\hat{W}, j_W\circ\alpha)$, hence (W, α) is an essential extension of V.

Conversely let (W, α) be an essential extension of V and $(\hat{V}, j_V)$ the Dedekind completion of V. Since $\hat{V}$ is injective, there exists a contractive linear map β of W into $\hat{V}$ with $\beta\circ\alpha=\mathrm{id}_V$. Then the essentiality of (W, α) implies that β is an order injection. Hence (W, α), being contained in the regular extension $(\hat{V}, j_V)$, is regular. q.e.d.

§ 3. The main theorem.

This section is devoted to the proof of the following:

THEOREM 3.1. *Any C*-algebra* A *has a regular monotone completion* $\bar{A}$ *(resp. maximal regular extension* $\tilde{A}$*) which is unique up to the equivalence relation* $\sim$*, and we have canonical inclusion maps* $A\hookrightarrow\bar{A}\hookrightarrow\tilde{A}\hookrightarrow I(A)$. *Moreover* $\tilde{A}$ *is* *monotone complete and the respective inclusion maps are sup-preserving.*

For the proof we need several lemmas. The first one is a modification of [**6**; the proof of Theorem 3.4].

LEMMA 3.2. *With A and $I(A)$ as above let p be a seminorm on $I(A)_{s.a.}$ such that $p(x)\leqq\|x\|$, $p(u^*xu)=p(x)$ and $p(a)=\|a\|$ for all $x\in I(A)_{s.a.}$, $u\in A_u$ and $a\in A_{s.a.}$. Then $p(x)=\|x\|$ for all $x\in I(A)_{s.a.}$.*

PROOF. Take a family $\{f_i\}$ of pure states of A such that the direct sum $\sum_i^{\oplus}\{\pi_{f_i}, H_{f_i}\}$ of the cyclic representations $\{\pi_{f_i}, H_{f_i}\}$ of A induced by f_i is faithful. We apply the Hahn-Banach theorem to obtain a state extension g_i of f_i to $I(A)$ such that $|g_i(x)|\leqq p(x)$ for all $x\in I(A)_{s.a.}$. Let $\{\pi, H\}=\sum_i^{\oplus}\{\pi_{g_i}, H_{g_i}\}$ be the direct sum of the cyclic representations $\{\pi_{g_i}, H_{g_i}\}$ of $I(A)$ and let E be the projection of H onto $\sum_i^{\oplus}A_{g_i}\subset\sum_i^{\oplus}H_{g_i}=H$. (Since f_i is pure, $A_{f_i}=H_{f_i}$, and so $A_{g_i}\subset H_{g_i}$, being isometric to A_{f_i}, is closed.) Then $E\in\pi(A)'$ (the commutant of $\pi(A)$), $\sum_i^{\oplus}\{\pi_{f_i}, H_{f_i}\}\cong\{\pi(\cdot)E|_A, EH\}$ (the representation of A restricted to EH) and $\pi(A)$ acts irreducibly on $A_{g_i}\subset H_{g_i}$ since $g_i|_A=f_i$ is pure. So defining $\phi: I(A)\to E\pi(I(A))E$ by $\phi(x)=E\pi(x)E$ we get a *-isomorphism $\phi|_A: A\to\phi(A)=\pi(A)E$ and its inverse $\psi=(\phi|_A)^{-1}: \pi(A)E\to A$. Since $I(A)$ is injective, there exists a completely positive extension $\tilde{\psi}: E\pi(I(A))E\to I(A)$ of ψ. Then $\tilde{\psi}\circ\phi: I(A)\to I(A)$ is a completely positive map with $\tilde{\psi}\circ\phi|_A=\mathrm{id}_A$, so that $\tilde{\psi}\circ\phi=\mathrm{id}_{I(A)}$.

We show that $\|\phi(x)\|\leqq p(x)$ for all $x\in I(A)_{s.a.}$. Given an $\varepsilon>0$ and an $x\in I(A)_{s.a.}$ choose a family $\{a_i\}$ of elements of A such that $\|\sum_i(a_i)_{g_i}\|=1$ and

$$|(\pi(x)\textstyle\sum_i(a_i)_{g_i}, \sum_i(a_i)_{g_i})|\geqq\|E\pi(x)E\|-\varepsilon=\|\phi(x)\|-\varepsilon\,.$$

Since $\pi(A)$ acts irreducibly on A_{g_i}, the transitivity theorem implies the existence of a unitary element u_i of A such that $(u_i)_{g_i}=\pi(u_i)1_{g_i}=\|(a_i)_{g_i}\|^{-1}(a_i)_{g_i}$. Hence

$$\begin{aligned}|(\pi(x)\textstyle\sum_i(a_i)_{g_i}, \sum_i(a_i)_{g_i})| &= \textstyle\sum_i\|(a_i)_{g_i}\|^2|g_i(u_i^*xu_i)|\\ &\leqq \textstyle\sum_i\|(a_i)_{g_i}\|^2p(u_i^*xu_i)\\ &= \|\textstyle\sum_i(a_i)_{g_i}\|^2p(x)=p(x)\,.\end{aligned}$$

Therefore $\|\phi(x)\|\leqq p(x)$, so that $\|x\|=\|\tilde{\psi}\circ\phi(x)\|\leqq\|\phi(x)\|\leqq p(x)$ and $p(x)=\|x\|$.

q. e. d.

Now let V be the Dedekind completion of the partially ordered linear space $A_{s.a.}$ with order unit 1, where we identify $A_{s.a.}$ with its image in V and so we consider $A_{s.a.}\subset V$. By Lemma 2.3, V is the injective envelope (as a real Banach space) of $A_{s.a.}$. Hence the set

$$\Phi=\{\text{contractive linear maps } \phi \text{ of } I(A)_{s.a.} \text{ into } V \text{ with } \phi|_{A_{s.a.}}=\mathrm{id}_{A_{s.a.}}\}$$

is nonvoid. We apply Lemma 3.2 to show the following:

LEMMA 3.3. *For $x\in I(A)_{s.a.}$, $\phi(x)\geqq0$ for all $\phi\in\Phi$ implies $x\geqq0$.*

PROOF. We observe that the seminorm p on $I(A)_{s.a.}$ defined by $p(x)=\sup\{\|\phi(x)\|:\phi\in\Phi\}$ satisfies the conditions of Lemma 3.2. In fact, fix $x\in I(A)_{s.a.}$, $u\in A_u$ and $\phi\in\Phi$, and define a map $T_u: I(A)_{s.a.}\to I(A)_{s.a.}$ by $T_u(x)=u^*xu$. Then, since V is the injective envelope of $A_{s.a.}$, the linear isometry $T_u|_{A_{s.a.}}: A_{s.a.}\to A_{s.a.}$ extends to a linear isometry $S_u: V\to V$. Hence $S_{u^*}\circ\phi\circ T_u\in\Phi$ and $p(x)\geqq\|S_{u^*}\circ\phi\circ T_u(x)\|=\|\phi\circ T_u(x)\|=\|\phi(u^*xu)\|$. Thus $p(x)\geqq p(u^*xu)$, so that $p(x)=p(u^*xu)$. The other conditions are clearly satisfied. Therefore by Lemma 3.2,

$$\sup\{\|\phi(x)\|:\phi\in\Phi\}=p(x)=\|x\| \quad \text{for all } x\in I(A)_{s.a.},$$

and the left-hand side$=\sup\{|f\circ\phi(x)|:\phi\in\Phi, f\in S(V)\}$, so that the weak* closed convex hull of $\{f\circ\phi:\phi\in\Phi, f\in S(V)\}=S(I(A))$, the state space of $I(A)$. Hence if $\phi(x)\geqq 0$ for all $\phi\in\Phi$ then $g(x)\geqq 0$ for all $g\in S(I(A))$ and consequently $x\geqq 0$.

q. e. d.

LEMMA 3.4. *For* $x\in I(A)_{s.a.}$,

$$\{\phi(x):\phi\in\Phi\}=\{v\in V:\sup_V(-\infty, x]_A\leqq v\leqq\inf_V[x, +\infty)_A\}.$$

(The both sides in the inequalities exist since V is a boundedly complete vector lattice.)

PROOF. Since each contractive linear map $\psi: A_{s.a.}+\boldsymbol{R}x\to V$ with $\psi|_{A_{s.a.}}=\mathrm{id}_{A_{s.a.}}$ extends to an element of Φ, we have

$$\{\phi(x):\phi\in\Phi\}=\{\psi(x):\psi \text{ is a contractive linear map of } A_{s.a.}+\boldsymbol{R}x \text{ into } V \text{ with } \psi|_{A_{s.a.}}=\mathrm{id}_{A_{s.a.}}\}.$$

A $v\in V$ belongs to the right-hand side if and only if

$$-\|x+a\|\leqq v+a\leqq\|x+a\| \quad \text{for all } a\in A_{s.a.},$$

i. e.,

$$\sup_V\{-\|x+a\|-a: a\in A_{s.a.}\}\leqq v\leqq\inf_V\{\|x+a\|-a: a\in A_{s.a.}\}.$$

Clearly $\sup_V\{-\|x+a\|-a: a\in A_{s.a.}\}=w$ (say)$\leqq\sup_V(-\infty, x]_A$. Moreover from the foregoing $w=\phi(x)$ for some $\phi\in\Phi$, so that $(-\infty, x]_A=\phi((-\infty, x]_A)\leqq\phi(x)=w$ and $\sup_V(-\infty, x]_A\leqq w$. Hence $\sup_V\{-\|x+a\|-a: a\in A_{s.a.}\}=\sup_V(-\infty, x]_A$ and similarly $\inf_V\{\|x+a\|-a: a\in A_{s.a.}\}=\inf_V[x, +\infty)_A$. q. e. d.

