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Orbits on affine symmetric spaces under the action of the isotropy subgroups

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Introduction.

Let G be a connected Lie group, σ an involutive automorphism of G and H a closed subgroup of G satisfying $(G_\sigma)_0 \subset H \subset G_\sigma$ where $G_\sigma=\{x\in G \mid \sigma(x)=x\}$ and $(G_\sigma)_0$ is the identity component of G_σ. Then the triple (G, H, σ) is called an affine symmetric space ([7, p. 223 and p. 225]). Suppose that G is real semi-simple, and consider the double coset decomposition $H\backslash G/H$.

In the case of a Riemannian symmetric space (G, K, θ) of noncompact type the double coset decomposition is the Cartan decomposition $G=KA_{\mathfrak{p}}K$. Secondly consider an affine symmetric space $(G\times G, \Delta G, \sigma)$ where G is real semi-simple, ΔG denotes the diagonal of $G\times G$, and σ is the mapping $(x, y)\to(y, x)$ $(x, y\in G)$. If we identify $G\times G/\Delta G$ with G in the natural way, then the structure of $\Delta G\backslash G\times G/\Delta G$ is, for the most part, known by the following Harish-Chandra's theorem (see [3, p. 102], [4, p. 556] and [12, p. 113]).

THEOREM. *Let G' be the set of regular elements in G, $\{\mathfrak{j}_i \mid i=1, \cdots, r\}$ representatives of conjugacy classes of Cartan subalgebras in $\mathfrak{g}$, and J_i the Cartan subgroup associated with $\mathfrak{j}_i$. Then*

$$G'=\bigcup_{i=1}^{r}\bigcup_{x\in G} xJ_i'x^{-1}$$

where $J_i'=J_i\cap G'$.

In this paper we will extend this theorem to an arbitrary affine symmetric space (G, H, σ) such that G is real semisimple.

Let φ be the mapping of G into G defined by $\varphi(g)=g\sigma(g)^{-1}$ for $g\in G$ (see [1], [8, p. 182]). Then G/G_σ and $\varphi(G)$ are diffeomorphic by this mapping, and the H-orbits on G/G_σ correspond to the H-orbits on $\varphi(G)$ under the action $(h, x)\to hxh^{-1}$ $(h\in H, x\in\varphi(G))$. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively, and let the automorphism σ of $\mathfrak{g}$ be the one induced by the automorphism σ of G. Put $\mathfrak{q}=\{X\in\mathfrak{g} \mid \sigma(X)=-X\}$. A subspace $\mathfrak{a}_{\mathfrak{q}}$ of $\mathfrak{q}$ is called an A-subspace if the following two conditions are satisfied: (i) $\mathfrak{a}_{\mathfrak{q}}$ is maximal abelian in $\mathfrak{q}$; (ii) Every element of $\mathfrak{a}_\mathfrak{q}$ is a semi-simple element of $\mathfrak{g}$. The centralizer $A_\mathfrak{q}$ of $\mathfrak{a}_\mathfrak{q}$ in $\varphi(G)$ is called the A-subset associated with $\mathfrak{a}_\mathfrak{q}$. Then every connected component of $A_\mathfrak{q}$ is diffeomorphic to $\exp \mathfrak{a}_\mathfrak{q}$ (§ 3, Proposition 1). Consider the polynomial

$$\det(t-\mathrm{ad}(X))=\sum_{i=0}^{n} d_i(X)t^i, \qquad X\in\mathfrak{q},$$

where d_i are polynomial functions on $\mathfrak{q}$ and $n=\dim\mathfrak{g}$. Let k be the least integer such that $d_k\not\equiv 0$. Then the elements of $\mathfrak{q}'=\{X\in\mathfrak{q}\mid d_k(X)\neq 0\}$ are called the $\mathfrak{q}$-regular elements. Next consider the polynomial

$$\det(1+t-\mathrm{Ad}(x))=\sum_{i=0}^{n} D_i(x)t^i, \qquad x\in\varphi(G).$$

Then the elements of $\varphi(G)'=\{x\in\varphi(G)\mid D_k(x)\neq 0\}$ are called the $\varphi(G)$-regular elements. Let $\{\mathfrak{a}_{\mathfrak{q}i}\mid i\in I\}$ be a set of representatives of H-conjugacy classes of A-subspaces and $A_{\mathfrak{q}i}$ the A-subsets associated with $\mathfrak{a}_{\mathfrak{q}i}$. Then we have the following theorem (§ 6, Theorem 2).

THEOREM. (i) $\mathfrak{q}'=\bigcup_{i\in I}\mathrm{Ad}(H)\mathfrak{a}'_{\mathfrak{q}i}$ *(disjoint union) where* $\mathfrak{a}'_{\mathfrak{q}i}=\mathfrak{a}_{\mathfrak{q}i}\cap\mathfrak{q}'$. *The mapping* $\zeta_i\colon H/Z_H(\mathfrak{a}_{\mathfrak{q}i})\times\mathfrak{a}'_{\mathfrak{q}i}\to\mathfrak{q}'$ *defined by* $\zeta_i(hZ_H(\mathfrak{a}_{\mathfrak{q}i}), Y)=\mathrm{Ad}(h)Y$ $(h\in H,$ $Y\in\mathfrak{a}'_{\mathfrak{q}i})$ *is an everywhere regular* $|W(\mathfrak{a}_{\mathfrak{q}i}, H)|$*-to-one mapping where* $W(\mathfrak{a}_{\mathfrak{q}i}, H)$ $=N_H(\mathfrak{a}_{\mathfrak{q}i})/Z_H(\mathfrak{a}_{\mathfrak{q}i})$.

(ii) *Put* $\varphi(G)_i=\bigcup_{h\in H}hA'_{\mathfrak{q}i}h^{-1}$ *where* $A'_{\mathfrak{q}i}=A_{\mathfrak{q}i}\cap\varphi(G)'$. *Then* $\varphi(G)'=\bigcup_{i\in I}\varphi(G)_i$ *(disjoint union). The mapping* $\eta_i\colon H/Z_H(A_{\mathfrak{q}i})\times A'_{\mathfrak{q}i}\to\varphi(G)'$ *defined by* $\eta_i(hZ_H(A_{\mathfrak{q}i}), y)=hyh^{-1}$ $(h\in H,\ y\in A'_{\mathfrak{q}i})$ *is an everywhere regular* $|W(A_{\mathfrak{q}i}, H)|$*-to-one mapping where* $W(A_{\mathfrak{q}i}, H)=N_H(\mathfrak{a}_{\mathfrak{q}i})/Z_H(A_{\mathfrak{q}i})$.

Moreover we will prove the following results. For every affine symmetric space (G, H, σ), there exists a (finite) covering group G_2 of G such that $G_2/(G_2)_\sigma\cong G/H$ and $(G_2)_\sigma\backslash G_2/(G_2)_\sigma\cong H\backslash G/H$ (Lemma 3).

