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An accessibility proof of ordinal diagrams

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This is a sequel to our previous work [2], in which we defined the fundamental sequences of ordinal diagrams. Here we conclude our first objective: an accessibility proof of ordinal diagrams.

Let S be a set with a linear ordering $<$. An accessibility proof of the system (S, $<$) is a "concrete" proof which establishes that there is no (strictly) $<$-decreasing, infinite sequence of elements of S. When there is an accessibility proof of S, (S, $<$) is said to be accessible, or S is said to be $<$-accessible.

At present there are no means to completely characterize the "concreteness" of accessibility proofs. An example of a "concrete" accessibility proof is seen in the theory of eliminators for ε_0 (cf. [1]).

In the accessibility proof of ordinal diagrams which is to be presented in this article, the notion of strong accessibility plays an essential role. This is not an elementary concept and the inductive definition involving higher order quantifiers is introduced in the course of the proof. Therefore our proof does not deserve a claim of concreteness in strict sense. It is nevertheless an improvement from the set-theoretical proof, which employs the transfinite induction (as a general principle), as in our proof the difficulty concentrates in one part, and it is a preparation for the more constructive proof in a subsequent paper.

For our present approach to the accessibility of ordinal diagrams, it is necessary to generalize the theory of ordinal diagrams, namely we shall consider systems of ordinal diagrams with three basic well-ordered sets rather than two of them. An account of such systems will be given in the first part of this article. It is also there that a sequence of systems of ordinal diagrams is studied. This materializes the notion of strong accessibility, thereby supplying us with a means to carry out the accessibility proof (the second part). Our major task is to show that every ordinal diagram is strongly accessible, and the crucial part of the proof depends heavily on the construction of fundamental sequences.

In the course of our proof, we often use the terms "accessible" and "well-ordered" for a set with a linear ordering. The full meaning of the former is that the set has a primitive recursive linear ordering for which fundamental

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sequences are defined, and there is a "good" accessibility proof for it. The latter means the set theoretical well-ordering. Full acquaintance with the references [1] (§ 26) and [2] is assumed for the reader, and we shall take over much of the notational convention from them. Here, however, we shall relinquish the notational distinction between an ordinal diagram and an occurrence of it in another ordinal diagram. The distinction be made from the content.

I. Extended systems of ordinal diagrams.

We shall define and develop a theory which is a generalization of the theory of ordinal diagrams as presented in § 26 of [1]. We follow § 26 of [1], hence quote the numbers of definitions and propositions there. For example, [Definition 26.1] will mean Definition 26.1 in § 26 of [1].

§ 1. An extended system of ordinal diagrams.

DEFINITION 1.1. (cf. [Definition 26.1]) Let I and A be non-empty, "accessible" sets; namely each of I and A is (at most countable and) primitive recursive, has a primitive recursive well-ordering for which a good "accessibility proof" exists. (This includes the existence of fundamental sequences which are defined primitive recursively; cf. [2].) Let S be a non-empty, (at most) countable, well-ordered set for which fundamental sequences are defined; namely, there is a specific way to define a sequence of elements converging to a limit element from below. Note that no constructive nature is imposed on the well-ordering of S. We shall henceforth assume those requisites for I, A and S respectively throughout. We also assume that the notion of fundamental sequences applies to an entire set. Namely, if I has a limit order type, then there is a primitive recursive, increasing sequence from I, say $\{i_m\}_m$, such that for any i in I, $i<i_m$ for some m. With A and S likewise, save that for S primitive recursiveness is not required.

The system of ordinal diagrams (o.d.'s) $\boldsymbol{O}$(I, A, S) based on I, A and S is defined as follows.

1) Every member of S is an o.d. (of $\boldsymbol{O}$(I, A, S)).

2) Let i be an element of I (an indicator), a of A and α of $\boldsymbol{O}$(I, A, S) (which has already been defined). Then (i, a, α) is an o.d. (of $\boldsymbol{O}$(I, A, S)).

An o.d. whose last application of the formation rule is either 1) or 2) is said to be connected.

3) Let $n>1$ and suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are connected o.d.'s which have already been defined. Then $\alpha_1 \# \alpha_2 \# \cdots \# \alpha_n$ is an o.d., which is not connected. Each of $\alpha_1, \alpha_2, \cdots, \alpha_n$ is called a component of the o.d. thus defined.

We often deal with a situation where I and A are held constant while S varies. For this reason we may write $\boldsymbol{O}$(S) for $\boldsymbol{O}$(I, A, S).

DEFINITION 1.2. 1) Equality relation (=) for o.d.'s is defined as [Definition 26.4].

2) The ordering $<_i$ for every i, where i is either an element of I or a symbol ∞, is defined as for $\boldsymbol{O}$(I, A) except for the two bases (cf. [Definition 26.7]).

2.1) The elements of S are ordered by the original ordering of S for every i.

2.2) Let s be an element of S and let α be an o.d. with at least one application of 2) in Definition 1.1 (i, e. α contains ( )). Then $s<_i\alpha$.

DEFINITION 1.3. The notions of i-sections and i-subsections of α for each i in I (an indicator) are defined as in [Definition 26.5]. An element of S does not have any section and has a sole subsection, that element itself, for every i.

The notions of "i-active", "connected to a sub-o.d." etc. are defined as in [Definition 26.5].

REMARK. Although the notion of o.d.'s in the present content is wider than the original one, defined in [**1**], we shall call both kinds simply o.d.'s.

From now on we assume that I, A, and S are given, and that an o.d. means that of $\boldsymbol{O}$(I, A, S). All elementary propositions concerning o.d.'s can be proved in the same manner as in § 26 of [**1**]. Needless to say, this excludes the accessibility proof. We shall here note only a few points.

PROPOSITION 1.1. (cf. [Proposition 26.9]). *If β is an o.d. and α is an i-section of β, then $\alpha<_i\beta$.*

PROOF. We may assume that β is connected. There is an application of 2) of Definition 1.1 to β. Let γ be a component of α. If γ is an element of S, then $\gamma<_i\beta$ by definition. If γ is not an element of S, then γ and β are both connected and not in S. α is an i-section of β and $\gamma\leqq_i\alpha$ by definition, hence by definition $\gamma<_i\beta$, from which follows $\alpha<_i\beta$.

PROPOSITION 1.2. (cf. [Proposition 26.8]) *$\boldsymbol{O}$(I, A, S) is linearly ordered for every $<_i$, where i is either an element of* I *or* ∞.

PROOF. For the elements of S, $<_i$ is the ordering of S. Suppose $\alpha<_i\beta$ and $\beta<_i\gamma$, where α is an element of S while γ is not. Then $\alpha<_i\gamma$ by definition. Other cases involving the elements of S are dealt with similarly. The proof for other cases is the same as for $\boldsymbol{O}$(I, A).

PROPOSITION 1.3. (cf. [Proposition 26.10]) *Let β be a connected o.d. If α is an i-subsection of β which is distinct from β, then $\alpha<_l\beta$ for every $l\leqq i$.*

PROPOSITION 1.4. *A successor element of $\boldsymbol{O}$*(I, A, S) *has a component s_0, where s_0 is the least element of* S, *and every o.d. has a successor common to all i's. Namely the successor of α is $\alpha \# s_0$, which we denote by $\alpha+1$. Here the order of components in $\alpha \# s_0$ is immaterial.*

§ 2. Valuation and approximation.

In this section we assume that α is a connected o.d. of $\boldsymbol{O}$(I, A, S) and j is an element of I (i.e. an indicator). We are to define valuations and approximations for α and j. See § 26 of [**1**] for various notions and notations.

DEFINITION 2.1. 1) The 0th j-valuation of α, $v_0(j, \alpha)$, is the maximum j-active value of α, presuming that α has a value (cf. [Definition 26.42]).

Note that any connected o.d. has the 0th j-valuation unless α is an element of S.

2) Kernels are defined as in [Definition 26.41]. In particular, the elements of S occurring j-active in α are j-kernels of α.

PROPOSITION 2.1. (cf. [Proposition 26.43]). *If* $v_0(j, \beta) < v_0(j, \alpha) = (i, a)$, *then* $\beta <_j \alpha$.

PROOF. Let (i, a, γ) be any j-subsection of α whose outermost value is (i, a). We prove that

(*) $\eta <_l (i, a, \gamma)$ for every η and l, where η is a j-subsection of β and $l \geqq j$.

If η is an element of S (hence is a j-kernel), then $\eta <_l (i, a, \gamma)$ by the definition of $<_l$. Otherwise the proof for $\boldsymbol{O}$(I, A) goes through.

DEFINITION 2.2. $apr(0, j, \alpha) = \alpha_0$ is defined as for $\boldsymbol{O}$(I, A); the notion that "β j-omits α_0" also ([Definition 26.44]).

PROPOSITION 2.2. (cf. [Lemma 26.45]) *If* η *j-omits* α_0 (*in* α), *then* $\eta <_l \alpha_0$ *for every* $l \geqq j$.