LEMMA 3.5. *The sup-closure* $s\text{-cl}_{I(A)}A_{s.a.}$ *of* $A_{s.a.}$ *in* $I(A)_{s.a.}$ *is a real linear subspace of* $I(A)_{s.a.}$.

PROOF. We show $s\text{-cl}_{I(A)}A_{s.a.}=\{x\in I(A)_{s.a.}:\phi(x)=\psi(x) \text{ for all } \phi, \psi\in\Phi\}=W$, say. This will complete the proof since W is linear. If $x\in s\text{-cl}_{I(A)}A_{s.a.}$ then $x=\sup_{I(A)}(-\infty, x]_A$. Put $v=\sup_V(-\infty, x]_A\in V$; then $v=\inf_V\mathcal{F}$ for some $\mathcal{F}\subset A_{s.a.}$ by the order density of $A_{s.a.}$ in V. Hence $I(A)_{s.a.}\supset(-\infty, x]_A\leqq v\leqq\mathcal{F}\subset I(A)_{s.a.}$, $x=\sup_{I(A)}(-\infty, x]_A\leqq\mathcal{F}$, and $\mathcal{F}\subset[x, +\infty)_A$. Then $\sup_V(-\infty, x]_A=v=\inf_V\mathcal{F}\geqq\inf_V[x, +\infty)_A$ and so $\sup_V(-\infty, x]_A=\inf_V[x, +\infty)_A$. Hence $x\in W$ by Lemma

3.4. Conversely let $x\in W$. If $I(A)_{s.a.}\ni y\geqq(-\infty, x]_A$ then $\phi(y)\geqq\phi((-\infty, x]_A)=(-\infty, x]_A$ for all $\phi\in\Phi$, so that $\phi(y)\geqq\sup_V(-\infty, x]_A=\phi(x)$ by Lemma 3.4, and $\phi(y-x)\geqq 0$. Lemma 3.3 implies $y-x\geqq 0$, $y\geqq x$. Thus $x=\sup_{I(A)}(-\infty, x]_A$, i.e., $x\in s\text{-cl}_{I(A)}A_{s.a.}$. q.e.d.

LEMMA 3.6. *The inclusion map* $A\hookrightarrow I(A)$ *is sup-preserving.*

PROOF. Suppose that $\sup_A\mathcal{F}=a$ for some $\mathcal{F}\subset A_{s.a.}$ and $a\in A_{s.a.}$. Then $I(A)_{s.a.}\ni x\geqq\mathcal{F}$ implies $\phi(x)\geqq\phi(\mathcal{F})=\mathcal{F}$ for all $\phi\in\Phi$, and $\phi(x)\geqq\sup_V\mathcal{F}=\sup_A\mathcal{F}=a$ since the inclusion map $A_{s.a.}\hookrightarrow V$ is sup-preserving. Hence $\phi(x-a)=\phi(x)-a\geqq 0$ for all $\phi\in\Phi$ and $x\geqq a$ by Lemma 3.3, so that $\sup_{I(A)}\mathcal{F}=a$. q.e.d.

LEMMA 3.7. *Let* (B, κ) *be a regular extension of* A *and* $I(B)$ *the injective envelope of* B (B *being considered as a* C^**-subalgebra of* $I(B)$). *Then the extension* $(I(B), \kappa)$ *of* A *is the injective envelope of* A.

PROOF. To see this we need only show that if $\phi\circ\kappa=\kappa$ with ϕ a completely positive map of $I(B)$ into itself then $\phi=\mathrm{id}_{I(B)}$. By the regularity of (B, κ) and Lemma 3.6 with A replaced by B we have for each $x\in B_{s.a.}$,

$$x=\sup_B(-\infty, x]_{\kappa(A)}=\sup_{I(B)}(-\infty, x]_{\kappa(A)}\,.$$

Now $x\geqq(-\infty, x]_{\kappa(A)}$ implies $\phi(x)\geqq\phi((-\infty, x]_{\kappa(A)})=(-\infty, x]_{\kappa(A)}$, which in turn implies $\phi(x)\geqq\sup_{I(B)}(-\infty, x]_{\kappa(A)}=x$. Similarly $\phi(-x)\geqq -x$ and $\phi(x)\leqq x$. Hence $\phi(x)=x$ for all $x\in B_{s.a.}$ and, since $I(B)$ is the injective envelope of B, $\phi=\mathrm{id}_{I(B)}$. q.e.d.

PROOF OF THEOREM 3.1. Define $\bar{A}$ (resp. $\tilde{A}$) as the monotone (resp. sup-) closure of A in its injective envelope $I(A)$: $\bar{A}=m\text{-cl}_{I(A)}A$ (resp. $\tilde{A}=s\text{-cl}_{I(A)}A$). Then Lemmas 1.4 and 1.6, together with Lemma 3.5, imply that $\bar{A}$ and $\tilde{A}$ are both monotone closed C^*-subalgebras of $I(A)$, hence also that $\bar{A}\subset\tilde{A}$. Thus $\bar{A}$ and $\tilde{A}$ are regular extensions of A. Moreover $I(A)$, being injective, is monotone complete [**20**; Theorem 7.1], so that $\bar{A}$ and $\tilde{A}$ are monotone complete. Since $I(\bar{A})=I(\tilde{A})=I(A)$, Lemma 3.6 implies that the inclusion maps $A\hookrightarrow\bar{A}\hookrightarrow\tilde{A}\hookrightarrow I(A)$ are sup-preserving.

To see the maximality of $\tilde{A}$ and the uniqueness of $\bar{A}$ and $\tilde{A}$ take a regular extension (B, κ) of A. Then $(I(B), \kappa)$ is the injective envelope of A (Lemma 3.7), so that by the uniqueness of the injective envelope there exists a *-isomorphism ι of $I(B)$ onto $I(A)$ with $\iota\circ\kappa=\mathrm{id}_A$. As seen in the proof of Lemma 3.7, $x=\sup_{I(B)}(-\infty, x]_{\kappa(A)}$ for all $x\in B_{s.a.}$ and so

$$\iota(x)=\sup_{I(A)}\iota((-\infty, x]_{\kappa(A)})=\sup_{I(A)}(-\infty, \iota(x)]_A\in\tilde{A}\,.$$

Hence $\iota(B)\subset\tilde{A}$, i.e., $(B, \kappa)\prec\tilde{A}$, and if in addition (B, κ) is a maximal regular extension (resp. regular monotone completion) of A then $(B, \kappa)\sim\tilde{A}$ [resp. $\iota(B)=\iota(m\text{-cl}_B\kappa(A))=\iota(m\text{-cl}_{I(B)}\kappa(A))=m\text{-cl}_{I(A)}A=\bar{A}$; hence $(B, \kappa)\sim\bar{A}$]. Thus $\tilde{A}$ (resp. $\bar{A}$) is a unique maximal regular extension (resp. regular monotone completion) of A. q.e.d.

REMARK 3.7. With A and $I(A)$ as above take the monotone σ-closure $\sigma\text{-cl}_{I(A)}A$ of A in $I(A)$. Then $\sigma\text{-cl}_{I(A)}A\subset\bar{A}\subset\tilde{A}$ and so $\sigma\text{-cl}_{I(A)}A$ is identified with the regular σ-completion $\hat{A}$ of A in the sense of J. D. M. Wright (Lemma 1.4); hence $A\subset\hat{A}\subset\bar{A}\subset\tilde{A}\subset I(A)$. Moreover $Z_A\subset Z_{\hat{A}}\subset Z_{\bar{A}}\subset Z_{\tilde{A}}\subset Z_{I(A)}$ (cf. [**6**; Corollary 4.3]), and if A is simple then so are $\hat{A}$, $\bar{A}$, $\tilde{A}$ and $I(A)$; hence $\bar{A}$, $\tilde{A}$ and $I(A)$ are AW^*-factors (cf. [**6**; Proposition 4.15]).

COROLLARY 3.8. *If A is a separable, infinite dimensional, simple C^*-algebra then its injective envelope $I(A)$ is a σ-finite, injective, non W^*, AW^*-factor of type* III.

PROOF. As noted above $A\subset\hat{A}\subset I(A)$ and $I(A)$ is an injective AW^*-factor. By [**23**; Theorem N] $\hat{A}$ is a monotone complete non W^*, AW^*-factor of type III and so $\hat{A}=\bar{A}$. Since $\bar{A}$ is monotone closed in $I(A)$ and is non W^*, $I(A)$ is also non W^*. Moreover $I(A)$ is of type III since it is a simple AW^*-factor and 1 is an infinite projection of $\hat{A}$, hence of $I(A)$. By the separability of A and the construction of $I(A)$ [**6**; Theorem 5.1] we may assume that A is a C^*-subalgebra of $B(H)$ with H a separable Hilbert space and $I(A)$ is completely order isomorphic to a self-adjoint linear subspace, containing A, of $B(H)$. Since $B(H)$ has a faithful state, V hence $I(A)$ also has a faithful state. Hence $I(A)$ is σ-finite. q. e. d.