An element X of $\mathfrak{g}$ is called semi-simple (resp. nilpotent) when $\mathrm{ad}(X)$ is a semi-simple (resp. nilpotent) endomorphism of $\mathfrak{g}$. An element x of G is called semi-simple (resp. unipotent) when $\mathrm{Ad}(x)$ is a semi-simple endomorphism of $\mathfrak{g}$ (resp. $x=\exp X$ with a nilpotent element X of $\mathfrak{g}$). Then every semi-simple element in $\mathfrak{q}$ (resp. $\varphi(G)$) is contained in some A-subspace (resp. A-subset) (Corollary to Theorem 2). Moreover every element in $\mathfrak{q}$ (resp. $\varphi(G)$) is decomposed to the semi-simple component and the nilpotent (resp. unipotent) component contained in $\mathfrak{q}$ (resp. $\varphi(G)$) with respect to the Jordan decomposition (§ 5, Proposition 2).

The determination of H-conjugacy classes of A-subspaces is equivalent to that of K_+-conjugacy classes of $\bar\theta$-stable maximal abelian subspaces of $\bar{\mathfrak{q}}$ (§ 7), where $(\bar{\mathfrak{g}}, \bar{\mathfrak{k}}, \bar\theta)$ is the symmetric Lie algebra dual to $(\mathfrak{g}, \mathfrak{h}, \sigma)$ (cf. [**2**, p. 111]),

$K_+=K\cap H$ and $K=G_\theta$. This is studied in [**9**, § 2]. Theorem 3 which is a corollary to [**9**, Theorem 2] gives an explicit construction of representatives of H-conjugacy classes of A-subspaces.

Notations.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be subsets of $\mathfrak{g}$ and let S_1, S_2 and S be subsets of G. Then

$$\mathfrak{z}_{\mathfrak{s}_1}(\mathfrak{s}_2)=\{X\in\mathfrak{s}_1 \mid [Y, X]=0 \text{ for all } Y\in\mathfrak{s}_2\}$$

$$\mathfrak{z}_{\mathfrak{s}_1}(S_2)=\{X\in\mathfrak{s}_1 \mid \mathrm{Ad}(y)X=X \text{ for all } y\in S_2\},$$

$$Z_{S_1}(\mathfrak{s}_2)=\{x\in S_1 \mid \mathrm{Ad}(x)Y=Y \text{ for all } Y\in\mathfrak{s}_2\},$$

$$Z_{S_1}(S_2)=\{x\in S_1 \mid xyx^{-1}=y \text{ for all } y\in S_2\},$$

$$N_{S_1}(\mathfrak{s}_2)=\{x\in S_1 \mid \mathrm{Ad}(x)\mathfrak{s}_2=\mathfrak{s}_2\},$$

$$N_{S_1}(S_2)=\{x\in S_1 \mid xS_2x^{-1}=S_2\}.$$

Let σ be an automorphism of G. Then

$$S_\sigma=\{x\in S \mid \sigma(x)=x\}.$$

Let H be a closed subgroup of G. Then H_0 denotes the identity component of H. Let W be a finite set. Then $|W|$ denotes the number of the elements of W.

§ 1. Definitions.

Let G be a connected Lie group, σ an involutive automorphism of G, and H a closed subgroup of G satisfying $(G_\sigma)_0\subset H\subset G_\sigma$. Then the triple (G, H, σ) is called an *affine symmetric space*. Let $\mathfrak{g}$ be a Lie algebra, σ an involutive automorphism of $\mathfrak{g}$. Put $\mathfrak{h}=\{X\in\mathfrak{g} \mid \sigma(X)=X\}$. Then the triple $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is called a *symmetric Lie algebra*. To every affine symmetric space (G, H, σ) there corresponds a symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively, and the automorphism σ of $\mathfrak{g}$ is the one induced by the automorphism σ of G. We assume that G and $\mathfrak{g}$ are semisimple for every affine symmetric space (G, H, σ) and symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ appeared in this paper.

Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a symmetric Lie algebra. Put $\mathfrak{q}=\{X\in\mathfrak{g} \mid \sigma(X)=-X\}$. Then $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ (direct sum).

DEFINITION. Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a symmetric Lie algebra. Then a subspace $\mathfrak{a}_\mathfrak{q}$ of $\mathfrak{q}$ is called an *A-subspace* if the following two conditions are satisfied:

(i) $\mathfrak{a}_{\mathfrak{q}}$ is a maximal abelian subspace of $\mathfrak{q}$;

(ii) Every element of $\mathfrak{a}_{\mathfrak{q}}$ is a semi-simple element of $\mathfrak{g}$.

Let (G, H, σ) be an affine symmetric space and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ the corresponding symmetric Lie algebra. Define a mapping φ of G into G by

$$\varphi(g)=g\sigma(g)^{-1}, \qquad g\in G.$$

If $\varphi(g_1)=\varphi(g_2)$ for $g_1, g_2\in G$, then $g_2^{-1}g_1=\sigma(g_2^{-1}g_1)$. Hence $g_2^{-1}g_1\in G_\sigma$ and so the mapping φ gives an injection of G/G_σ into G. For each element x of G, define a transformation $a(x)$ of G by

$$a(x)y=xy\sigma(x)^{-1}, \qquad y\in G.$$

Clearly G acts transitively on $\varphi(G)$ under the action of a and we have

$$\varphi(xy)=a(x)\varphi(y), \qquad x, y\in G. \tag{1.1}$$

Thus the H-orbit structure on G/G_σ is identified with the H-orbit structure on $\varphi(G)$ under the action $(h, y)\to hyh^{-1}$ $(h\in H, y\in\varphi(G))$.

Let θ be a Cartan involution of $\mathfrak{g}$ commutative with σ ([**2**], [8, p. 153, Theorem 2.1]) and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition. Put $\mathfrak{k}_+=\mathfrak{k}\cap\mathfrak{h}$, $\mathfrak{k}_-=\mathfrak{k}\cap\mathfrak{q}$, $\mathfrak{p}_+=\mathfrak{p}\cap\mathfrak{h}$, and $\mathfrak{p}_-=\mathfrak{p}\cap\mathfrak{q}$. Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$. Then the mapping $(k, X, Y)\to k\exp X\exp Y$ is an analytic diffeomorphism of $K\times\mathfrak{p}_-\times\mathfrak{p}_+$ onto G ([8, p. 161], [**10**]). Thus we have

$$\varphi(G)=a(K)\exp\mathfrak{p}_-. \tag{1.2}$$

Though the following three lemmas are proved also in [8], we will give them with proofs for the sake of completeness.

