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Approximation problem restricted by an incidence matrix

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§ 0. Introduction.

A matrix $E=(e_{ij})_{j=0,\cdots,s}^{i=1,\cdots,k}$ is called an incidence matrix if $e_{ij}=0$ or 1. Let $e=\{(i, j)\,;\, e_{ij}=1\}$ and $|e|=\sum e_{ij}$. In this paper we consider both the "algebraic case" and the "trigonometric case", simultaneously. Thus, through this paper we assume that $s=\max\{j\,;\,(i, j)\in e\}$ and that

$$\bar{e}=\begin{cases} |e|-1 & \text{in the algebraic case,} \\ [(|e|-1)/2] & \text{in the trigonometric case,} \end{cases}$$

where $[x]$ is the largest integer such that $[x]\leqq x$. Let Π_n denote the algebraic or trigonometric polynomials of degree n or less. Let A denote an interval $[0, 1]$ or unit circle $K=[-\pi, \pi)$. Given k distinct points $x_1, \cdots, x_k\in A$ and a polynomial $P\in\Pi_{\bar{e}}$. If $P^{(j)}(x_i)=0$ for $(i, j)\in e$ implies $P=0$, we said that the scheme $S=(E\,;\,\{x_i\})$ is poised. If the scheme S is poised for all choices of nodes $\{x_i\}$, E is called a poised matrix. In the algebraic case, a wide class of poised matrices has been found. In order to mention them, we need several definitions. Given an incidence matrix E, we define

$$m_j=\sum_{i=1}^{k} e_{ij} \quad\text{and}\quad M_p=\sum_{j=0}^{p} m_j\,, \qquad j,\, p=0, \cdots, s\,.$$

An incidence matrix E is said to satisfy the Pólya conditions if

$$M_p\geqq p+1\,, \qquad p=0, \cdots, s\,. \tag{0.1}$$

A sequence of 1's in a row of E;

$$e_{ij}=e_{ij+1}=\cdots=e_{ij+r-1}=1\,, \tag{0.2}$$

is called a block if its length r is maximum. A block is even or odd according as its length r is even or odd. A block (0.2) is called a Hermite block if $j=0$.

THEOREM 0.1. (Ferguson [1], Atkinson and Sharma [2]) *In the algebraic polynomial class $\Pi_{\bar{e}}$, an incidence matrix satisfying* (0.1) *is poised if its interior rows contain no odd blocks of non Hermite data.*

We have also a wide class of poised matrices for the trigonometric polynomials.

THEOREM 0.2. *In the trigonometric polynomial class* $\Pi_{\bar{e}}$, *an incidence matrix satisfying* $m_0\neq 0$ *is poised if it contains no odd blocks of non Hermite data.*

We need the following lemma in order to prove the theorem.

LEMMA 0.1. *Let* E *be a Hermite matrix, that is, all of the blocks in* E *be Hermite. Then* E *is poised.*

PROOF. We may assume that for each i

$$e_{i0}=\cdots=e_{ij_i-1}=1, \qquad e_{ij_i}=0 \qquad \text{and} \qquad j_i>0\,.$$

Then we have $\sum_{i=1}^{k} j_i=\sum e_{ij}$. If $T\in\Pi_{\bar{e}}$ is a polynomial which satisfies $T^{(j)}(x_i)=0$ for $(i, j)\in e$, T has a zero of order at least j_i at x_i. Thus, T must have $\bar{e}$ zeroes. Thus, T is identically zero. (q. e. d.)

PROOF OF THEOREM 0.2. We use induction in r, where r is the number of even blocks of non Hermite data in E. If $r=0$, the theorem follows from Lemma 0.1. We assume that the theorem is true for $r<q$, and that E has q even blocks of non Hermite data. Let $T\in\Pi_{\bar{e}}$ satisfy $T^{(j)}(x_i)=0$ for $(i, j)\in e$. Define

$$j_0=\min\{j\,;\, e_{ij}=\cdots=e_{ij_1}=1 \text{ is even block, } j\neq 0\}\,.$$

From Rolle's theorem we see that $T^{(j_0-1)}$ has M_{j_0-1} distinct zeroes. Thus, we have the set R_{j_0} which consists of M_{j_0-1} Rolle zeroes of $T^{(j_0)}$. Let K_{j_0} be the set which consists of m_{j_0} zeroes specified by the scheme $S=(E\,;\,\{x_i\})$.

If $K_{j_0}\cap R_{j_0}=\emptyset$, $T^{(j_0)}$ interpolates the scheme S_0 which has nodes $\{x_i\}\cup R_{j_0}$. Since S_0 has at most $q-1$ even blocks of non Hermite data, it is poised. Thus, $T^{(j_0)}=0$. Since $m_0\neq 0$, we have $T=0$.

If $K_{j_0}\cap R_{j_0}\neq\emptyset$, let $R_{j_0}\backslash K_{j_0}=\{x_{k+1}, \cdots, x_m\}$. We will consider a new scheme S_1 that $T^{(j_0)}$ interpolates. Define $S_1=(E_1\,;\,\{x_i\}_{i=1}^{m})$. Here matrix $E_1=(f_{ij})_{j=0,\cdots,s-j_0+1}^{i=1,\cdots,m}$ as follows: If $x_i\in K_{j_0}\backslash R_{j_0}$, we define $f_{ij}=e_{ij+j_0}$, $j=0, \cdots, s-j_0$, and $f_{is-j_0+1}=0$. If $x_i\in R_{j_0}\backslash K_{j_0}$, we define $f_{i0}=1$ and $f_{i1}=\cdots=f_{is-j_0+1}=0$. If $x_i\in K_{j_0}\cap R_{j_0}$, we have the even block; $e_{ij_0}=\cdots=e_{ij_1}=1$. Since x_i is Rolle zero of $T^{(j_0)}$, we have $T^{(j_1+1)}(x_i)=0$. Then we define $f_{i0}=\cdots=f_{ij_1-j_0+1}=1$ and $f_{ij}=e_{ij+j_0}$, $j=j_1-j_0+2, \cdots, s-j_0$, $f_{is-j_0+1}=0$. Thus the polynomial $T^{(j_0)}$ interpolate the scheme S_1. Since S_1 has at most $q-1$ even blocks of non Hermite data, it is poised. Thus $T^{(j_0)}=0$. Since $m_0\neq 0$, we have $T=0$. (q. e. d.)

The space $X=C^s[A]$ with a norm;

$$\|f\|_X=\max_{0\leq j\leq s}\|f^{(j)}\|_C=\max_{0\leq j\leq s}\max_{x\in A}|f^{(j)}(x)| \qquad \text{for } f\in X, \tag{0.3}$$

is a Banach space. For each $F\in X$ and a scheme S, we consider a closed subset;

$$\Pi_n(S\,;\,f)=\{P\in\Pi_n\,;\, P^{(j)}(x_i)=f^{(j)}(x_i),\ (i, j)\in e\}\,.$$

We can approximate f by $P\in\Pi_n(S;f)$, $n\geqq\bar{e}$.

In section 1 we consider the best approximant for a function in X. By the well known methods we can show existence and characterization of a best approximant for each $f\in X$. However, uniqueness of best approximant is not realized without conditions. We shall provide a function $f\in X$ with many best approximants. But, if $f\in C^{(s+1)}(A)$, for a kind of incidence matrices we can show uniqueness of best approximant. In section 2 we show that our approximation problem is appropriate, that is, the degree of approximation of f tends to zero. In this case we shall obtain a bounded linear polynomial operator $L_m^{[S]}$ on X. Then the degree of approximation of $f\in X$ by the operator $L_m^{[S]}$ is expressed by means of Bernstein or Jackson operator. In section 3 we apply our results in section 2 to the approximation problem that has been investigated by Carroll and Mclaughlin [**3**].

In section 4 we estimate the degree of approximation by the operator $L_m^{[S]}$ to the function f such that $f^{(s)}\in \mathrm{Lip}_M(\alpha;A)$. Let $f\in X$ and $S=(E;\{x_i\})$ be a scheme. Then we define

(0.4) $$E_n^S(f)=\inf_{Q\in\Pi_n(S;f)}\|f-Q\|_X .$$

In section 5 we estimate $E_n^S(f)$, and as an application we extend the result by Wayne T. Ford and John A. Roulier in [**4**, Theorem 2]. In section 6 we deal with the comonotone polynomial approximation ([**5**]). In section 7 we consider the derivative of the polynomial of best approximation. But, it does not relate to our main theme. We will give an application of Lemma 5.2 or (5.13), there.

§ 1. Existence, characterization and uniqueness of best approximant.

Let $S=(E;\{x_i\})$ be a scheme, and let S be poised. For each $f\in X$ we consider a subset $\Pi_n(S;f)$. We shall approximate f by $\Pi_n(S;f)$. We can prove the following theorem by means of the well known methods ([**9**], p. 17, Lemma 1).

THEOREM 1.1. *For each $f\in X$ there is a best approximant to f.*

For each $f\in X$ we must provide some definitions to characterize the best approximant $P\in\Pi_n(S;f)$. Define

(1.1) $$A_{fP}=\{(x,j);\ |f^{(j)}(x)-P^{(j)}(x)|=\|f-P\|_X,\ \ 0\leqq x\leqq 1,\ j=0,\cdots,s\}.$$

If we consider a Cartesian product $A\times J$ of A and $J=\{0,\cdots,s\}$ with a norm

$$\|(x,j)\|=(x^2+j^2)^{1/2},$$

A_{fP} becomes a compact subset of $A\times J$. Let

(1.2) $$L(f)(x,j)=f^{(j)}(x),$$

then for each $f\in X$, $L(f)$ is continuous on $A\times J$. Notice that we may consider $L(f)$ for $j>s$ if there is $f^{(j)}(x)$, and that $L(f)$ is linear with respect to f.

Our characterization is stated as follows.

THEOREM 1.2. *Let $f\in X$ and $P\in\Pi_n(S;f)$. P is a polynomial of best approximant for f if and only if for each polynomial $Q\in\Pi_n(S;0)$,*

$$\max_{(x,j)\in A_{fP}} L(f-P)(x,j)\cdot L(Q)(x,j)\geqq 0. \tag{1.3}$$

PROOF. We use the methods in the proof of [9, p. 18, Theorem 2].