PROOF. If η is an element of S, then $\eta <_l \alpha_0$ by definition. Otherwise the proof in [**1**] goes through.

PROPOSITION 2.3. (cf. [Proposition 26.46]) *Suppose* $v_0(j, \alpha) = v_0(j, \beta) = (i, a)$ *and* $apr(0, j, \beta) <_i apr(0, j, \alpha)$. *Then* $\beta <_j \alpha$.

PROOF. We can show that for any j-subsection of β, say η, $\eta <_l \alpha_0$ for all $l \geqq j$; for if η is an element of S, then $\eta <_l \alpha_0$ by definition; otherwise the proof in [**1**] goes through.

DEFINITION 2.3. $\alpha_1 = apr(1, j, \alpha)$ and the notion that β j-omits α_1 are defined as in [Definition 26.48] and [Definition 26.50].

PROPOSITION 2.4. *Let* η *be a j-subsection of* α *which satisfies that either* η *j-omits* α_0 *or* η *contains* (*j-active*) α_0 *and, for each occurrence of* α_0 *in* η, *say* $\hat{\alpha}_0$, *there is an indicator connected to* $\hat{\alpha}_0$ *which is less than i. Let* $\alpha_0^1, \cdots, \alpha_0^m$ *be all the occurrences of* α_0 *in* η *and* q_k *be the least such indicator connected to* α_0^k. *Let* $q = q(\eta) = \max(q_1, \cdots, q_m)$. *If* η *omits* α_0, *let* $q = j$. *Then* $\eta <_l \alpha_0$ *if* $q < l \leqq i$.

The proof in [Proposition 26.47] goes through. Note that any element of S occurring in α which is not contained in α_0 omits α_0.

PROPOSITION 2.5. *If η omits α_1 (in α), then $\eta <_l \alpha_1$ for any l such that $j \leqq l \leqq i$* (cf. [Proposition 26.51]).

Note that any element of S (in α) is $<_l \alpha_1$ by definition.

PROPOSITION 2.6. *Suppose $v_0(j, \alpha) = v_0(j, \beta) = (i, a)$ and $\alpha_0 = \beta_0$. If $\beta_1 <_i \alpha_1$, then $\beta <_j \alpha$* (cf. [Proposition 26.52]).

DEFINITION 2.4. 1) Suppose $(i_0, \alpha_0), (i_1, \alpha_1), \cdots, (i_n, \alpha_n)$ have been defined. Then (i_{n+1}, α_{n+1}) is defined as in [Definition 26.53]. The notion that β j-omits α_n is defined as in [Definition 26.55].

PROPOSITION 2.7. $n \geqq 1$. *Suppose $\alpha_n = \beta_n$ and $v_{n+1}(j, \beta) < v_{n+1}(j, \alpha)$. Then $\beta <_j \alpha$. If $\alpha_n = \beta_n$, $v_{n+1}(j, \alpha) = v_{n+1}(j, \beta) = i_{n+1}$ and $\beta_{n+1} <_{i_{n+1}} \alpha_{n+1}$, then $\beta <_j \alpha$* ([Proposition 26.56]).

DEFINITION 2.5. $\alpha_{(n, k)}$ is defined as in [Definition 26.57].

PROPOSITION 2.8. *Either $\alpha_{(n, k)}$ is α_{n+1} or $\alpha_{(n, k)}$ occurs as a component of the δ in a sub-o. d. of the form (i_{n+1}, c, δ)* ([Proposition 26.58]).

PROPOSITION 2.9. *If $\beta_{(n, k-1)} = \alpha_{(n, k-1)}$, $v_{n+1}(j, \beta) = v_{n+1}(j, \alpha) = i_{n+1}$ and $\beta_{(n, k)} <_{i_{n+1}+1} \alpha_{(n, k)}$, then $\beta_{n+1} <_{i_{n+1}} \alpha_{n+1}$* ([Proposition 26.61]).

§ 3. Fundamental sequences.

Given an indicator j_0 and a connected o.d. of $\boldsymbol{O}$(I, A, S) which is not the least element of S, say $\tilde{\alpha}$, we can define the fundamental sequence for $(j_0, \tilde{\alpha})$, following the material in [2]. We shall remark only a few points.

DEFINITION 3.1. The scanned pairs (sp), the marked places, and other concepts defined in § 1 of [2] for $\boldsymbol{O}$(I, A) are defined for our $(j_0, \tilde{\alpha})$ exactly parallel to the original definition. We have only to add the following case to Definition 1.1 in [2]:

(0) γ is an element of S. Stop. This is the last reduction place.

Recall that if α is an o.d., then its successor, which will be denoted by $\alpha+1$, is $\alpha \# s_0$, where s_0 is the least element of S. So, when s is a successor of t in S, then $t+1 = t \# s_0 <_i s$, hence s is not the successor of t in $\boldsymbol{O}$(I, A, S).

As for the consequences of the definition, see § 1.1 of [2].

DEFINITION 3.2. Reduction of the last reduction place. See Definition 1.3 of [2]. We note here only the following cases. Let (t, ν) be the last reduction pair.

(0) ν is an element of S, say s. If s is a limit element of S, then there is a fundamental sequence for s in the ordering of S, namely an increasing sequence of elements of S which converges to s from below. This is defined to be the reduction sequence of s in $\boldsymbol{O}$(I, A, S). If s is a successor element of S and t is its predecessor, then let $t_m = t \# \cdots \# t$ (with $m+1$ components). Let $\{t_m\}_m$ be the reduction sequence of s.

We shall use the symbol 0 for the least element of I as well as of A.

b. 3) $\nu=(0, 0, s_0)$.

Case 1. ε is the maximal element of S.

Let $\nu_m=\varepsilon \# \varepsilon \# \cdots \# \varepsilon$.

Case 2. $s_m \uparrow$ S. Let $\nu_m=s_m$.

e. 4) Note that the α in this case is not an element of S.

Now we can follow the rest of the material in [2]. In the proof given in § 2.1 of [2], only the following change is needed: (0) is a new case (which causes no trouble) and in b. 3) either $\nu_m=\varepsilon \# \cdots \# \varepsilon$ or $\nu_m=s_m$. It is easy to see that $\{\nu_m\}_m$ satisfies the condition. For the proofs of Propositions 2.3 and 2.4 in [2], those cases are irrelevant.

In § 3.2 of [2], the case (0) must be added, and b. 3) needs a change.

(0) Suppose $s_m \uparrow s$ in S and $\beta <_i s$. Then by definition of the ordering $<_i$, all the components of β are elements of S, say $t_1, \cdots, t_k$. $t_1, \cdots, t_k < s$ in S and there is an m such that $t_1, \cdots, t_k < s_m$. So $\beta <_i s_m$. Suppose s is the successor of t in S and $\beta <_i s$. Then the components of $\beta \leqq t$, so $\beta <_i t \# \cdots \# t$ for an appropriate number of t's. b. 3) Suppose $\beta <_i (0, 0, s_0)$. Then β cannot contain ( ). Therefore all the components of β are elements of S, hence $\beta <_i \varepsilon \# \cdots \# \varepsilon$ for some number of ε's for Case 1 and $\beta <_i s_m$ for some m for Case 2.

REMARK. Suppose (j, γ) is a sp (of $(j_0, \tilde{\alpha})$) and γ is of the form $(i, a, \alpha \# s)$, where s is an element of S and is a least component of $\alpha \# s$ and $s \neq s_0$. We wish to clarify the forms of γ_m for (j, γ).

Due to the condition on s, $\alpha \# s$ is not s_0 and not a successor (in $\boldsymbol{O}$(I, A, S)). Since s is an element of S, it is not the case that all the components of $\alpha \# s$ are marked. This fact eliminates certain cases. (j, γ) is not the last reduction pair. Only (3.3), (4.3), (5.2) and (5.3) are possible and [1°] is not possible for these cases. Therefore by Definition 1.4 of [2] $\gamma_m=(i, a, \alpha \# s_m)$ for all possible cases, where $\{s_m\}_m$ is the reduction sequence of s in $\boldsymbol{O}$(I, A, S).

§ 4. Equivalence of two theories of o.d.'s.

Although it is not necessary for the accessibility proof, we shall show that two theories of o.d.'s, one with two basic sets and one with three, are equivalent, provided that the constructive aspect of the basic sets are sacrificed. Namely, given I, A and S, $\boldsymbol{O}$(I, A) can be embedded in $\boldsymbol{O}$(I, A, S) as a subsystem, and, in particular, if S is a singleton, then two systems are isomorphic.

On the other hand, $\boldsymbol{O}$(I, A, S) can be embedded in a system $\boldsymbol{O}$(I, B) for an appropriate B. Since constructivity is not assumed for the ordering of S, it cannot be imposed on B either.

PROPOSITION 4.1. *$\boldsymbol{O}$(I, A) can be emedded in $\boldsymbol{O}$(I, A, S) as a subsystem for any* S. *In particular $\boldsymbol{O}$(I, A) is isomorphic to $\boldsymbol{O}$(I, A, $\{s_0\}$).*

PROOF. Replace 0 in any o.d. of $\boldsymbol{O}(\mathrm{I}, \mathrm{A})$ by s_0, the least element of S. This defines a natural, order-preserving mapping from $\boldsymbol{O}(\mathrm{I}, \mathrm{A})$ into $\boldsymbol{O}(\mathrm{I}, \mathrm{A}, \mathrm{S})$ regardless of S.