REMARK 3.9. The following problem was left open in [**6**]: If A is a C^*-algebra and is embedded in an injective C^*-algebra B as a C^*-subalgebra, containing the unit, of B then can we take the injective envelope of A as a C^*-subalgebra of B? (The injective envelope of A is completely order isomorphic to some self-adjoint linear subspace of B and is *-isomorphic to a quotient C^*-algebra of some C^*-subalgebra of B.) This problem is affirmative in case A is commutative, but is negative in the general case. In fact let A be a UHF algebra acting on a Hilbert space so that the von Neumann algebra B generated by A is a hyperfinite II_1-factor. Then B is injective and the injective envelope $I(A)$ of A is an AW^*-factor of type III by Corollary 3.8. Hence $I(A)$ cannot be *-isomorphic to any C^*-subalgebra of B.

COROLLARY 3.10. *If A is a C^*-algebra and M_n is the C^*-algebra of all $n\times n$ matrices over $\boldsymbol{C}$ then $(A\otimes M_n)^{-}=\bar{A}\otimes M_n$. In particular if A is monotone complete then so is $A\otimes M_n$, too.*

PROOF. By Theorem 3.1, $\bar{A}$ and $(A\otimes M_n)^{-}=B$, say, exist. If e is a minimal projection of M_n then $B\cong(1\otimes e)B(1\otimes e)\otimes M_n$ and $A\cong(1\otimes e)(A\otimes M_n)(1\otimes e)$, so that $\bar{A}\cong(1\otimes e)(A\otimes M_n)^{-}(1\otimes e)=(1\otimes e)B(1\otimes e)$ (Proposition 1.11). Hence $B\cong\bar{A}\otimes M_n$.
q. e. d.

If A is a monotone complete C^*-algebra then $A=\bar{A}$ is a monotone closed C^*-subalgebra of $I(A)$ and, in particular, it is an AW^*-subalgebra of $I(A)$. More generally we consider the following problem: If A is an AW^*-algebra then is

A an AW^*-subalgebra of $I(A)$? This is the case, as will be seen below, for a finite AW^*-algebra A; whereas the general case remains open. For a C^*-algebra A and a subset $\mathcal{F}$ of A_p we denote the supremum (resp. infimum) of $\mathcal{F}$ in A_p by $\vee_A \mathcal{F}$ (resp. $\wedge_A \mathcal{F}$) if it exists. The existence of $\vee_A \mathcal{F}$ need not imply that of $\sup_A \mathcal{F}$ (the supremum in $A_{s.a.}$) (see Example 4.7 below), but the converse is true:

LEMMA 3.11. *With notation as above suppose that* $\sup_A \mathcal{F}$ *exists. Then* $\vee_A \mathcal{F}$ *exists and equals* $\sup_A \mathcal{F}$.

PROOF. We need only check that $\sup_A \mathcal{F}$ is a projection. Put $x = \sup_A \mathcal{F}$. Since $\mathcal{F} \leqq 1$, $e \leqq x \leqq 1$ for all $e \in \mathcal{F}$; hence e and x commute and so $e = e^2 \leqq x^2$. Thus $0 \leqq x = \sup_A \mathcal{F} \leqq x^2 \leqq 1$ and $x = x^2$. q. e. d.

LEMMA 3.12. *If A is a finite AW^*-algebra and $\mathcal{F}$ is an increasing net in A_p then* $\sup_A \mathcal{F}$ *exists and* $\vee_A \mathcal{F} = \sup_A \mathcal{F} = \sup_{\tilde{A}} \mathcal{F} = \vee_{\tilde{A}} \mathcal{F}$.

PROOF. Regard A as a C^*-subalgebra of $\tilde{A}$. Since $\tilde{A}$ is monotone complete, $\sup_{\tilde{A}} \mathcal{F}$ exists and $\sup_{\tilde{A}} \mathcal{F} = \vee_{\tilde{A}} \mathcal{F} = \tilde{e}$, say (Lemma 3.11). Put $e = \vee_A \mathcal{F} \in A_p$ (this exists since A is AW^*). We need only show that $e = \tilde{e}$ since it follows then that $\sup_{\tilde{A}} \mathcal{F} \in A_p$ and $\sup_A \mathcal{F} = \sup_{\tilde{A}} \mathcal{F} = e$. Since A is order dense in $\tilde{A}$, so is eAe in $e\tilde{A}e$ (Lemma 1.10). Hence if a seminorm p on $(e\tilde{A}e)_{s.a.}$ satisfies $p(x) \leqq \|x\|$ and $p(a) = \|a\|$ for all $x \in (e\tilde{A}e)_{s.a.}$ and $a \in (eAe)_{s.a.}$ then $p(x) = \|x\|$ for all $x \in (e\tilde{A}e)_{s.a.}$ (Proposition 2.6). Define a seminorm p on $(e\tilde{A}e)_{s.a.}$ by $p(x) = \sup\{\|xf\| : f \in \mathcal{F}\}$. Clearly $p(x) \leqq \|x\|$ for all $x \in (e\tilde{A}e)_{s.a.}$ and $p(e - \tilde{e}) = 0$. Hence if we show

$$p(a) = \|a\| \quad \text{for all } a \in (eAe)_{s.a.} \tag{3.1}$$

the proof is complete since we have then $\|e - \tilde{e}\| = p(e - \tilde{e}) = 0$ and $e = \tilde{e}$.

Proof of (3.1): Suppose on the contrary that there exist an $a \in (eAe)_{s.a.}$ and an $\varepsilon > 0$ such that $p(a) \leqq \|a\| - \varepsilon$. Then $afa \leqq \|afa\| = \|af\|^2 \leqq p(a)^2 \leqq (\|a\| - \varepsilon)^2$ for all $f \in \mathcal{F}$. We may assume that $\|a\|$ is in the spectrum of a (if necessary, replace a by $-a$). Then we can take a nonzero projection g in a maximal commutative *-subalgebra, containing a, of eAe so that

$$\|ag - \|a\|g\| < \delta \text{ and } \delta > 0 \text{ satisfies } (\|a\| - \varepsilon)^2 + 2\|a\|\delta < \|a\|^2.$$

Hence for each $f \in \mathcal{F}$ we have

$$gafag \leqq (\|a\| - \varepsilon)^2 g$$

and

$$\begin{aligned} \|gafag - \|a\|^2 gfg\| &= \|(ga - \|a\|g)fag + \|a\|gf(ag - \|a\|g)\| \\ &\leqq 2\|a\|\,\|ag - \|a\|g\| < 2\|a\|\delta\,. \end{aligned}$$

Then

$$\|a\|^2 gfg < gafag + 2\|a\|\delta g \leqq \{(\|a\| - \varepsilon)^2 + 2\|a\|\delta\}\, g\,,$$

$$\|f\wedge g\|=\|g(f\wedge g)g\|\leqq\|gfg\|\leqq(1/\|a\|^2)\{(\|a\|-\varepsilon)^2+2\|a\|\delta\}\|g\|<1$$

and so $f\wedge g=0$. Hence by the continuity of the lattice operation in finite AW^*-algebras [**11**; Theorem 6.5] $g=e\wedge g=(\vee_A\mathcal{F})\wedge g=\vee_A\{f\wedge g: f\in\mathcal{F}\}=0$. This is a contradiction. q. e. d.

From Lemma 3.12 and the fact that $\tilde{A}$, being monotone closed in $I(A)$, is an AW^*-subalgebra of $I(A)$ we deduce the following:

PROPOSITION 3.13. *A finite AW^*-algebra A is an AW^*-subalgebra of its injective envelope $I(A)$.*

REMARK 3.14. The above argument shows that if A is an AW^*-algebra for which the conclusion of Lemma 3.12, i. e.,

(3.2) $\sup_A\mathcal{F}$ exists for any increasing net $\mathcal{F}$ in A_p,

holds then A is an AW^*-subalgebra of the monotone complete C^*-algebra $\tilde{A}$ (or $\bar{A}$ or $I(A)$). Conversely it is readily seen that if A is an AW^*-subalgebra of some monotone complete C^*-algebra then (3.2) holds.

§4. Normal and sup-preserving embeddings.

In this section some embeddings of C^*-algebras into another C^*-algebras are shown to be normal or sup-preserving. In the remainder of the paper the C^*-tensor product of two C^*-algebras A and B will always mean the minimal C^*-tensor product and will be denoted by $A\otimes B$.

PROPOSITION 4.1. *For any C^*-algebras A and B the map $A\ni x\mapsto x\otimes 1\in A\otimes B$ is normal.*

PROOF. Representing the injective envelope $I(A)$ of A and B faithfully on some Hilbert spaces H and K respectively, we may assume that $A\subset I(A)\subset B(H)$ and $B\subset B(K)$, hence that $A\otimes B\subset I(A)\otimes I(B)\subset B(H)\otimes B(K)\subset B(H\otimes K)$. Since $I(A)$ is injective, we have a projection ϕ of norm one from $B(H)$ onto $I(A)$; hence there is a projection $\phi\otimes 1$ of norm one from $B(H)\otimes B(K)$ onto $I(A)\otimes B(K)$ such that $(\phi\otimes 1)(x\otimes y)=\phi(x)\otimes y$ for all $x\in B(H)$ and $y\in B(K)$ [**19**; Theorem 1]. Let $\mathcal{F}$ be a bounded increasing net in $A_{s.a.}$ with $\sup_A\mathcal{F}=x\in A_{s.a.}$. Since $B(H)\supset A$ is W^*, $\sup_{B(H)}\mathcal{F}=$strong limit of $\mathcal{F}=y\in B(H)$, say. Then we have

$$\phi(y)=\sup_{I(A)}\mathcal{F}=\sup_A\mathcal{F}=x$$

([**20**; the proof of Theorem 7.1] and Theorem 3.1) and strong limit of $\mathcal{F}\otimes 1$ in $B(H\otimes K)=y\otimes 1\in B(H)\otimes B(K)$. Hence if $(A\otimes B)_{s.a.}\ni z\geqq\mathcal{F}\otimes 1$ then $z\geqq y\otimes 1$, so that $z=(\phi\otimes 1)(z)\geqq(\phi\otimes 1)(y\otimes 1)=\phi(y)\otimes 1=x\otimes 1$. Therefore $x\otimes 1=\sup_{A\otimes B}\mathcal{F}\otimes 1$. q. e. d.

