J. Math. Soc. Japan
Vol. 33, No. 1, 1981

On presentations of the fundamental group of the 3-sphere associated with Heegaard diagrams

By Takeshi KANETO

(Received Jan. 8, 1979)
(Revised March 29, 1979)

1. Introduction.

We argue a property on presentations of the fundamental group of the 3-sphere S^3 associated with Heegaard diagrams. Few years ago, Prof. T. Homma and M. Ochiai made many examples of homology 3-spheres of Heegaard genus 2 by an electronic computer and picked up simply connected ones among them. In the process, they found an interesting fact that in case of 3-sphere, one of two relators of the presentation is included completely in the other one as a subword. Using this property, they showed splendidly the triviality of the group. In this paper, we show that this fact is true in certain sense for the general case of Heegaard genus n. Our proof is based on Suzuki's result [4].

2. Statement of a result.

To state our result precisely, we need some definitions about presentations of groups. Let $\langle a_1, \cdots, a_n; r_1, \cdots, r_m\rangle$ denote a presentation of a finitely generated group with generators $a_1, \cdots, a_n$ and relators $r_1, \cdots, r_m$. It will be noticed that the relator r_i is a word in the alphabet $a_1, \cdots, a_n$.

DEFINITION 1. (*Simple transformation*): We call the following transformations of relators of a presentation *simple transformations*, cf. [2]. Let b_q $(q=0, 1, \cdots, k)$ denote a letter in the alphabet $a_1, \cdots, a_n$, that is, an element of the set $\{a_1, \cdots, a_n, a_1^{-1}, \cdots, a_n^{-1}\}$.

S_1) (*Cyclic reduction*); If a relator r_i is of the form $(b_1 \cdots b_j)b_0b_0^{-1}(b_{j+1} \cdots b_k)$ or $b_0(b_1 \cdots b_k)b_0^{-1}$, we say that the relator $r_i'=b_1 \cdots b_k$ is obtained from r_i by a cyclic reduction. Replace r_i by r_i'. (cf. Note 1.)

S_2) (*Cyclic permutation*); Replace a relator $r_i=b_1 \cdots b_k$ by the cyclically permuted one $r_i'=b_2 \cdots b_k b_1$.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 454003), Ministry of Education.

S_3) (*Inversion*); Replace a relator $r_i=b_1 \cdots b_k$ by the inverted one $r_i^{-1}=b_k^{-1} \cdots b_1^{-1}$.

S_4) (*Substitution*); If there are two relators r_i, r_j $(i \neq j)$ such that $r_i=w_1 r_j w_2$, where w_1 and w_2 are words in the alphabet $a_1, \cdots, a_n$, then we have a new relator $r_i'=w_1 w_2$. Replace r_i by r_i'.

NOTE 1. In the case of $r_i=(b_1 \cdots b_j)b_0 b_0^{-1}(b_{j+1} \cdots b_k)$, S_1) is called a *free reduction*. In this case we say that r_i is obtained from r_i' by a *free expansion*. A cyclic reduction can be obtained by a free reduction and S_2) but a cyclic reduction itself is natural and convenient for a transformation of a relator.

We use a notation $\{r_i \mid i=1, \cdots, m\} \searrow \{r_i' \mid i=1, \cdots, m\}$ if a set $\{r_i \mid i=1, \cdots, m\}$ of relators can be transformed to another one $\{r_i' \mid i=1, \cdots, m\}$ by finite applications of simple transformations.

DEFINITION 2. A presentation $\langle a_1, \cdots, a_n ; r_1, \cdots, r_n \rangle$ is said to be *simply trivial* if $\{r_i \mid i=1, \cdots, n\} \searrow \{a_i \mid i=1, \cdots, n\}$.

A relator r_i is called to be *reduced* if it is not possible to apply any cyclic reduction to r_i.

DEFINITION 3. (*Simple equivalence of presentations*): Two presentations $\langle a_1, \cdots, a_n ; r_1, \cdots, r_m \rangle$ and $\langle a_1, \cdots, a_n ; r_1', \cdots, r_m' \rangle$ are said to be *simply equivalent* if there exists a presentation $\langle a_1, \cdots, a_n ; r_1^*, \cdots, r_m^* \rangle$ such that each r_i^* is reduced, $i=1, \cdots, m$, and both $\{r_i \mid i=1, \cdots, m\}$ and $\{r_i' \mid i=1, \cdots, m\}$ can be transformed to $\{r_i^* \mid i=1, \cdots, m\}$ by finite applications of simple transformations of type S_1), S_2) and S_3).

Now, we can state our result as follows;

THEOREM. *For a presentation $\langle a_1, \cdots, a_n ; r_1, \cdots, r_n \rangle$ of $\Pi_1(S^3)$ associated with a Heegaard diagram of genus n, there exists a simply equivalent presentation $\langle a_1, \cdots, a_n ; r_1', \cdots, r_n' \rangle$ which is simply trivial.*

NOTE 2. In general, the required simply trivial presentation $\langle a_1, \cdots, a_n ; r_1', \cdots, r_n' \rangle$ can be derived from $\langle a_1, \cdots, a_n ; r_1, \cdots, r_n \rangle$ by finite applications of free expansion. So, the $\langle a_1, \cdots, a_n ; r_1', \cdots, r_n' \rangle$ is not always a presentation associated with a Heegaard diagram for S^3.

Next example will illustlate the assertion of Theorem more clearly.

EXAMPLE 1. A presentation $\langle a_1, a_2, a_3, a_4 ; r_1, r_2, r_3, r_4 \rangle$, where $r_1=a_4 a_3 a_2 a_1^3$, $r_2=a_2 a_1 a_2$, $r_3=a_3 a_2 a_1^2 a_3$ and $r_4=a_4 a_3 a_2 a_1 a_2^{-1} a_3^{-1} a_1$, is not simply trivial. But there exists a simply equivalent presentation $\langle a_1, a_2, a_3, a_4 ; r_1', r_2', r_3', r_4' \rangle$, where $r_1'=a_4 a_3 a_2 a_1^2 a_3 a_3^{-1} a_1$, $r_2'=r_2$, $r_3'=r_3$ and $r_4'=a_4 a_3 a_2 a_1^2 a_3 a_3^{-1} a_1^{-1} a_2^{-1} a_3^{-1} a_1$, which is simply trivial.