PROPOSITION 4.2. *Given* $\boldsymbol{O}(\mathrm{I}, \mathrm{A}, \mathrm{S})$, *let* B *be* $\mathrm{S} \cup \mathrm{A}$, *where the ordering of* B *is induced from those of* A *and* S *by placing the elements of* S *below the elements of* A. *Then* $\boldsymbol{O}(\mathrm{I}, \mathrm{A}, \mathrm{S})$ *is embedded in* $\boldsymbol{O}(\mathrm{I}, \mathrm{B})$ (*for every ordering* $<_i$).

PROOF. If s is an element of S, then assign $(0, s, 0)$ (an o.d. of $\boldsymbol{O}(\mathrm{I}, \mathrm{B})$) to it. For a compound element of $\boldsymbol{O}(\mathrm{I}, \mathrm{A}, \mathrm{S})$, the induction hypotheses apply: to (i, a, α) assign (i, a, α^*) where α^* is assigned to α, and to $\alpha_1 \# \cdots \# \alpha_m$ assign $\alpha_1^* \# \cdots \# \alpha_m^*$. We can easily establish that the number of components of α and that of α^* are the same, and the assignment is one-to-one.

Also, if σ is a j-section of α, then σ^* is a j-section of α^*.

The order-preserving property of the map is established by induction on $\omega \cdot l + \iota$ (cf. § 26 of [1]).

Compare, as an example, the images of s and (i, a, α), viz. $(0, s, 0)$ and (i, a, α^*); here s is an element of S. $0 \leqq i$ and $s < a$ in B, so $(0, s, 0) <_\infty (i, a, \alpha^*)$, from which follows $(0, s, 0) <_l (i, a, \alpha^*)$ for every $l > 0$. The 0-section of $(0, s, 0)$ is 0, so it is $<_0 (i, a, \alpha^*)$. Therefore $(0, s, 0) <_0 (i, a, \alpha^*)$. Thus the relations $s <_l (i, a, \alpha)$ is preserved under the assignment for every l.

As another example, consider $(i, a, \alpha) <_p (k, b, \beta)$, where $<_p$ holds with the condition that there is a p-section of (k, b, β), say σ, such that $(i, a, \alpha) \leqq_p \sigma$. By the induction hypothesis $(i, a, \alpha^*) \leqq_p \sigma^*$. σ^* is a p-section of (k, b, β^*). So $(i, a, \alpha^*) <_p (k, b, \beta^*)$. Other cases can be dealt with in a similar manner.

Therefore the mapping is order preserving for every $<_i$.

It should be noted that with the assignment defined above the image of $\alpha + 1$ is not the successor of α^*, for $(\alpha+1)^* = (\alpha \# s_0)^* = \alpha^* \# (0, s_0, 0)$.

§ 5. #-extension of a well-ordered system.

DEFINITION 5.1. Let J be a well-ordered set with an ordering $<$. $\mathrm{J}^\#$ is defined as follows. The elements of $\mathrm{J}^\#$ are those of the form $a_1 \# \cdots \# a_m$, $m \geqq 1$, where each of $a_1, \cdots, a_m$ is an element of J. Each of $a_1, \cdots, a_m$ is called a component of $a_1 \# \cdots \# a_m$. Two elements of $\mathrm{J}^\#$ are identified when they share exactly the same components (including repetition). Suppose $\alpha = a_1 \# \cdots \# a_m$ and $\beta = b_1 \# \cdots \# b_n$, where the components are arranged in the non-increasing order. Then $\alpha < \beta$ (in $\mathrm{J}^\#$) if and only if $a_1 = b_1, \cdots, a_m = b_m$ and $m < n$ or there is an l such that $1 \leqq l \leqq m, n$ and $a_1 = b_1, \cdots, a_{l-1} = b_{l-1}$ and $a_l < b_l$.

Note that here we are assuming that each element of J has a name and by an element of J we mean its name.

DEFINITION 5.2. Let S be a set well-ordered by $<$. If there is a sequence of elements of S which is $<$-decreasing and whose first (greatest) element is α, then we say that the sequence is led by α.

PROPOSITION 5.1. *The ordering* $<$ *defined in Definition* 5.1 *is a linear ordering of* $J^{\#}$. *It is also a well-ordering.*

PROOF. Linearity of $<$ is obvious. For well-ordering, we shall prove the following statement:

(*) Suppose there is an infinite, (strictly) $<$-decreasing sequence of elements of $J^{\#}$ led by α. Then we can construct an infinite, decreasing sequence of elements of J whose leading element is $\leqq$ a maximum component of α. We can then conclude that there can be no such sequence from $J^{\#}$.

The proof of (*) is by transfinite induction on the maximum component of α. The detail is exactly parallel to the proof of Lemma 26.19 of [**1**].

As a corollary of Proposition 5.1, we can claim the following.

PROPOSITION 5.2. *Let* $\boldsymbol{O}$(I, A, S) *be a system of o.d.'s and let i be an element of* I *or* ∞. *Suppose there is an infinite* $<_i$*-decreasing sequence of o.d.'s from* $\boldsymbol{O}$(I, A, S). *Then we can construct an infinite,* $<_i$*-decreasing sequence of connected o.d.'s.*

PROOF. Let J be the set of all connected o.d.'s of $\boldsymbol{O}$(I, A, S). Then $\boldsymbol{O}$(I, A, S) $=J^{\#}$. For every $<_i$, the ordering of $J^{\#}$ defined as in Definition 5.1 coincides with the $<_i$ of $\boldsymbol{O}$(I, A, S). Therefore this proposition is a case of Proposition 5.1.

REMARK. In the proof of Proposition 5.1 (adopted from Lemma 26.19 of [**1**]), we used transfinite induction freely. This is, however, only for the simplicity of argument. We can rewrite the proof in a more constructive manner by considering the least element of J for which (*) does not hold, provided that J is "accessible" with regards to $<$ and that a decreasing sequence as in (*) is concretely given.

This remark is valid also for similar cases in the subsequent content. In particular, if we take $\boldsymbol{O}$(I, A) for $\boldsymbol{O}$(I, A, S) in Proposition 5.2, then the accessibility of $\boldsymbol{O}$(I, A) (for an ordering $<_i$) is reduced to that of the connected ordinal diagrams, since the transition can be made "constructive".

§ 6. Successive extensions of a system of ordinal diagrams.

DEFINITION 6.1. Let $J_0, J_1, \cdots, J_n$ be non-empty, well-ordered, (at most) countable, disjoint sets, where fundamental sequences are defined for each system. Define $J^n = J_0 * J_1 * \cdots * J_n$ to be the set $J_0 \cup J_1 \cup \cdots \cup J_n$ with the ordering induced from the given orderings of $J_0, J_1, \cdots, J_n$ as follows; the elements of J_i, $0 \leqq i \leqq n$, are ordered by the ordering of J_i, and when $i<j$ every element of J_i precedes the elements of J_j. Note that no constructive nature is imposed on those sets.

PROPOSITION 6.1. 1) J^n *is well-ordered by the ordering defined as in Definition* 6.1. *Fundamental sequences for* J^n *are constructed from those for* J_0, J_1, $\cdots$, J_n *in a natural manner.*

2) $*$ *is associative.*

PROOF. 1) The well-ordering property is obvious. Let $i>0$ and let λ be the least element of J_i. If J_{i-1} has a limit order type, then there is an increasing sequence from J_{i-1}. This serves as the fundamental sequence for λ in J^n.

2) Obvious.

DEFINITION 6.2. J^n $(n\geqq 1)$ is called an extension of J_0 (by $J_1 * \cdots * J_n$).

This makes sense, for J^n can be identified with $J_0 * (J_1 * \cdots * J_n)$ (cf. 2) of Proposition 6.1).

DEFINITION 6.3. Let $\{J_n\}_n$, $n=0, 1, 2, \cdots$ be a sequence of non-empty, well-ordered, (at most) countable, disjoint sets, where fundamental sequences are defined for each system. Then $\{J_n\}_n$ is called an annexing sequence (of sets, for J_0).

From now on $\{J_n\}_n$ is assumed to be an annexing sequence whenever it is considered.

PROPOSITION 6.2. *Suppose* $0\leqq n<m$. *There is a natural embedding of* $\boldsymbol{O}(J^n)$ *into* $\boldsymbol{O}(J^m)$; *or* $\boldsymbol{O}(J^n)$ *is a subsystem of* $\boldsymbol{O}(J^m)$ *with regards to every* $<_i$. *Consequently the embedding of* $\boldsymbol{O}(J^n)$ *into* $\boldsymbol{O}(J^m)$ *is an extension of the one into* $\boldsymbol{O}(J^l)$ *when* $n<l<m$.