Necessity: Assume that P is a best approximant to f. Let $\|f-P\|_X=D$. If (1.3) is not true, there exists a polynomial $Q\in\Pi_n(S;0)$ such that

$$\max_{(x,j)\in A_{fP}} L(f-P)(x,j)\cdot L(Q)(x,j)=-2\varepsilon<0$$

for some $\varepsilon>0$. By the continuity of $L(f)$, there exists an open subset G of $A\times J$ such that

$$G\supset A_{fP},\qquad L(f-P)(x,j)\cdot L(Q)(x,j)<-\varepsilon$$

for $(x,j)\in G$. Let $P_1=P-\lambda Q$, $\lambda>0$, and $M=\|Q\|_X$, then we obtain $P_1\in\Pi_n(S;f)$ and for $(x,j)\in G$

$$\begin{aligned}&|L(f-P_1)(x,j)|^2\\&=|L(f-P)(x,j)+L(Q)(x,j)|^2\\&=|L(f-P)(x,j)|^2+2\lambda L(f-P)(x,j)\cdot L(Q)(x,j)+\lambda^2|L(Q)(x,j)|^2\\&<D^2-2\lambda\varepsilon+\lambda^2M^2.\end{aligned}$$

If we take $\lambda<M^{-2}\varepsilon$, then $\lambda^2M^2<\lambda\varepsilon$, and we have

$$|L(f-P_1)(x,j)|^2<D^2-\lambda\varepsilon\qquad\text{for } (x,j)\in G. \tag{1.4}$$

In order to consider the points $(x,j)\notin G$ we define $H=G^c$ $(\subset A\times J)$. We can find some $\delta>0$ such that

$$|L(f-P)(x,j)|<D-\delta\qquad\text{for } (x,j)\in H.$$

Thus, if we take λ so small that $0<\lambda<(2M)^{-1}\delta$, we have

$$\begin{aligned}&|L(f-P_1)(x,j)|\leqq|L(f-P)(x,j)|+\lambda|L(Q)(x,j)|\\&\leqq D-\delta+\delta/2=D-\delta/2\end{aligned} \tag{1.5}$$

for $(x,j)\in H$. Thus, (1.4) and (1.5) contradict for P to be a best approximant.

Sufficiency: Assume that (1.3) holds for each $Q\in\Pi_n(S;0)$. Taking an arbitrary polynomial $P_1\in\Pi_n(S;f)$, we see $Q+P-P_1\in\Pi_n(S;0)$. Since there is a point $(x, j)\in A_{fP}$ such that

$$L(f-P)(x, j)\cdot L(Q)(x, j)\geqq 0,$$

thus,

$$\begin{aligned}&|L(f-P_1)(x, j)|^2\\&=|L(f-P)(x, j)|^2+2L(f-P)(x, j)\cdot L(Q)(x, j)+|L(Q)(x, j)|^2\\&\geqq\|f-P\|_X^2.\end{aligned}$$

Consequently, we have

$$\|f-P\|_X^2\leqq|L(f-P_1)(x, j)|^2,$$

thus, we see that P is a best approximant to f. (q. e. d.)

The uniqueness theorem with respect to a best approximant is not true in general. The following examples prove it.

EXAMPLES. Algebraic case. Let

$$E=\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 1 & 0 & 0\end{pmatrix} \quad \text{and} \quad x_1=-1, \quad x_2=0, \quad x_3=1.$$

Let $f(x)=\int_0^x\int_0^t f^{(2)}(s)dsdt$, where

$$f^{(2)}(x)=\begin{cases}1, & x=0,\ 1,\\ 0, & x=1/8,\ 3/8,\ 5/8,\ 7/8,\\ -1/2, & x=1/4,\ 3/4,\\ \text{linear}, & \text{otherwise in } [0, 1],\\ \text{even}. & \end{cases}$$

Now, we approximate f by $\Pi_3(S; f)$, where $S=(E; \{x_i\})$. Then we see that $P(x)=ax(x^2-1)$, $|a|\leqq 1/2$, are the best approximants to f.

Trigonometric case. Let $E=\begin{pmatrix}10\\10\end{pmatrix}$ and $x_1=-\pi$, $x_2=0$, and let $f(x)=\int_0^x f^{(1)}(t)dt$, where

$$f^{(1)}(x)=\begin{cases}0, & x\in[0, \pi/3] \text{ or } x=\pi,\\ 1, & x=\pi/2,\\ -1, & x=5\pi/6,\\ \text{linear}, & \text{otherwise in } [0, \pi],\\ \text{even}. & \end{cases}$$

We approximate f by $\Pi_1(S;f)$. Each $T\in\Pi_1(S;f)$ is of the form $T(x)=a\cdot\sin x$. Thus, $T(x)=a\cdot\sin x$, $0\leqq a\leqq 1$, are the best approximants to f.

If $f\in C^{(s+1)}[A]$, we can prove the uniqueness theorem to some class of incidence matrices. We put the following assumptions.

(1.6) The incidence matrix $E=(e_{ij})_{j=0,\cdots,s}^{i=1,\cdots,k}$ satisfies $\bar{e}=s$ and (0.1). Let $0\leqq x_1<\cdots<x_k\leqq 1$. In this case if $0<x_i<1$ then i-th row of the scheme $S=(E;\{x_i\})$ has only Hermite block or even blocks.

In the trigonometric case, our assumption is

(1.7) the incidence matrix $E=(e_{ij})_{j=0,\cdots,s}^{i=1,\cdots,k}$ satisfies that $m_0\neq 0$ and its blocks are Hermite or even.

When A_{fP} is finite, we need to define some incidence matrix E_{fP} decided by the set A_{fP}. Let

$$B_{fP}=\{y;(y,j)\in A_{fP} \text{ for some } j=0,\cdots,s\}=\{y_t\}_{t=1}^{m}. \tag{1.8}$$

Then we define a scheme $S_{fP}=(E_{fP};\{y_t\})$ such that the incidence matrix $E_{fP}=(e_{tj'})_{j'=0,\cdots,s+1}^{t=1,\cdots,m}$, satisfies that

$$e_{tj'}=1 \quad \text{if } (y_t, j')\in A_{fP}, \tag{1.9}$$

$$e_{tj'+1}=1 \text{ if } (y_t, j')\in A_{fP} \text{ and, in the algebraic case, } 0<y_t<1, \tag{1.10}$$

and otherwise $e_{tj'}=0$.

For the proof of the uniqueness theorem we need two lemmas. We state only the algebraic case. Then, as its analogy we can obtain the corresponding results in the trigonometric case.

LEMMA 1.3. *Let $f\in X$ and a scheme S satisfy the assumption* (1.6). *When both P and P_1 in $\Pi_n(S;f)$ are the best approximants to f, define*

$$R=(P+P_1)/2, \qquad \|f-P\|_X=\|f-P_1\|_X=D.$$

Then we obtain the following (i) *and* (ii).

(i) *R is also a best approximant to f, and*

$$A_{fR}\subset A_{fP}\cap A_{fP_1}. \tag{1.11}$$

(ii) *For $(y, j)\in A_{fR}$ we have*

$$L(f-P)(y, j)=L(f-P_1)(y, j)=\pm D, \tag{1.12}$$

furthermore if $0<y<1$ and there is $f^{(j+1)}(y)$, then

$$L(P)(y, j+1)=L(P_1)(y, j+1). \tag{1.13}$$

We omit the proof of Lemma 1.3 as it is easy. If there is $f^{(s+1)}(y)$ for each $(y, s)\in A_{fP}$, (1.9) corresponds to the points where $|f^{(j)}(x)-P^{(j)}(x)|$ attains its maximum $\|f-P\|_X$, and in this case (1.10) corresponds to the points $(y, j+1)$. Thus, an incidence matrix E_{fP} is well defined, that is, 1's in (1.9) or (1.10) don't overlap and t-th row of the scheme S_{fP} has only even blocks if $0<y_t<1$.

LEMMA 1.4. *Let P be a best approximant to f. If A_{fP} is finite, our incidence matrix $E_{fP}=(e_{tj'})$ satisfies*

$$\sum e_{tj'}\geqq n-s+1\,. \tag{1.14}$$

PROOF. We assume that $\sum e_{tj'}\leqq n-s$. We may define a scheme

$$S_1=((E,\ E_{fP})\,;\,(\{x_i\},\ \{y_t\}))\,,\qquad E_1=(E,\ E_{fP})\,, \tag{1.15}$$

for the incidence matrices (E, E_{fP}) and the nodes $(\{x_i\}, \{y_t\})$. Here notice that matrix E_1 is of $(k+m)\times(s+2)$ type. Since

$$n'=\sum_{e_{ij}\in E_1} e_{ij}\leqq(s+1)+(n-s)=n+1$$

and E_1 is poised for the polynomials in $\Pi_{n'-1}$, there is a unique polynomial Q such that $L(Q)(x_i, j)=-\|f-P\|_X$ if (i, j) is given by (1.9), and $L(Q)(x, j)$ vanishes on the points (x, j) corresponding to other $e_{ij}=1$. But, because $Q\in\Pi_n(S\,;0)$ we have a contradiction with Theorem 1.2. (q. e. d.)

When A_{fP} is finite, we can define a scheme S_1 as (1.15), then from Lemma 1.4 we see that S_1 has at least $n+2$ 1's, thus S_1 is poised in Π_{n+1}. Of course, S_1 is poised in Π_n.

If $f\in C^{(s+1)}[0, 1]$, we can prove the uniqueness of the best approximant to f.

THEOREM 1.5. *Let $f\in C^{(s+1)}[0, 1]$ and the scheme $S=(E\,;\,\{x_i\})$ satisfy the assumption* (1.6). *Then the best approximant to f is unique in $\Pi_n(S\,;f)$ for $n\geqq k$.*

PROOF. Let P be a best approximant to f. If A_{fP} is infinite, from Lemma 1.3 (1.12) we see that the best approximant is unique. Thus, we assume that A_{fP} is finite.

Now we assume that there is the best approximant P_1 different from P. In this case we may assume that

$$A_{fP}=A_{fP_1}\,. \tag{1.16}$$

In fact, define

$$R=aP+bP_1\,;\qquad a,\ b\geqq0,\qquad a+b=1\,,$$

then from Lemma 1.3, R is also the best approximant to f, and we have the distinct best approximants if the points (a, b) are distinct. Thus, from Lemma 1.3 (1.11) we can select two distinct polynomials R_1 and R_2 such that $A_{fR_1}=A_{fR_2}$. Thus we can take P and P_1 satisfying (1.16).

From Lemma 1.3 we see that $R=(P+P_1)/2$ is a best approximant to f, and we have an equality (1.12) on A_{fR}. Let $A_{fR}=\{y_t\}_{t=1}^{m}$, then we have an equality (1.13) if $0<y_t<1$. From Lemma 1.3, Lemma 1.4 and its remarks, we see that the polynomial $P-P_1$ vanishes on the scheme $S_1=((E, E_{fR});(\{x_i\}, \{y_t\}))$, thus $P=P_1$. This contradicts with $P\neq P_1$. (q. e. d.)

By the same method as the proof of Theorem 1.5, we can prove the uniqueness theorem in the trigonometric case.

THEOREM 1.6. *Let* $f\in C^{(s+1)}[K]$, *and let* $S=(E;\{x_i\})$ *and* E *satisfy the assumption* (1.7). *Then the best approximant to* f *is unique in* $\Pi_n(S;f)$, *where* $n\geqq[(k-1)/2]$.

§ 2. Approximability.

In this section we shall see that the degree of approximation to f by $\Pi_n(S;f)$ tends to zero. For this cause we examine the special case when

$$E=(e_{ij});\ e_{ij}=1 \qquad \text{for all } i=1, \cdots, k,\ j=0, \cdots, s. \tag{2.1}$$

But we must suppose that n is sufficiently large.