LEMMA 1. *$\varphi(G)$ is a closed submanifold of G.*

PROOF. If $x\in\varphi(G)$, then $\sigma(x)=x^{-1}$. Thus there exists a neighborhood V of the identity in G such that $\varphi(G)\cap V=\exp\mathfrak{q}\cap V$. Since G acts transitively on $\varphi(G)$ under the action of a, and since $a(y)$ is a transformation of G for every $y\in G$, it follows that $\varphi(G)$ is a regular submanifold.

Next we will prove that $\varphi(G)$ is closed in G. Let Z denote the center of G. Since $\varphi(Z)$ and $\mathfrak{p}_-$ are closed in K and $\mathfrak{p}$, respectively, it follows from the Cartan decomposition $G=K\exp\mathfrak{p}$ that $\varphi(Z)\exp\mathfrak{p}_-$ is a closed subset of G. Let x be an element of G which is not contained in $\varphi(G)$. Then for every $k\in K$, there exist a neighborhood V of x in G and a neighborhood W of the identity in K such that

$$a(W)V\cap a(Zk)\exp\mathfrak{p}_-=\emptyset.$$

Hence

$$V\cap a(ZW^{-1}k)\exp\mathfrak{p}_-=\emptyset.$$

Since K/Z is compact, there exists a neighborhood V' of x in G such that

$$V' \cap a(K) \exp \mathfrak{p}_- = \emptyset .$$

This implies that $\varphi(G)$ is closed in G (see (1.2)). q. e. d.

LEMMA 2. *The number of connected components of G_σ is finite.*

PROOF. Since $G_\sigma = K_\sigma \exp \mathfrak{p}_+$, we have only to prove that the number of connected components of K_σ is finite. Let $\mathfrak{c}$ be the center of $\mathfrak{k}$ and put $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$. Then $\mathfrak{k} = \mathfrak{c} + \mathfrak{k}'$ (direct sum). Let D and K' be the analytic subgroups of G corresponding to $\mathfrak{c}$ and $\mathfrak{k}'$, respectively. Then $K = DK'$, K' is compact and $D \cap K'$ is a finite set. Since D, K' and $D \cap K'$ are σ-stable, σ acts on $D/D \cap K'$ and $K'/D \cap K'$. Since the natural mapping $K \to D/D \cap K' \times K'/D \cap K'$ is a finite covering, it is enough to prove that the numbers of connected components of $(D/D \cap K')_\sigma$ and $(K'/D \cap K')_\sigma$ are finite, which is easy to prove. q. e. d.

LEMMA 3. *There exists a covering group G_2 ($\pi_2 : G_2 \to G$ the convering map) such that σ lifts to G_2 and that $\pi_2^{-1}(H) = (G_2)_\sigma$. Moreover π_2 can be taken to be a finite covering map.*

PROOF. Let G_1 be the universal covering group of G and $\pi_1 : G_1 \to G$ the covering map. Then σ lifts to G_1. Let Z denote the center of G_1. Put $\Gamma = \varphi(\pi_1^{-1}(H)) = \{g\sigma(g^{-1}) \mid g \in \pi_1^{-1}(H)\}$. Then $\Gamma \subset \pi_1^{-1}(1) \subset Z$ and therefore Γ is a subgroup of $\pi_1^{-1}(1)$. Put $G_2 = G_1/\Gamma$ and let $\pi_2 : G_2 \to G$ be the covering map induced from π_1. Since Γ is σ-stable, σ lifts to G_2. We will prove $\pi_2^{-1}(H) = (G_2)_\sigma$. If $g\Gamma \subset \pi_1^{-1}(H)$, then $g\Gamma\sigma(g\Gamma)^{-1} = g\sigma(g)^{-1}\Gamma = \varphi(g)\Gamma = \Gamma$ and therefore $g\Gamma \in (G_2)_\sigma$. Hence $\pi_2^{-1}(H) \subset (G_2)_\sigma$. Conversely suppose $g\Gamma \in (G_2)_\sigma$. $(G_1)_\sigma = \varphi^{-1}(1)$ is connected. In fact, let K_1 be the analytic subgroup of G_1 for $\mathfrak{k}$ and define K_1' and D as in Lemma 2. Then $K_1 = K_1' \times D_1$ where K_1' and D are σ-stable. Since $(G_1)_\sigma = (K_1)_\sigma \exp \mathfrak{p}_+$, we have only to prove that $(K_1)_\sigma = (K_1')_\sigma \times (D_1)_\sigma$ is connected. Since D_1 is a vector group, $(D_1)_\sigma$ is connected. For the connectedness of $(K_1')_\sigma$, see [**5**, P. 272]. Thus we have $\pi_1^{-1}(H) \supset (G_1)_\sigma$ and therefore $\pi_1^{-1}(H) = \varphi^{-1}(\Gamma)$. Since $\varphi(g) \in \Gamma$, $g \in \pi_1^{-1}(H)$. Hence $(G_2)_\sigma \subset \pi_2^{-1}(H)$.

Note that $\pi_1^{-1}(G_\sigma) = \{g \in G_1 \mid \varphi(g) \in \pi_1^{-1}(1)\}$ and suppose that there exists a σ-stable subgroup Γ' of $\pi_1^{-1}(1)$ such that $\pi_1^{-1}(1)/\Gamma'$ is finite and that $\varphi(\pi_1^{-1}(G_\sigma)) \cap \Gamma' = \varphi(\pi_1^{-1}(H))$. Then we can easily prove in the same way that $G_3 = G_1/\Gamma'$ satisfies the conditions of the last half of Lemma 3. Such a Γ' is given as follows. Since $\pi_1^{-1}(1)/\varphi(\pi_1^{-1}(H))$ is a finitely generated abelian group and since $\varphi(\pi_1^{-1}(G_\sigma))/\varphi(\pi_1^{-1}(H))$ is finite (Lemma 2), it follows from the fundamental theorem for finitely generated abelian group that there exists a subgroup Γ_1 of $\pi_1^{-1}(1)$ such that $\pi_1^{-1}(1)/\Gamma_1$ is finite and $\varphi(\pi_1^{-1}(G_\sigma)) \cap \Gamma_1 = \varphi(\pi_1^{-1}(H))$. $\Gamma' = \Gamma_1 \cap \sigma(\Gamma_1)$ is a desired subgroup of $\pi_1^{-1}(1)$. q. e. d.

The (finite) covering map $\pi_2 : G_2 \to G$ obtained by Lemma 3 induces a diffeomorphism $G_2/(G_2)_\sigma \xrightarrow{\sim} G/H$ and a bijection $(G_2)_\sigma\backslash G_2/(G_2)_\sigma \xrightarrow{\sim} H\backslash G/H$.

DEFINITION. Let (G, H, σ) be an affine symmetric space and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ the corresponding symmetric Lie algebra. Let $\mathfrak{a}_\mathfrak{q}$ be an A-subspace. Then the centralizer $A_\mathfrak{q}$ of $\mathfrak{a}_\mathfrak{q}$ in $\varphi(G)$ is called the *A-subset* associated with $\mathfrak{a}_\mathfrak{q}$.