DEFINITION 6.4. Given an annexing sequence $\{J_n\}_n$, define J^∞ to be the set $\bigcup_{n=0}^{\infty} J_n$ ordered as follows: for every n the elements of J_n are ordered by the ordering of J_n, and every element of J_n precedes the elements of J_m if $n<m$. We write J^∞ also as $\cup * J_n$.

PROPOSITION 6.3. J^∞ *is well-ordered by the ordering defined as above, and* J^n *is an initial segment of* J^∞ *for every* n. J^∞ *can be regarded as an extension of* J_0, *for* J^∞ *is isomorphic to* $J_0 * (\bigcup_{n=1}^{\infty} * J_n)$, *which makes sense. Fundamental sequences for* J^∞ *are naturally induced from these for* $\{J_n\}_n$. *For* J^∞ *itself, take the diagonal sequence: let* $\{j_{n,m}\}_m$ *be the fundamental sequence for* J_n. *Then* $\{j_{n,n}\}_n$ *serves as one for* J^∞.

PROPOSITION 6.4. *There is a natural embedding of* $\boldsymbol{O}(J^n)$ *into* $\boldsymbol{O}(J^\infty)$ *for every* n, *and for any element of* $\boldsymbol{O}(J^\infty)$ *there is an* n *such that it belongs to* $\boldsymbol{O}(J^n)$ (*up to the embedding*). *The natural embedding of* $\boldsymbol{O}(J^n)$ *into* $\boldsymbol{O}(J^\infty)$ *is an extension of the one of* $\boldsymbol{O}(J^m)$ (*into* $\boldsymbol{O}(J^\infty)$) *if* $m<n$.

DEFINITION 6.5. Let α and β be o.d.'s of $\boldsymbol{O}(J^n)$ and $\boldsymbol{O}(J^m)$ respectively, where m and n may or may not be equal, and ∞ is allowed. Then $\alpha <_i^* \beta$ is defined to be the relation $\alpha <_i \beta$ in $\boldsymbol{O}(J^l)$ where $l=\max(m, n)$. (See Propositions 6.2 and 6.4).

$<_i^*$ is a relation defined for the elements of $\bigcup_n \boldsymbol{O}(\mathrm{J}^n)$ (when $\boldsymbol{O}(\mathrm{J}^n)$ is regarded as a subsystem of $\boldsymbol{O}(\mathrm{J}^m)$ for any n, m where $n<m$). We shall often omit the $*$ of $<_i^*$ and shall call this relation as the "$<_i$ for the sequence $\{\boldsymbol{O}(\mathrm{J}^n)\}_n$".

PROPOSITION 6.5. *$<_i^*$ is a linear ordering of $\bigcup_n \boldsymbol{O}(\mathrm{J}^n)$ and is an extension of the $<_i$ for $\boldsymbol{O}(\mathrm{J}^n)$ for every n. There is a natural isomorphism between $\bigcup_n \boldsymbol{O}(\mathrm{J}^n)$ with $<_i^*$ and $\boldsymbol{O}(\mathrm{J}^\infty)$ with $<_i$.*

This is an immediate consequence of Proposition 6.4 and Definition 6.5.

Propositions 6.2, 6.4 and 6.5 claim that $\boldsymbol{O}(\mathrm{J}^\infty)$ can be regarded as the limit of $\{\boldsymbol{O}(\mathrm{J}^n)\}_n$, or $\boldsymbol{O}(\mathrm{J}^\infty)$ is isomorphic to $\bigcup_n \boldsymbol{O}(\mathrm{J}^n)$ where the latter has the ordering $<_i^*$ for every i. Let us state this fact as a definition.

DEFINITION 6.6. "$\boldsymbol{O}(\mathrm{J}^\infty)$ is isomorphic to $\bigcup_n \boldsymbol{O}(\mathrm{J}^n)$" will mean that Propositions 6.2, 6.4 and 6.5 hold.

PROPOSITION 6.6. *Let $\mathrm{J}*\mathrm{K}$ be an extension of J and let β be an o.d. of $\boldsymbol{O}(\mathrm{J}*\mathrm{K})$. Then there is an o.d. of $\boldsymbol{O}(\mathrm{J})$ (indeed an o.d. of $\boldsymbol{O}(\{s_0\})$), say α, satisfying $\beta<_i\alpha$ for every i (in $\boldsymbol{O}(\mathrm{J}*\mathrm{K})$), and α can be so chosen that it is connected if and only if β is.*

PROOF. Replace each occurrence of an element of $\mathrm{J}*\mathrm{K}$ in β by $(0, 0, s_0)$ to obtain α, where s_0 is the least element of J.

As an immediate consequence of this and Proposition 5.2, we have

PROPOSITION 6.7. *Given a well-ordered set J, the following two statements are equivalent for every i.*

(1) *There is an extension of J, say $\mathrm{J}*\mathrm{K}$, and there is a $<_i$-decreasing (infinite) sequence from $\boldsymbol{O}(\mathrm{J}*\mathrm{K})$ whose initial entry is an o.d. of $\boldsymbol{O}(\mathrm{J})$.*

(2) *There is an extension of J, say $\mathrm{J}*\mathrm{L}$, and there is a $<_i$-decreasing (infinite) sequence of connected o.d.'s in $\boldsymbol{O}(\mathrm{J}*\mathrm{L})$ whose initial entry is an o.d. of $\boldsymbol{O}(\mathrm{J})$.*

II. Accessibility proof.

§7. Strong accessibility and conclusion.

DEFINITION 7.1. Let S be a set with linear ordering $<$, and let α be an element of S. α is said to be $<$-accessible if it can be "concretely" shown that there is no infinite (strictly) $<$-decreasing sequence from S whose initial entry is α.

DEFINITION 7.2. Given (a well-ordered set) J, let α be an o.d. of $\boldsymbol{O}(\mathrm{I}, \mathrm{A}, \mathrm{J})$ $(=\boldsymbol{O}(\mathrm{J}))$. α is said to be strongly accessible if for any extension of J, say $\mathrm{J}*\mathrm{K}$, there is no infinite $<_0$-decreasing sequence from $\boldsymbol{O}(\mathrm{J}*\mathrm{K})$ whose initial entry is α.

More precisely, α is strongly accessible if, whenever we suppose that there is a $J*K$ an extension of J and there is a sequence from $O(J*K)$ whose initial entry is α and which is $<_0$-decreasing, then we can demonstrate that the sequence actually stops at a finite stage.

Let us note that strong accessibility is not a concrete notion inasmuch as the choice of K is not necessarily concrete.

The definition makes sense due to Proposition 6.2.

We are to show the accessibility of o.d.'s (of $O(I, A)$ for any given I and A where I and A are "accessible") for every $<_i$, undertaking the following process.

1°. Let W be the set of strongly accessible elements of $O(I, A)$. Then W is $<_i$-accessible for every i.

2°. $W=O(I, A)$

It is, however, more convenient if we deal with the generalized version of o.d.'s, namely a system $O(I, A, J)$ with J as a parameter. Therefore our objective is to establish 1° and 2° above for $O(I, A, J)$.

We shall henceforth use the term "accessible" for elements of $O(J)$ when they are accessible "modulo J". Namely the well-foundedness is established in a manner that it will become an accessibility proof when J is accessible in the original sense.

PROPOSITION 7.1. *A strongly accessible o.d. of* $O(I, A, J)$ $(=O(J))$ *is* $<_0$-*accessible in* $O(J)$.

PROOF. Suppose not: there is a $<_0$-decreasing, infinite sequence from $O(J)$ whose initial entry is α a strongly accessible o.d. Let $K=\{a\}$ (the singleton a, where a is a dummy symbol). Then the sequence becomes a $<_0$-decreasing sequence from $O(J*K)$ whose initial entry is α, contradicting the strong accessibility of α.

Note that the proof can be "concrete" when the argument is restricted to the case where J is "accessible" in a concrete sense; in particular that is so when $J=\{s_0\}$.

DEFINITION 7.3. Given $O(I, A, J)$, let $W(J)$ denote the set of strongly accessible elements of $O(I, A, J)$. When J needs not be specified, we may omit J from $W(J)$.

PROPOSITION 7.2. *If* α *and* β *are o.d.'s of* $O(J)$ *satisfying* $\beta<_0\alpha$ *and* α *is an element of* $W(J)$ (*i.e.* α *is strongly accessible*), *then* β *is also. This means that* $W(J)$ *forms an initial segment of* $<_0$ *in* $O(J)$.

PROPOSITION 7.3. $W=W(J)$ *is not empty for any* J. *In fact* $J^{\#}\subseteqq W$.

PROOF. It suffices to show that $(0, 0, s_0)$ belongs to W, where s_0 is the least element of J (cf. Proposition 7.2 and the proof of Proposition 5.1). But this follows from the fact that $\beta<_0(0, 0, s_0)$ in $O(J*K)$ means that β is an element of $(J*K)^{\#}$.

PROPOSITION 7.4. *If* α *is strongly accessible and* β *is a sub-o.d. of* α, *then*

β is strongly accessible.