COROLLARY 4.2. *Let A and B be C^*-algebras and let A_1 and B_1 be extensions of A and B respectively. Then*

$$m\text{-cl}_{A_1\otimes B_1}A\otimes 1=(m\text{-cl}_{A_1}A)\otimes 1$$

and

$$(m\text{-cl}_{A_1}A)\otimes(m\text{-cl}_{B_1}B)\subset m\text{-cl}_{A_1\otimes B_1}A\otimes B\,.$$

PROOF. Since the map $A_1\ni x\mapsto x\otimes 1\in A_1\otimes B_1$ is normal, $(m\text{-cl}_{A_1}A)\otimes 1$ is monotone closed in $A_1\otimes B_1$ and so $m\text{-cl}_{A_1\otimes B_1}A\otimes 1\subset(m\text{-cl}_{A_1}A)\otimes 1\subset A_1\otimes 1$. Hence $m\text{-cl}_{A_1\otimes B_1}A\otimes 1$ is monotone closed in $A_1\otimes 1$. So if ϕ denotes the *-isomorphism of A_1 onto $A_1\otimes 1$ given by $\phi(x)=x\otimes 1$ then $\phi^{-1}(m\text{-cl}_{A_1\otimes B_1}A\otimes 1)\supset A$ is monotone closed in A_1; hence $\phi^{-1}(m\text{-cl}_{A_1\otimes B_1}A\otimes 1)\supset m\text{-cl}_{A_1}A$. Thus $m\text{-cl}_{A_1\otimes B_1}A\otimes 1\supset\phi(m\text{-cl}_{A_1}A)$ $=(m\text{-cl}_{A_1}A)\otimes 1$ and $m\text{-cl}_{A_1\otimes B_1}A\otimes 1=(m\text{-cl}_{A_1}A)\otimes 1$. By symmetry $m\text{-cl}_{A_1\otimes B_1}1\otimes B=$ $1\otimes(m\text{-cl}_{B_1}B)$. On the other hand,

$$\begin{aligned}(m\text{-cl}_{A_1}A)\otimes(m\text{-cl}_{B_1}B)&=C^*((m\text{-cl}_{A_1}A)\otimes 1\cup 1\otimes(m\text{-cl}_{B_1}B))\\&=C^*(m\text{-cl}_{A_1\otimes B_1}A\otimes 1\cup m\text{-cl}_{A_1\otimes B_1}1\otimes B)\\&\subset m\text{-cl}_{A_1\otimes B_1}A\otimes B\quad\text{(Lemma 1.4)}.\end{aligned}$$

q. e. d.

COROLLARY 4.3. *Let A and B be C*-algebras with A monotone complete. Then $A\otimes 1$ is a monotone closed C*-subalgebra of $(A\otimes B)^{\bar{}}$.*

PROOF. Immediate from Corollary 4.2 and the fact that the inclusion map $A\otimes B\hookrightarrow(A\otimes B)^{\bar{}}$ is sup-preserving, hence normal. q. e. d.

This corollary shows that if A is a simple monotone complete non W^*, AW^*-factor then $(A\otimes B)^{\bar{}}$ is also a simple non W^*, AW^*-factor for any simple C^*-algebra B since $A\otimes B$ is simple (Remark 3.7).

PROPOSITION 4.4. *Let A and B be C*-algebras. If B is commutative then the maps $A\ni x\mapsto x\otimes 1\in A\otimes B$, $B\ni y\mapsto 1\otimes y\in A\otimes B$ are sup-preserving.*

PROOF. Put $\phi(x)=x\otimes 1\in A\otimes B$ for $x\in A$. For $g\in S(B)$ let L_g be the linear map of $A\otimes B$ into A such that $L_g(\sum_{i=1}^{n}x_i\otimes y_i)=\sum_{i=1}^{n}g(y_i)x_i$ (cf. [20]); then L_g is positive and $L_g\circ\phi=\mathrm{id}_A$. Moreover $L_g(z)\geqq 0$, $z\in(A\otimes B)_{s.a.}$, for all $g\in S(B)$ implies $z\geqq 0$. In fact, since B is commutative, the weak* closed convex hull of $S(A)\otimes S(B)$ in $(A\otimes B)^*=S(A\otimes B)$ [18; Proposition 1]. Hence if $L_g(z)\geqq 0$ for all $f\in S(A)$ and $g\in S(B)$ then $(f\otimes g)(z)=f(L_g(z))\geqq 0$ for all $f\in S(A)$ and $g\in S(B)$, so that $h(z)\geqq 0$ for all $h\in S(A\otimes B)$. Therefore $z\geqq 0$.

Let $\mathcal{F}$ be a subset of $A_{s.a.}$ with $\sup_A\mathcal{F}=x\in A_{s.a.}$. If $(A\otimes B)_{s.a.}\ni z\geqq\phi(\mathcal{F})$ then $A_{s.a.}\ni L_g(z)\geqq L_g\circ\phi(\mathcal{F})=\mathcal{F}$ and $L_g(z)\geqq\sup_A\mathcal{F}=x$. Hence $L_g(z-\phi(x))=$ $L_g(z)-x\geqq 0$ for all $g\in S(B)$ and from the foregoing $z-\phi(x)\geqq 0$, $z\geqq\phi(x)$. Thus $\sup_{A\otimes B}\phi(\mathcal{F})=\phi(x)$. Similarly for the map $B\ni y\mapsto 1\otimes y\in A\otimes B$. q. e. d.

We give an example which shows that Proposition 4.4 is not true for general C^*-algebras A and B. First we show the following:

LEMMA 4.5. *For a Hilbert space H and a family $\mathcal{F}$ consisting of projections of $B(H)$ we have $\sup_{B(H)}\mathcal{F}=1$ if and only if the set $D=\{\xi\in H:\ \|\xi\|=1,\ p\xi=\xi$*

for some $p\in\mathcal{F}\}$ *is dense in the unit sphere of H.*

PROOF. Sufficiency: Suppose that D is dense in the unit sphere of H. Then $B(H)_{s.a.}\ni x\geqq\mathcal{F}$ implies $(x\xi,\xi)\geqq(p\xi,\xi)=(\xi,\xi)$ for all $\xi\in D$ and $p\in\mathcal{F}$ with $p\xi=\xi$, and so $(x\xi,\xi)\geqq(\xi,\xi)$ for all unit vectors ξ of H. Hence $x\geqq 1$ and $\sup_{B(H)}\mathcal{F}=1$.

Necessity: Suppose that $\sup_{B(H)}\mathcal{F}=1$ but that D is not dense in the unit sphere of H. Then there exist a unit vector $\xi_0\in H$ and an $\varepsilon>0$ such that $\|\xi_0-\xi\|\geqq\varepsilon$ for all $\xi\in D$. Let p_0 be the projection of H onto $C\xi_0$. We have $\|p\xi_0\|\leqq 1-\varepsilon^2/2$ for all $p\in\mathcal{F}$ since $\varepsilon^2\leqq\|\xi_0-(p\xi_0/\|p\xi_0\|)\|^2=2(1-\|p\xi_0\|)$. Hence for $p\in\mathcal{F}$,

$$p_0pp_0=\|p_0pp_0\|p_0 \quad\text{and}\quad \|p_0pp_0\|=(p\xi_0,\xi_0)=\|p\xi_0\|^2\leqq(1-\varepsilon^2/2)^2,$$

so that $p_0\mathcal{F}p_0\leqq(1-\varepsilon^2/2)^2p_0$. On the other hand, by Lemma 1.9, $\sup_{B(H)}p_0\mathcal{F}p_0=p_0(\sup_{B(H)}\mathcal{F})p_0=p_0$, a contradiction. q. e. d.

EXAMPLE 4.6. The map $M_2\ni x\mapsto x\otimes 1\in M_2\otimes M_2$ is not sup-preserving, where M_2 denotes the C^*-algebra of 2×2 matrices over $\boldsymbol{C}$.