For every $X\in\mathfrak{q}$, consider the eigenpolynomial

$$\det(t-\operatorname{ad}(X))=\sum_{i=0}^{n} d_i(X)t^i$$

of the endomorphism $\operatorname{ad}(X)$ of $\mathfrak{g}$ where t is an indeterminate, $n=\dim\mathfrak{g}$ and the d_i are polynomial functions on $\mathfrak{q}$. Let k be the least integer such that $d_k\not\equiv 0$.

DEFINITION. An element $X\in\mathfrak{q}$ is said to be *$\mathfrak{q}$-regular* if $d_k(X)\neq 0$. The set of $\mathfrak{q}$-regular elements is denoted by $\mathfrak{q}'$.

For every $x\in\varphi(G)$, put

$$\det(t+1-\operatorname{Ad}(x))=\sum_{i=0}^{n} D_i(x)t^i\,.$$

Then D_i are analytic functions on $\varphi(G)$ and $D_k\not\equiv 0$.

DEFINITION. An element $x\in\varphi(G)$ is said to be *$\varphi(G)$-regular* if $D_k(x)\neq 0$. The set of $\varphi(G)$-regular elements is denoted by $\varphi(G)'$.

REMARK. Let G_C be a connected complex semi-simple Lie group, σ a complex analytic involutive automorphism of G_C, and H_C a closed subgroup of G_C satisfying $(G_{C\sigma})_0\subset H_C\subset G_{C\sigma}$. Then H_C is a complex subgroup of G_C and (G_C, H_C, σ) is an affine symmetric space. Let $(\mathfrak{g}_C, \mathfrak{h}_C, \sigma)$ be the corresponding symmetric Lie algebra. It is well known that a complex endomorphism f (such as $\operatorname{ad}X$ for $X\in\mathfrak{g}_C$ or $\operatorname{Ad}x$ for $x\in G_C$) of $\mathfrak{g}_C$ is semi-simple if and only if it is semi-simple when $\mathfrak{g}_C$ is regarded as a $2n$-dimensional real vector space ($n=\dim_C\mathfrak{g}_C$). As for the $\mathfrak{q}$-regularity and the $\varphi(G)$-regularity a similar statement holds.

DEFINITION. Such an affine symmetric space (G_C, H_C, σ) is called a *complex affine symmetric space* and $(\mathfrak{g}_C, \mathfrak{h}_C, \sigma)$ is called a *complex symmetric Lie algebra*.

Let (G, H, σ) be an affine symmetric space and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ the corresponding symmetric Lie algebra. Let $\mathfrak{g}_C$ and $\mathfrak{h}_C$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and extend σ to the complex linear automorphism of $\mathfrak{g}_C$. The inclusion mapping of $\mathfrak{g}$ into $\mathfrak{g}_C$ is denoted by ι. Let G_C be a connected complex Lie group with Lie algebra $\mathfrak{g}_C$ such that the mappings $\iota:\mathfrak{g}\to\mathfrak{g}_C$ and $\sigma:\mathfrak{g}_C\to\mathfrak{g}_C$ lift to Lie group homomorphisms $\iota: G\to G_C$ and $\sigma: G_C\to G_C$, respectively. A complex affine symmetric space (G_C, H_C, σ) satisfying the above conditions will be called a *complexification* of the affine symmetric space (G, H, σ). Every affine symmetric space (G, H, σ) has at least one complexification $(G_C, (G_C)_\sigma, \sigma)$ where $G_C=\operatorname{Int}(\mathfrak{g}_C)$.

§2. $\mathfrak{a}_{\mathfrak{q}}$-regular elements.

For an A-subspace $\mathfrak{a}_{\mathfrak{q}}$ of $(\mathfrak{g}, \mathfrak{h}, \sigma)$, let $\mathfrak{a}_{\mathfrak{q}_C}$ be the complexification of $\mathfrak{a}_{\mathfrak{q}}$. Then $\mathfrak{a}_{\mathfrak{q}_C}$ is an A-subspace of $(\mathfrak{g}_C, \mathfrak{h}_C, \sigma)$. Let $\Phi(\mathfrak{a}_{\mathfrak{q}_C})$ denote the root system of the pair $(\mathfrak{g}_C, \mathfrak{a}_{\mathfrak{q}_C})$. Then

$$\mathfrak{g}_C=\mathfrak{z}_{\mathfrak{h}_C}(\mathfrak{a}_{\mathfrak{q}_C})+\mathfrak{a}_{\mathfrak{q}_C}+\sum_{\lambda\in\Phi(\mathfrak{a}_{\mathfrak{q}_C})}\mathfrak{g}_{C\lambda} \qquad \text{(direct sum)} \tag{2.1}$$

where $\mathfrak{g}_{C\lambda}=\{X\in\mathfrak{g}_C \mid [Y, X]=\lambda(Y)X \text{ for all } Y\in\mathfrak{a}_{\mathfrak{q}_C}\}$.

DEFINITION. An element $Y\in\mathfrak{a}_{\mathfrak{q}_C}$ (resp. $\mathfrak{a}_{\mathfrak{q}}$) is said to be *$\mathfrak{a}_{\mathfrak{q}_C}$-regular* (resp. *$\mathfrak{a}_{\mathfrak{q}}$-regular*) if $\lambda(Y)\neq 0$ for all $\lambda\in\Phi(\mathfrak{a}_{\mathfrak{q}_C})$. The set of $\mathfrak{a}_{\mathfrak{q}_C}$-regular (resp. $\mathfrak{a}_{\mathfrak{q}}$-regular) elements is denoted by $\mathfrak{a}'_{\mathfrak{q}_C}$ (resp. $\mathfrak{a}'_{\mathfrak{q}}$).

Retain the above notations and put $\tilde{\mathfrak{g}}_C=\sum_{\lambda\in\Phi(\mathfrak{a}_{\mathfrak{q}_C})}\mathfrak{g}_{C\lambda}$, $\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_C\cap\mathfrak{g}$, $\tilde{\mathfrak{h}}=\tilde{\mathfrak{g}}\cap\mathfrak{h}$ and $\tilde{\mathfrak{q}}=\tilde{\mathfrak{g}}\cap\mathfrak{q}$. Then it follows from (2.1) that

$$\begin{cases} \mathfrak{h}=\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}_{\mathfrak{q}})+\tilde{\mathfrak{h}} & \text{(direct sum)} \\ \mathfrak{q}=\mathfrak{a}_{\mathfrak{q}}+\tilde{\mathfrak{q}} & \text{(direct sum)}. \end{cases} \tag{2.2}$$