At first, we deal with the algebraic case. Let

$$L_i(x)=\Omega(x)/(x-x_i)\Omega'(x_i),\ \Omega(x)=(x-x_1)\cdots(x-x_k),\quad i=1, \cdots, k, \tag{2.2}$$

and $f\in C[0, 1]$, then the polynomial $\sum_{i=1}^{k} f(x_i)L_i(x)$ is the Lagrange polynomial of the degree $k-1$, which interpolates f at the points $x_1, \cdots, x_k$. If we take k polynomials P_i, $i=1, \cdots, k$, satisfying

$$P_i(x_i)=1,\qquad i=1, \cdots, k, \tag{2.3}$$

we also obtain the polynomial

$$\sum_{i=1}^{k} f(x_i)P_i(x)L_i(x) \tag{2.4}$$

which interpolates f at $x_1, \cdots, x_k$, but belongs to Π_{m+k-1} if $P_i\in\Pi_m$, $i=1, \cdots, k$.

When E satisfy (2.1) we have the following theorem.

THEOREM 2.1. *Let* $f\in X$. *Then for each* $m=0, 1, 2, \cdots$, *there is a bounded linear polynomial operator on* X *such that*

$$L_m^{[S]}(f)\in \Pi_{\max[m,\, ks]+k-1}(S;f) \tag{2.5}$$

and

$$\exists M_s>0;\ \|L_m^{[S]}(f)-f\|_X\leqq M_s\|B_m(f)-f\|_X, \tag{2.6}$$

where $B_m(f)$ *is Bernstein polynomial of degree* m

$$B_m(f)=\sum_{r=0}^{m} f(r/m)\binom{m}{r}x^r(1-x)^{m-r}.$$

PROOF. First we assume that $f(x_i)\neq 0$, $i=1, \cdots, k$. Then we take P_{mi}, $i=1, \cdots, k$, for (2.3) such that

$$(2.7)\qquad P_{mi}(x)=(1/f(x_i))(B_m(f)(x)-B_m(f)(x_i)+f(x_i)+q_i(x)(x-x_i)),\quad i=1, \cdots, k.$$

Here $q_i(x)$, $i=1, \cdots, k$, are the polynomials of degree $ks-1$, and are defined by (2.9) below. For each $i=1, \cdots, k$, $t=1, \cdots, s$ we define the following polynomials and constants;

$$\lambda_{it}(x)=\prod_{p=1}^{k}(x-x_p)^{s+1}/(x-x_i)^{s-t+1},$$

$$C_i^{[t]}=[\lambda_{it}^{(t)}(x_i)]^{-1}=[t\,!\prod_{j\neq i}(x_i-x_j)^{s+1}]^{-1},$$

$$(2.8)\qquad \varepsilon_{mi}^{[t]}=f^{(t)}(x_i)-B_m^{(t)}(f)(x_i)+\sum_{j=1}^{k}[B_m(f)(x_j)-f(x_j)]L_j^{(t)}(x_i),$$

$$\delta_{mi}^{[t-1]}=\begin{cases}0 & (t=1),\\ \sum_{\eta=1}^{t-1}C_i^{[\eta]}[\varepsilon_{mi}^{[\eta]}-\delta_{mi}^{[\eta-1]}]\lambda_{i\eta}^{(t)}(x_i) & (2\leqq t\leqq s).\end{cases}$$

Let $q_i(x)=0$, $i=1, \cdots, k$, if $s=0$, and if $s\geqq 1$ we define

$$(2.9)\qquad q_i(x)=(\Omega'(x_i)/\Omega(x))\sum_{t=1}^{s}C_i^{[t]}[\varepsilon_{mi}^{[t]}-\delta_{mi}^{[t-1]}]\lambda_{it}(x),\qquad i=1, \cdots, k,$$

then we have

$$(2.10)\qquad \sum_{i=1}^{k}f(x_i)P_{mi}(x)L_i(x)=\sum_{i=1}^{k}[B_m(f)(x)-B_m(f)(x_i)+f(x_i)]L_i(x)+\sum_{i=1}^{k}\sum_{t=1}^{s}C_i^{[t]}[\varepsilon_{mi}^{[t]}-\delta_{mi}^{[t-1]}]\lambda_{it}(x).$$

By the definition, $\varepsilon_{mi}^{[t]}$ and $\delta_{mi}^{[t]}$ are linear with respect to f. Thus,

$$(2.11)\qquad L_m^{[S]}(f)(x)=\sum_{i=1}^{k}f(x_i)P_{mi}(x)L_i(x)$$

is a bounded linear polynomial operator. We show that $L_m^{[S]}(f)$ satisfies (2.5). It is trivial that the degree of $L_m^{[S]}(f)$ equals to $\max[m, ks]+k-1$. Let $i'=1, \cdots, k$, $j'=0, \cdots, s$, then

$$L_m^{[S](j')}(f)(x_{i'})=B_m^{(j')}(f)(x_{i'})-\sum_{i=1}^{k}[B_m(f)(x_i)-f(x_i)]L_i^{(j')}(x_{i'})+\sum_{i=1}^{k}\sum_{t=1}^{s}C_i^{[t]}[\varepsilon_{mi}^{[t]}-\delta_{mi}^{[t-1]}]\lambda_{it}^{(j')}(x_{i'}).$$

Since $\lambda_{it}^{(j')}(x_{i'})=0$ if $j'<t$ or $i\neq i'$, the third term in the right side of (2.12) equals to

$$\sum_{t=1}^{j'} C_{i'}^{[t]}[\varepsilon_{mi'}^{[t]}-\delta_{mi'}^{[t-1]}]\lambda_{i't}^{(j')}(x_{i'})$$

$$=\sum_{t=1}^{j'-1} C_{i'}^{[t]}[\varepsilon_{mi'}^{[t]}-\delta_{mi'}^{[t-1]}]\lambda_{i't}^{(j')}(x_{i'})+[\varepsilon_{mi'}^{[j']}-\delta_{mi'}^{[j'-1]}]=\varepsilon_{mi'}^{[j']}.$$

Thus, by the definition of $\varepsilon_{mi'}^{[j']}$, the equality (2.12) means

$$L_m^{[S](j')}(f)(x_{i'})=f^{(j')}(x_{i'}).$$

Next, we state the degree of approximation. For $t=1,\cdots,s$ we have

$$\text{(2.13)}\qquad \exists M_t'>0\,;\ |\varepsilon_{mi'}^{[t]}-\delta_{mi'}^{[t-1]}|=M_t'\max_{0\leq j\leq t}\|B_m^{(j)}(f)-f^{(j)}\|_C,\qquad i'=1,\cdots,k.$$

In order to show this we use the inductive methods with respect to t. If $t=1$ we have

$$|\varepsilon_{mi'}^{[1]}-\delta_{mi'}^{[0]}|=|\varepsilon_{mi'}^{[1]}|\leqq|f^{(1)}(x_{i'})-B_m^{(1)}(f)(x_{i'})|+\sum_{i=1}^{k}|B_m(f)(x_i)|\,|L_i^{(1)}(x_{i'})|$$

$$\leqq(1+\sum_{i=1}^{k}|L_i^{(1)}(x_{i'})|)\max_{0\leq j\leq 1}\|B_m^{(j)}(f)-f^{(j)}\|_C$$

$$\leqq M_1'\max_{0\leq j\leq 1}\|B_m^{(j)}(f)-f^{(j)}\|_C.$$

Now, if we have the inequalities (2.13) for all $j\leqq t$ $(1\leqq t<s)$, then

$$|\varepsilon_{mi'}^{[t+1]}-\delta_{mi'}^{[t]}|\leqq|\varepsilon_{mi'}^{[t+1]}|+|\delta_{mi'}^{[t]}|$$

$$\leqq|f^{(t+1)}(x_{i'})-B_m^{(t+1)}(f)(x_{i'})|+\sum_{i=1}^{k}|B_m(f)(x_i)-f(x_i)|\,|L_i^{(t+1)}(x_{i'})|$$

$$+\sum_{\eta=1}^{k}|C_{i'}^{[\eta]}|\,|\varepsilon_{mi'}^{[\eta]}-\delta_{mi'}^{[\eta-1]}|\,|\lambda_{i'\eta}^{(t+1)}(x_{i'})|$$

$$\leqq(1+\sum_{i=1}^{k}|L_i^{(t+1)}(x_{i'})|+\sum_{\eta=1}^{t}|C_{i'}^{[\eta]}|\,|\lambda_{i'\eta}^{(t+1)}(x_{i'})|M_\eta')\max_{0\leq j\leq t+1}\|B_m^{(j)}(f)-f^{(j)}\|_C$$

$$\leqq M_{t+1}'\max_{0\leq j\leq t+1}\|B_m^{(j)}(f)-f^{(j)}\|_C.$$

Thus, we obtain (2.13). Thus, from (2.12) and (2.13) we have

$$\|L_m^{[S](j')}(f)-f^{(j')}\|_C\leqq\|B_m^{(j')}(f)-f^{(j')}\|_C+\|B_m(f)-f\|_C\sum_{i=1}^{k}\|L_i^{(j')}\|_C$$

$$+\sum_{i=1}^{k}\sum_{t=1}^{s}|C_i^{[t]}|M_t'\max_{0\leq j\leq t}\|B_m^{(j)}(f)-f^{(j)}\|_C\|L_i^{(j')}\|_C$$

$$\leqq(1+\sum_{i=1}^{k}\|L_i^{(j')}\|_C+\sum_{i=1}^{k}\sum_{t=1}^{s}|C_i^{[t]}|M_t'\|L_i^{(j')}\|_C)\|B_m(f)-f\|_X$$
$$=M_s\|B_m(f)-f\|_X\,.$$

Consequently, we have the inequality (2.6).

If $f(x_i)=0$ for some $i=1, \cdots, k$, we take a constant c such that $f(x_i)+c>0$ for all $i=1, \cdots, k$. Let $F(x)=f(x)+c$, then $F(x_i)\neq 0$ for all $i=1, \cdots, k$. Further we see that

$$L_m^{[S]}(F)-F=L_m^{[S]}(f)-f \quad \text{and} \quad B_m(F)-F=B_m(f)-f\,,$$

since $L_m^{[S]}$ and B_m are linear with respect to f. Thus, for all $f\in X$ the theorem is true. (q. e. d.)

From $\|B_m(f)-f\|_X\to 0$ as $m\to\infty$, our approximation is appropriate.

We can prove the approximability to the trigonometric case as an analogy of the algebraic case. We assume that the incidence matrix E satisfies a special condition;

$$E=(e_{ij})_{j=0,\cdots,s}^{i=0,\cdots,2k},\ e_{ij}=1 \qquad \text{for all } (i,\ j)\,.$$

It is trivial that the approximability to the general case follows immediately from our theorem. Let

$$J_n(f)(x)=\int_{-\pi}^{\pi}f(x+t)K_n(t)dt$$

be Jackson operator of degree n.

LEMMA 2.1. *For $f\in X$ and $j=0, \cdots, s$, we have*

$$J_n^{(j)}(f)(x)=\int_{-\pi}^{\pi}f^{(j)}(x+t)K_n(t)dt=J_n(f^{(j)})(x)\,.$$

Thus, for each $j=0, \cdots, s$

$$\|J_n^{(j)}(f)-f^{(j)}\|_C\leqq \mathrm{const}\cdot w(f^{(j)},\ 1/n)\,,$$

where $w(f^{(j)}, \cdot)$ is the modulus of continuity of $f^{(j)}$.