As an immediate consequence of Proposition 7.1, we have

PROPOSITION 7.5. *$\boldsymbol{W}$ is accessible for $<_0$.*

PROPOSITION 7.6. *Let α be an o.d. of $\boldsymbol{O}(\mathrm{J})$ and let $\mathrm{J} * \mathrm{K}$ be an extension of* J. *Then α is an element of $\boldsymbol{W}(\mathrm{J})$ if and only if α is an element of $\boldsymbol{W}(\mathrm{J} * \mathrm{K})$. As a consequence, there is a natural embedding of $\boldsymbol{W}(\mathrm{J})$ into $\boldsymbol{W}(\mathrm{J} * \mathrm{K})$ and* $\boldsymbol{O}(\mathrm{J}) \cap \boldsymbol{W}(\mathrm{J} * \mathrm{K}) = \boldsymbol{W}(\mathrm{J})$.

Also, if α is strongly accessible in $\boldsymbol{O}(\mathrm{J})$ and if β is an o.d. of $\boldsymbol{O}(\mathrm{J} * \mathrm{K})$ and $\beta \leqq_0 \alpha$, then β is strongly accessible.

PROOF. Suppose α belongs to $\boldsymbol{W}(\mathrm{J})$ but not to $\boldsymbol{W}(\mathrm{J} * \mathrm{K})$. Then there is an extension of $\mathrm{J} * \mathrm{K}$, say $(\mathrm{J} * \mathrm{K}) * \mathrm{L}$, relative to which α is not strongly accessible. But $(\mathrm{J} * \mathrm{K}) * \mathrm{L}$ is isomorphic to $\mathrm{J} * (\mathrm{K} * \mathrm{L})$ (2) of Proposition 6.1), so there is an infinite $<_0$-decreasing sequence from $\boldsymbol{O}(\mathrm{J} * (\mathrm{K} * \mathrm{L}))$ led by α, which means that α is not strongly accessible as an o.d. of $\boldsymbol{O}(\mathrm{J})$, yielding a contradiction.

Suppose α does not belong to $\boldsymbol{W}(\mathrm{J})$. Then there is an extension of J, say $\mathrm{J} * \mathrm{L}$, such that there is a $<_0$-decreasing sequence from $\boldsymbol{O}(\mathrm{J} * \mathrm{L})$ led by α. But this sequence can be regarded as a sequence from $\boldsymbol{O}((\mathrm{J} * \mathrm{K}) * \mathrm{L})$. So α does not belong to $\boldsymbol{W}(\mathrm{J} * \mathrm{K})$.

PROPOSITION 7.7. *Let $\{\mathrm{J}_n\}_n$ be an annexing sequence. Then $\boldsymbol{W}(\mathrm{J}^\infty)$ and $\bigcup_n \boldsymbol{W}(\mathrm{J}^n)$ are isomorphic with regards to the isomorphism between $\boldsymbol{O}(\mathrm{J}^\infty)$ and $\bigcup_n \boldsymbol{O}(\mathrm{J}^n)$: cf. Proposition* 6.5. *More precisely:* 1) *Suppose α is an element of $\boldsymbol{W}(\mathrm{J}^\infty)$. There is an n such that α can be regarded as an o.d. of $\boldsymbol{O}(\mathrm{J}^n)$ and for this n α is an element of $\boldsymbol{W}(\mathrm{J}^n)$.*

2) *Suppose conversely that α is an element of $\boldsymbol{W}(\mathrm{J}^n)$. Then α is an element of $\boldsymbol{W}(\mathrm{J}^\infty)$.*

PROOF. 1) Suppose α is an element of $\boldsymbol{W}(\mathrm{J}^\infty)$ and α belongs to $\boldsymbol{O}(\mathrm{J}^n)$. Suppose α is not an element of $\boldsymbol{W}(\mathrm{J}^n)$. Then there is an extension of J^n, say $\mathrm{J}^n * \mathrm{K}$, such that there is an infinite, $<_0$-decreasing sequence from $\boldsymbol{O}(\mathrm{J}^n * \mathrm{K})$ led by α. But this sequence can be regarded as a sequence from $\boldsymbol{O}(\mathrm{J}^\infty * \mathrm{K})$, yielding a contradiction.

2) Let α be an element of $\boldsymbol{W}(\mathrm{J}^n)$. Suppose it is not strongly accessible in $\boldsymbol{O}(\mathrm{J}^\infty)$. Then there is an extension of J^∞, say $\mathrm{J}^\infty * \mathrm{K}$, such that there is an infinite, $<_0$-decreasing sequence from $\boldsymbol{O}(\mathrm{J}^\infty * \mathrm{K})$ whose initial entry is α. $\mathrm{J}^\infty * \mathrm{K}$ is isomorphic to $\mathrm{J}^n * ((\bigcup_{i=n+1}^{\infty} * \mathrm{J}_i) * \mathrm{K})$, which is an extension of J^n. This means that α is not strongly accessible as an o.d. of $\boldsymbol{O}(\mathrm{J}^n)$, yielding a contradiction.

§8. Accessibility of W.

Here we shall show that the set of strongly accessible elements is accessible for every $<_i$.

PROPOSITION 8.1. *Given* J, $W=W(\mathrm{J})$ *is accessible for every* $<_i$, *where* $i=\infty$ *is included.*

PROOF. W is accessible for $<_0$ (Proposition 7.5). Suppose that the proposition is not true and let i be the least element of $\mathrm{I}\cup\{\infty\}$ for which W is not accessible. $i>0$. Suppose first i is an indicator (viz., i is not ∞). Let α be an element of W and (k, b, β) be an i-subsection of α such that $k<i$ (presuming that such exists). First we shall show that

(1) the set K$'$ of all such connected sub-o.d.'s of α for all α the elements of W is well-ordered by $<_i$.

(1) is proved as follows. Let (k, b, β) and (p, c, γ) be two such o.d.'s. $(k, b, \beta)<_i(p, c, \gamma)$ if and only if $(k, b)<(p, c)$ or $(k, b)=(p, c)$ and $\beta<_k\gamma$. By Proposition 7.4, β and γ belong to W. $k<i$. So by the choice of i, there cannot be an infinite, $<_k$-decreasing sequence from W led by γ. From this follows (1).

Now introduce a new symbol for each element of K$'$ and call the set of those new symbols K. Let $\prec$ be its well-ordering induced by $<_i$. Let α be an element of $W=W(\mathrm{J})$ and let α' be the figure obtained from α by replacing every element of K$'$ by its name in K. By identifying an element of J occurring in α' with its corresponding element in $\mathrm{J}\cup\mathrm{K}$, α' can be regarded as an o.d. of $O(\mathrm{J}*\mathrm{K})$. It can be easily shown that if $l<i$, then there is no l-section of α'. Let $l\geqq i$. If σ is an l-section of α, then σ' is an l-section of α'. (σ' is defined since σ is an i-subsection of α.) If δ is an l-section of α', then there is an l-section of α, say σ, such that $\sigma'=\delta$.

Next we wish to show $\alpha'\leqq_0\alpha$ in $O(\mathrm{J}*\mathrm{K})$, regarding α as an o.d. of $O(\mathrm{J}*\mathrm{K})$. In fact we shall prove that for every j

(2) $\alpha'\leqq_j\alpha$ and, for any j-section of α', say δ, there is a j-section of α, say σ, such that $\delta\leqq_j\sigma$ in $O(\mathrm{J}*\mathrm{K})$.

First consider the case where α does not contain any indicator which is less than i. Then α' is α, hence $\alpha'=\alpha$ in $O(\mathrm{J}*\mathrm{K})$. Next consider the case where α is of the form (k, b, β) with $k<i$. Let α' be its name in K. Then $\alpha'<_j\alpha$ in $O(\mathrm{J}*\mathrm{K})$ is obvious for every j. Suppose $\alpha=(k, b, \beta)$ where $k\geqq i$ and there is an (i-active) indicator $<i$ in α. Then $\alpha'=(k, b, \beta')$ and $\beta'\leqq_k\beta$ by the induction hypothesis, so $\alpha'\leqq_j\alpha$ if $j\geqq k$. Let $j<k$ and let δ be a j-section of α'. Then δ is a j-section of β', hence the induction hypothesis applies and there is a j-section of β, say σ, such that $\delta\leqq_j\sigma$. σ is a j-section of α, so $\delta\leqq_j\sigma<_j\alpha$. This together with $\alpha'\leqq_k\alpha$ implies that $\alpha'\leqq_j\alpha$ for every $j<k$. In case α is not

connected, apply the induction hypothesis to each component. This proves (2).

Thirdly, we wish to show that

(3) for every $j \geqq i$ $\alpha <_j \beta$ implies $\alpha' <_j \beta'$, and $\alpha <_i \beta$ implies $\alpha' <_j \beta'$ for every $j \leqq i$.

It suffices to prove (3) for connected α and β.

(3.1) α and β are elements of J. Obvious.

(3.2) α is an element of J and β is an element of K'. α' is an element of J and β' is an element of K. So $\alpha' <_j \beta'$ for every j.