Before ending this section, we touch on our plan for the following sections, briefly. After establishing a Heegaard diagram for a 3-manifold M in Section 3, we formulate the associated presentation of the fundamental group $\Pi_1(M)$ in Section 4. In Section 5 we discuss such the presentations for the 3-sphere and prove key lemmata, and in Section 6 we prove our Theorem.

3. Heegaard splittings, diagrams and sewings.

We sketch the definitions of Heegaard splittings, diagrams and sewings and the relationships among them. (To know them in detail, see [**1**].) Let T denote a solid torus of genus n and T_i ($i=1, 2$) denote a copy of T. Suppose that M is a connected orientable closed 3-manifold. A triplet $(M; M_1, M_2)$ is called a *Heegaard splitting* of M with *genus* n if M_1 and M_2 are homeomorphic to T satisfying $M=M_1\cup M_2$ and $M_1\cap M_2=\partial M_1\cap\partial M_2=\partial M_1=\partial M_2=F$, the *Heegaard surface*.

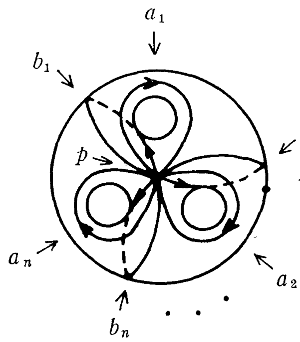
Let D_1 (respectively D_2) be a *complete system of meridian disks* of M_1 (resp. M_2), that is, D_1 (resp. D_2) consists of n mutually disjoint proper disks in M_1 (resp. M_2) such that M_1 (resp. M_2) cut open along D_1 (resp. D_2) is a 3-ball. We say that $C_1=\partial D_1$ (resp. $C_2=\partial D_2$) is a *complete system of meridian curves* of $F=\partial M_1$ for M_1 (resp. $F=\partial M_2$ for M_2). Then we call a triplet $(F; C_1, C_2)$ a *Heegaard diagram of genus* n for M. Conversely, suppose that C_1' and C_2' are two *complete system of curves* on F, that is, C_1' (resp. C_2') consists of n mutually disjoint simple closed curves such that F cut open along C_1' (resp. C_2') is a $2n$-punctured 2-sphere. Then there are self-homeomorphisms h_1 and h_2 on F such that $h_1(C_1)=C_1'$ and $h_2(C_2)=C_2'$. And we have an orientable closed 3-manifold M' by attaching M_1 and M_2 to both sides of F using h_1 and h_2 respectively, (M' is distinct from M in general). $(F; C_1', C_2')$ is a Heegaard diagram for M' and $(M'; M_1', M_2')$ is a Heegaard splitting of M', where M_1' (resp. M_2') is the image of M_1 (resp. M_2) in M' by the attaching.

By the definition, for a Heegaard splitting $(M; M_1, M_2)$, there are two homeomorphism $f_i: T_i\to M_i$ ($i=1, 2$). A sewing space $T_1\cup_\phi T_2$ by the sewing map $\phi=f_1^{-1}f_2|\partial T_2: \partial T_2\to\partial T_1$ is homeomorphic to M. In fact, $f_1\cup f_2: T_1\cup_\phi T_2\to M$ is a homeomorphism between triplets $(T_1\cup_\phi T_2; T_1, T_2)$ and $(M; M_1, M_2)$. We may assume that ϕ is orientation-preserving, (if necessary, exchange f_1 for f_1r, where $r: T_1\to T_1$ is an orientation-reversing homeomorphism). Conversely, for an orientation-preserving homeomorphism $\phi: \partial T_2=\partial T\to\partial T=\partial T_1$, the sewing space $T_1\cup_\phi T_2$ has a natural Heegaard splitting $(T_1\cup_\phi T_2; T_1, T_2)$. A map ϕ is called a *Heegaard sewing*.

We may identify a triplet with another triplet if they are homeomorphic as triplets. Hence hereafter, we prefer to use a Heegaard sewing ϕ, splitting $(T_1\cup_\phi T_2; T_1, T_2)$ and diagram $(\partial T; C_0, \phi(C_0))$ for the technical reason. Here, C_0 is a standard complete system of meridian curves on ∂T $(=\partial T_1=\partial T_2)$ as shown in Fig. 2.

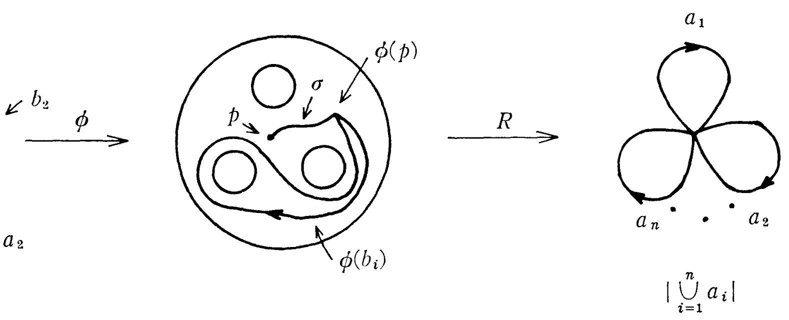
4. A presentation $\Pi_1(\phi)$ associated with a Heegaard diagram.

Now, for a given Heegaard sewing ϕ or Heegaard diagram $(\partial T; C_0, \phi(C_0))$, we give a method of calculating a presentation $\Pi_1(\phi)$ of the fundamental group $\Pi_1(T_1 \cup_\phi T_2)$. Take a fixed point $p \in \partial T$ and a fixed system $\{a_i, b_i \mid i=1, \cdots, n\}$ of p-based curves on ∂T as shown in Fig. 1.(a). We use the same notation a_i (or b_i) to indicate the homotopy class of a_i (or b_i) for convenience sake. $\{a_i, b_i \mid i=1, \cdots, n\}$ makes a set of generators for $\Pi_1(\partial T, p)$. Let R be a retraction of T onto a bouquet $B = |\bigcup_{i=1}^{n} a_i|$, and let σ be an arc from p to $\phi(p)$.