LEMMA 2.2. (i) *Let $s_r(x)=\sin^r(x/2)$, $r=1, 2, \cdots$, then we have*

$$s_r^{(j)}(0)\begin{cases} =0 & \text{for } j=0, \cdots, r-1\,, \\ \neq 0 & \text{for } j=r\,. \end{cases}$$

(ii) *Let $S_r(x)=\sin^r x$, $r=1, 2, \cdots$, then we have*

$$S_r^{(j)}(0)\begin{cases} =0 & \text{for } j=0, \cdots, r-1\,, \\ \neq 0 & \text{for } j=r\,. \end{cases}$$

PROOF. We prove only (i), and the proof of (ii) is the same as one of (i).

We use induction in r. If $r=1$, the lemma is trivial. We assume that the lemma is true for $r\leqq k$. If $j<k$

$$s_{k+1}^{(j)}(0)=\sum_{i=0}^{j}\binom{j}{i}s_{k}^{(i)}(0)s_{1}^{(j-i)}(0)=0\,.$$

If $j=k$

$$s_{k+1}^{(k)}(0)=\sum_{i=0}^{k-1}\binom{k}{i}s_{k}^{(i)}(0)s_{1}^{(k-i)}(0)+s_{k}^{(k)}(0)s_{1}(0)=0\,.$$

If $j=k+1$, we have

$$\begin{aligned} s_{k+1}^{(k+1)}(0)&=\sum_{i=0,\, i\neq k}^{k+1}\binom{k+1}{i}s_{k}^{(i)}(0)s_{1}^{(k+1-i)}(0)+s_{k}^{(k)}(0)s_{1}^{(1)}(0)\\ &=(1/2)s_{k}^{(k)}(0)\\ &\neq 0\,. \end{aligned}$$ (q. e. d.)

LEMMA 2.3. *$L_i(x)=\prod_{p\neq i}s_1(x-x_p)/s_1(x_i-x_p)$, $i=0,\cdots,2k$, are the trigonometric polynomials of degree k, and satisfy*

$$L_i(x_p)=\delta_{ip}\,,\qquad i=0,\cdots,2k,\ p=0,\cdots,2k\,,$$

where

$$\delta_{ip}=\begin{cases}1\,, & i=p\,,\\ 0\,, & i\neq p\,.\end{cases}$$

Let

$$s_{pr}(x)=s_r(x-x_p),\quad S_{pr}(x)=S_r(x-x_p)\,;\quad p=0,\cdots,2k,\ r=1,\cdots,s+1\,.$$

From the above lemmas we have the following main lemma.

LEMMA 2.4. *Let*

$$\lambda_{it}(x)=S_{it}(x)\prod_{p\neq i}s_{ps+1}(x)\,,$$

$$C_i^{[t]}=[\lambda_{it}^{(t)}(x_i)]^{-1}=[S_{it}^{(t)}(x_i)\prod_{p\neq i}s_{ps+1}(x_i)]^{-1}\,,$$

$$\varepsilon_{mi}^{[t]}=f^{(t)}(x_i)-J_m^{(t)}(f)(x_i)+\sum_{j=0}^{2k}[J_m(f)(x_j)-f(x_j)]L_j^{(t)}(x_i)\,,$$

$$\delta_{mi}^{[t-1]}=\begin{cases}0\,, & t=1\,,\\ \displaystyle\sum_{\eta=1}^{t-1}C_i^{[\eta]}[\varepsilon_{mi}^{[\eta]}-\delta_{mi}^{[\eta-1]}]\lambda_{i\eta}^{(t)}(x_i)\,, & t=2,\cdots,s\,,\end{cases}$$

for $i=0,\cdots,2k$, $t=1,\cdots,s$, $m=0,1,\cdots$.

Then we have

(i) $\lambda_{it}^{(j')}(x_{i'})=0$ if $0\leqq j'<t\leqq s$ or $i\neq i'$,

(ii) $\lambda_{it}^{(t)}(x_i)=0$, thus $|C_i^{[t]}|<\infty$,

(iii) both $\varepsilon_{mi}^{[t]}$ and $\delta_{mi}^{[t-1]}$ are linear with respect to f, and

(iv) for each $t=1, \cdots, s$, there is a constant $M'_t>0$ such that

$$|\varepsilon_{mi'}^{[t]}-\delta_{mi'}^{[t-1]}|\leqq M'_t \max_{0\leqq j\leqq t} \|J_m^{(j)}(f)-f^{(j)}\|_C, \qquad i=0, \cdots, 2k.$$

The following result is obtained as an analogy of Theorem 2.1. We omit its proof.

THEOREM 2.2. *Let $f\in X$. For each $m=0, 1, \cdots$, there exists a bounded linear polynomial operator $L_m^{[S]}$ on X such that*

$$L_m^{[S]}(f)\in \prod_{\max[m,(k+1)s]+k}(S;f)$$

and

$$\exists M_s>0;\ \|L_m^{[S]}(f)-f\|_X\leqq M_s\|J_m(f)-f\|_X.$$

From Lemma 2.1 and Theorem 2.2, we see that our approximation is possible.

§ 3. Applications.

The space $C_1[0, 1]$, consisting of all continuous real valued functions on $[0, 1]$, is a normed space if $f\in C_1[0, 1]$ has this norm

$$\|f\|_1=\int_0^1|f(x)|\,dx.$$

Let the space Π_{1n} be a subspace of $C_1[0, 1]$ that consists of all polynomials of degree n or less. Carroll and Mclaughlin [**3**] have investigated the polynomials $q\in\Pi_{1n}$ such that

$$\|f_1-q\|_1+\|f_2-q\|_1=\inf_{p\in\Pi_{1n}}[\|f_1-p\|_1+\|f_2-p\|_1]. \tag{3.1}$$

They have solved the questions of existence and characterization of q satisfying (3.1). Furthermore they have given the following result.

THEOREM 3.1. (Carroll and Mclaughlin [**3**]) *For $f_1, f_2\in C_1[0, 1]$, let $q\in\Pi_{1n}$ satisfy* (3.1). *If one of them satisfies that*

$$\exists\bar{x}\in[0, 1];\ [f_1(\bar{x})-q(\bar{x})][f_2(\bar{x})-q(\bar{x})]>0, \tag{3.2}$$

then q is unique.

In this section we give some concrete methods to determine whether q satisfying (3.1) is unique when $f_1, f_2\in C_1[0, 1]$ are given. For $f_1, f_2\in C_1[0, 1]$, define

$$Z_{f_1-f_2}=\{x\,;\, f_1(x)-f_2(x)=0,\ x\in[0,1]\},$$

$$D_{f_1f_2}=\{(x,y)\,;\, f_1(x)<y<f_2(x) \text{ or } f_1(x)>y>f_2(x),\ x\in[0,1]\},$$

$$F_{f_1-f_2}=\{(x,y)\,;\, y=f_1(x)=f_2(x),\ x\in[0,1]\}.$$

We need the following lemma.

LEMMA 3.1. *Let* $f_1, f_2\in C_1[0, 1]$. *If there are two polynomials satisfying* (3.1), *then* $F_{f_1-f_2}$ *consists of at most* n *points.*

PROOF. Let q and $\bar{q}$ be two distinct best approximants satisfying (3.1). Let $Z_{q-\bar{q}}=\{x_1, \cdots, x_m\}$, then $m\leqq n$, but let $Z_{q-\bar{q}}$ be empty if $m=0$. If some $q_0=aq+b\bar{q}$; $a, b\geqq 0$, $a+b=1$, satisfies (3.2), then we obtain a contradiction to Theorem 3.1 because q_0 is also a best approximant to (f_1, f_2). Thus, on $[0, 1]$

$$[f_1-q_0][f_2-q_0]\leqq 0 \qquad \text{for each } q_0 .$$

That is, two graphs of $y=f_1(x)$ and $y=f_2(x)$ are outside $D_{q\bar{q}}$, and they are opposite with respect to $D_{q\bar{q}}$ with each other. Thus, $F_{f_1-f_2}$ consists of at most m points, and from $m\leqq n$ we complete the proof. (q. e. d.)

The following is a corollary to Theorem 3.1. Let $\bar{D}$ be the closure of D.

COROLLARY 3.1. *Let* $f_1, f_2\in C_1[0, 1]$ *be given. The best approximant to* (f_1, f_2) *is unique in* Π_{1n} *if and only if the following* (i) *or* (ii) *is realized.*

(i) $F_{f_1-f_2}$ *consists of at least* $n+1$ *points.*

(ii) $\bar{D}_{f_1f_2}$ *doesn't contain two distinct polynomials in* Π_{1n}.

PROOF. Necessity: Let (i) be not true. If $\bar{D}_{f_1f_2}$ contains two distinct polynomials in Π_{1n}, each of them gives the degree of best approximation

$$\int_0^1|f_1-f_2|\,dx=\int_0^1|f_1-q|\,dx+\int_0^1|f_2-q|\,dx .$$

This contradicts with our assumption.

Sufficiency: Let (i) be realized. Then from Lemma 3.1 the best approximant to (f_1, f_2) is unique. We assume that $F_{f_1-f_2}$ consists of at most n points, and (ii) is realized. Then except at most one polynomial, any other polynomial satisfies (3.2). Thus, from Theorem 3.1 the best approximant to (f_1, f_2) is unique. (q. e. d.)

In order to classify the points in $F_{f_1-f_2}$, define the following closed sets $\bar{U}_\varepsilon^1(x_0, y_0)$ and $\bar{U}_\varepsilon^2(x_0, y_0)$. Let $\bar{C}_\varepsilon(x_0, y_0)$ be the closed disc with center at (x_0, y_0) and radius ε. Define two lines

$$L^+(x)=a(x-x_0)+y_0,\quad L^-(x)=b(x-x_0)+y_0, \qquad \text{where } a>b .$$

Then we define

$$\bar{U}^1_\varepsilon(x_0, y_0)=\{(x, y)\,;\ y\leqq L^+(x),\ y\leqq L^-(x)\}\cap\bar{C}_\varepsilon(x_0, y_0)\,,$$
(3.3)
$$\bar{U}^2_\varepsilon(x_0, y_0)=\{(x, y)\,;\ (y-L^+(x))(y-L^-(x))\leqq 0\}\cap\bar{C}_\varepsilon(x_0, y_0)\,.$$

Now we classify the points of $F_{f_1-f_2}$. A point $(x_0, y_0)\in F_{f_1-f_2}$ is called of the first class if there are $\varepsilon>0$ and a, b such that

$$\bar{D}_{f_1f_2}\cap\bar{C}_\varepsilon(x_0, y_0)\subset\bar{U}^1_\varepsilon(x_0, y_0)\,,$$

and is called of the second class if there are $\varepsilon>0$ and a, b such that

$$\bar{U}^2_\varepsilon(x_0, y_0)\subset\bar{D}_{f_1f_2}\,.$$

Then our criteria is simple.