We identify M_2 with $B(\boldsymbol{C}^2)$, where $\boldsymbol{C}^2$ is the two-dimensional Hilbert space. Suppose that the map ϕ: $M_2\ni x\mapsto x\otimes 1\in M_2\otimes M_2$ is sup-preserving and let $\mathcal{F}$ be the family of all minimal projections in M_2. Then $\sup_{M_2}\mathcal{F}=1$ by Lemma 4.5 and so $\sup_{M_2\otimes M_2}\mathcal{F}\otimes 1=\sup_{M_2\otimes M_2}\phi(\mathcal{F})=\phi(1)=1\otimes 1$. By symmetry $\sup_{M_2\otimes M_2}1\otimes\mathcal{F}=1\otimes 1$. Hence by Lemma 1.9,

$$\sup_{M_2\otimes M_2}\mathcal{F}\otimes q=(1\otimes q)(\sup_{M_2\otimes M_2}\mathcal{F}\otimes 1)(1\otimes q)=1\otimes q$$

for all $q\in\mathcal{F}$, so that

$$\begin{aligned}\sup_{M_2\otimes M_2}\mathcal{F}\otimes\mathcal{F}&=\sup_{M_2\otimes M_2}\cup\{\mathcal{F}\otimes q:\ q\in\mathcal{F}\}\\&=\sup_{M_2\otimes M_2}\{\sup_{M_2\otimes M_2}\mathcal{F}\otimes q:\ q\in\mathcal{F}\}\\&=\sup_{M_2\otimes M_2}\{1\otimes q:\ q\in\mathcal{F}\}\\&=\sup_{M_2\otimes M_2}1\otimes\mathcal{F}=1\otimes 1,\end{aligned}$$

which implies again by Lemma 4.5 that the set $D=\{\xi\otimes\eta\in\boldsymbol{C}^2\otimes\boldsymbol{C}^2:\ \xi,\eta\in\boldsymbol{C}^2,\ \|\xi\|=\|\eta\|=1\}$ is dense in the unit sphere of $\boldsymbol{C}^2\otimes\boldsymbol{C}^2$. But this is a contradiction since the unit vector $(\xi_1\otimes\xi_1+\xi_1\otimes\xi_2+\xi_2\otimes\xi_1-\xi_2\otimes\xi_2)/2$ with $\{\xi_1,\xi_2\}$ an orthonormal basis in $\boldsymbol{C}^2$ does not belong to the closed set D.

We give another example, which will be used later.

EXAMPLE 4.7. Let $A=\left\{\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\in M_2:\ \alpha,\beta\in\boldsymbol{C}\right\}\subset M_2$. Then the inclusion map $A\hookrightarrow M_2$ is not sup-preserving.

In fact $\sup_A\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ since A is commutative. But

(4.1) $$\sup_{M_2}\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\} \text{ does not exist}$$

since if it does then it must be a projection (Lemma 3.11), and so

$$\sup_{M_2}\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};$$

while $$\begin{pmatrix}2 & \sqrt{2}\\ \sqrt{2} & 2\end{pmatrix}\geqq\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

but $$\begin{pmatrix}2 & \sqrt{2}\\ \sqrt{2} & 2\end{pmatrix}\not\geqq\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

The next lemma is motivated by the observation due to E. G. Effros [**5**; (2.2)]:

LEMMA 4.8. *Let A be a C^*-algebra and p a projection of A. Then for $x\in A_{s.a.}$ we have $x\geqq 0$ if and only if $pxp\geqq 0$, $(1-p)x(1-p)\geqq 0$ and $pxp+\varepsilon p\geqq px(1-p)\{(1-p)x(1-p)+\varepsilon\}^{-1}(1-p)xp$ for all $\varepsilon>0$.*

PROOF. For $\varepsilon>0$ put $x_\varepsilon=x+\varepsilon$ and $(x_\varepsilon)_{ij}=p_ix_\varepsilon p_j$, $i, j=1, 2$, where $p_1=p$ and $p_2=1-p$. Then $x\geqq 0$ if and only if $x_\varepsilon\geqq 0$ for all $\varepsilon>0$ if and only if $(x_\varepsilon)_{11}\geqq 0$, $(x_\varepsilon)_{22}\geqq 0$ and

(4.2) $$\begin{aligned}&1+(x_\varepsilon)_{11}^{-1/2}(x_\varepsilon)_{12}(x_\varepsilon)_{22}^{-1/2}+(x_\varepsilon)_{22}^{-1/2}(x_\varepsilon)_{21}(x_\varepsilon)_{11}^{-1/2}\\ &=\{(x_\varepsilon)_{11}^{-1/2}+(x_\varepsilon)_{22}^{-1/2}\}x_\varepsilon\{(x_\varepsilon)_{11}^{-1/2}+(x_\varepsilon)_{22}^{-1/2}\}\geqq 0\end{aligned}$$

for all $\varepsilon>0$, where $(x_\varepsilon)_{ii}^{-1/2}$ [$=$ the $-1/2$ power of $(x_\varepsilon)_{ii}$ in p_iAp_i]$=p_i(p_ixp_i+\varepsilon)^{-1/2}p_i$, $i=1, 2$. Moreover (4.2) holds if and only if

$$\{(x_\varepsilon)_{11}^{-1/2}(x_\varepsilon)_{12}(x_\varepsilon)_{22}^{-1/2}\}\{(x_\varepsilon)_{11}^{-1/2}(x_\varepsilon)_{12}(x_\varepsilon)_{22}^{-1/2}\}^*\leqq p_1,$$

i. e., $$px(1-p)\{(1-p)x(1-p)+\varepsilon\}^{-1}(1-p)xp\leqq pxp+\varepsilon p.$$

In fact, putting $y=(x_\varepsilon)_{11}^{-1/2}(x_\varepsilon)_{12}(x_\varepsilon)_{22}^{-1/2}$, $1+y+y^*\geqq 0$ implies $p_1-yy^*=(p_1-y)(1+y+y^*)(p_1-y)^*\geqq 0$, and conversely $yy^*\leqq p_1$ implies $1+y+y^*=(p_2+y)(p_2+y)^*+p_1-yy^*\geqq 0$. q. e. d.

PROPOSITION 4.9. *For a C^*-algebra A and a projection p of A the inclusion map $pAp\hookrightarrow A$ is sup-preserving.*

PROOF. Let $\mathcal{F}$ be a subset of $(pAp)_{s.a.}$ with $\sup_{pAp}\mathcal{F}=x\in(pAp)_{s.a.}$. By Lemma 4.8, $A_{s.a.}\ni y\geqq a$ for all $a\in\mathcal{F}$ if and only if $pyp\geqq a$, $(1-p)y(1-p)\geqq 0$ and $pyp-a+\varepsilon p\geqq py(1-p)\{(1-p)y(1-p)+\varepsilon\}^{-1}(1-p)yp$ for all $a\in\mathcal{F}$ and $\varepsilon>0$, which imply $pyp\geqq x$, $(1-p)y(1-p)\geqq 0$ and $pyp-x+\varepsilon p\geqq py(1-p)\{(1-p)y(1-p)+\varepsilon\}^{-1}(1-p)yp$ for all $\varepsilon>0$, so that $y\geqq x$ again by Lemma 4.8. Hence $\sup_A\mathcal{F}=x$.

q. e. d.

From this proposition a sharpening of Lemma 1.9 follows (compare with [**10**; Lemma 2.1]):

COROLLARY 4.10. *Let A be a C^*-algebra and a any element of A. If $\mathcal{F}$ is a bounded subset of $A_{s.a.}$ such that* $\sup_A \mathcal{F}$ *exists then* $\sup_A a^*\mathcal{F}a = a^*(\sup_A \mathcal{F})a$.

PROOF. Embed A in its regular monotone completion $\bar{A}$. Since $\bar{A}$ is AW^*, a has the polar decomposition $a=wr$, where $r=(a^*a)^{1/2}\in A$ and w is a partial isometry of $\bar{A}$ such that $w^*w=\mathrm{RP}(a)$ and $ww^*=\mathrm{LP}(a)$ [1; p. 133, Proposition 2]. Put $e=w^*w$ and $f=ww^*$. By Theorem 3.1, $\sup_A \mathcal{F}=\sup_{\bar{A}}\mathcal{F}$, and by Lemma 1.9 and Proposition 4.9,

$$\begin{aligned} w^*(\sup_{\bar{A}}\mathcal{F})w &= w^*f(\sup_{\bar{A}}\mathcal{F})fw = w^*(\sup_{\bar{A}} f\mathcal{F}f)w \\ &= w^*(\sup_{f\bar{A}f} f\mathcal{F}f)w = \sup_{e\bar{A}e} w^*\mathcal{F}w \\ &= \sup_{\bar{A}} w^*\mathcal{F}w\,, \end{aligned}$$

where we used also the facts that if $\sup_{\bar{A}}\mathcal{G}$ with $\mathcal{G}\subset(f\bar{A}f)_{s.a.}$ exists then $\sup_{\bar{A}}\mathcal{G}=\sup_{f\bar{A}f}\mathcal{G}$ and that the map $f\bar{A}f\ni x\mapsto w^*xw\in e\bar{A}e$ is a *-isomorphism. Hence again by Lemma 1.9,

$$\begin{aligned} a^*(\sup_A \mathcal{F})a &= rw^*(\sup_{\bar{A}}\mathcal{F})wr = r(\sup_{\bar{A}} w^*\mathcal{F}w)r \\ &= \sup_{\bar{A}} rw^*\mathcal{F}wr = \sup_A a^*\mathcal{F}a\,. \end{aligned}$$

q. e. d.

We characterize the sup-preserving unital *-monomorphism of a commutative AW^*-algebra into another C^*-algebra.