$T_2 \supset \partial T_2 = \partial T$

Fig. 1.(a)



$\partial T = \partial T_1 \subset T_1$

$|\bigcup_{i=1}^{n} a_i|$ bouquet

Fig. 1.(b)

DEFINITION 4. A *presentation* $\Pi_1(\phi)$ *of* $\Pi_1(T_1 \cup_\phi T_2, p)$ *associated with a Heegaard sewing* ϕ is $\langle a_1, \cdots, a_n; r_1, \cdots, r_n \rangle$ such that each r_i is a word $r_i(a_1, \cdots, a_n)$ determined by $[R(\sigma \cdot \phi(b_i) \cdot \sigma^{-1})] \in \Pi_1(B, p) = \langle a_1, \cdots, a_n \rangle$, where $[\ ]$ denotes a homotopy class and $\cdot$ indicates a composition of curves, (cf. Fig. 1.(b)).

In fact, we can easily check that $\Pi_1(\phi)$ is a presentation of $\Pi_1(T_1 \cup_\phi T_2, p)$ by van Kampen's theorem. Since $\Pi_1(B, p)$ is a free group $\langle a_1, \cdots, a_n \rangle$, a word $r_i(a_1, \cdots, a_n)$ is uniquely determined up to free reduction. By this fact and next Lemma 1, $\Pi_1(\phi)$ is well-defined for ϕ up to simple equivalence.

REMARK 1. Since $\Pi_1(B, p) = \langle a_1, \cdots, a_n \rangle$ is isomorphic to $\langle a_1, \cdots, a_n, b_1, \cdots, b_n; \prod_{i=1}^{n} (b_i^{-1} a_i^{-1} b_i a_i), b_1, \cdots, b_n \rangle = \Pi_1(\partial T, p)/N(b_1, \cdots, b_n)$, where $N(\ )$ means a normal closure, the relator r_i can be obtained as follows: Let $\bar{r}_i(a_1, \cdots, a_n, b_1, \cdots, b_n)$ be a word determined by $[\sigma \cdot \phi(b_i) \cdot \sigma^{-1}] \in \Pi_1(\partial T, p)$, then $r_i = \bar{r}_i(a_1, \cdots, a_n, 1, \cdots, 1)$.

REMARK 2. Let $C_0 = \{c_1, \cdots, c_n\}$ be the standard complete system of meridian curves on ∂T as shown in Fig. 2. By reading signed geometric intersections $C_0 \cap \phi(c_i)$ along $\phi(c_i)$, we have a word $\bar{r}_i(c_1, \cdots, c_n)$. Put $r'_i = \bar{r}_i(a_1, \cdots, a_n)$

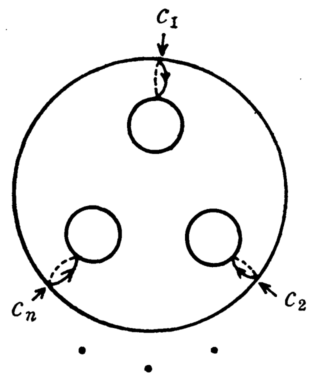


Fig. 2.

by replacing c_j by a_j, then $\langle a_1, \cdots, a_n; r'_1, \cdots, r'_n\rangle$ is called a *presentation* of $\Pi_1(T_1\cup_\phi T_2, p)$ *associated with a Heegaard diagram* $(\partial T; C_0, \phi(C_0))$. This definition coincides with Definition 4 up to simple equivalence, because each c_i is isotopic to b_i, (cf. Proof of Lemma 2, below).

LEMMA 1. *A presentation* $\Pi_1(\phi)$ *does not depends on the choice of an arc* σ *up to simple equivalence.*

PROOF. Let σ' be another arc from p to $\phi(p)$, then the corresponding relator r'_i is presented by a homotopy class

$$\begin{aligned} r'_i &= [R(\sigma'\cdot\phi(b_i)\cdot\sigma'^{-1})] = [R(\sigma'\cdot\sigma^{-1}\cdot\sigma\cdot\phi(b_i)\cdot\sigma^{-1}\cdot\sigma\cdot\sigma'^{-1})] \\ &= [R(\sigma'\cdot\sigma^{-1})][R(\sigma\cdot\phi(b_i)\cdot\sigma^{-1})][R(\sigma\cdot\sigma'^{-1})] \\ &= [R(\sigma'\cdot\sigma^{-1})]r_i[R(\sigma'\cdot\sigma^{-1})]^{-1}, \end{aligned}$$

therefore, r'_i and r_i have the same reduced form.

LEMMA 2. *If* ϕ *is isotopic to* ϕ', $\Pi_1(\phi)$ *is simply equivalent to* $\Pi_1(\phi')$.

PROOF. Since $\phi(b_i)$ is freely homotopic to $\phi'(b_i)$, the difference between $[R(\sigma\cdot\phi(b_i)\cdot\sigma^{-1})]$ and $[R(\sigma'\cdot\phi'(b_i)\cdot\sigma'^{-1})]$ is an inner-automorphism of $\Pi_1(B, p)$ which can be removed by cyclic reductions.

5. $\Pi_1(\phi)$ for $T_1\cup_\phi T_2\approx S^3$.