THEOREM 3.2. *Let $f_1, f_2\in C_1[0, 1]$ be given.*

(i) *If $F_{f_1-f_2}$ consists of at least $k+1$ points or contains at least one first class point, the best approximant to (f_1, f_2) is unique in Π_{1k}.*

(ii) *If $F_{f_1-f_2}$ is empty or consists at most k points of the second class, the best approximant to (f_1, f_2) is not unique in Π_{1n} for n sufficiently large.*

PROOF. (i) follows immediately from Theorem 3.1 and Corollary 3.1 (i). We prove (ii). When $F_{f_1-f_2}$ is empty, let $f=(f_1+f_2)/2$ then if we approximate uniformly f by Π_n, for n sufficiently large, there are two distinct polynomials $p, q\in\Pi_n$ such that

$$f_1\lneqq p\,,\quad q\lneqq f_2\,.$$

Here both p and q are the best approximants to (f_1, f_2).

Now let $F_{f_1-f_2}=\{(x_i, y_i)\,;\ i=1, \cdots, k\}$, $x_1<\cdots<x_k$, and let (x_i, y_i), $i=1, \cdots, k$, be of the second class. By the definitions, for each $i=1, \cdots, k$ we get a closed set $\bar{U}^2_\varepsilon(x_i, y_i)\subset\bar{D}_{f_1f_2}$ such that it is obtained by two line segments

$$y=L_i^+(x)\,,\qquad y=L_i^-(x)\,;\qquad x_i-\delta\leqq x\leqq x_i+\delta,\ \ \delta>0\,.$$

In this case we can find a curve $y=f_0(x)$ in $C^1[0, 1]$ such that its graph is contained in $\bar{D}_{f_1f_2}$, and equals to the line segment $y_i=L_i^+(x)+L_i^-(x)$ in each interval $[x_i-\delta, x_i+\delta]$, $i=1, \cdots, k$. Let the incidence matrix E be $E=(e_{ij})_{j=0,1}^{i=1,\cdots,k}$ and $e_{ij}=1$ for all (i, j), then we consider the scheme $S=(E\,;\ \{x_i\})$. If we approximate f_0 by $\Pi_n(S\,;f_0)$, where of course the norm is $\|\cdot\|_X$, then for n sufficiently large we have

$$f_1(x)\lneqq L_n^{[S]}(f_0)(x)\lneqq f_2(x)\,,\qquad 0\leqq x\leqq 1\,. \tag{3.4}$$

In fact, if (3.4) is not true for some x_i, $i=1, \cdots, k$, there is a sequence $\{x^{[n]}\}$ such that

$$x^{[n]}\longrightarrow x_i\,,\qquad \{L_n^{[S]}(f_0)(x^{[n]})-y_i\}/(x^{[n]}-x_i)\longrightarrow y_i'|_{x=x_i}=f_0'(x_i) \tag{3.5}$$

as $n\to\infty$. Since

$$\| L_n^{[S]}(f_0)-f_0\|_X \longrightarrow 0 \qquad \text{as } n\to\infty ,$$

we have

$$\| L_n^{[S]'}(f_0)-f_0'\|_C \longrightarrow 0 \qquad \text{as } n\to\infty .$$

Thus,

$$\forall\varepsilon_0>0,\ \exists N;\ n\geqq N \Rightarrow |L_n^{[S]'}(f_0)(x)-f_0'(x)|<\varepsilon_0/2 \qquad \text{for all } x\in[0, 1],$$

$$\exists\delta_0;\ |x-x_i|<\delta_0 \Rightarrow |f_0'(x)-f_0'(x_i)|<\varepsilon_0/2 .$$

Thus, if $n\geqq N$ and $|x-x_i|<\delta_0$, then

$$\begin{aligned}|L_n^{[S]'}(f_0)(x)-f_0'(x_i)| &\leqq |L_n^{[S]'}(f_0)(x)-f_0'(x)|+|f_0'(x)-f_0'(x_i)| \\ &<\varepsilon_0 .\end{aligned}$$

Since there is an $\eta^{[n]}$ such that

$$\{L_n^{[S]}(f_0)(x^{[n]})-y_i\}/(x^{[n]}-x_i)=L_n^{[S]'}(\eta^{[n]}), \quad x^{[n]}\lesseqgtr\eta^{[n]}\lesseqgtr x_i, \quad n=1, 2, \cdots ,$$

we have

$$|\{L_n^{[S]}(f_0)(x^{[n]})-y_i\}/(x^{[n]}-x_i)-f_0'(x_i)|<\varepsilon_0 ,$$

if $n\geqq N$ and $|\eta^{[n]}-x_i|<\delta_0$. Since ε_0 is arbitrary it contradicts with (3.5).

Thus, we have (3.4) and for n sufficiently large we conclude the non-uniqueness of the best approximant to (f_1, f_2). (q. e. d.)

We investigate further the points of second class minutely.

THEOREM 3.3. *Let f_1 and f_2 be two continuous functions. Suppose that there exist k nonnegative integers p_i, $i=1, \cdots, k$, and k distinct points x_i, $i=1, \cdots, k$, in A such that*

(a) *$f_1^{(j)}(x_i)=f_2^{(j)}(x_i)$ for $j=0, \cdots, p_i$, $i=1, \cdots, k$,*

(b) *for each $i=1, \cdots, k$, $(x_i, y_i)\in F_{f_1^{(p_i)}-f_2^{(p_i)}}$ means*

$$\bar{U}_\varepsilon^2(x_i, y_i)\subset \bar{D}_{f_1^{(p_i)}f_2^{(p_i)}} \qquad \textit{for some } \varepsilon>0 \textit{ and } a, b,$$

(c) *$f_1(x)\neq f_2(x)$ if $x\in A$ and $x\neq x_i$, $i=1, \cdots, k$.*

Then for n sufficiently large there exists a polynomial P_n satisfying

$$P_n^{(j)}(x_i)=f_1^{(j)}(x_i)=f_2^{(j)}(x_i) \qquad \textit{for } j=0, \cdots, p_i,\ i=1, \cdots, k, \tag{3.6}$$

$$f_1^{(p_i)}(x)\gtrless P_n^{(p_i)}(x)\gtrless f_2^{(p_i)}(x) \qquad \textit{if } x\in A \textit{ and } x\neq x_i,\ i=1, \cdots, k, \tag{3.7}$$

$$f_1(x)\gtrless P_n(x)\gtrless f_2(x) \qquad \textit{if } x\in A \textit{ and } x\neq x_i,\ i=1, \cdots, k. \tag{3.8}$$

PROOF. Let $E=(e_{ij})_{j=0,\cdots,s}^{i=1,\cdots,k}$, where $s=\max\{p_i;\ i=1, \cdots, k\}$, be the incidence matrix such that

$$e_{ij}=\begin{cases} 1, & j=0, \cdots, p_i \\ 0, & j=p_i+1, \cdots, s \end{cases} \qquad i=1, \cdots, k .$$

Define the scheme $S=(E;\{x_i\})$, and let $g=(f_1+f_2)/2$. We approximate g by $\Pi_n(S;g)$. By its definition $P_n\in\Pi_n(S;g)$ satisfies (3.6). From the assumption (b) and Theorem 3.2, for n sufficiently large we have (3.7). By induction, it is easy to prove (3.8) using (3.6) and (3.7). (q. e. d.)

§4. Degree of approximation by the operator $L_m^{[S]}$.

Let $0<\alpha\leqq 1$. In this section we estimate $\|L_m^{[S]}(f)-f\|_X$ for $f\in \mathrm{Lip}_M(\alpha, A)$. We use Theorem 2.1 and Theorem 2.2. The following lemma is well known.

LEMMA 4.1. *Let $0<\alpha\leqq 1$, $X=x(1-x)$ and let M be a constant.*

(i) *If $f\in\mathrm{Lip}_M(\alpha, C[0,1])$, we have*

$$|B_n(f,x)-f(x)|\leqq M(X/n)^{\alpha/2} \quad \text{for } x\in[0,1].$$

(ii) *If $f'\in\mathrm{Lip}_M(\alpha, C[0,1])$, we have*

$$|B_n(f,x)-f(x)|\leqq M(X/n)^{(1+\alpha)/2} \quad \text{for } x\in[0,1].$$

LEMMA 4.2. *Let $0<\alpha\leqq 1$ and s be a nonnegative integer.*

(i) *If $f^{(s)}\in\mathrm{Lip}_M(\alpha, C[0,1])$, there is a constant $M(f,\alpha,s)$ depending on f, α and s such that*

$$\|B_n(f)-f\|_X\leqq M(f,\alpha,s)n^{(-\alpha)/2} \quad \text{for } n\geqq s+1.$$

(ii) *If $f^{(s+1)}\in\mathrm{Lip}_M(\alpha, C[0,1])$, there is a constant $M(f,\alpha,s)$ depending on f, α and s such that*

$$\|B_n(f)-f\|_X\leqq M(f,\alpha,s)n^{(-1-\alpha)/2} \quad \text{for } n\geqq s+1.$$

PROOF. If $s=0$, Lemma 4.2 follows from Lemma 4.1. Let $s\geqq 1$.

(i) Let $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$ and

$$|f^{(s)}(x)-f^{(s)}(y)|\leqq M|x-y|^\alpha \quad \text{for } x, y\in[0,1].$$

We know ([7])

$$B_n^{(s)}(f,x)=1\{1-1/n\}\cdots\{1-(s-1)/n\}\sum_{k=0}^{n-s}f^{(s)}(k/n+\eta_k s/n)p_{n-s,k}(x),$$
$$0<\eta_k<1.$$

Thus, we have

$$(4.1) \qquad |B_n^{(s)}(f,x)-f^{(s)}(x)|\leqq\left|\sum_{k=0}^{n-s}\{f^{(s)}(k/(n-s))-f^{(s)}(x)\}p_{n-s,k}(x)\right|$$
$$+[1-1\{1-1/n\}\cdots\{1-(s-1)/n\}]\sum_{k=0}^{n-s}|f^{(s)}(k/(n-s))|p_{n-s,k}(x)$$

$$+\sum_{k=0}^{n-s}|f^{(s)}(k/n+\eta_k s/n)-f^{(s)}(k/(n-s))|p_{n-s,k}(x)$$

$$=\Sigma_1+\Sigma_2+\Sigma_3\,.$$

From Lemma 4.1 there is a constant $A(\alpha, s)$ depending on α and s such that

$$\Sigma_1\leqq M(n-s)^{(-\alpha)/2}\leqq A(\alpha, s)n^{(-\alpha)/2} \qquad \text{for } n\geqq s+1\,.$$

It is trivial that there is a constant $A'(f, s)$, which depending on f and s, such that

$$\Sigma_2\leqq A'(f, s)n^{-1} \qquad \text{for } n\geqq s+1\,.$$

We have

$$\Sigma_3\leqq M\sum_{k=0}^{n-s}|s/n+\eta_k s/n-k/(n-s)|^{\alpha}p_{n-s,k}(x)$$

$$\leqq M\sum_{k=0}^{n-s}|sk/n(n-s)+s/n|^{\alpha}p_{n-s,k}(x)$$

$$\leqq M(2s)^{\alpha}n^{-\alpha}=A''(\alpha, s)n^{-\alpha} \qquad \text{for } n\geqq s+1\,.$$