LEMMA 4.11. *If A is a monotone closed C^*-subalgebra of a commutative AW^*-algebra C then the inclusion map $A\hookrightarrow C$ is sup-preserving.*

PROOF. By hypothesis the inclusion map $A\hookrightarrow C$ is normal. If $\mathcal{F}$ is a subset of $A_{s.a.}$ with $\sup_A \mathcal{F}=x\in A_{s.a.}$ then, for a fixed $a_0\in\mathcal{F}$, $\mathcal{G}=\{\sup_A(\mathcal{F}'\cup\{a_0\}) : \mathcal{F}'$ is a finite subset of $\mathcal{F}\}$ is a bounded increasing net in $A_{s.a.}$ with $\sup_A \mathcal{G}=x$. (Note that $A_{s.a.}$ is a lattice.) Hence $\sup_C \mathcal{G}=x$. Moreover, since $\sup_A(\mathcal{F}'\cup\{a_0\}) =\sup_C(\mathcal{F}'\cup\{a_0\})\leqq\sup_C\mathcal{F}$, we have $\sup_C\mathcal{G}=\sup_C\mathcal{F}$. Thus $\sup_A\mathcal{F}=\sup_A\mathcal{G}=\sup_C\mathcal{G} =\sup_C\mathcal{F}$. q. e. d.

LEMMA 4.12. *Let B be an AW^*-algebra and e_1, e_2 two orthogonal projections of B. If* $\sup_B\{e_1, e_2\}=e_1+e_2$ *then* $C(e_1)C(e_2)=0$, *where $C(p)$ with p a projection denotes the central cover of p in B.*

PROOF. By the comparability theorem [**1**; p. 80, Corollary 1] there exists a central projection h such that

$$he_1 \prec he_2 \quad \text{and} \quad (1-h)e_1 \succ (1-h)e_2\,.$$

Suppose $C(e_1)C(e_2)\neq 0$. Then $he_1\neq 0$ or $(1-h)e_2\neq 0$; e.g., let $he_1\neq 0$. We have a projection $p\in B$ and a partial isometry $w\in B$ such that $he_1=w^*w\sim p=ww^*\leqq he_2$. It follows from $\sup_B\{e_1, e_2\}=e_1+e_2$ that

(4.3) $$\sup_B\{he_1, p\}=he_1+p\,.$$

In fact if $B_{s.a.}\ni x\geqq he_1, p$ then

$$x+(1-h)e_1+e_2-p\geqq\begin{cases} x+(1-h)e_1\geqq e_1 \\ x+e_2-p\geqq e_2\,; \end{cases}$$

hence $x+(1-h)e_1+e_2-p\geqq e_1+e_2$ and $x\geqq he_1+p$. Since the algebra $\boldsymbol{C}he_1+\boldsymbol{C}w+\boldsymbol{C}w^*+\boldsymbol{C}p\subset B$ is *-isomorphic to M_2 under the *-isomorphism which sends he_1, w, w^*, p to $\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}$ respectively, (4.3) means that $\sup_{M_2}\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, contradictory to (4.1) in Example 4.7.

q. e. d.

LEMMA 4.13. *Let A be a commutative AW^*-algebra and B an extension of A which is AW^* and contains A as a monotone closed C^*-subalgebra. Then the inclusion map $A\hookrightarrow B$ is sup-preserving if and only if $A\subset Z_B$.*

PROOF. Sufficiency: Let $\sup_A\mathcal{F}=a_0$ for some $\mathcal{F}\subset A_{s.a.}$ and $a_0\in A_{s.a.}$. Then we must show that $B_{s.a.}\ni x\geqq\mathcal{F}$ implies $x\geqq a_0$. But, since $A\subset Z_B$, there exists a maximal commutative *-subalgebra C of B which contains A and x. Then by Lemma 4.11, $x\geqq\sup_C\mathcal{F}=\sup_A\mathcal{F}=a_0$.

Necessity: Suppose that the inclusion map $A\hookrightarrow B$ is sup-preserving and let $e\in A_p$. Then $\sup_B\{e, 1-e\}=\sup_A\{e, 1-e\}=1$, and by Lemma 4.12, $C(e)C(1-e)=0$; hence $e\leqq C(e)\leqq 1-C(1-e)\leqq 1-(1-e)=e$, $e=C(e)\in Z_B$. Thus $A\subset Z_B$.

q. e. d.

PROPOSITION 4.14. *Let A be a commutative AW^*-algebra and B an extension of A. Then the inclusion map $A\hookrightarrow B$ is sup-preserving if and only if it is normal and $A\subset Z_B$.*

PROOF. Since the inclusion map $B\hookrightarrow\bar{B}$ is sup-preserving, the inclusion map $A\hookrightarrow B$ is sup-preserving if and only if $A\hookrightarrow\bar{B}$ is. Moreover, since $Z_B\subset Z_{\bar{B}}$ (Remark 3.7) and so $Z_B=Z_{\bar{B}}\cap B$, $A\subset Z_B$ if and only if $A\subset Z_{\bar{B}}$. On the other hand, since A is monotone complete, $A\hookrightarrow B$ is normal if and only if A is monotone closed in $\bar{B}$. Hence Lemma 4.13 applied to A and $\bar{B}$ completes the proof. q. e. d.

REMARK 4.15. Let A be a commutative AW^*-algebra and B an extension of A which is an AW^*-algebra. Then A is monotone closed in B if and only if it is an AW^*-subalgebra of B. Necessity is clear. Sufficiency: Take a maximal commutative *-subalgebra C of B which contains A. Then C is monotone closed in B since $\sup_B\mathcal{F}=x$ with $\mathcal{F}$ a bounded increasing net in $C_{s.a.}$ and $x\in B_{s.a.}$ implies that $u^*xu=\sup_B u^*\mathcal{F}u=\sup_B\mathcal{F}=x$ for all $u\in C_u$, hence that $x\in C'\cap B=C$. Moreover it is readily seen that an AW^*-subalgebra of a commutative AW^*-algebra is monotone closed.

§ 5. Regular extensions of C^*-tensor products.

In this section we consider the regular extensions of the C^*-tensor products of special C^*-algebras.

PROPOSITION 5.1. *Let A be a commutative C^*-algebra and $B(H)$ the type* I *W^*-factor with* $\dim H=\aleph$. *Then $(A\otimes B(H))^{-}$ is the $\aleph$-homogeneous type* I *AW^*-algebra with center isomorphic to $\bar{A}$. Conversely an $\aleph$-homogeneous type* I *AW^*-algebra B is of the form $B=(Z\otimes B(H))^{-}$ with Z the center of B and* $\dim H=\aleph$.

PROOF. Take an orthogonal family $\{p_i\}_{i\in I}$ of minimal projections in $B(H)$ with $\bigvee_{B(H)}\{p_i: i\in I\}=1$ and $\bar{\bar{I}}=\aleph$. Then $\{\sum_{i\in J}p_i: J\subset I$ finite subsets$\}$ is an increasing net with supremum 1 and so by Proposition 4.1,

$$\sup_{(A\otimes B(H))^-}\{\textstyle\sum_{i\in J}1\otimes p_i: J\subset I \text{ finite subsets}\}=1\otimes 1;$$

hence

$$\bigvee_{(A\otimes B(H))^-}\{1\otimes p_i: i\in I\}=1\otimes 1.$$

By Proposition 1.11 we have

$$(1\otimes p_i)(A\otimes B(H))^{-}(1\otimes p_i)=\{(1\otimes p_i)(A\otimes B(H))(1\otimes p_i)\}^{-}$$
$$=(A\otimes p_i)^{-}\cong\bar{A}$$

with $\bar{A}$ commutative, so that the $1\otimes p_i$ are mutually equivalent abelian projections of $(A\otimes B(H))^{-}$ with supremum $1\otimes 1$. Hence $(A\otimes B(H))^{-}$ is an $\aleph$-homogeneous type I AW^*-algebra with center $\bar{A}\otimes 1$. The second statement of the theorem is obvious from the first one and the uniqueness of the $\aleph$-homogeneous type I AW^*-algebra for the given center and $\aleph$ [**13**; Theorem 1]. q. e. d.

The next result shows that with notation as above we have $(A\otimes B(H))^{-}=(A\otimes B(H))^{\sim}=I(A\otimes B(H))$.

PROPOSITION 5.2. *Any* type I *AW^*-algebra is injective.*

PROOF. Let A be a type I AW^*-algebra. We may and will assume that A is homogeneous since each type I AW^*-algebra is a C^*-direct sum of homogeneous ones and a C^*-direct sum is injective whenever each direct summand is. Then by Proposition 5.1 we have $A=(Z\otimes B(K))^{-}$ with Z a commutative AW^*-algebra and K a Hilbert space. We assume that Z is a C^*-subalgebra, containing the unit, of $B(H)$ with H a Hilbert space. Since Z is injective, there exists a completely positive projection ϕ of $B(H)$ onto Z. Let $\{\eta_\alpha\}_{\alpha\in I}$ be an orthonormal basis in K, $\bar{\bar{I}}=\aleph$ and $J_\alpha: H\to H\otimes K$ the linear isometry defined by $J_\alpha\xi=\xi\otimes\eta_\alpha$. Then each $x\in B(H\otimes K)$ has the matrix representation $x=[x_{\alpha\beta}]$ with $x_{\alpha\beta}=J_\alpha^* xJ_\beta\in B(H)$, α, $\beta\in I$. Then the map

$$\phi\otimes 1: B(H\otimes K)\to B(H\otimes K),\ (\phi\otimes 1)([x_{\alpha\beta}])=[\phi(x_{\alpha\beta})]$$