We discuss a presentation $\Pi_1(\phi)$ of $\Pi_1(T_1\cup_\phi T_2, p)$ in the case that $T_1\cup_\phi T_2$ is homeomorphic to S^3. By Suzuki [4], the isotopy class group of all orientation-preserving self-homeomorphisms on T has a finite set of generators which is $\mathfrak{G}_n=\{[\rho], [\rho_{12}], [\omega_1], [\tau_1], [\theta_{12}], [\xi_{12}]\}$ for $n>2$, $\mathfrak{G}_2=\{[\rho], [\omega_1], [\tau_1], [\theta_{12}], [\xi_{12}]\}$ for $n=2$ or $\mathfrak{G}_1=\{[\omega_1], [\tau_1]\}$ for $n=1$, where $[f]$ is an isotopy class of f and ρ, ρ_{12}, ω_1, τ_1, θ_{12}, ξ_{12} are self-homeomorphisms on T defined in [4]. Let $\dot{\mathfrak{G}}_n^{\pm}$ denote a set $\{\dot\rho, \dot\rho^{-1}, \dot\rho_{12}, \dot\rho_{12}^{-1}, \dot\omega_1, \dot\omega_1^{-1}, \dot\tau_1, \dot\tau_1^{-1}, \dot\theta_{12}, \dot\theta_{12}^{-1}, \dot\xi_{12}, \dot\xi_{12}^{-1}\}$ for $n>2$ (similarly for $n=2$, 1), where $\dot\rho=\rho|\partial T$, $\dot\rho_{12}=\rho_{12}|\partial T$ and so on. Suppose that ϕ_0 is a *standard*

Heegaard sewing for S^3 which maps a standard meridian curve system to a standard longitudinal curve system, (cf. ϕ_0 is $\mu_1 \cdots \mu_n$ in [4]).

Next proposition is due to Suzuki [4],

PROPOSITION 1. *For each homeomorphism f of* $\dot{\mathfrak{G}}_n^{\pm} \cup \{\phi_0\}$, *the induced* Π_1-*isomorphism* $f_\# : \Pi_1(\partial T, p) \to \Pi_1(\partial T, p)$ *is given as the following table, where* $\{a_1, \cdots, a_n, b_1, \cdots, b_n\}$ *is a set of generators for* $\Pi_1(\partial T, p)$ *defined as Fig.* 1.(*a*) *and the trivial case of* $f_\#(a_i) = a_i$ *or* $f_\#(b_j) = b_j$ *is omitted from the table;*

1) *Cyclic transformation of handles* $\dot{\rho} = f, f^{-1}$

$$\dot{\rho}_\# : \begin{cases} a_i \longrightarrow a_{i+1} & i=1, \cdots, n \pmod{n} \\ b_i \longrightarrow b_{i+1} & i=1, \cdots, n \pmod{n} \end{cases}$$

$$\dot{\rho}_\#^{-1} : \begin{cases} a_i \longrightarrow a_{i-1} & i=1, \cdots, n \pmod{n} \\ b_i \longrightarrow b_{i-1} & i=1, \cdots, n \pmod{n} \end{cases}$$

2) *Interchanging knobs* $\dot{\rho}_{12} = f, f^{-1}$

$$\dot{\rho}_{12\#} : \begin{cases} a_1 \longrightarrow (b_1^{-1}a_1^{-1}b_1a_1)a_2(b_1^{-1}a_1^{-1}b_1a_1)^{-1} \\ a_2 \longrightarrow a_1 \\ b_1 \longrightarrow (b_1^{-1}a_1^{-1}b_1a_1)b_2(b_1^{-1}a_1^{-1}b_1a_1)^{-1} \\ b_2 \longrightarrow b_1 \end{cases}$$

$$\dot{\rho}_{12\#}^{-1} : \begin{cases} a_1 \longrightarrow a_2 \\ a_2 \longrightarrow (a_2^{-1}b_2^{-1}a_2b_2)a_1(a_2^{-1}b_2^{-1}a_2b_2)^{-1} \\ b_1 \longrightarrow b_2 \\ b_2 \longrightarrow (a_2^{-1}b_2^{-1}a_2b_2)b_1(a_2^{-1}b_2^{-1}a_2b_2)^{-1} \end{cases}$$

3) *Twisting a knob* $\dot{\omega}_1 = f, f^{-1}$

$$\dot{\omega}_{1\#} : \begin{cases} a_1 \longrightarrow a_1^{-1}b_1^{-1}a_1^{-1}b_1a_1 \\ b_1 \longrightarrow a_1^{-1}b_1^{-1}a_1 \end{cases}$$

$$\dot{\omega}_{1\#}^{-1} : \begin{cases} a_1 \longrightarrow b_1^{-1}a_1^{-1}b_1 \\ b_1 \longrightarrow b_1^{-1}a_1^{-1}b_1^{-1}a_1b_1 \end{cases}$$

4) *Twisting a handle* $\dot{\tau}_1 = f, f^{-1}$

$$\dot{\tau}_{1\#} : \{ a_1 \longrightarrow b_1^{-1}a_1$$

$$\dot{\tau}_{1\#}^{-1} : \{ a_1 \longrightarrow b_1a_1$$

5) *Sliding a handle* $\dot{\theta}_{12}, \dot{\xi}_{12} = f, f^{-1}$

$$\dot{\theta}_{12\#}: \begin{cases} a_1 \longrightarrow a_1 b_2^{-1} a_2^{-1} b_2 \\ b_2 \longrightarrow a_2 b_2 a_1^{-1} b_1 a_1 b_2^{-1} a_2^{-1} b_2 \end{cases}$$

$$\dot{\theta}_{12\#}^{-1}: \begin{cases} a_1 \longrightarrow b_1 a_1 b_2^{-1} a_2 b_2 a_1^{-1} b_1^{-1} a_1 \\ b_2 \longrightarrow b_2 a_1^{-1} b_1^{-1} a_1 \end{cases}$$

$$\dot{\xi}_{12\#}: \begin{cases} a_1 \longrightarrow b_1 a_1 b_2^{-1} a_2^{-1} b_2^{-1} a_2 b_2 a_1^{-1} b_1^{-1} a_1 \\ a_2 \longrightarrow a_2 b_2 a_1^{-1} b_1^{-1} a_1 b_2^{-1} \end{cases}$$

$$\dot{\xi}_{12\#}^{-1}: \begin{cases} a_1 \longrightarrow a_1 b_2^{-1} a_2^{-1} b_2 a_2 b_2 \\ a_2 \longrightarrow b_2^{-1} a_2 b_2 a_1^{-1} b_1 a_1 b_2^{-1} a_2^{-1} b_2 a_2 \end{cases}$$

6) *Standard sewing map for* S^3 $\phi_0 = f$

$$\phi_{0\#}: \begin{cases} a_i \longrightarrow a_i^{-1} b_i a_i & i=1, \cdots, n \\ b_i \longrightarrow a_i^{-1} & i=1, \cdots, n \end{cases}$$

Next corollary is derived from Proposition 1 by elavorate observations of the above table. The retraction $R: \partial T \to B$ induces the Π_1-homeomorphism $R_\#: \Pi_1(\partial T, p) \to \Pi_1(B, p)$ presented by $R_\#(a_i) = a_i$ and $R_\#(b_i) = 1$, $i=1, \cdots, n$, where 1 means the unit of $\Pi_1(B, p)$.