LEMMA 4. *The mapping ζ of $H/Z_H(\mathfrak{a}_{\mathfrak{q}})\times\mathfrak{a}'_{\mathfrak{q}}$ into $\mathfrak{q}$ defined by*

$$\zeta(hZ_H(\mathfrak{a}_{\mathfrak{q}}), Y)=\mathrm{Ad}(h)Y, \qquad h\in H, \quad Y\in\mathfrak{a}'_{\mathfrak{q}}$$

is an everywhere regular $|W(\mathfrak{a}_{\mathfrak{q}}, H)|$-to-one mapping onto $\mathrm{Im}\,\zeta$, *where*

$$W(\mathfrak{a}_{\mathfrak{q}}, H)=N_H(\mathfrak{a}_{\mathfrak{q}})/Z_H(\mathfrak{a}_{\mathfrak{q}}).$$

PROOF. If $h\in H$, $X\in\tilde{\mathfrak{h}}$, $Y\in\mathfrak{a}'_{\mathfrak{q}}$, $Y_1\in\mathfrak{a}_{\mathfrak{q}}$ and $t\in\boldsymbol{R}$, then

$$\mathrm{Ad}(h\exp tX)(Y+tY_1)=\mathrm{Ad}(h)Y+t\,\mathrm{Ad}(h)([X, Y]+Y_1)+o(t).$$

Since Y is $\mathfrak{a}_{\mathfrak{q}}$-regular, the mapping $-\mathrm{ad}\,Y|_{\tilde{\mathfrak{h}}}: \tilde{\mathfrak{h}}\to\tilde{\mathfrak{q}}$ is a bijection. Hence the regularity of ζ follows from (2.2).

Assume $\mathrm{Ad}(h_1)Y_1=\mathrm{Ad}(h_2)Y_2$, $h_1, h_2\in H$ and $Y_1, Y_2\in\mathfrak{a}'_{\mathfrak{q}}$. Since $\mathfrak{z}_{\mathfrak{q}}(Y_1)=\mathfrak{z}_{\mathfrak{q}}(Y_2)=\mathfrak{a}_{\mathfrak{q}}$, it follows that $\mathrm{Ad}(h_2^{-1}h_1)\mathfrak{a}_{\mathfrak{q}}=\mathfrak{a}_{\mathfrak{q}}$ and that $h_2^{-1}h_1\in N_H(\mathfrak{a}_{\mathfrak{q}})$.

q. e. d.

It follows from Lemma 4 that $\mathrm{Ad}(H)(\mathfrak{a}'_{\mathfrak{q}})$ is an open subset of $\mathfrak{q}$. Since $\mathfrak{q}'$ is an open dense subset of $\mathfrak{q}$, $\mathfrak{q}'\cap\mathrm{Ad}(H)(\mathfrak{a}'_{\mathfrak{q}})\neq\emptyset$. Since $\mathfrak{q}'$ is $\mathrm{Ad}(H)$-invariant, we have $\mathfrak{q}'\cap\mathfrak{a}'_{\mathfrak{q}}\neq\emptyset$. Let Γ be an element of $\mathfrak{q}'\cap\mathfrak{a}'_{\mathfrak{q}}$. If $Y\in\mathfrak{a}'_{\mathfrak{q}}$, then $\mathfrak{z}_{\mathfrak{g}}(Y)=\mathfrak{z}_{\mathfrak{g}}(\Gamma)=\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})$. Hence $Y\in\mathfrak{q}'$ and $\mathfrak{a}'_{\mathfrak{q}}\subset\mathfrak{q}'$. It follows easily that $\mathfrak{a}_{\mathfrak{q}}\cap\mathfrak{q}'=\mathfrak{a}'_{\mathfrak{q}}$.

§3. Connected components of $A_{\mathfrak{q}}$.

LEMMA 5. *Let $\mathfrak{a}_{\mathfrak{q}}$ be an A-subspace of a symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$. Then there exists a Cartan involution θ of $\mathfrak{g}$ satisfying* (i) *θ is commutative with*

σ *and* (ii) $\mathfrak{a}_{\mathfrak{q}}$ *is θ-stable.*

PROOF. If $Y\in\mathfrak{a}_{\mathfrak{q}}'$, then $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})=\mathfrak{z}_{\mathfrak{g}}(Y)$ is reductive in $\mathfrak{g}$ ([**10**, p. 105]). Hence $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})=\mathfrak{c}+\mathfrak{l}$ where $\mathfrak{c}$ is the center of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})$ and $\mathfrak{l}=[\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}}), \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})]$. Then $\mathfrak{a}_{\mathfrak{q}}\subset\mathfrak{c}$ and $\mathfrak{l}\subset\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}_{\mathfrak{q}})$. Let $\mathfrak{a}_{\mathfrak{l}}$ be a Cartan subalgebra of $\mathfrak{l}$. Then $\mathfrak{a}=\mathfrak{c}+\mathfrak{a}_{\mathfrak{l}}=\mathfrak{a}_{\mathfrak{h}}+\mathfrak{a}_{\mathfrak{q}}$ $(\mathfrak{a}_{\mathfrak{h}}=\mathfrak{a}\cap\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}$ and is σ-stable. Let θ' be a Cartan involution of $\mathfrak{g}$ such that $\mathfrak{a}$ is θ'-stable ([**12**, Proposition 1.3.1.1]). Since $\mathfrak{a}$ is $\sigma\theta'$-stable, there exists an $x\in\mathrm{Int}(\mathfrak{g})$ such that $x\theta'x^{-1}$ is commutative with σ and that $x(\mathfrak{a})=\mathfrak{a}$ ([**9**, Lemma 3]). Put $\theta=x\theta'x^{-1}$. Then $\theta(\mathfrak{a})=\mathfrak{a}$. Since $\theta\sigma=\sigma\theta$, then $\theta(\mathfrak{a}_{\mathfrak{q}})=\mathfrak{a}_{\mathfrak{q}}$. q.e.d.

REMARK. Let θ be a Cartan involution of $\mathfrak{g}$ commutative with σ. Since every Cartan involution of $\mathfrak{g}$ commutative with σ can be written as $h\theta h^{-1}$, with an $h\in\mathrm{Ad}(H_0)$ ([8, p. 153, Theorem 2.1]), it follows from Lemma 5 that every A-subspace is H_0-conjugate to a θ-stable A-subspace.