(3.3) α is of the form (p, a, δ) where $p<i$, and β is (q, b, γ) where $q<i$. Then $\alpha <_l \beta$ for one $l \geqq i$ implies $\alpha <_j \beta$ for every $j \geqq i$. In particular $\alpha <_i \beta$, hence $\alpha' \prec \beta'$ in K. Therefore $\alpha' <_j \beta'$ for every j.

(3.4) α is an element of J or of K', and β is (q, b, γ) where $q \geqq i$. Then α' is an element of J$\cup$K and β is (q, b, γ'), hence $\alpha' <_j \beta'$ for every j.

(3.5) α is (p, a, δ) where $p \geqq i$ and β is (q, b, γ) where $q \geqq i$. α' is (p, a, δ') and β' is (q, b, γ'). Suppose $\alpha <_\infty \beta$. Then either $(p, a)<(q, b)$, in which case $\alpha' <_\infty \beta'$ is obvious, or $(p, a)=(q, b)$ and $\delta <_p \gamma$, in which case $\delta' <_p \gamma'$ by the induction hypothesis, hence $\alpha' <_\infty \beta'$. Let j be an indicator such that $j \geqq i$ and $\alpha <_j \beta$. Then there are two cases. Let ι be the least indicator such that $\iota > j$ and there is an ι-section of α and/or β if such exists; if there is no such section, then let ι be ∞.

1°. $\alpha <_\iota \beta$ and, for every j-section of α, say σ, $\sigma <_j \beta$. By the induction hypothesis, $\alpha' <_\iota \beta'$. Let ρ be a j-section of α'. Then there is a j-section of α, say σ, such that $\sigma' = \rho$. $\sigma <_j \beta$, hence $\rho = \sigma' <_j \beta'$ by the induction hypothesis. If $j<l<\iota$, then there is no l-section of α' or β'. So $\alpha' <_j \beta'$.

2°. There is a j-section of β, say σ, such that $\alpha \leqq_j \sigma$. σ' is a j-section of β' and $\alpha' \leqq_j \sigma'$ by the induction hypothesis. So $\alpha' <_j \beta'$.

Suppose $\alpha <_i \beta$. Then $\alpha' <_i \beta'$ (proved above), and, since there is no j-section of α' if $j<i$, this implies $\alpha' <_j \beta'$ for every $j<i$.

Now suppose there is a $<_i$-decreasing sequence of elements of $\boldsymbol{W}$ $(=\boldsymbol{W}(\text{J}))$, $\alpha_0\, {}_i\!> \alpha_1\, {}_i\!> \alpha_2\, {}_i\!> \cdots$. Then by (2) and (3) $\alpha_0\, {}_0\!\geqq \alpha'_0\, {}_0\!> \alpha'_1\, {}_0\!> \alpha'_2\, {}_0\!> \cdots$ in $\boldsymbol{O}(\text{J} * \text{K})$. α_0 is strongly accesible, therefore such a sequence must be finite. This means that the original sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$ is also finite. Therefore, $\boldsymbol{W}$ must be $<_i$-accessible.

When $i=\infty$, we consider each component of an element of $\boldsymbol{W}$ and give a new name to it. The rest of the proof for i an indicator goes through.

With Proposition 8.1, we have completed the first objective in our project: see 1° in §7.

Note that, if J is "accessible", then the accessibility proof can be made concrete; see the comment subsequent to Proposition 7.1.

DEFINITION 8.1. Let α be an o.d. of $\boldsymbol{O}(\text{J})=\boldsymbol{O}(\text{I, A, J})$. We say that α is strongly accessible in connected o.d.'s (abbreviated to c-strongly accessible), if α is connected and for any extension of J, say $J*K$, there is no infinite, $<_0$-decreasing sequence of connected o.d.'s from $\boldsymbol{O}(J*K)$ led by α.

PROPOSITION 8.2. 1) *If α is connected, then the strong accesibility of α implies the strong accessibility in connected o. d.'s.*

2) *The elements of* J *are c-strongly accessible.*

3) *Suppose every connected o. d. of $\boldsymbol{O}(J)$ is c-strongly accessible. Then for every $J*K$ an extension of J every connected o. d. of $\boldsymbol{O}(J*K)$ is also c-strongly accessible.*

4) *Under the same assumption as in* 3), *the set of connected o. d.'s of $\boldsymbol{O}(J*K)$ is well-ordered by $<_0$, or is accessible for $<_0$, for every $J*K$ an extension of* J.

PROOF. 2) By Proposition 7.3 and 1) above.

3) Let α be a connected o.d. of $\boldsymbol{O}(J*K)$. If α is not c-strongly accessible, then there is an extension of $J*K$, say $(J*K)*L$, such that there is an infinite, $<_0$-decreasing sequence of connected o.d.'s from $\boldsymbol{O}((J*K)*L)$ led by α. By Proposition 6.6, there is a connected o.d., say γ, of $\boldsymbol{O}(J)$ satisfying $\gamma_0>\alpha$. So adding γ to the sequence as the initial entry, we see that γ is not c-strongly accessible (since $(J*K)*L$ is isomorphic to $J*(K*L)$). This contradicts the assumption.

4) Under the assumption, every connected o.d. of $\boldsymbol{O}(J*K)$ is c-strongly accessible (cf. 3)). Then the $<_0$-accessibility of the set of c-strongly accessible elements of $\boldsymbol{O}(J*K)$ is proved as Proposition 7.1.

PROPOSITION 8.3. *Suppose every connected o. d. of $\boldsymbol{O}(J)$ is c-strongly accessible and let $J*K$ be an extension of* J. *Then $\boldsymbol{O}(J*K)$ is well-ordered by $<_0$.*

PROOF. By 4) of Proposition 8.2, and Proposition 5.2.

Proposition 8.3 has an immediate consequence:

PROPOSITION 8.4. *Suppose every connected o. d. of $\boldsymbol{O}(J)$ is c-strongly accessible. Then every o. d. of $\boldsymbol{O}(J)$ is strongly accessible, viz., $\boldsymbol{W}=\boldsymbol{W}(J)=\boldsymbol{O}(J)$.*

In the light of Proposition 8.4, we now know that in order to establish Step 2° of §7 it suffices to show that every connected o.d. of $\boldsymbol{O}(J)$ is c-strongly accessible. Let us state it explicitly.

3°. Every connected o.d. of $\boldsymbol{O}(J)$ is c-strongly accessible.

This then is our goal and the rest of this article is devoted to the proof of 3°.

PROPOSITION 8.5. *Let $\{J_n\}_n$ be an annexing sequence and let i be an indicator. If $\{\alpha_n\}_n$ is a sequence of o. d.'s satisfying that α_n is an o. d. of $\boldsymbol{W}(J^n)$ (namely α_n is strongly accessible) for every n, and $\alpha_{n+1}<_i\alpha_n$ for every n, where $<_i$ is the ordering for $\{\boldsymbol{O}(J^n)\}_n$ (cf. Definition* 6.5), *then the sequence $\{\alpha_n\}_n$ is in fact finite.*

PROOF. We have the following facts at our disposal.

1) $\bigcup_n \boldsymbol{W}(\mathrm{J}^n)$ is isomorphic to $\boldsymbol{W}(\mathrm{J}^\infty)$ (Proposition 7.7).

2) $\boldsymbol{W}(\mathrm{J}^\infty)$ is well-ordered by $<_i$ for every i (Proposition 8.1 applied to J^∞).

By virtue of 1), we may regard the sequence $\{a_n\}_n$ as a sequence from $\boldsymbol{W}(\mathrm{J}^\infty)$ satisfying $\alpha_{n+1}<_i\alpha_n$ for every n. Namely $\{\alpha_n\}_n$ is a $<_i$-decreasing sequence from $\boldsymbol{W}(\mathrm{J}^\infty)$, contradicting 2). Therefore $\{\alpha_n\}_n$ must be finite.

PROPOSITION 8.6. *The set of c-strongly accessible elements of* $\boldsymbol{O}(\mathrm{J})$ *is well-ordered by* $<_0$.

PROPOSITION 8.7. *If* α *is a c-strongly accessible element of* $\boldsymbol{O}(\mathrm{J})$ *and* β *is a connected o.d. of* $\boldsymbol{O}(\mathrm{J}*\mathrm{K})$ *satisfying* $\beta\leqq_0\alpha$, *then* β *is also c-strongly accessible.*

PROOF. Follow the proof of Proposition 7.6.

PROPOSITION 8.8. α *is strongly accessible if and only if every component of* α *is c-strongly accessible.*

PROOF. It can be easily shown that each component of a strongly accessible o.d. is strongly accessible. Therefore by 1) of Proposition 8.2 each component of α is c-strongly accessible. This proves the "only if" part.