COROLLARY 1. (1) *For any homeomorphism* $f \in \dot{\mathfrak{H}}_n^\pm$, *the following properties hold:*

(a) $\{f_\#(b_i) \mid i=1, \cdots, n\} \searrow \{b_i \mid i=1, \cdots, n\}$.

(b) $\{R_\# f_\#(a_i) \mid i=1, \cdots, n\} \searrow \{a_i \mid i=1, \cdots, n\}$.

(c) $R_\# f_\#(b_i)$ *is transformed to* 1 *by free reductions for each* $i=1, \cdots, n$.

(2) $\langle a_1, \cdots, a_n ; R_\# \phi_{0\#}(b_1), \cdots, R_\# \phi_{0\#}(b_n) \rangle = \langle a_1, \cdots, a_n ; a_1^{-1}, \cdots, a_n^{-1} \rangle$ *is simply trivial.*

Let ϕ be a Heegaard sewing of genus n with $\phi(p) = p$ and $\phi_\#$ be the induced Π_1-isomorphism. It will be noticed that for each $\phi_\#(a_i)$ (or $\phi_\#(b_i)$), $i=1, \cdots, n$, there are infinitely many words in the alphabet $a_1, \cdots, a_n, b_1, \cdots, b_n$, which represent $\phi_\#(a_i)$ (or $\phi_\#(b_i)$). If $2n$ representative words $r'_1, \cdots, r'_n, r''_1, \cdots, r''_n$ for $\phi_\#(a_1), \cdots, \phi_\#(a_n), \phi_\#(b_1), \cdots, \phi_\#(b_n)$ are given, a representative word for $\phi_\#(w)$ is uniquely determined for any word w in the alphabet $a_1, \cdots, a_n, b_1, \cdots, b_n$. Then we have the presentation $\langle a_1, \cdots, a_n ; R_\# \phi_\#(b_1), \cdots, R_\# \phi_\#(b_n) \rangle$ as $\Pi_1(\phi)$ by Remark 1 for we may choose the constant map as an arc σ in the definition of $\Pi_1(\phi)$. Hereafter, in case of a Heegaard sewing ϕ of genus n with $\phi(p) = p$, we assume that $2n$ certain representative words for $\phi_\#(a_1), \cdots, \phi_\#(a_n), \phi_\#(b_1), \cdots, \phi_\#(b_n)$ are given, and by $\Pi_1(\phi_\#)$ we denote the corresponding presentation $\langle a_1, \cdots, a_n ; R_\# \phi_\#(b_1), \cdots, R_\# \phi_\#(b_n) \rangle$. We write these representative words for $\phi_\#(a_1), \cdots, \phi_\#(a_n), \phi_\#(b_1), \cdots, \phi_\#(b_n)$ explicitely only if necessary. Let $\phi_1, \cdots, \phi_k$ be k Heegaard sewing of genus n with p fixed. Since $(\phi_1 \cdots \phi_k)_\# = \phi_{1\#} \cdots \phi_{k\#}$,

$2n$ representative words for $(\psi_1 \cdots \psi_k)_\#(a_i)$ and $(\psi_1 \cdots \psi_k)_\#(b_i)$, $i=1, \cdots, n$, are uniquely determined by those of $\psi_{j\#}(a_1), \cdots, \psi_{j\#}(a_n), \psi_{j\#}(b_1), \cdots, \psi_{j\#}(b_n)$, $j=1, \cdots, k$. We denote the corresponding presentation $\Pi_1((\psi_1 \cdots \psi_k)_\#)$ by $\Pi_1(\psi_{1\#} \cdots \psi_{k\#})$.

Using Corollary 1, we can show next key lemmata.

LEMMA 3. *Let ψ be a Heegaard sewing of genus n with $\psi(p)=p$. Suppose that for any homeomorphism $f \in \dot{\mathfrak{H}}_n^{\pm}$, $2n$ representative words for $f_\#(a_i)$ and $f_\#(b_i)$, $i=1, \cdots, n$, are given by the table of Proposition 1. Then $\{R_\# \psi_\# f_\#(b_i) \mid i=1, \cdots, n\} \searrow \{R_\# \psi_\#(b_i) \mid i=1, \cdots, n\}$ holds for any choice of $2n$ representative words for $\psi_\#(a_i)$ and $\psi_\#(b_i)$, $i=1, \cdots, n$.*

PROOF. Suppose that $f_\#(b_i)$ is presented as a word $r_i(a_1, \cdots, a_n, b_1, \cdots, b_n)$ in the alphabet $a_1, \cdots, a_n, b_1, \cdots, b_n$. According to $R_\# \psi_\# f_\#(b_i) = r_i(R_\# \psi_\#(a_1), \cdots, R_\# \psi_\#(a_n), R_\# \psi_\#(b_1), \cdots, R_\# \psi_\#(b_n))$ and (a) of Corollary 1.(1), we have $\{R_\# \psi_\# f_\#(b_i) \mid i=1, \cdots, n\} \searrow \{R_\# \psi_\#(b_i) \mid i=1, \cdots, n\}$ because of the commutativity of simple transformations and homomorphisms.