Let $\mathfrak{a}_{\mathfrak{q}}$ be an A-subspace and θ a Cartan involution of $\mathfrak{g}$ such that $\sigma\theta=\theta\sigma$ and that $\theta(\mathfrak{a}_{\mathfrak{q}})=\mathfrak{a}_{\mathfrak{q}}$ (Lemma 5). Let $A_{\mathfrak{q}}$ be the A-subset associated with $\mathfrak{a}_{\mathfrak{q}}$. Let $\mathfrak{k}$, $\mathfrak{k}_+$, $\mathfrak{k}_-$, $\mathfrak{p}$, $\mathfrak{p}_+$, $\mathfrak{p}_-$ and K be as in §1 and put $\mathfrak{a}_{\mathfrak{k}_-}=\mathfrak{a}_{\mathfrak{q}}\cap\mathfrak{k}_-$ and $\mathfrak{a}_{\mathfrak{p}_-}=\mathfrak{a}_{\mathfrak{q}}\cap\mathfrak{p}_-$. Then $\mathfrak{a}_{\mathfrak{q}}=\mathfrak{a}_{\mathfrak{k}_-}+\mathfrak{a}_{\mathfrak{p}_-}$ (direct sum). Let x be an element of $A_{\mathfrak{q}}$. Following (1.2), x can be written as $x=k\exp X\sigma(k)^{-1}$ where $k\in K$ and $X\in\mathfrak{p}_-$. If $Y\in\mathfrak{a}_{\mathfrak{k}_-}$, then

$$\mathrm{Ad}(k\exp X\sigma(k)^{-1})Y=Y.$$

Hence

$$\mathrm{Ad}(\exp X)\mathrm{Ad}(\sigma(k)^{-1})Y=\mathrm{Ad}(k^{-1})Y.$$

Since $\mathrm{Ad}(\sigma(k)^{-1})Y\in\mathfrak{k}$ and $\mathrm{Ad}(k^{-1})Y\in\mathfrak{k}$, it follows that $[X, \mathrm{Ad}(\sigma(k)^{-1})Y]=0$ ([**12**, p. 28, Lemma 1.1.3.7]). Therefore

$$\mathrm{Ad}(k\sigma(k)^{-1})Y=Y.$$

When $Y\in\mathfrak{a}_{\mathfrak{p}_-}$, we have the same result. Summarizing, we have

$$A_{\mathfrak{q}}=\{k\exp X\sigma(k)^{-1}\mid k\in K,\ X\in\mathfrak{p}_-,\ \varphi(k)\in Z_{\varphi(K)}(\mathfrak{a}_{\mathfrak{q}}),\ X\in\mathfrak{z}_{\mathfrak{p}_-}(\mathrm{Ad}(k)^{-1}\mathfrak{a}_{\mathfrak{q}})\}. \tag{3.1}$$

Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}_C$ such that $\sigma(\mathfrak{u})=\mathfrak{u}$, and τ the conjugation of $\mathfrak{g}_C$ with respect to $\mathfrak{u}$, which is a Cartan involution of $\mathfrak{g}_C$. Put $\mathfrak{h}_{\mathfrak{u}}=\mathfrak{u}\cap\mathfrak{h}_C$ and $\mathfrak{q}_{\mathfrak{u}}=\mathfrak{u}\cap\mathfrak{q}_C$. Let (G_C, H_C, σ) be a complexification of (G, H, σ) and let U denote the analytic subgroup of G_C corresponding to $\mathfrak{u}$.

LEMMA 6. $\varphi(U)=\exp\mathfrak{q}_{\mathfrak{u}}$.

PROOF. In the compact symmetric space U/U_σ, every geodesic starting from the origin is of the form $\exp(tX)U_\sigma$ with an $X\in\mathfrak{q}_{\mathfrak{u}}$ $(t\in R)$. Since U/U_σ is complete, $U=\exp(\mathfrak{q}_{\mathfrak{u}})U_\sigma$. Hence $\varphi(U)\subset\exp\mathfrak{q}_{\mathfrak{u}}$. The reverse inclusion is clear. q.e.d.

LEMMA 7. *A maximal torus contained in* $\exp\mathfrak{q}_{\mathfrak{u}}$ *is a maximal abelian subset of* $\exp\mathfrak{q}_{\mathfrak{u}}$.

The proof of this Lemma is the same as that of [**5**, p. 247, Corollary 2.7].

Let $\mathfrak{a}_{\mathfrak{q}_C}$ be a τ-stable A-subspace with respect to the symmetric Lie algebra $(\mathfrak{g}_C, \mathfrak{h}_C, \sigma)$ and $A_{\mathfrak{q}_C}$ the associated A-subset with respect to (G_C, H_C, σ). Put $\mathfrak{a}_{\mathfrak{q}_u}=\mathfrak{a}_{\mathfrak{q}_C}\cap\mathfrak{u}$. Then $\mathfrak{a}_{\mathfrak{q}_C}=\mathfrak{a}_{\mathfrak{q}_u}+\sqrt{-1}\mathfrak{a}_{\mathfrak{q}_u}$.

It follows from Lemma 6 and Lemma 7 that $Z_{\varphi(U)}(\mathfrak{a}_{\mathfrak{q}_C})=Z_{\varphi(U)}(\mathfrak{a}_{\mathfrak{q}_u})=\exp\mathfrak{a}_{\mathfrak{q}_u}$. Hence it follows from (3.1) that $A_{\mathfrak{q}_C}$ is connected. On the other hand $\exp\mathfrak{a}_{\mathfrak{q}_C}=\exp\mathfrak{a}_{\mathfrak{q}_u}\exp\sqrt{-1}\mathfrak{a}_{\mathfrak{q}_u}$ is a closed subgroup of G_C since $\exp\mathfrak{a}_{\mathfrak{q}}$ is closed.

LEMMA 8. $$A_{\mathfrak{q}_C}=\exp\mathfrak{a}_{\mathfrak{q}_C}.$$

PROOF. It is clear that $A_{\mathfrak{q}_C}\supset\exp\mathfrak{a}_{\mathfrak{q}_C}$. Since $a(\exp\mathfrak{a}_{\mathfrak{q}_C})$ acts on $A_{\mathfrak{q}_C}$ and acts transitively on $\exp\mathfrak{a}_{\mathfrak{q}_C}$ and since $A_{\mathfrak{q}_C}$ is connected, it suffices to show that there exists a neighborhood V of the identity in G_C such that $A_{\mathfrak{q}_C}\cap V\subset\exp\mathfrak{a}_{\mathfrak{q}_C}$. Let V' be a neighborhood of the origin in $\mathfrak{q}_C$ such that the restriction of the exponential map to V' is a diffeomorphism onto $V\cap\varphi(G_C)$ for some neighborhood V of the identity in G_C. Let Y be an element of V' such that $\exp Y\in A_{\mathfrak{q}_C}\cap V$. Then $e^{\mathrm{ad}\,Y}Y_1=Y_1$ for all $Y_1\in\mathfrak{a}_{\mathfrak{q}_C}$. If V' is sufficiently small, it follows that $[Y, Y_1]=0$ for all $Y_1\in\mathfrak{a}_{\mathfrak{q}_C}$ which implies $Y\in\mathfrak{a}_{\mathfrak{q}_C}$. q. e. d.