Thus, (4.1) means

$$|B_n^{(s)}(f, x)-f^{(s)}(x)|\leqq M'(f, \alpha, s)n^{(-\alpha)/2} \qquad \text{for } n\geqq s+1\,, \tag{4.2}$$

where $M'(f, \alpha, s)$ depends on f, α and s. When $0\leqq j<s$ we have

$$|f^{(j)}(x)-f^{(j)}(y)|=\left|\int_y^x f^{(j+1)}(t)dt\right|\leqq\|f^{(j+1)}\|_C|x-y|.$$

Thus, from (4.2)

$$|B_n^{(j)}(f, x)-f^{(j)}(x)|\leqq M'(f, j)n^{(-1)/2} \qquad \text{for } n\geqq j+1\,, \tag{4.3}$$

where $M'(f, j)$ depends on f and j. By (4.2) and (4.3) there is a constant $M(f, \alpha, s)$ depending on f, α and s such that

$$\|B_n(f)-f\|_X\leqq M(f, \alpha, s)n^{(-\alpha)/2} \qquad \text{for } n\geqq s+1\,.$$

(ii) We use

$$f^{(s)}(x)-f^{(s)}(y)=(x-y)\{f^{(s+1)}(\lambda)-f^{(s+1)}(y)\}+(x-y)f^{(s+1)}(y)\,.$$

$$x\gtrless\lambda\gtrless y\,.$$

We have

$$|B_n^{(s)}(f, x)-f^{(s)}(x)|\leqq\left|\sum_{k=0}^{n-s}\{f^{(s)}(k/(n-s))-f^{(s)}(x)\}p_{n-s,k}(x)\right|$$

$$+[1-1\{1-1/n\}\cdots\{1-(s-1)/n\}]\sum_{k=0}^{n-s}|f^{(s)}(k/(n-s))|p_{n-s,k}(x)$$

$$(4.4)\qquad +\left|\sum_{k=0}^{n-s}\{k/n+\eta_k s/n-k/(n-s)\}\{f^{(s+1)}(\lambda_k)-f^{(s+1)}(k/(n-s))\}p_{n-s,k}(x)\right|$$

$$+\left|\sum_{k=0}^{n-s}\{k/n+\eta_k s/n-k/(n-s)\}f^{(s+1)}(k/(n-s))p_{n-s,k}(x)\right|,$$

$$k/n+\eta_k s/n\gtrless\lambda_k\gtrless k/(n-s)$$

$$=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4.$$

From Lemma 4.1

$$\Sigma_1\leqq M(n-s)^{(-1-\alpha)/2}\leqq A(\alpha, s)n^{(-1-\alpha)/2}\qquad \text{for } n\geqq s+1,$$

where $A(\alpha, s)$ depends on α and s. It is easy to show

$$\Sigma_2\leqq A'(f, s)n^{-1}\qquad \text{for } n\geqq s+1,$$

where $A'(f, s)$ is a constant depending on f and s. We have

$$\Sigma_3\leqq M\sum_{k=0}^{n-s}|k/n+\eta_k s/n-k/(n-s)|^{1+\alpha}p_{n-s,k}(x)$$

$$\leqq M\sum_{k=0}^{n-s}|sk/n(n-s)+s/n|^{1+\alpha}p_{n-s,k}(x)$$

$$\leqq M(2s)^{1+\alpha}n^{-(1+\alpha)}$$

$$=A''(\alpha, s)n^{-(1+\alpha)}\qquad \text{for } n\geqq s+1,$$

and

$$\Sigma_4=\|f^{(s+1)}\|_C\sum_{k=0}^{n-s}|sk/n(n-s)+s/n|p_{n-s,k}(x)$$

$$\leqq\|f^{(s+1)}\|_C(2s)n^{-1}\qquad \text{for } n\geqq s+1.$$

Thus, (4.4) means

$$(4.5)\qquad |B_n^{(s)}(f, x)-f^{(s)}(x)|\leqq M'(f, \alpha, s)n^{(-1-\alpha)/2}\qquad \text{for } n\geqq s+1,$$

where $M'(f, \alpha, s)$ is a constant depending on f, α and s. When $0\leqq j<s$, we have

$$|f^{(j+1)}(x)-f^{(j+1)}(y)|=\left|\int_y^x f^{(j+2)}(t)dt\right|\leqq\|f^{(j+2)}\|_C|x-y|.$$

By (4.5)

$$(4.6)\qquad |B_n^{(j)}(f, x)-f^{(j)}(x)|\leqq M''(f, j)n^{-1}\qquad \text{for } n\geqq j+1,$$

where $M''(f, j)$ depends on f and j. Consequently, by (4.5) and (4.6) there is a constant $M(f, \alpha, s)$ depending on f, α and s such that

$$\|B_n(f)-f\|_X\leqq M(f, \alpha, s)n^{(-1-\alpha)/2}\qquad \text{for } n\geqq s+1.\qquad \text{(q. e. d.)}$$

Next, we consider the trigonometric case.

LEMMA 4.3. *Let* $0<\alpha\leqq 1$ *and* $f^{(s)}\in \mathrm{Lip}_M(\alpha, K)$. *For some constant* $M(f, \alpha, s)$ *depending on* f, α *and* s, *we have*

$$\|f-J_n(f)\|_X\leqq M(f, \alpha, s)n^{-\alpha}.$$

PROOF. For each $j=0, \cdots, s$, we have

$$\|J_n^{(j)}(f)-f^{(j)}\|_C\leqq M(j)w(f^{(j)}, 1/n),$$

where $M(j)$ is a constant depending only on j, and $w(f^{(j)}, \cdot)$ is the modulus of continuity of $f^{(j)}$. Since we have

$$w(f^{(j)}, 1/n)\leqq\|f^{(j+1)}\|_C n^{-1} \quad \text{if } 0\leqq j<s,$$

and

$$w(f^{(s)}, 1/n)\leqq Mn^{-\alpha},$$

there is a constant $M'(f, \alpha, s)$ depending on f, α and s such that

$$\|J_n(f)-f\|_X\leqq M(f, \alpha, s)n^{-\alpha}. \qquad \text{(q. e. d.)}$$

From Theorem 2.1 and Lemma 4.2, and Theorem 2.2 and Lemma 4.3 we have the following theorems:

THEOREM 4.1. *Let* $0<\alpha\leqq 1$ *and* s *be a nonnegative integer. Let* $M(f, \alpha, s, \{x_i\})$ *be a constant depending on* f, α, s, $\{x_i\}$.

(i) *If* $f^{(s)}\in \mathrm{Lip}_M(\alpha, C[0, 1])$, *there is a constant* $M(f, \alpha, s, \{x_i\})$ *such that*

$$\|L_n^{[S]}(f)-f\|_X\leqq M(f, \alpha, s, \{x_i\})n^{(-\alpha)/2} \quad \text{for } n\geqq s+1.$$

(ii) *If* $f^{(s+1)}\in \mathrm{Lip}_M(\alpha, C[0, 1])$, *there is a constant* $M(f, \alpha, s, \{x_i\})$ *such that*

$$\|L_n^{[S]}(f)-f\|_X\leqq M(f, \alpha, s, \{x_i\})n^{(-1-\alpha)/2} \quad \text{for } n\geqq s+1.$$

THEOREM 4.2. *Let* $0<\alpha\leqq 1$. *If* $f^{(s)}\in \mathrm{Lip}_M(\alpha, K)$, *there is a constant* $M(f, \alpha, s, \{x_i\})$ *depending on* f, α, s *and* $\{x_i\}$ *such that*

$$\|L_n^{[S]}(f)-f\|_X\leqq M(f, \alpha, s, \{x_i\})n^{-\alpha} \quad \text{for } n\geqq ks+k+s.$$

§ 5. Estimation of the degree $E_n^S(f)$.

Wayne T. Ford and John A. Roulier get the following theorem with respect to "monotone approximation".

THEOREM 5.1. ([4, Theorem 2]) *Let* $k_1<k_2<\cdots<k_p$ *be fixed positive integers and let* $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p$ *be fixed signs (i. e.,* $\varepsilon_i=\pm 1$). *Suppose* $f\in C^k[a, b]$ *and* $k_p\leqq k$. *Assume*

$$\varepsilon_i f^{(k_i)}(x)>0 \quad \text{for } a\leqq x\leqq b \text{ and } i=1, \cdots, p.$$

Suppose $m+1$ points are given so that

$$a\leqq x_0<x_1<\cdots<x_m\leqq b\,.$$

Then for n sufficiently large there are polynomials P_n of degree less than or equal to n for which

$$\varepsilon_j P_n^{(k_j)}(x)>0 \qquad \text{on } [a, b],\ j=1, \cdots, p,$$

$$P_n(x_i)=f(x_i), \qquad i=0, \cdots, m, \tag{5.1}$$

$$\|f-P_n\|_{C[a,b]}\leqq Cn^{-k}w(f^{(k)}, 1/n),$$

where C is a constant depending only on $x_0, \cdots, x_m$, and w is the modulus of continuity of $f^{(k)}$ on $[a, b]$.

If we define

$$E=(e_{ij})_{j=0}^{i=1,\cdots,m}, \quad e_{ij}=1 \qquad \text{for all } (i, j),$$

then the polynomial P_n in (5.1) belongs to $\Pi_n(S;f)$, where $S=(E;\{x_i\})$. We consider this problem to more general incidence matrices. We use the methods in [4]. Through this section we assume that the incidence matrix $E=(e_{ij})_{j=0,\cdots,s}^{i=1,\cdots,k}$ is poised. Then we define a scheme $S=(E;\{x_i\})$, where $x_i\in A$, $i=1, \cdots, k$.

THEOREM 5.2. *Let $f\in C^m[a, b]$, where $m\geqq s$. For each $n\geqq\bar{e}$, there is an algebraic polynomial P_n in $\Pi_n(S;f)$ such that*

$$\|f-P_n\|_X\leqq M(m, I, S)n^{s-m}w(f^{(m)}, 1/n), \qquad I=[a, b], \tag{5.2}$$

where $M(m, I, S)$ is a constant depending on m, I and the scheme S.

We may assume that the interval $[a, b]$ satisfies $-1<a<b<1$ and $b=-a$. We need the following lemma.