LEMMA 4. *Let ψ be a Heegaard sewing of genus n with $\psi(p)=p$. If $\Pi_1(\psi_\#) = \langle a_1, \cdots, a_n ; R_\# \psi_\#(b_1), \cdots, R_\# \psi_\#(b_n) \rangle$ is simply trivial, $\Pi_1(f_\# \psi_\#)$ is also simply trivial for any homeomorphism $f \in \dot{\mathfrak{H}}_n^{\pm}$, where $2n$ representative words for $f_\#(a_i)$ and $f_\#(b_i)$, $i=1, \cdots, n$, are given by the table of Proposition 1.*

PROOF. The set of relators of $\Pi_1(f_\# \psi_\#)$ is $\{R_\# f_\# \psi_\#(b_i) \mid i=1, \cdots, n\}$. Let $r_i(a_1, \cdots, a_n, b_1, \cdots, b_n)$ be the given representative word for $\psi_\#(b_i)$ in the alphabet $a_1, \cdots, a_n, b_1, \cdots, b_n$, where $i=1, \cdots, n$. Because of $R_\# f_\# \psi_\#(b_i) = r_i(R_\# f_\#(a_1), \cdots, R_\# f_\#(a_n), R_\# f_\#(b_1), \cdots, R_\# f_\#(b_n))$ and (c) of Corollary 1.(1), $R_\# f_\# \psi_\#(b_i)$ is transformed to $r_i(R_\# f_\#(a_1), \cdots, R_\# f_\#(a_n), 1, \cdots, 1)$ by free reductions. Since $R_\# \psi_\#(b_i) = r_i(a_1, \cdots, a_n, 1, \cdots, 1)$ for each $i=1, \cdots, n$, $\{r_i(a_1, \cdots, a_n, 1, \cdots, 1) \mid i=1, \cdots, n\}$ is the set of relators for $\Pi_1(\psi_\#)$. Since $\Pi_1(\psi_\#)$ is simply trivial from our assumption, it holds that $\{R_\# f_\# \psi_\#(b_i) \mid i=1, \cdots, n\} \searrow \{R_\# f_\#(a_i) \mid i=1, \cdots, n\}$. Therefore, by (b) of Corollary 1.(1), we can conclude that $\Pi_1(f_\# \psi_\#)$ is also simply trivial.

The following is due to Waldhausen [5] and Suzuki [4].

PROPOSITION 2. *A sewing space $T_1 \cup_\phi T_2$ by a Heegaard sewing ϕ of genus n is homeomorphic to S^3 if and only if ϕ is isotopic to a composed homeomorphism $f_k \cdots f_1 \phi_0 g_1 \cdots g_m$, where $f_i, g_j \in \dot{\mathfrak{H}}_n^{\pm}$.*

PROOF. The sufficiency of the condition is obvious because two homeomorphisms $f=f_k \cdots f_1$ and $g=g_1 \cdots g_m$ can be extended to self-homeomorphisms on T respectively. We show the necessity of it. Suppose that $h' : T_1 \cup_{\phi_0} T_2 \to S^3$ and $h'' : T_1 \cup_\phi T_2 \to S^3$ are homeomorphisms, then $(S^3 ; h'(T_1), h'(T_2))$ and $(S^3 ; h''(T_1), h''(T_2))$ are Heegaard splittings of S^3 with genus n. By the uniqueness of Heegaard splittings of S^3 due to Waldhausen [5], there is a homeomorphism $h : S^3 \to S^3$ such that $hh''(T_i) = h'(T_i)$, $i=1, 2$. A homeomorphism $\bar{h} = h'^{-1} h h'' : T_1 \cup_\phi T_2 \to T_1 \cup_{\phi_0} T_2$ satisfies $\bar{h}(T_i) = T_i$ $(i=1, 2)$ and so $\phi = (\bar{h}^{-1} | \partial T_1)$

$\phi_0(\bar{h}|\partial T_2)$. We may assume that $\bar{h}$ is orientation-preserving (if not, change h'), that is, $\bar{h}^{-1}|T_1: T_1 \to T_1$ and $\bar{h}|T_2: T_2 \to T_2$ is orientation-preserving. By Suzuki [4], $\bar{h}^{-1}|\partial T_1$ and $\bar{h}|\partial T_2$ are isotopic to composed homeomorphisms $f_k \cdots f_1$ and $g_1 \cdots g_m$ respectively, where $f_i, g_j \in \dot{\mathfrak{G}}_n^{\pm}$.

6. Proof of Theorem.

Let ϕ be a sewing map such that $T_1 \cup_\phi T_2$ is homeomorphic to S^3. Because of the last of Section 3 and Remark 2, we may assume that $\langle a_1, \cdots, a_n ; r_1, \cdots, r_n\rangle$ is $\Pi_1(\phi)$. By Proposition 2 and Lemma 2, $\Pi_1(\phi)$ is simply equivalent to $\Pi_1(f_{k\#} \cdots f_{1\#}\phi_{0\#}g_{1\#} \cdots g_{m\#})$, where $f_i, g_j \in \dot{\mathfrak{G}}_n^{\pm}$. By Lemma 3, the set of relators of $\Pi_1(f_{k\#} \cdots f_{1\#}\phi_{0\#}g_{1\#} \cdots g_{m\#})$ is transformed to that of $\Pi_1(f_{k\#} \cdots f_{1\#}\phi_{0\#})$ by simple transformations. By Corollary 1.(2) and Lemma 4, $\Pi_1(f_{k\#} \cdots f_{1\#}\phi_{0\#})$ is simply trivial. Therefore Theorem is proved.

7. Supplemental Remark.

Our Theorem is a kind of existence theorem. It is natural to think of the possibility of making it algorithmic. For example, we may ask if it is possible to omit the permission of free expansions at first step according to "up to simple equivalence". One natural way is to make the notion of "simply trivial" stronger by ordering the application of simple transformations as follows;

DEFINITION 5. A presentation $\langle a_1, \cdots, a_n ; r_1, \cdots, r_n\rangle$ is said to be *strongly simply trivial* if $\{r_i | i=1, \cdots, n\}$ can be transformed to $\{a_i | i=1, \cdots, n\}$ by applying a finite sequence of simple transformations which satisfies the condition that S_1) must always be applied before the other three simple transformations S_2), S_3) and S_4) being done if it is applicable.