For the "if" part, suppose every component of α is c-strongly accessible but α is not strongly accessible. Then there is an extension of J, say $\mathrm{J}*\mathrm{K}$, and there is an infinite, $<_0$-decreasing sequence from $\boldsymbol{O}(\mathrm{J}*\mathrm{K})$ led by α. Let $\boldsymbol{C}$ be the set of c-strongly accessible elements of $\boldsymbol{O}(\mathrm{J}*\mathrm{K})$. $\boldsymbol{C}$ is well-ordered by $<_0$ (Proposition 8.6). Therefore $\boldsymbol{C}^\#$ is well-ordered by $<_0$ (Proposition 5.1). If $\beta<_0\alpha$ for a β from $\boldsymbol{O}(\mathrm{J}*\mathrm{K})$, then each component of β is c-strongly accessible (Proposition 8.7). This is true for every entry of the infinite sequence assumed as above. Therefore each entry of the sequence belongs to $\boldsymbol{C}^\#$. This means that there is an infinite, $<_0$-decreasing sequence from $\boldsymbol{C}^\#$, contradicting the well-ordered property of $\boldsymbol{C}^\#$. Thus, α must be strongly accessible.

PROPOSITION 8.9. *If* α *is c-strongly accessible and* β *is a connected sub-o.d. of* α, *then* β *is c-strongly accessible.*

PROOF. By Proposition 8.7.

§ 9. Accessibility proof.

We are now in the position to complete our program, namely the proof of 3° in § 8. We restate what must be proved as a proposition.

PROPOSITION. *Given any system of o.d.'s* $\boldsymbol{O}(\mathrm{I}, \mathrm{A}, \mathrm{J})$, *every connected o.d. of* $\boldsymbol{O}(\mathrm{I}, \mathrm{A}, \mathrm{J})$ *is c-strongly accessible.*

PROOF. We suppose the contrary and deduce a contradiction.

Suppose for some I, A, and J there is a connected o.d. of $\boldsymbol{O}(\mathrm{I}, \mathrm{A}, \mathrm{J})$ which is not c-strongly accessible. Let I and A be accessible sets for which the supposition holds. Fix such I and A. The proof will be split into several steps.

I. There is a value (i, a) (of (I, A)) such that there is a J and an o.d. α of

$\boldsymbol{O}$(I, A, J) which satisfy that every component of α is c-strongly accessible (hence α is strongly accessible: Proposition 8.8) but (i, a, α) is not c-strongly accessible.

For, otherwise, for every (i, a) and every J, if each component of α an o.d. of $\boldsymbol{O}$(I, A, J) is c-strongly accessible, then so is (i, a, α). By 2) of Proposition 8.2, each element of J is c-strongly accessible. Therefore, for every J, every connected o.d. of $\boldsymbol{O}$(I, A, J) is c-strongly accessible. But this contradicts our assumption.

II. Let (i, a) be the least value (of (I, A)) which satisfies I above; there are a J and an o.d. of $\boldsymbol{O}$(I, A, J), say α, such that every component of α is c-strongly accessible and (i, a, α) is not c-strongly accessible.

Let us fix a J and an α satisfying II and let us denote (i, a, α) by $\tilde{\alpha}$. Since I and A will be fixed throughout the rest of the proof, we shall write $\boldsymbol{O}$(J) for $\boldsymbol{O}$(I, A, J).

III. α is strongly accessible (Proposition 8.8).

IV. We can construct an annexing sequence $\{J_n\}_n$ where $J_0=J$ and an infinite, $<_0^*$-decreasing sequence of connected o.d.'s from $\{\boldsymbol{O}(J^n)\}_n$, say $\{\tilde{\alpha}_n\}_n$, where $\tilde{\alpha}_0=\tilde{\alpha}$ and $\tilde{\alpha}_n$ is an o.d. of $\boldsymbol{O}(J^n)$ which is not c-strongly accessible (See Definition 6.5 for $<_0^*$.), such that $\tilde{\alpha}_n$ is of the form (i, a, α_n) where each component of α_n is c-strongly accessible (hence α_n is strongly accessible) and $\{\alpha_n\}_n$ is a $<_i^*$-decreasing sequence from $\{\boldsymbol{O}(J^n)\}_n$; namely α_n is an o.d. of $\boldsymbol{W}(J^n)$, $\alpha_0=\alpha$, and $\alpha_{n+1}<_i\alpha_n$ in $\boldsymbol{O}(J^{n+1})$.

Proposition 8.5 claims that the sequence $\{\alpha_n\}_n$ must stop at a finite stage. In other words an infinite sequence satisfying IV cannot exist. Therefore our supposition is erroneous, or the proposition must hold.

This means that the proof of IV will complete the proof of Proposition, hence our task.

Proof of IV. Put $J_0=J$ and $\tilde{\alpha}_0=\tilde{\alpha}=(i, a, \alpha)$, hence $\alpha_0=\alpha$. Suppose we have constructed J_n and $\tilde{\alpha}_n=(i, a, \alpha_n)$ so as to satisfy the condition. $\tilde{\alpha}_n$ is an o.d. of $\boldsymbol{O}(J^n)$ which is not c-strongly accessible while each component of α_n is c-strongly accessible. Since $\tilde{\alpha}_n$ is not c-strongly accessible, there is an extension of J^n, say J^n*K, and an infinite, $<_0$-decreasing sequence of connected o.d.'s from $\boldsymbol{O}(J^n*K)$ led by $\tilde{\alpha}_n$, say $\{\beta_m\}_m$. $\beta_0=\tilde{\alpha}_n$. Let K be J_{n+1}. $\beta_1<_0\tilde{\alpha}_n=(i, a, \alpha_n)$. β_1 is connected and not c-strongly accessible, hence is not strongly accessible (Proposition 8.8). Let us rewrite $\tilde{\alpha}_n$ as $\tilde{\eta}=(i, a, \eta)$. $\tilde{\eta}$ is not strongly accessible as an element of $\boldsymbol{O}(J^{n+1})$ (cf. Proposition 7.6). Let $\{\tilde{\eta}_m\}_m$ be the fundamental sequence of $\tilde{\eta}$ in $\boldsymbol{O}(J^{n+1})$ with regards to $<_0$ (cf. Definition 1.3 of [2] and Definition 3.2 of this article). Since $\beta_1<_0\tilde{\eta}$, there is an m such that $\beta_1<_0\tilde{\eta}_m$.

V. $\tilde{\eta}_m$ is not strongly accessible since β_1 is not.

We shall give a prescription to construct an o.d. of $\boldsymbol{O}(J^{n+1})$ of the form

(i, a, δ), where $(i, a, \delta)<_0(i, a, \eta)=\tilde{\eta}$, $\delta<_i\eta$, δ is strongly accessible (as an element of $O(J^{n+1})$) and (i, a, δ) is not c-strongly accessible. Put $\tilde{\alpha}_{n+1}=(i, a, \delta)$ and $\alpha_{n+1}=\delta$. $\tilde{\alpha}_{n+1}$ satisfies the condition in IV.

The construction of (i, a, δ) depends on the form of $\tilde{\eta}_m$.

VI. For every sub-o.d. of η, say γ, γ is strongly accessible as an o.d. of $O(J^n)$ as well as one of $O(J^{n+1})$ (cf. Propositions 7.4 and 7.6). It is so in particular for a proper scanned o.d. of $\tilde{\eta}$ with regards to $<_0$.

VII. For any proper scanned o.d. of $(0, \tilde{\eta})$, say γ, γ_m is strongly accessible, where $\{\gamma_m\}$ is the reduction sequence for γ.

For, let j be the indicator corresponding to γ. γ is strongly accessible (cf. VI), and $\gamma_m<_l\gamma$ for every l such that $h=h(\gamma)\leqq l\leqq j$ (cf. Proposition 2.1 of [2]). Here $h=0$. So $\gamma_m<_0\gamma$, and hence γ_m is strongly accessible (Proposition 7.2).

VIII. There is a connected sub-o.d. of $\tilde{\eta}_m$, say υ, such that υ is of the form (i, a, δ), satisfying that $\upsilon<_0\tilde{\eta}$, $\delta<_i\eta$, δ is strongly accessible and υ is not c-strongly accessible. This υ satisfies the requirement in V.

To find the υ as desired, we only need to go over all the cases in the definition of reduction sequences; those are seen in Definitions 1.3 and 1.4 of [2] and Definition 3.2 of this article. (j, γ) or (t, ν) in those definitions is here $(j, \tilde{\eta})$ for some j and the symbol 0 (an o.d.) there is s_0 (the least element of J^{n+1}) here.

IX. j in $(j, \tilde{\eta})$ above satisfies that $j\leqq i$ and $h=h(\tilde{\eta})=0$.

In the beginning $j=0$. But the actual construction of a reduction sequence may be executed on $(j, \tilde{\eta})$ where $j>0$, due to the possibility of (2′) and (5.3). According to the definition of (2′) and (5.3), however, $j\leqq i$ is secured. $h=0$ is obvious.

In the sequel, we quote the numbering in Definitions 1.3 (for (t, ν)) and 1.4 (for (j, γ)) of [2]; see also Definition 3.2 of this article for (0), b. 3) and e. 4). For the reader's convenience, we first list the case descriptions (cf. Definition 3.1 of this article and Definition 1.1 of [2]: See also Corollary of Proposition 1.5 in [2].).