LEMMA 9. *Let $\mathfrak{a}_{\mathfrak{q}}$ be an A-subspace and $A_{\mathfrak{q}}$ the associated A-subset. Then every element of $A_{\mathfrak{q}}$ is semi-simple and $\mathfrak{z}_{\mathfrak{g}}(A_{\mathfrak{q}})=\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})$.*

PROOF. Let (G_C, H_C, σ) be a complexification of (G, H, σ), $\mathfrak{a}_{\mathfrak{q}_C}$ the complexification of $\mathfrak{a}_{\mathfrak{q}}$ and $A_{\mathfrak{q}_C}$ the associated A-subset. Let $y\in A_{\mathfrak{q}}$. Then $\mathrm{Ad}(y)=\mathrm{Ad}(\iota(y))\in\mathrm{Ad}(A_{\mathfrak{q}_C})$. Since $A_{\mathfrak{q}_C}=\exp\mathfrak{a}_{\mathfrak{q}_C}$, it follows that y is semi-simple. Let $X\in\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})$. Then $\mathrm{Ad}(y)X=\mathrm{Ad}(\iota(y))X=X$ because $A_{\mathfrak{q}_C}=\exp\mathfrak{a}_{\mathfrak{q}_C}$. This implies $X\in\mathfrak{z}_{\mathfrak{g}}(A_{\mathfrak{q}})$. Hence $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})\subset\mathfrak{z}_{\mathfrak{g}}(A_{\mathfrak{q}})$. The reverse inclusion is clear. q. e. d.

It follows from (3.1) that every connected component of $A_{\mathfrak{q}}$ contains an element of $\varphi(K)$. Let $A_{\mathfrak{q}j}$ $(j\in J)$ be the connected components of $A_{\mathfrak{q}}$ and let k_j $(j\in J)$ be an element of $A_{\mathfrak{q}j}\cap\varphi(K)$. Note that the subgroup $\exp\mathfrak{a}_{\mathfrak{q}}$ of G is closed in G. This follows from the fact that the Lie algebra of the closure of $\exp\mathfrak{a}_{\mathfrak{q}}$ is an abelian subspace of $\mathfrak{q}$ and the fact that $\mathfrak{a}_{\mathfrak{q}}$ is maximal abelian in $\mathfrak{q}$.

PROPOSITION 1. $A_{\mathfrak{q}j}=k_j\exp\mathfrak{a}_{\mathfrak{q}}$ $(j\in J)$. *If G is of finite center then J is a finite set.*

PROOF. It is clear that $A_{\mathfrak{q}j}\supset k_j\exp\mathfrak{a}_{\mathfrak{q}}$, so it suffices to prove $A_{\mathfrak{q}j}\subset k_j\exp\mathfrak{a}_{\mathfrak{q}}$. Since $a(y)$ with a $y\in\exp\mathfrak{a}_{\mathfrak{q}}$ stabilizes $A_{\mathfrak{q}j}$ and since $a(\exp\mathfrak{a}_{\mathfrak{q}})$ acts transitively on $k_j\exp\mathfrak{a}_{\mathfrak{q}}$, we have only to prove that there exists a neighborhood V of the identity in G such that $k_jV\cap A_{\mathfrak{q}j}\subset k_j\exp\mathfrak{a}_{\mathfrak{q}}$.

Let V' be an open neighborhood of the origin in $\mathfrak{g}$ such that:

(i) $V'=-V'=\sigma(V')$;

(ii) the restriction of the exponential mapping to V' is a diffeomorphism into G;

(iii) $\exp(V'\cap\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}}))=\exp V'\cap Z_G(\mathfrak{a}_{\mathfrak{q}})$.

Putting $V=\exp V'$, every element x of $V\cap k_j^{-1}A_{\mathfrak{q}j}$ can be uniquely written as $x=\exp X$ with an $X\in V'\cap\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})$ since $k_j^{-1}A_{\mathfrak{q}j}\subset Z_G(\mathfrak{a}_{\mathfrak{q}})$. Then

$$x^{-1}k_j^{-1}=\sigma(k_jx)=k_j^{-1}\sigma(x)\,.$$

Since $\mathrm{Ad}(k_j)X=X$ (Lemma 9), $\exp(-X)=\exp\sigma(X)$. Hence $X\in\mathfrak{q}$, which proves $X\in\mathfrak{a}_{\mathfrak{q}}$. Thus we have $k_jV\cap A_{\mathfrak{q}j}\subset k_j\exp\mathfrak{a}_{\mathfrak{q}}$.

We have an open covering $C=\{a(y)(k_jV\cap\varphi(K))\mid y\in\exp\mathfrak{a}_{\mathfrak{t}_-},\ j\in J\}$ of $A_{\mathfrak{q}}\cap\varphi(K)$ in $\varphi(K)$. If J is an infinite set, there exist no finite subsets of C which cover $A_{\mathfrak{q}}\cap\varphi(K)$. Hence $A_{\mathfrak{q}}\cap\varphi(K)$ is non-compact. On the other hand, if G is of finite center, then K is compact, so are $\varphi(K)$ and $A_{\mathfrak{q}}\cap\varphi(K)$, which is a contradiction. q. e. d.

§ 4. $A_{\mathfrak{q}}$-regular elements.

Let (G, H, σ) be an affine symmetric space and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ the corresponding symmetric Lie algebra. Let $\mathfrak{a}_{\mathfrak{q}}$ be an A-subspace and $A_{\mathfrak{q}}$ the associated A-subset. Let Z be the center of G and L denote the analytic subgroup of G corresponding to $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})$. Since the Lie algebra of $Z_G(A_{\mathfrak{q}})$ is $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})$ (Lemma 9), $Z_G(A_{\mathfrak{q}})\supset ZL$. On the other hand $N_G(\mathfrak{a}_{\mathfrak{q}})\subset N_G(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}}))=N_G(L)$. Since $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})$ is reductive in $\mathfrak{g}$ and since $\mathrm{rank}(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}}))=\mathrm{rank}\,\mathfrak{g}$, it follows from [**12**, Proposition 1.4.2.4] that $N_G(L)/ZL$ is finite. Hence $N_G(\mathfrak{a}_{\mathfrak{q}})/Z_G(A_{\mathfrak{q}})$ is finite, so is $N_H(\mathfrak{a}_{\mathfrak{q}})/Z_H(A_{\mathfrak{q}})$. Put

$$W(A_{\mathfrak{q}}, H)=N_H(\mathfrak{a}_{\mathfrak{q}})/Z_H(A_{\mathfrak{q}})$$

and put $A_{\mathfrak{q}}'=A_{\mathfrak{q}}\cap\varphi(G)'$. Then $A_{\mathfrak{q}}'$ is an open dense subset of $A_{\mathfrak{q}}$.