EXAMPLE 2. (1) A presentation $\langle a_1, a_2 ; a_1a_2a_2^{-1}a_1a_2, a_2a_1a_2^{-1}a_1^2a_2\rangle$ is strongly simply trivial.

(2) The presentation $\langle a_1, a_2, a_3, a_4 ; r_1', r_2', r_3', r_4'\rangle$ of Example 1 is simply trivial but not strongly simply trivial.

At first, we have attacked the following questions.

QUESTION 1. Is any presentation associated with a Heegaard diagram for S^3 strongly simply trivial?

A weak form of the above question is the following;

QUESTION 2. Is any presentation associated with a Heegaard diagram for S^3 simply trivial?

As noticed before, $\Pi_1(\phi) = \langle a_1, \cdots, a_n ; r_1, \cdots, r_n\rangle$ is not uniquely determined for a Heegaard sewing ϕ but its reduced form $\langle a_1, \cdots, a_n ; \tilde{r}_1, \cdots, \tilde{r}_n\rangle$, where each $\tilde{r}_i$ is obtained from r_i by cyclic reductions, is unique for ϕ up to S_2) and S_3). Next question is stronger than Question 2 and weaker than Question 1.

QUESTION 3. Is the reduced form $\tilde{\Pi}_1(\phi)$ for any Heegaard sewing ϕ simply trivial if $T_1\cup_\phi T_2$ is homeomorphic to S^3?

Unfortunately, these are false in case of genus $n>2$. In case of genus 2, which is the original case of T. Homma and M. Ochiai's discovery, it is open yet in the sense of the above Question 1 although we expect "yes".

A counter example in the case $n=3$ is the following;

EXAMPLE 3. A Heegaard sewing ϕ is a composed homeomorphism on ∂T of genus 3 defined as $\phi=f\phi_0 g$, where f and g are the composed homeomorphisms of $\dot{\mathfrak{G}}_3^{\pm}$ given as follows:

$$\begin{aligned} f=(&\dot\rho\dot\rho_{12}^{-1}\dot\rho^{-1}\dot\rho_{12}^{-1}\dot\rho^{-1}\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}\dot\rho\dot\rho_{12}\dot\rho\dot\rho_{12}\dot\rho^{-1}\dot\rho_{12}^{-1}\dot\rho^{-1}\\ &\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}\dot\rho\dot\rho_{12}\dot\rho^{-1}\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}\dot\tau_1^{-5}\dot\rho\dot\rho_{12}^{-1}\dot\rho^{-1}\dot\theta_{12}^{-1}\\ &\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}^{2}\dot\theta_{12}\dot\rho\dot\rho_{12}\dot\rho^{-1}\dot\tau_1^{-2}\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}\dot\theta_{12}\dot\xi_{12}^{-1}\dot\theta_{12}^{-1}\\ &\dot\xi_{12}\dot\theta_{12}\dot\rho\dot\rho_{12}^{-1}\dot\rho^{-1}\dot\theta_{12}^{-1}\dot\xi_{12}^{-2}\dot\theta_{12}^{-1}\dot\xi_{12}\dot\theta_{12}\dot\rho\dot\rho_{12}\dot\rho^{-1}\dot\tau_1^{-1}\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\\ &\dot\theta_{12}\dot\xi_{12}\dot\theta_{12}\dot\rho)\ (\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}^{-1}\dot\tau_1^{2}\dot\theta_{12}^{-1}\dot\xi_{12}\dot\theta_{12}) \end{aligned}$$

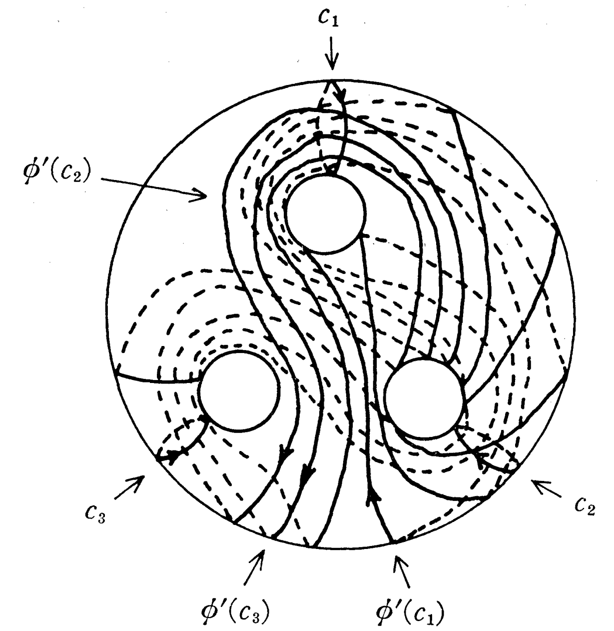
$$\begin{aligned} g=(&\dot\rho^{2}\dot\omega_1^{-1}\dot\rho^{-1}\dot\omega_1^{-1}\dot\rho^{-1}\dot\omega_1^{-1}\dot\rho^{-1}\dot\tau_1^{-2}\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\tau_1^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}^{-1}\\ &\dot\xi_{12}\dot\theta_{12}\dot\rho\dot\rho_{12}^{-1}\dot\rho^{-1}\dot\tau_1^{-2}\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}^{-1}\dot\theta_{12}^{-1}\dot\xi_{12}\dot\theta_{12}\dot\rho\dot\rho_{12}\dot\rho^{-1}\\ &\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}\dot\theta_{12}^{-1}\dot\xi_{12}\dot\theta_{12}\dot\tau_1^{-6}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}\dot\theta_{12}\dot\rho\dot\rho_{12}^{-1}\dot\rho^{-1}\\ &\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\tau_1^{-2}\dot\xi_{12}^{-2}\dot\theta_{12}^{-1}\dot\xi_{12}\dot\theta_{12}\dot\rho\dot\rho_{12}\dot\rho^{-1}\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}\dot\theta_{12}^{-1}\\ &\dot\xi_{12}\dot\theta_{12}\dot\tau_1\dot\rho)(\dot\rho^{-1}\dot\omega_1^{-1}\dot\tau_1^{2}\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}^{-1}\dot\theta_{12}^{-1}\dot\xi_{12}\dot\theta_{12}\dot\rho\\ &\dot\rho_{12}^{-1}\dot\rho^{-1}\dot\tau_1^{-2}\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}\dot\theta_{12}\dot\rho\dot\rho_{12}\dot\rho^{-1}\dot\theta_{12}^{-1}\dot\xi_{12}^{-1}\dot\theta_{12}\dot\xi_{12}\dot\omega_1\dot\rho) \end{aligned}$$