LEMMA 5.1. (John A. Roulier [4]) *Let $f\in C^m[a, b]$, and let $w(f^{(m)}, \cdot)$ be the modulus of continuity of $f^{(m)}$ on $[a, b]$. f may be extended to a function $F\in C^m[-1, 1]$ in such a way that the modulus of continuity $w(F^{(m)}, \cdot)$ satisfies*

$$w(F^{(m)}, h)\leqq w(f^{(m)}, h) \qquad \text{for } h\leqq b-a\,.$$

LEMMA 5.2. *Let $F\in C^m[-1, 1]$. If for a sequence of polynomials $\{P_n\}$, where $P_n\in\Pi_n$, the condition*

$$\|F-P_n\|_{C[-1,1]}\leqq A(m)n^{-m}w(F^{(m)}, 1/n), \qquad n=1, 2, \cdots, \tag{5.3}$$

where $A(m)$ is a constant depending on m, is satisfied, then there is a constant $A'(m, b)$ depending on m and b such that

$$\|F^{(j)}-P_n^{(j)}\|_{C[a,b]}\leqq A'(m, b)n^{j-m}w(F^{(m)}, 1/n), \qquad n=1, 2, \cdots. \tag{5.4}$$

PROOF. Let $0<b<b_1<1$, $a_1=-b_1$. By Malzemov [7], we have a sequence of polynomials $\{Q_n\}$, where $Q_n \in \Pi_n$, such that

$$\|F^{(j)}-Q_n^{(j)}\|_{C[-1,1]} \leqq M(m)\varepsilon_n(x)^{m-j} w(F^{(m)}, \varepsilon_n(x)) \qquad \text{for } j=0, \cdots, m,$$

where $M(m)$ is a constant depending on m, and $\varepsilon_n(x)=(1/n)\{(1-x^2)^{1/2}+1/n\}$. Thus,

$$\|F-Q_n\|_{C[-1,1]} \leqq 2^{m+1} M(m) n^{-m} w(F^{(m)}, 1/n),$$

$$\|F^{(1)}-Q_n^{(1)}\|_{C[-1,1]} \leqq 2^m M(m) n^{1-m} w(F^{(m)}, 1/n).$$

By the assumption

$$\|F-P_n\|_{C[-1,1]} \leqq A(m) n^{-m} w(F^{(m)}, 1/n).$$

Then there is a constant $A_1(m)$ depending on m such that

$$\|P_n-Q_n\|_{C[-1,1]} \leqq A_1(m) n^{-m} w(F^{(m)}, 1/n).$$

From Bernstein's inequality [**9**, p. 39, Theorem 3] we have

$$|P_n^{(1)}(x)-Q_n^{(1)}(x)| \leqq A_1(m)(1-x^2)^{-1/2} n^{1-m} w(F^{(m)}, 1/n), \quad -1<x<1.$$

Then there is a constant $A_2(m, b_1)$ depending on m and b_1, such that

$$\|P_n^{(1)}-Q_n^{(1)}\|_{C[a_1, b_1]} \leqq A_2(m, b_1) n^{1-m} w(F^{(m)}, 1/n),$$

where $0<b_1<1$ and $a_1=-b_1$. Thus, we have

$$\|F^{(1)}-P_n^{(1)}\|_{C[a_1, b_1]} \leqq A_3(m, b_1) n^{1-m} w(F^{(m)}, 1/n), \tag{5.5}$$

where $A_3(m, b_1)$ is a constant depending on m and b_1.

Let $0<b<b_2<1$ and $a_2=-b_2$. Define

$$G(x)=F^{(1)}(b_1 x)=F^{(1)}(y) \quad \text{and} \quad R_n(x)=P_n^{(1)}(b_1 x)=P_n^{(1)}(y),$$
$$-1 \leqq x \leqq 1, \quad y=b_1 x.$$

Then,

$$\|G-R_n\|_{C[-1,1]}=\|F^{(1)}-P_n^{(1)}\|_{C[a_1, b_1]} \leqq A_3(m, b_1) n^{1-m} w(F^{(m)}, 1/n).$$

By the same way as we got (5.5), we have

$$\|G^{(1)}-R_n^{(1)}\|_{C[a_2, b_2]} \leqq A_4(m, b_1, b_2,) n^{2-m} w(F^{(m)}, 1/n),$$

where $A_4(m, b_1, b_2)$ is a constant depending on m, b_1 and b_2. From $G^{(1)}(x)-R_n^{(1)}(x)=b_1(F^{(2)}(y)-P_n^{(2)}(y))$, we have

$$\|F^{(2)}-P_n^{(2)}\|_{C[b_1 a_2, b_1 b_2]} \leqq A_5(m, b_1, b_2) n^{2-m} w(F^{(m)}, 1/n),$$

where $A_5(m, b_1, b_2)$ is a constant depending on m, b_1 and b_2. If we continue in this manner, we have

$$\|F^{(j)}-P_n^{(j)}\|_{C[b_1\cdots b_{j-1}a_j,\, b_1\cdots b_{j-1}b_j]}$$
$$\leqq A_{j+3}(m, b_1, \cdots, b_j)n^{j-m}w(F^{(m)}, 1/n) \qquad \text{for } j=1, \cdots, m,$$

where $0<b<b_j<1$, $a_j=-b_j$ and $A_{j+3}(m, b_1, \cdots, b_j)$ is a constant depending on $m, b_1, \cdots, b_j$. Thus, if we take $b_1, \cdots, b_m$ such as $b\leqq b_1\cdots b_m<1$ we have (5.4).

(q. e. d.)

LEMMA 5.3. *Let $f\in C^m[a, b]$ and suppose there is a sequence of algebraic polynomials $\{P_n\}$, where $P_n\in\Pi_n$, and also a sequence of positive numbers $\{\varepsilon_n\}$ satisfying*

$$\|f-P_n\|_X\leqq\varepsilon_n .$$

Then there is a sequence of polynomials $\{Q_n\}_{n=\bar{e}}^{\infty}$, for which

$$Q_n\in\Pi_n(S; f) \quad \text{and} \quad \|f-Q_n\|_X\leqq B\varepsilon_n ,$$

where B is a constant.

PROOF. Let $n\geqq\bar{e}$ and define

$$b_{ij}=f^{(j)}(x_i)-P_n^{(j)}(x_i) \qquad \text{for } (i, j)\in e .$$

Let $R_{ij}(x)$ be the polynomial in $\Pi_{\bar{e}}$ such as

$$R_{ij}^{(j')}(x_{i'})=\delta_{(i,j),(i',j')} ,$$

where

$$\delta_{(i,j),(i',j')}=\begin{cases} 1, & (i, j)=(i', j') \\ 0, & (i, j)\neq(i', j') . \end{cases}$$

Since E is poised, there exists such a polynomial R_{ij}. If we define $R(x)=\sum_{(i,j)\in e} b_{ij}R_{ij}(x)$, then we have

$$\|R\|_X\leqq \sum_{(i,j)\in e}|b_{ij}|\,\|R_{ij}\|_X\leqq C(S)\varepsilon_n ,$$

where $C(S)$ is a constant depending on the scheme S. Then define

(5.6) $$Q_n=P_n+R ,$$

and we have

$$\|f-Q_n\|_X\leqq\|f-P_n\|_X+\|R\|_X\leqq(1+C(S))\varepsilon_n .$$

When $(i, j)\in e$, we have

$$Q_n^{(j)}(x_i)=P_n^{(j)}(x_i)+\{f^{(j)}(x_i)-P_n^{(j)}(x_i)\}=f^{(j)}(x_i) .$$

Thus, $Q_n\in\Pi_n(S; f)$. Let $B=1+C(S)$. (q. e. d.)

PROOF OF THEOREM 5.2. Extend f to a function $F\in C^m[-1, 1]$ as in Lemma 5.1. For each n let P_n be the polynomial of best approximation to F on $[-1, 1]$.

By Jackson's theorem there is a constant $A(m)$ depending on m such that

$$\|F-P_n\|_{C[-1,1]}\leqq A(m)n^{-m}w(F^{(m)}, 1/n).$$

From Lemma 5.2, we see that there is a constant $A'(m, b)$ depending on m and b such that

$$\|f^{(j)}-P_n^{(j)}\|_{C[a,b]}\leqq A'(m, b)n^{j-m}w(f^{(m)}, 1/n) \qquad \text{for } j=0, \cdots, m. \tag{5.7}$$

Thus, we have

$$\|f-P_n\|_X\leqq A'(m, b)n^{s-m}w(f^{(m)}, 1/n).$$

By Lemma 5.3, we have a sequence of polynomials $\{Q_n\}_{n=\bar{e}}^{\infty}$ such that

$$Q_n\in \Pi_n(S; f) \quad \text{and} \quad \|f-Q_n\|_X\leqq M(m, b, S)n^{s-m}w(f^{(m)}, 1/n), \tag{5.8}$$

where $M(m, b, S)$ is a constant depending on m, b, S. (q. e. d.)

So far, we have defined the norm $\|f\|_X$ of f with

$$\|f\|_X=\max_{0\leq j\leq s}\|f^{(j)}\|_C, \qquad \text{where } s=\max\{j; (i, j)\in e\}.$$

We see that Theorem 5.2 is correct if we substitute the norm $\|f\|_X$ with the norm $\|f\|_{\bar{e}}$;

$$\|f\|_{\bar{e}}=\max_{0\leq j\leq \bar{e}}\|f^{(j)}\|_{C[a,b]}.$$

The following result is obtained by using Theorem 5.2 to the norm $\|\cdot\|_{\bar{e}}$. In its proof we use (5.6), (5.7), (5.8), but in (5.8) we must substitute s with $\bar{e}$, and $\|\cdot\|_X$ with $\|\cdot\|_{\bar{e}}$.

THEOREM 5.3. *Let $f\in C^m[a, b]$, where $m\geqq\bar{e}$. Let $0<k_1<\cdots<k_p\leqq m$ be the fixed integers and let $\varepsilon_1, \cdots, \varepsilon_p$ be the fixed signs (i. e., $\varepsilon_j=\pm 1$). If f satisfies*

$$\varepsilon_i f^{(k_i)}(x)>0 \qquad \textit{for } a\leqq x\leqq b \textit{ and } i=1, \cdots, p, \tag{5.9}$$

for n sufficiently large we have a polynomial Q_n such that

$$Q_n\in \Pi_n(S; f), \quad \varepsilon_i Q_n^{(k_i)}(x)>0 \qquad \textit{for } a\leqq x\leqq b,\ i=1, \cdots, p, \tag{5.10}$$

and

$$\|f-Q_n\|_{\bar{e}}\leqq M(m, b, S)n^{\bar{e}-m}w(f^{(m)}, 1/n), \tag{5.11}$$

where $M(m, b, S)$ is a constant depending on m, b and the scheme S.

PROOF. If we take Q_n in (5.8), from (5.6) $Q_n=P_n+R_n$. For $j>\bar{e}$ we have $Q_n^{(j)}=P_n^{(j)}$. Thus, we have $f^{(j)}-Q_n^{(j)}=f^{(j)}-P_n^{(j)}$. By (5.7) and (5.8) we have

$$Q_n^{(j)}(x)\longrightarrow f^{(j)}(x), \qquad \text{uniformly in } [a, b], \text{ for all } j=0, \cdots, m.$$

If we take n sufficiently large, from (5.9) we have (5.10). (5.11) follows from (5.8). (q. e. d.)

Next, we consider the trigonometric case. We obtained the same estimation as (5.2) in this case, as well.

THEOREM 5.4. *Let* $f\in C^m[K]$, *where* $m\geqq s$. *For each* $n\geqq\bar{e}$ *there is a trigonometric polynomial* $T_n\in\Pi_n(S;f)$ *such that*

$$\|f-T_n\|_X\leqq M(m, S)n^{s-m}w(f^{(m)}, 1/n), \tag{5.12}$$

where $M(m, S)$ *is a constant depending on* m *and the scheme* S.