As we follow the notation in [2], there may be some notational incoherence, for which we must be excused.

Let (j, γ) be a scanned pair under consideration.

(0) γ is an element of J^{n+1}.

(1) γ is of the form (i, a, s_0).

(2) $\gamma=(i, a, \alpha+1)$ where α is not empty and it is not the case that $a=0$, $j<i$, $\gamma=apr((n, k+1), j, \gamma)$ where $n>0$ and $v_{n+1}(j, \gamma)<i$. The excluded case is named as (2′).

In the subsequent cases, we assume that γ is of the form (i, a, α) where α is a limit o.d.

(3) $\gamma=apr(0, j, \gamma)$.

(3.1) and (3.2) α is connected and is marked. $j>i$ (Proposition 1.7 of [2]).

(3.3) Not all the components of α are marked.

(4) $\gamma=apr((0, k+1), j, \gamma)$.

(4.1) α is connected and marked, $a=0$, $j\leqq i$ and i is a successor.

(4.2) α is connected and marked and (4.1) is not the case.

(4.3) Not all the components of α are marked.

(5) $\gamma=apr((n, k+1), j, \gamma)$ where $n>0$.

(5.1) $i=i_{n+1}=v_{n+1}(j, \gamma)$ and α is connected and marked.

(5.2) $i=i_{n+1}$ and not all the components of α are marked.

(5.3) $i_{n+1}<i$.

[1°] (for (3.3), (4.3) and (5.2)) $\gamma=(t, b, \lambda \# \nu)$ where ν is a least, with respect to $<_t$, component of $\lambda \# \nu$ and is of the form $(i, 0, \alpha)$ and (3.2) applies to the pair (t, ν).

[2°] [1°] is not the case.

Now the construction of υ.

(0) ν is an element of J^{n+1}. This does not apply to $\tilde{\eta}$, since $\tilde{\eta}$ is not strongly accessible.

(1) $\nu=(i, a, s_0)$.

a) $a\neq 0$.

a. 1) $a_m \uparrow a$ and $\nu_m=(i, a_m, s_0)$. s_0 is strongly accessible and (i, a_m) is less than (i, a). So ν_m is strongly accessible (II), contradicting V. Therefore this case is non-applicable.

a. 2) $a=b+1$, $t\leqq i$ and $\nu_m=(i, b, \cdots, (i, b, s_0)\cdots)$. Non-applicable since $(i, b) <(i, a)$. (The proof is by induction on m.)

a. 3) $a=b+1$ and $t>i$. Non-applicable since $t\leqq i$ is secured (IX).

b) $a=0$.

b. 1) $i_m \uparrow i$ and $\nu_m=(i_m, 0, s_0)$. Non-applicable since $(i_m, 0)<(i, a)$.

b. 2) $i=i_0+1$.

b. 2.1) $a_m \uparrow \mathrm{A}$ and $\nu_m=(i_0, a_m, s_0)$. Non-applicable.

b. 2.2) A has the maximal element e.

b. 2.2 ; 1) $t\geqq i$ and $h\geqq i=i_0+1$. Non-applicable, since $h=0$ is the case here (IX).

b. 2.2 ; 2) $t\geqq i$ and $h<i$. Non-applicable, since neither (2′) nor (5.3) applies to this case, hence $t=0$ must hold, while here $t\geqq i=i_0+1$.

b. 2.2 ; 3) $t<i$ and $\nu_m=(i_0, e, \cdots, (i_0, e, s_0)\cdots)$. Non-applicable since $(i_0, e) <(i, a)$.

b. 3) $i=0$.

Case 1. J^{n+1} has the maximal element ε and $\nu_m=\varepsilon \# \cdots \# \varepsilon$.

Case 2. $s_m \uparrow J^{n+1}$ and $\nu_m=s_m$.

Both are non-applicable, since in either case ν_m is strongly accessible (Proposition 7.3), contradicting V.

(2) $\gamma=(i, a, \alpha+1)$.

(2.1) $a=0$, $j\leqq i$, $i=i_0+1$ and there is an $(i_0, 0)$-dominant of γ in α.

(2.1; 1) $j<i$ and $\gamma_m=(i, 0, \alpha_m \# (i_0, 0, \mu))$, where $\mu=(i, 0, \alpha)$. α is strongly accessible.

Case 1. μ is strongly accessible. Then $(i_0, 0, \mu)$ is strongly accessible (since $(i_0, 0) < (i, 0)$) and α_m is strongly accessible (VII), hence $\alpha_m \# (i_0, 0, \mu)$ also. $\alpha_m \# (i_0, 0, \mu)<_i\alpha+1$. Take $\gamma_m(=\tilde{\eta}_m)$ as the desired υ. This in particular implies υ is not c-strongly accessible (V).

Case 2. μ is not strongly accessible. $\mu<_0\tilde{\eta}$ $(=\gamma)$ and $\alpha<_i\alpha+1=\eta$. So take μ as the υ.

(2.1; 2) $j=i$ and $\gamma_m=\mu \# \cdots \# \mu \# (i, 0, \alpha_m \# (i_0, 0, \mu))$.

Case 1. μ is strongly accessible. Since γ_m $(=\tilde{\eta}_m)$ is not strongly accessible $(i, 0, \alpha_m \# (i_0, 0, \mu))$ is not c-strongly accessible (Proposition 8.8). Put $\upsilon=(i, 0, \alpha_m \# (i_0, 0, \mu))$.

Case 2. μ is not strongly accessible. Put $\upsilon=\mu$.

(2.2) Not (2.1).

c. 1) $t>i$. Non-applicable.

c. 2) $t\leqq i$.

c. 2.1) $a_m \uparrow a$ and $\nu_m=(i, a_m, \mu)$. If μ is strongly accessible, then so is $\tilde{\eta}_m$ since $(i, a_m)<(i, a)$, yielding a contradiction. So μ is not strongly accessible. Take μ as υ.

c. 2.2) $a=b+1$ and $\nu_m=(i, b, \cdots, (i, b, \mu)\cdots)$. Take μ as υ.

c. 2.3) $a=0$, $t=i$ and $\nu_m=\mu \# \cdots \# \mu \# \rho_m$. Either μ or ρ_m is not strongly accessible. If μ is strongly accessible, then so is ρ_m, since ρ_m is either empty, (i_0, a_m, μ) or $(i_0, e, \cdots, (i_0, e, \mu \# \cdots \# \mu)\cdots)$, where (i_0, a_m), $(i_0, e)<(i, a)$. So μ is not strongly accessible. Take μ as υ.

c. 2.4) $i_m \uparrow i$ and $\nu_m=(i_m, 0, \mu)$. Take μ as υ.

c. 2.5) $i=i_0+1$ and $\nu_m=(i_0, a_m, \mu)$ or $\nu_m=(i_0, e, \cdots, (i_0, e, \mu \# \cdots \# \mu)\cdots)$ where (i_0, a_m), $(i_0, e)<(i, a)$. Take μ as υ.

In the following γ is of the form (i, a, α).

(3.1) and (3.2) $j>i$. Non-applicable.

(3.3) [1°] $\gamma=(t, b, \lambda \# \nu)$ and $\gamma_m=\kappa_m \# \mu_m$ where $\mu_m=(t, b, \lambda \# \nu_m)$. $\lambda \# \nu$ is strongly accessible and the values in κ_m connected to $\lambda \# \nu$ are less than (t, b). So κ_m is strongly accessible, and hence μ_m must not be strongly accessible. ν_m is strongly accessible (VII), hence so is $\lambda \# \nu_m$. $\lambda \# \nu_m<_t\lambda \# \nu$. Take μ_m as υ.

[2°] $\upsilon=\nu_m$ $(=\tilde{\eta}_m)$.

(4.1) $i=i_0+1$.

(4.1; 1) $j<i$ and $\gamma_m=(i, 0, \alpha_m \# (i_0, 0, \alpha))$.

(4.1; 2) $j=i$ and $\gamma_m=\alpha \# \cdots \# \alpha \# (i, 0, \alpha_m \# (i_0, 0, \alpha))$.

α is strongly accessible, hence so are α_m and $(i_0, 0, \alpha)$. $\alpha_m \# (i_0, 0, \alpha)<_i\alpha$.

Take $(i, 0, \alpha_m \# (i_0, 0, \alpha))$ as υ for both cases.

(4.2) A component of ν_m is either α or ρ_m, where in ρ_m all the values connected to α are less than (i, a). So ν_m is strongly accessible in any case. Therefore (4.2) is non-applicable.

(4.3) See (3.3).

(5.1.1) See (4.1).

(5.1.2) See (4.2).

(5.2) See (3.3).

(5.3) and (2′) are only transitive steps, hence irrelevant in the construction of υ.

Note that when a case is not applicable for the reason that $\tilde{\eta}_m$ is strongly accessible, $\tilde{\eta}$ is strongly accessible to begin with.

This completes the proof of VIII, hence of the proposition.

References

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