is a well-defined completely positive projection. In fact let p_α be the projection of K onto $\boldsymbol{C}\eta_\alpha$ and let $\{q_\gamma\}$ be the family of all finite sums of the p_α. Then, since ϕ is completely positive, for each $x \in B(H\otimes K)$ we have

$$\sup_\gamma \|(1\otimes q_\gamma)[\phi(x_{\alpha\beta})](1\otimes q_\gamma)\| \leqq \sup_\gamma \|(1\otimes q_\gamma)[x_{\alpha\beta}](1\otimes q_\gamma)\| = \|x\|,$$

where $(1\otimes q_\gamma)[\phi(x_{\alpha\beta})](1\otimes q_\gamma)$ denotes the matrix $[y_{\alpha\beta}] \in B(H\otimes K)$ such that $y_{\alpha\beta} = \phi(x_{\alpha\beta})$ if $p_\alpha q_\gamma = p_\alpha$ and $p_\beta q_\gamma = p_\beta$; $=0$ otherwise. This implies that the element in $B(H\otimes K)$ with the matrix representation $[\phi(x_{\alpha\beta})]$ exists and that $\phi\otimes 1$ is a well-defined contractive projection. Moreover, replacing K in the above argument by the direct sum of n copies of K $(n=1, 2, \cdots)$, we see that $\phi\otimes 1$ is completely positive. Hence $(\phi\otimes 1)(B(H\otimes K)) = V$, say, is an injective operator system and so V, equipped with the multiplication given by $x\circ y = (\phi\otimes 1)(xy)$, is an injective C^*-algebra which contains $Z\otimes B(K)$ as a C^*-subalgebra [**3**; Theorem 3.1]. As in Proposition 5.1 we have

$$(1\otimes p_{\alpha'})\circ V\circ(1\otimes p_{\alpha'}) = \{[x_{\alpha\beta}] \in V:\ x_{\alpha\beta}=0 \text{ if } \alpha\neq\alpha' \text{ or } \beta\neq\alpha'\} \cong Z$$

and $\bigvee_V 1\otimes p_\alpha = \sup_V 1\otimes q_\gamma = (\phi\otimes 1)(\sup_{B(H\otimes K)} 1\otimes q_\gamma) = 1\otimes 1$, so that V is an $\aleph$-homogeneous type I AW^*-algebra with center Z. Hence $A\cong V$ is injective. q. e. d.

PROPOSITION 5.3. *Let A be an arbitrary C^*-algebra and suppose that* (i) *B is a commutative C^*-algebra or that* (ii) *B is a type* I *AW^*-algebra. Then $\tilde{A}\otimes\tilde{B}$ is a regular extension of $A\otimes B$, i. e., $\tilde{A}\otimes\tilde{B} \subset (A\otimes B)^\sim$.*

PROOF. (i) We first show that $A\otimes B$ is order dense in $\tilde{A}\otimes B$. If $x\in\tilde{A}_{s.a.}$ then $x=\sup_{\tilde{A}}\mathcal{F}$ for some $\mathcal{F}\subset A_{s.a.}$, and by Proposition 4.4, $x\otimes 1 = \sup_{\tilde{A}\otimes B}\mathcal{F}\otimes 1$. Hence for $y\in B^+$,

$$\begin{aligned} x\otimes y &= (1\otimes y^{1/2})(x\otimes 1)(1\otimes y^{1/2}) \\ &= \sup_{\tilde{A}\otimes B}(1\otimes y^{1/2})(\mathcal{F}\otimes 1)(1\otimes y^{1/2}) \\ &= \sup_{\tilde{A}\otimes B}\mathcal{F}\otimes y \qquad \text{(Lemma 1.9).} \end{aligned}$$

The set consisting of the elements of the form $x_1\otimes y_1 + \cdots + x_n\otimes y_n$ with $x_1, \cdots, x_n \in \tilde{A}_{s.a.}$ and $y_1, \cdots, y_n \in B^+$ is norm dense in $(\tilde{A}\otimes B)_{s.a.}$, and from the foregoing $x_i\otimes y_i = \sup_{\tilde{A}\otimes B}\mathcal{G}_i$ for some $\mathcal{G}_i \subset (A\otimes B)_{s.a.}$; hence

$$x_1\otimes y_1 + \cdots + x_n\otimes y_n = \sup_{\tilde{A}\otimes B}(\mathcal{G}_1 + \cdots + \mathcal{G}_n).$$

Thus $A\otimes B$ is order dense in $\tilde{A}\otimes B$. A similar process shows that $\tilde{A}\otimes B$ is order dense in $\tilde{A}\otimes\tilde{B}$, so that $A\otimes B$ is order dense in $\tilde{A}\otimes\tilde{B}$ (Lemma 2.5).

(ii) Since B is type I AW^*, there exists an orthogonal family $\{p_i\}_{i\in I}$ of nonzero abelian projections of B with supremum 1. The family $\{q_J\}$ of all finite sums $q_J=\sum_{i\in J}p_i$ ($J\subset I$ finite subsets) forms an increasing net and we have $\sup_B q_J=1$ since B is monotone complete and $\sup_B q_J$ is a projection (Lemma 3.11). Hence by Proposition 4.1, $\sup_{\tilde{A}\otimes B}1\otimes q_J=1\otimes 1$, and for each $x\in\tilde{A}^+$,

$$x\otimes 1=(x^{1/2}\otimes 1)(1\otimes 1)(x^{1/2}\otimes 1)=\sup_{\tilde{A}\otimes B}x\otimes q_J\,.$$

Moreover, since p_iBp_i is commutative, by (i) and Proposition 4.9 we have

$$x\otimes p_i=\sup_{\tilde{A}\otimes p_iBp_i}\mathcal{F}_i=\sup_{\tilde{A}\otimes B}\mathcal{F}_i$$

for some $\mathcal{F}_i\subset(A\otimes p_iBp_i)_{s.a.}$, so that

$$x\otimes q_J=\sum_{j\in J}x\otimes p_i=\sup_{\tilde{A}\otimes B}\sum_{i\in J}\mathcal{F}_i\,.$$

Hence we have

$$x\otimes 1=\sup_{\tilde{A}\otimes B}\{\sum_{i\in J}\mathcal{F}_i: J\subset I \text{ finite subsets}\},$$

and so for $x\in\tilde{A}_{s.a.}$,

$$x\otimes 1=(x+\|x\|)\otimes 1-\|x\|(1\otimes 1)=\sup_{\tilde{A}\otimes B}\{\mathcal{G}-\|x\|(1\otimes 1)\}$$

with $\mathcal{G}\subset(A\otimes B)_{s.a.}$. Since $\tilde{B}=B$ by Proposition 5.2, the reasoning as in (i) completes the proof. q. e. d.

COROLLARY 5.4. *Under the same hypothesis as above we have* $\bar{A}\otimes\bar{B}\subset(A\otimes B)^{-}$.

PROOF. Note by the construction that $\bar{A}=m\text{-cl}_{\tilde{A}}A$ and $\bar{B}=m\text{-cl}_{\tilde{B}}B$. Since $\tilde{A}\otimes\tilde{B}\subset(A\otimes B)^{\sim}$ and the inclusion map $\tilde{A}\otimes\tilde{B}\hookrightarrow(A\otimes B)^{\sim}$ is normal, $m\text{-cl}_{\tilde{A}\otimes\tilde{B}}A\otimes B\subset m\text{-cl}_{(A\otimes B)^{\sim}}A\otimes B=(A\otimes B)^{-}$. Moreover by Corollary 4.2,

$$\bar{A}\otimes\bar{B}=(m\text{-cl}_{\tilde{A}}A)\otimes(m\text{-cl}_{\tilde{B}}B)\subset m\text{-cl}_{\tilde{A}\otimes\tilde{B}}A\otimes B\,.$$

Hence $\bar{A}\otimes\bar{B}\subset(A\otimes B)^{-}$. q. e. d.

COROLLARY 5.5. *For any C*-algebra A we have* $(A\otimes M_n)^{\sim}=\tilde{A}\otimes M_n$.

PROOF. We have $\tilde{A}\otimes M_n\subset(A\otimes M_n)^{\sim}$. Since $1\otimes M_n\subset(A\otimes M_n)^{\sim}$, $(A\otimes M_n)^{\sim}$ is of the form $B\otimes M_n$ with $B\supset A$ a C^*-algebra. Take a minimal projection e of M_n. Then $(1\otimes e)(A\otimes M_n)(1\otimes e)\cong A$ and $(1\otimes e)(B\otimes M_n)(1\otimes e)\cong B$, so that A is order dense in B (Lemma 1.10), i.e., $A\subset B\subset\tilde{A}$. Hence $(A\otimes M_n)^{\sim}=B\otimes M_n\subset\tilde{A}\otimes M_n$ and so $(A\otimes M_n)^{\sim}=\tilde{A}\otimes M_n$. q. e. d.

§6. The type I direct summand of the injective envelope.

In this section we see that for any C^*-algebra A the maximum type I direct summands of $\bar{A}$ and $I(A)$ coincide (Corollary 6.5). Hence the study of $I(A)$ is reduced to those of $\bar{A}$ with A any C^*-algebra and of $I(A)$ with A a continuous monotone complete AW^*-algebra.

We prepare some lemmas.