PROOF. We need a generalization of Jackson's operator ([**9**, p. 57, (7)]);

$$I_n(x)=\int_{-\pi}^{\pi}K_{nr}(t)\sum_{k=1}^{m+1}(-1)^k\binom{m+1}{k}f(x+kt)dt\,, \qquad \text{where } r=[(m+3)/2]\,.$$

For each $j=0, \cdots, m$, we have

$$\begin{aligned} |f^{(j)}(x)-I_n^{(j)}(x)| &\leqq M'(m)w_{m+1}(f^{(j)}, 1/n) \\ &\leqq M'(m)n^{j-m}w_{j+1}(f^{(m)}, 1/n) \\ &\leqq M'(m)2^j n^{j-m}w(f^{(m)}, 1/n) \\ &\leqq M''(m)n^{j-m}w(f^{(m)}, 1/n)\,, \end{aligned} \tag{5.13}$$

where $M'(m)$ and $M''(m)$ depend on m. Thus, we have

$$\|f-I_n\|_X\leqq M''(m)n^{s-m}w(f^{(m)}, 1/n)\,.$$

Let

$$b_{ij}=f^{(j)}(x_i)-I_n^{(j)}(x_i) \qquad \text{for } (i, j)\in e\,,$$

and let R_{ij} be a polynomial in $\Pi_{\bar{e}}$ satisfying

$$R_{ij}^{(j')}(x_{i'})=\delta_{(i,j),(i',j')} \qquad \text{for } (i, j)\in e\,.$$

Then the polynomial $R(x)=\sum_{(i,j)\in e} b_{ij}R_{ij}(x)$ satisfies

$$\begin{aligned} \|R\|_X &\leqq \sum_{(i,j)\in e}|b_{ij}|\,\|R_{ij}\|_X \\ &\leqq M'(S)\|f-I_n\|_X\,, \end{aligned}$$

where $M'(S)$ is a constant depending on the scheme S. Let $T_n=I_n+R$, then

$$\begin{aligned} \|f-T_n\|_X &\leqq \|f-I_n\|_X+\|R\|_X \\ &\leqq \{1+M'(S)\}\,\|f-I_n\|_X \\ &\leqq \{1+M'(S)\}\,M''(m)n^{s-m}w(f^{(m)}, 1/n)\,. \end{aligned}$$

When $(i', j')\in e$, we have

$$T_n^{(j')}(x_{i'})=I_n^{(j')}(x_{i'})+\sum_{(i,j)\in e} b_{ij}R_{ij}^{(j')}(x_{i'})$$
$$=I_n^{(j')}(x_{i'})+b_{i'j'}$$
$$=f^{(j')}(x_{i'}).$$

Thus, $T_n\in\Pi_n(S;f)$. Let $M(m, S)=\{1+M'(S)\}M''(m)$, then we have (5.12).
(q. e. d.)

REMARK. In Theorem 5.4, we may replace the norm $\|f\|_X$ by the norm $\|f\|_{\bar{e}}$, where

$$\|f\|_{\bar{e}}=\max_{0\leq j\leq \bar{e}}\|f^{(j)}\|_{C[K]},$$

and s by $\bar{e}$.

§ 6. Comonotone polynomial approximation.

f is said to be piecewise monotone if it has only a finite number of local maxima and minima in A. The local maxima and minima in (a, b) (or in K) are called the peaks of f. Let

$$E_n^*(f)=\inf\{\|f-P\|_C;\ P\in\Pi_n,\ P \text{ comonotone with } f\}.$$

Eli Passow, Louis Raymon and John A. Roulier [5] showed that if f is a piecewise monotone function with peaks at $x_1, \cdots, x_k$, and $f\in C^{(j+k+1)}[a, b]$, there exists d_j such that for $n>(2k+j)$

$$E_n^*(f)\leqq d_j(b-a)^{k+1}\|f^{(j+k+1)}\|_C n^{-j}.$$

Let $0\leqq j_1<j_2<\cdots<j_p$ be fixed integers, and let $\{x_i\}_{i=1}^k\subset A$. Assume that the function f satisfies the following conditions;

(a) there is a subset $\{x_{j_q,t}\}_{t=1}^{h_q}\subset\{x_i\}_{i=1}^k$, $q=1, \cdots, p$, such that $f^{(j_q)}$ is a piecewise monotone function with peaks at $\{x_{j_q,t}\}_{t=1}^{h_q}$,

(b) for each peaks $x_{j_q,t}$ there exists a positive integer $r_{j_q,t}$ such that

$$f^{(j_q+j)}(x_{j_q,t})=0 \qquad \text{for } j=1, \cdots, 2r_{j_q,t}-1,$$

and

$$f^{(j_q+2r_{j_q,t})}(x_{j_q,t})\neq 0.$$

Let $s=\max\{j_q+2r_{j_q,t};\ t=1, \cdots, h_q,\ q=1, \cdots, p\}$. Such a function f is said to be piecewise monotone of $(k; j_1, \cdots, j_p; s)$-type. We obtain the following theorem.

THEOREM 6.1. *Let f be of $(k; j_1, \cdots, j_p; s)$-type. If $f\in C^m[A]$, where $m\geqq s$, for n sufficiently large there exists a polynomial $P_n\in\Pi_n$ such that P_n is of $(k; j_1, \cdots, j_p; s)$-type and comonotone with f, and satisfies*

$$\|f-P_n\|_s \leqq M(m, I, S) n^{s-m} w(f^{(m)}, 1/n),$$

where $\|f\|_s = \max_{0 \leqq j \leqq s} \|f^{(j)}\|_C$, *and* $M(m, I, S)$ *is a constant depending on* m *and* S (*and* I *in the algebraic case*), *here* S *is a scheme decided by the conditions* (a) *and* (b), *and* $I=[a, b]$.

PROOF. We define a matrix $E=(e_{ij})_{j=0,\cdots,s}^{i=1,\cdots,k}$ as follows: Let

$$r_i = \max\{j_q + 2r_{j_q,t}\,;\ x_{j_q,t} = x_i \text{ for some } t \text{ and } q,\ 1 \leqq t \leqq h_q,\ 1 \leqq q \leqq p\}.$$

Define for each $i=1, \cdots, k$

$$e_{i0} = e_{i1} = \cdots = e_{ir_i} = 1, \qquad e_{ir_i+1} = \cdots = e_{is} = 0.$$

Then E is poised in $\Pi_{\bar{e}}$, where

$$\bar{e} = \begin{cases} \sum e_{ij} - 1 & \text{in the algebraic case,} \\ [(\sum e_{ij} - 1)/2] & \text{in the trigonometric case,} \end{cases}$$

since E is a Hermite matrix. From Theorem 5.2 or Theorem 5.4, we have

$$\exists P_n \in \Pi_n(S\,; f)\,;\ \|f-P_n\|_s \leqq M(m, I, S) n^{s-m} w(f^{(m)}, 1/n)$$

where $M(m, I, S)$ is a constant depending on m, I and the scheme S, but in the trigonometric case we omit I. Then P_n satisfies for each $t=1, \cdots, h_q$, $q=1, \cdots, p$

$$P_n^{(j_q+j)}(x_{j_q,t}) = 0 \qquad \text{for } j=1, \cdots, 2r_{j_q,t}-1,\ P_n^{(j_q+2r_{j_q,t})}(x_{j_q,t}) \neq 0.$$

Since for each $j=0, \cdots, s$ we have

$$\|P_n^{(j)} - f^{(j)}\|_C \longrightarrow 0 \qquad \text{as } n \to \infty,$$

there exists $\delta > 0$ such that for n sufficiently large, $P_n^{(j_q)}$ comonotone with $f^{(j_q)}$ in each interval $(x_{j_q,t}-\delta, x_{j_q,t}+\delta)$ for $t=1, \cdots, h_q$, $q=1, \cdots, p$. Thus, if we take larger n, we see that $P_n^{(j_q)}$ also comonotone with $f^{(j_q)}$ outside of interval $(x_{j_q,t}-\delta, x_{j_q,t}+\delta)$ for $t=1, \cdots, h_q$, $q=1, \cdots, p$. (q. e. d.)

§ 7. Derivative of best approximant.

Let $f \in C[-1, 1]$, and let $P_n \in \Pi_n$ be an algebraic polynomial of best approximation to f, that is,

$$\|f-P_n\|_{C[-1,1]} = \inf_{Q_n \in \Pi_n} \|f-Q_n\|_{C[-1,1]}.$$

John A. Roulier [6] showed that if $f \in C^m[-1, 1]$, for each k, where $2k \leqq m$, we have

$$\lim_{n\to\infty} \|f^{(k)} - P_n^{(k)}\|_{C[-1,1]} = 0. \tag{7.1}$$

Is (7.1) correct for all $k=0, \cdots, m$? Let $-1<a<b<1$. If we replace (7.1) by

$$\lim_{n\to\infty} \|f^{(k)}-P_n^{(k)}\|_{C[a,b]}=0, \tag{7.2}$$

we can show that (7.2) is correct for all $k=0, \cdots, m$. On the other hand, if we consider this problem in the trigonometric case, we need no restriction to the norm.

THEOREM 7.1. *Let $f\in C^m[-1, 1]$, and let $P_n\in\Pi_n$ be a polynomial of best approximation to f. Then if $-1<a<b<1$, there exists a constant $C(m, a, b)$ depending on m, a and b such that*

$$\|f^{(j)}-P_n^{(j)}\|_{C[a,b]}\leqq C(m, a, b)n^{j-m}w(f^{(m)}, 1/n), \quad \text{for } j=0, \cdots, m, \tag{7.3}$$

where $w(f^{(m)}, \cdot)$ is the modulus of continuity of $f^{(m)}$ on $[-1, 1]$.

PROOF. By Jackson's theorem

$$\|f-P_n\|_{C[-1,1]}\leqq M(m)n^{-m}w(f^{(m)}, 1/n),$$

where $M(m)$ is a constant depending on m. Thus, from Lemma 5.2 we have (7.3). (q. e. d.)

THEOREM 7.2. *Let $f\in C^m[K]$, and let $T_n\in\Pi_n$ be a polynomial of best approximation to f. Then there exists a constant $C(m)$ depending on m such that*

$$\|f^{(j)}-T_n^{(j)}\|_{C[K]}\leqq M(m)n^{j-m}w(f^{(m)}, 1/n), \quad j=0, \cdots, m,$$

where $w(f^{(m)}, \cdot)$ is the modulus of continuity of $f^{(m)}$ on K.

PROOF. From (5.13) there exists a constant $M(m)$ depending on m such that

$$\|f^{(j)}-I_n^{(j)}\|_{C[K]}\leqq M(m)n^{j-m}w(f^{(m)}, 1/n), \quad j=0, \cdots, m.$$

From Jackson's theorem

$$\|f-T_n\|_{C[K]}\leqq M'(m)n^{-m}w(f^{(m)}, 1/n),$$

where $M'(m)$ is a constant depending on m. Thus, we have

$$\|T_n-I_n\|_{C[K]}\leqq \{M'(m)+M(m)\}\, n^{-m}w(f^{(m)}, 1/n).$$

By Bernstein's inequality

$$\|T_n^{(j)}-I_n^{(j)}\|_{C[K]}\leqq \{M'(m)+M(m)\}\, n^{j-m}w(f^{(m)}, 1/n), \quad \text{for } j=0, \cdots, m.$$

Consequently, we have

$$\|f^{(j)}-T_n^{(j)}\|_{C[K]}\leqq \{M'(m)+2M(m)\}\, n^{j-m}w(f^{(m)}, 1/n), \quad \text{for } j=0, \cdots, m.$$

Let $C(m)=M'(m)+2M(m)$. (q. e. d.)

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