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A remark on inhomogeneity of Picard principle

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A nonnegative locally Hölder continuous function $P(z)$ on $0<|z|\leqq 1$ will be referred to as a *density* on the punctured unit disk Ω: $0<|z|<1$. We view Ω as the interior of the bordered Riemann surface: $0<|z|\leqq 1$; hence we consider the circle: $|z|=1$ the relative boundary (border) $\partial\Omega$ of Ω and $z=0$ the ideal boundary of Ω. The *elliptic dimension* of a density P on Ω at $z=0$, $\dim P$ in notation, is defined (cf. Nakai [7, 8]) to be 'the dimension' of the half module of nonnegative solutions of the equation $\Delta u=Pu$ on Ω with the vanishing boundary values on $\partial\Omega$. After Bouligand we say that the *Picard principle* is valid for P at $z=0$ if $\dim P=1$.

To illustrate the complexity of elliptic dimensions, Nakai [4] showed the following example of rather pathological nature, at least for the first sight: There exists a pair of rotation free densities P_j $(j=1, 2)$ (i.e. $P_j(z)=P_j(|z|)$) on Ω such that the Picard principle is valid for P_j $(j=1, 2)$ at $z=0$ but invalid for the density $P_0\equiv P_1+P_2$ at $z=0$. The purpose of this note is to show that *any* density P on Ω possesses a pair of densities P_j $(j=1, 2)$ with the above property. Namely we shall prove the following

THEOREM. *For any density P on Ω there exists a pair of densities P_j $(j=1, 2)$ such that the Picard principle is valid for P_j $(j=1, 2)$ at $z=0$ and $P=P_1+P_2$. If, moreover, P is rotation free, then P_j $(j=1, 2)$ can be chosen to be rotation free.*

Actually we will prove a bit more: For any density P on Ω and any integer $n\geqq 2$ there exists a finite set of densities P_j $(j=1, 2, \cdots, n)$ satisfying the following *condition* [C]: $P=\sum_{j=1}^{n} P_j$ and the Picard principle is valid for the density Q defined by the sum of any m $(1\leqq m<n)$ elements of $\{P_j\}_{j=1}^{n}$, especially $\dim P_j=1$ $(j=1, 2, \cdots, n)$. The construction of such a set of P_j $(j=1, 2, \cdots, n)$ will be given in nos. 2-4.

1. There have been given various practical sufficient conditions for the validity of the Picard principle (Nakai [3, 5, 6, 8], Kawamura-Nakai [1], Kawamura [2], etc.). Some of these conditions sufficient for the validity of Picard

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principle are of homogeneous character in the sense that if P_j $(j=1, 2, \cdots, n)$ satisfy one of these conditions, then $\sum_{j=1}^{n} P_j$ also satisfies the same condition. For example, $\int_{\Omega-E} P(z)\log|z|^{-1}dxdy<+\infty$ where E is a closed subset of Ω thin at $z=0$ ([**5**]); $\int_{\Omega} P(z)\,dxdy<+\infty$ ([**8**]); $P(z)=O(|z|^{-2})$ $(z\to 0)$ ([**2**]). On the other hand the existence of the densities P_j $(j=1, 2)$ in Nakai's example suggests us inhomogeneity of the Picard principle. The construction of his example is based on the P-unit criterion in [**1**]. For a given density P we will construct densities P_j $(j=1, 2, \cdots, n)$ in our theorem as an application of the theorem in 3.1 in [**2**].

2. To construct the required densities P_j $(j=1, 2, \cdots, n)$ we need to consider a finite set of C^1-functions f_j $(j=1, 2, \cdots, n)$, $f_j : [0, \infty)\to[0, 1]$, with the following properties (1) $\sum_{j=1}^{n} f_j=1$, (2) f_j $(j=1, 2, \cdots, n)$ are periodic functions with the same period, and (3) the zero set of the function g defined by the sum of any m $(1\leqq m<n)$ functions among f_j $(j=1, 2, \cdots, n)$ contains an infinite sequence of disjoint closed intervals with the constant positive length l. For example, $f_j(t)$ are periodic C^1-functions with the period $2n$ on $[0, +\infty)$ defined by $f_j(t)=1$ on $[2j-2, 2j-1]$, $f_j(t)=\phi(t-2j+1)$ on $[2j-1, 2j]$, $f_j(t)=0$ on $[2j, 2j+2n-3]$ and $f_j(t)=1-\phi(t-2j-2n+3)$ on $[2j+2n-3, 2j+2n-2]$, where ϕ is C^1-mapping of $[0, 1]$ into itself such that $\phi(0)=1$, $\phi(1)=0$ and $\phi'(0)=\phi'(1)=0$.

With the aid of these auxiliary functions f_j $(j=1, 2, \cdots, n)$ we successively define $h_j(z)=f_j(-\log|z|)$ and $P_j(z)=P(z)h_j(z)$ $(j=1, 2, \cdots, n)$. These are certainly densities on Ω and P_j is rotation free if P is rotation free. The property (1) for auxiliary functions f_j $(j=1, 2, \cdots, n)$ implies that $P=\sum_{j=1}^{n} P_j$. Therefore we only have to prove that $\{P_j\}_{j=1}^{n}$ satisfies the latter part of the condition [C].

3. Before proceeding to the proof of the latter part of the condition [C] we need to make some preparation. Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of disjoint annuli $A_k=\{z\in\Omega\,;\,a_k\leqq|z|\leqq b_k\}$, where $b_k>a_k>b_{k+1}$ and $\lim_{k\to\infty}a_k=0$. We say that $\{A_k\}_{k=1}^{\infty}$ satisfies the condition [A] if

$$\inf_k \mathrm{mod}\,A_k>0 \quad (\mathrm{mod}\,A_k=\log(b_k/a_k)).$$

It is known (the theorem in 3.1 in [**2**]) that if the density $P(z)$ on Ω satisfies $P(z)\leqq c|z|^{-2}$ on $\cup_{k=1}^{\infty} A_k$ with the condition [A] where c is some positive constant, then the Picard principle is valid for P at $z=0$.

4. We are ready to prove the assertion in the last part of no. 2. Observe that by the properties (2) and (3) for $\{f_j\}$ $(j=1, 2, \cdots, n)$, the inverse image $g^{-1}(0)$ of g defined by the sum of m functions of $\{f_j\}_{j=1}^{n}$ as in the property (3) contains an infinite sequence of disjoint closed intervals $[c_k, d_k]$ $(k=1, 2, \cdots)$ with constant positive length l. By setting $a_k=\exp(-d_k)$, $b_k=\exp(-c_k)$ and $A_k=\{z\in\Omega ; a_k\leqq|z|\leqq b_k\}$, we have that $Q^{-1}(0)\supset\cup_{k=1}^{\infty}A_k$ where $Q(z)$ is the density, corresponding to g, defined by the sum of m elements of $\{P_j\}_{j=1}^{n}$ $(1\leqq m<n)$. We deduce that $Q(z)\leqq|z|^{-2}$ (in reality $Q(z)\equiv 0$) on $\cup_{k=1}^{\infty}A_k$. Since $\mathrm{mod}\, A_k=l$, $\{A_k\}_{k=1}^{\infty}$ satisfies the condition [A]. Thus we conclude that the Picard principle is valid for Q at $z=0$.

The proof of the theorem is herewith complete.

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