and a corresponding Heegaard diagram $(\partial T; C_0, \phi(C_0))$ is as shown in Fig. 3 up to isotopy. Then both $\tilde{\Pi}_1(\phi)$ and the presentation associated with the Heegaard diagram of Fig. 3 coincide with $\langle a_1, a_2, a_3; a_1a_2^2, a_3a_1^{-1}a_3a_2^{-1}a_1^{-3}, a_3a_2^{-1}a_1^{-1}a_2a_1^{-1}\rangle$ which is not simply trivial, though $T_1\cup_\phi T_2$ is homeomorphic to S^3 by Proposition 2 and so Fig. 3 is a Heegaard diagram for S^3.

REMARK 3. (1) Such a counter example to our Questions was firstly found by M. Ochiai in the case $n=5$, though his aim is different from ours. Example 1 is also a counter example in the case $n=4$ made of Ochiai's one by slight modifications. Later, he attained his aim in case of genus 4 in [3], which is again available for our aim in the case $n=4$.

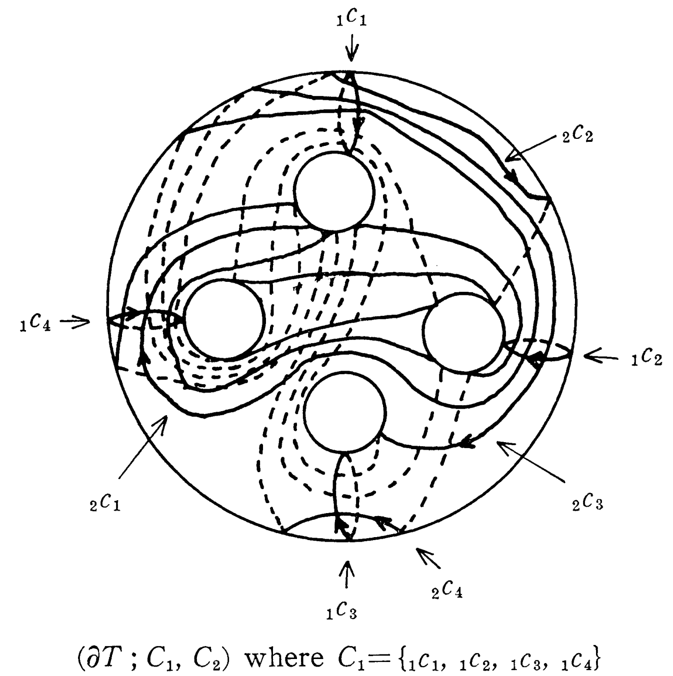
(2) Example 3 can be easily modified for $n>3$ by using connected sums with a standard Heegaard diagram for S^3 of genus 1.

A Heegaard diagram $(\partial T; C_1, C_2)$ has two presentations associated with it, that is, one is obtained by reading the intersections $C_1\cap C_2$ along curves of C_2



$(\partial T\,;C_0,\ \phi'(C_0))$ where $C_0=\{c_1,\ c_2,\ c_3\}$, ϕ' is isotopic to ϕ and $\phi'(C_0)=\{\phi'(c_1),\ \phi'(c_2),\ \phi'(c_3)\}$

Fig. 3.



$(\partial T\,;C_1,\ C_2)$ where $C_1=\{{}_1c_1,\ {}_1c_2,\ {}_1c_3,\ {}_1c_4\}$

$C_2=\{{}_2c_1,\ {}_2c_2,\ {}_2c_3,\ {}_2c_4\}$

Fig. 4.

and the dual one is got by reading them along those of C_1. Next example is a Heegaard diagram for S^3 of genus 4 such that both presentations associated with it are counter examples to Question 2.

EXAMPLE 4. A Heegaard diagram $(\partial T; C_1, C_2)$ is as shown in Fig. 4. Two presentations associated with it are $\langle a_1, a_2, a_3, a_4; a_4^3 a_1 a_2, a_1 a_4 a_1, a_3 a_1^{-1} a_4 a_1 a_2, a_3 a_4^{-1} a_3 a_2\rangle$ by reading along C_2 and $\langle a_1, a_2, a_3, a_4; a_2 a_3 a_1 a_3^{-1} a_2, a_3 a_1 a_4, a_4 a_3 a_4, a_1^2 a_4^{-1} a_2 a_3 a_1\rangle$ by reading along C_1 which are not simply trivial.

REMARK 4. Recently, O. Morikawa has made such an example for genus 3 in his master thesis.

References

[1] J. S. Birman, Heegaard Splittings, Diagrams and Sewings for Closed, Orientable 3-Manifolds, Lecture notes for CBMS conference at Blacksburg, Va., Oct. 8-12, 1977.

[2] W. Magnus, A. Karrass and D. Soliter, Combinatorial group theory, Interscience publishers, 1966.

[3] M. Ochiai, A counterexample to a conjecture of Whitehead and Volodin-Kuznetsov-Fomenko, J. Math. Soc. Japan, 31 (1979), 687-691.

[4] S. Suzuki, On homeomorphisms of a 3-dimensional handlebody, Canad. J. Math., 29 (1977), 111-124.

[5] F. Waldhausen, Heegaard-Zerlegungen der 3-Sphäre, Topology, 7 (1968), 197-203.

Takeshi KANEKO
Institute of Mathematics
University of Tsukuba
Sakura-mura, Niihari-gun
Ibaraki 305
Japan