LEMMA 6.1. *Let A be a monotone complete C^*-algebra, $I(A)$ its injective* *envelope and* e_1, e_2 *projections of* A. *Then we have*

$$\bigvee_A\{e_1, e_2\}=\bigvee_{I(A)}\{e_1, e_2\} \quad \text{and} \quad \bigwedge_A\{e_1, e_2\}=\bigwedge_{I(A)}\{e_1, e_2\}.$$

RROOF. Put $a=(e_1+e_2)/\|e_1+e_2\|\in A_{s.a.}$. Then $\sup_A\{a^{1/n}: n=1, 2, \cdots\}$ exists and equals $\bigvee_A\{e_1, e_2\}$. Similarly $\sup_{I(A)}\{a^{1/n}: n=1, 2, \cdots\}=\bigvee_{I(A)}\{e_1, e_2\}$. Hence by Theorem 3.1, $\bigvee_A\{e_1, e_2\}=\sup_A\{a^{1/n}: n=1, 2, \cdots\}=\sup_{I(A)}\{a^{1/n}: n=1, 2, \cdots\}=\bigvee_{I(A)}\{e_1, e_2\}$. Moreover $\bigwedge_A\{e_1, e_2\}=1-\bigvee_A\{1-e_1, 1-e_2\}=1-\bigvee_{I(A)}\{1-e_1, 1-e_2\}=\bigwedge_{I(A)}\{e_1, e_2\}$. q. e. d.

LEMMA 6.2. *Let* A *be a* C^**-algebra,* $I(A)$ *its injective envelope and* h *a central projection of* $I(A)$. *Then the injective envelope of* hA *is* $hI(A)$.

PROOF. We have $hA\subset hI(A)$, and $hI(A)$ is injective. Hence we need only show that if $\phi: hI(A)\rightarrow hI(A)$ is a completely positive map with $\phi|_{hA}=\mathrm{id}_{hA}$ then $\phi=\mathrm{id}_{hI(A)}$. But the map $\psi: I(A)\rightarrow I(A)$ defined by $\psi(x)=\phi(hx)+(1-h)x$, $x\in I(A)$ is completely positive and $\psi|_A=\mathrm{id}_A$, so that $\psi=\mathrm{id}_{I(A)}$ and $\phi=\mathrm{id}_{hI(A)}$ as desired. q. e. d.

THEOREM 6.3. *If* A *is monotone complete* C^**-algebra and* $I(A)$ *is its injective envelope then* $Z_A=Z_{I(A)}$.

PROOF. We know that $Z_A\subset Z_{I(A)}$ [**6**; Corollary 4.3]. To see the converse inclusion take a projection $h\in Z_{I(A)}$ and let $\mathcal{F}=\{e\in A_p: e\geqq h\}$. Put $h_1=\bigwedge_A\mathcal{F}\in A_p$; then $h_1\in Z_A$ and $h_1\geqq h$. In fact, since $u^*\mathcal{F}u=\mathcal{F}$ for all $u\in A_u$, we have $u^*h_1u=h_1$ for all $u\in A_u$. Moreover $\mathcal{F}$ is a decreasing net since for $e_1, e_2\in\mathcal{F}$ we have $\bigwedge_A\{e_1, e_2\}=\bigwedge_{I(A)}\{e_1, e_2\}\geqq h$ by Lemma 6.1, so that $h_1=\bigwedge_A\mathcal{F}=\inf_A\mathcal{F}=\inf_{I(A)}\mathcal{F}\geqq h$ (Lemma 3.11).

Define a *-homomorphism $\pi: h_1I(A)\rightarrow hI(A)$ by $\pi(x)=hx$; then $\pi|_{h_1A}$ is a *-isomorphism of h_1A onto hA. In fact, $\mathrm{Ker}(\pi|_{h_1A})$, being a two-sided ideal of the AW^*-algebra h_1A, is generated by its projections [**1**; p. 140, Proposition 5]. So if $\mathrm{Ker}(\pi|_{h_1A})\neq 0$ then there exists a nonzero projection $e\in\mathrm{Ker}(\pi|_{h_1A})$. Hence $he=\pi(e)=0$, $h\leqq h_1-e\lneqq h_1$ and $h_1-e\in A_p$, a contradiction.

Moreover by Lemma 6.2, $h_1I(A)$ and $hI(A)$ are injective envelopes of h_1A and hA respectively. Hence by the uniqueness of the injective envelope π is a *-isomorphism and so $\pi(h_1-h)=h(h_1-h)=0$ implies $h=h_1\in Z_A$. q. e. d.

COROLLARY 6.4. *The injective envelope of a monotone complete* AW^**-factor is also an* AW^**-factor.*

COROLLARY 6.5. *With notation as in Theorem* 6.3 *let* h *(resp.* h_1*) be the central projection of* A *(resp.* $I(A)$*) such that* hA *(resp.* $h_1I(A)$*) is the maximum type* I *direct summand of* A *(resp.* $I(A)$*). Then* $h=h_1$ *and* $hA=hI(A)$. *In particular* A *is discrete (resp. continuous) if and only if* $I(A)$ *is so.*

PROOF. We have $h, h_1\in Z_A=Z_{I(A)}$ and $hI(A)=I(hA)=hA$ is of type I (Lemma 6.2 and Proposition 5.2). Hence $h\leqq h_1$. On the other hand h_1A is a monotone closed, hence AW^*-subalgebra of $h_1I(A)$ with $Z_{h_1A}=Z_{h_1I(A)}$. Thus by

[**17**; Theorem 1] $h_1A=(h_1A)''$ (the double commutant of h_1A in $h_1I(A)$). Moreover $(h_1A)'=Z_{h_1I(A)}$ [**6**; Corollary 4.3], so that $h_1A=(Z_{h_1I(A)})'=h_1I(A)$ is of type I. Hence $h_1\leqq h$, and consequently $h=h_1$ and $hA=hI(A)$. q. e. d.

THEOREM 6.6. *If A is a* GCR-*algebra then $\bar{A}=\tilde{A}=I(A)$ and this is a type* I *AW^*-algebra.*

PROOF. It suffices to show that $\bar{A}$ is of type I since it will follow then from Proposition 5.2 that $\bar{A}=\tilde{A}=I(A)$. Let h be the central projection of $\bar{A}$ such that $h\bar{A}$ is the maximum type I direct summand of $\bar{A}$. Suppose $h\neq 1$. Then $(1-h)A$ is order dense in $(1-h)\bar{A}$ (Lemma 1.10). In particular $(1-h)A\neq 0$ and it, being *-isomorphic to a quotient C^*-algebra of A, is GCR. Hence by [**12**; Lemma 3] there exists a nonzero element $x\in((1-h)A)^+$ such that $x(1-h)Ax$ is commutative. Then $x(1-h)\bar{A}x$ is commutative. In fact, by Lemma 1.10, $x(1-h)Ax$ is order dense in $x(1-h)\bar{A}x$, and so is $B=$[the norm closure of $\boldsymbol{C}(1-h)+x(1-h)Ax$] in $C=$[the norm closure of $\boldsymbol{C}(1-h)+x(1-h)\bar{A}x$]. Hence C, being a regular extension of the commutative C^*-algebra B, is commutative. Since $x\neq 0$ is in the AW^*-algebra $(1-h)\bar{A}$, there exist a nonzero projection $p\in(1-h)\bar{A}$ and an element $y\in(1-h)\bar{A}$ such that $xy=yx=p$ [**1**; p. 42, Proposition 3]. Then $p\bar{A}p=xy\bar{A}yx\subset x(1-h)\bar{A}x$ is commutative. Hence p is a nonzero abelian projection $\leqq 1-h$, a contradiction. Thus $\bar{A}$ is of type I. q. e. d.

§7. The C^*-algebra whose regular monotone completion is an AW^*-factor.

A C^*-algebra is said to be *prime* if there are no nonzero closed two-sided ideals J and K such that $JK=0$.

THEOREM 7.1. *Given a C^*-algebra A its regular monotone completion $\bar{A}$ is an AW^*-factor if and only if A is prime.*

PROOF. Sufficiency: Suppose that $\bar{A}$ is not an AW^*-factor. Then there exists a central projection h of $\bar{A}$ with $0\neq h\neq 1$, and $J=A\cap hA$ and $K=A\cap(1-h)A$ are nonzero closed two-sided ideals of A with $JK=0$, i. e., A is not prime. In fact suppose $J=0$ and define a *-homomorphism $\pi:\bar{A}\to(1-h)\bar{A}$ by $\pi(x)=(1-h)x$. Then $\pi|_A$ is one-to-one, so that π is a *-isomorphism and $h=0$, a contradiction. Hence $J\neq 0$ and similarly $K\neq 0$.

Necessity: If A is not prime then there exist nonzero closed two-sided ideals J and K with $JK=0$. Put $h=\sup_{\bar{A}}\{\mathrm{LP}(x): x\in J^+\}$, where $\mathrm{LP}(x)$ denotes the left projection of x in $\bar{A}$ and h exists since $\{\mathrm{LP}(x): x\in J^+\}$ is an increasing net. Clearly $h\neq 0$, and it is a central projection of $\bar{A}$ since $u^*hu=\sup_{\bar{A}}\{\mathrm{LP}(u^*xu): x\in J^+\}=h$ for all $u\in A_u$ and so $h\in Z_{\bar{A}}$ [**6**; Corollary 4.3]. Take a nonzero $y\in K^+$. Then $yxy=0$ for all $x\in J^+$ implies $y\mathrm{LP}(x)y=0$, hence $yhy=0$ (Lemma 1.9). Therefore $h\neq 1$ and $\bar{A}$ is not an AW^*-factor. q. e. d.

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