LEMMA 10. *Let η be the mapping of $H/Z_H(A_{\mathfrak{q}})\times A_{\mathfrak{q}}'$ into $\varphi(G)'$ defined by $\eta(hZ_H(A_{\mathfrak{q}}), y)=hyh^{-1}$ $(h\in H,\ y\in A_{\mathfrak{q}}')$. Then* $\mathrm{Im}\,\eta$ *is an open subset of $\varphi(G)'$ and η is an everywhere regular $|W(A_{\mathfrak{q}}, H)|$-to-one mapping.*

PROOF. Suppose $h_1y_1h_1^{-1}=h_2y_2h_2^{-1}$ $(h_1, h_2\in H$ and $y_1, y_2\in A_{\mathfrak{q}}')$. Then $\mathrm{Ad}(h_2^{-1}h_1)\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})=\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{q}})$ and therefore $\mathrm{Ad}(h_2^{-1}h_1)\mathfrak{a}_{\mathfrak{q}}=\mathfrak{a}_{\mathfrak{q}}$. Hence $h_2^{-1}h_1\in N_H(\mathfrak{a}_{\mathfrak{q}})$. Conversely suppose $h\in N_H(\mathfrak{a}_{\mathfrak{q}})$. Then $hZ_G(\mathfrak{a}_{\mathfrak{q}})h^{-1}=Z_G(\mathfrak{a}_{\mathfrak{q}})$ and $h\varphi(G)h^{-1}=\varphi(G)$. Hence $hA_{\mathfrak{q}}h^{-1}=A_{\mathfrak{q}}$ and therefore $hA_{\mathfrak{q}}'h^{-1}=A_{\mathfrak{q}}'$. Thus η is a $|W(A_{\mathfrak{q}}, H)|$-to-one mapping.

Let $h\in H$, $X\in\tilde{\mathfrak{h}}$, $y\in A_{\mathfrak{q}}'$, $Y\in\mathfrak{a}_{\mathfrak{q}}$ and $t\in R$. Then

$$\begin{aligned}&h\exp(tX)y\exp tY(h\exp tX)^{-1}\\&\qquad=hyh^{-1}\exp(t\mathrm{Ad}(h)((\mathrm{Ad}(y^{-1})-1)X+Y)+o(t))\,.\end{aligned}$$

Since $y\in A_{\mathfrak{q}}'$, $(\mathrm{Ad}(y^{-1})-1)|_{\tilde{\mathfrak{g}}}:\tilde{\mathfrak{g}}\to\tilde{\mathfrak{g}}$ is a bijection. Thus it follows from (2.2) that η is regular and that $\mathrm{Im}\,\eta$ is an open subset of $\varphi(G)'$. q. e. d.

REMARK. Let θ be a Cartan involution of $\mathfrak{g}$ commutative with σ, $K=G_\theta$ and $K_+=K\cap H$. Let $\mathfrak{a}_\mathfrak{q}$ be a θ-stable A-subspace and $A_\mathfrak{q}$ the associated A-subset. Put

$$W(\mathfrak{a}_\mathfrak{q}, K_+)=N_{K_+}(\mathfrak{a}_\mathfrak{q})/Z_{K_+}(\mathfrak{a}_\mathfrak{q})$$

and

$$W(A_\mathfrak{q}, K_+)=N_{K_+}(\mathfrak{a}_\mathfrak{q})/Z_{K_+}(A_\mathfrak{q}).$$

Then $W(\mathfrak{a}_\mathfrak{q}, H)\cong W(\mathfrak{a}_\mathfrak{q}, K_+)$ and $W(A_\mathfrak{q}, H)\cong W(A_\mathfrak{q}, K_+)$. This is proved easily by using [**12**, Lemma 1.1.3.7].

The following lemma will be used in § 6.

LEMMA 11. *Let x be a semi-simple element of $\varphi(G)$. Let μ be the mapping of $H\times\mathfrak{z}_\mathfrak{q}(x)$ into $\varphi(G)$ defined by $\mu(h, X)=hx\exp Xh^{-1}$ ($h\in H$, $X\in\mathfrak{z}_\mathfrak{q}(x)$). Then the rank of μ at $(1, 0)$ equals the dimension of $\varphi(G)$.*

PROOF. Since the rank of the mapping $\mu: G\times\mathfrak{z}(x)\to G$ defined by $\mu(g, X)=gx\exp Xg^{-1}$ equals the dimension of G ([**12**, Lemma 1.4.3.1]) and since $\mathfrak{z}_\mathfrak{g}(x)=\mathfrak{z}_\mathfrak{h}(x)+\mathfrak{z}_\mathfrak{q}(x)$ (direct sum), we have only to prove the following. If a curve $\gamma_t=\mu(\exp tY, tX)$ ($Y\in\mathfrak{q}$, $X\in\mathfrak{z}_\mathfrak{h}(x)$) is tangent to $\varphi(G)$ at $t=0$, then the tangent vector to γ_t at $t=0$ is zero. This is proved as follows. The tangent vector to $\sigma(\gamma_t)^{-1}$ at $t=0$ is equal to that of γ_t. Then comparing

$$\begin{aligned}\sigma(\gamma_t)^{-1}&=(\exp(tY)x\exp tX\exp(-tY))^{-1}\\&=\exp(-tY)\exp(-tX)x\exp tY\\&=x\exp(t(-\mathrm{Ad}(x^{-1})Y+Y-X)+o(t))\end{aligned}$$

and

$$\gamma_t=x\exp(t(\mathrm{Ad}(x^{-1})Y-Y+X)+o(t)),$$

we have $\mathrm{Ad}(x^{-1})Y-Y+X=0$ and therefore the tangent vector to γ_t at $t=0$ is zero.

§ 5. Jordan decomposition.

Let $S_\mathfrak{g}$ denote the set of semi-simple elements in $\mathfrak{g}$ and $N_\mathfrak{g}$ the set of nilpotent elements in $\mathfrak{g}$. Let X be an element of $\mathfrak{q}$ and $X=X_s+X_n$ ($X_s\in S_\mathfrak{g}$, $X_n\in N_\mathfrak{g}$) be the Jordan decomposition of X. Then $-X_s-X_n=-X=\sigma(X)=\sigma(X_s)+\sigma(X_n)$. It follows from the uniqueness of the Jordan decomposition that $\sigma(X_s)=-X_s$ and $\sigma(X_n)=-X_n$. Hence $X_s, X_n\in\mathfrak{q}$.

Let S_G denote the set of semi-simple elements in G and N_G the set of unipotent elements in G. Let $x\in\varphi(G)$ and $x=x_sx_u$ ($x_s\in S_G$, $x_u\in N_G$) be the Jordan decomposition of x. Then $x_s^{-1}x_u^{-1}=x_u^{-1}x_s^{-1}=x^{-1}=\sigma(x)=\sigma(x_s)\sigma(x_u)$. It follows from the uniqueness of the Jordan decomposition that $\sigma(x_s)=x_s^{-1}$ and

$\sigma(x_u)=x_u^{-1}$. Moreover since $\exp|_{N_\mathfrak{g}}: N_\mathfrak{g}\to N_G$ is a bijection, $x_u=\exp X_n$ for some $X_n\in N_\mathfrak{g}\cap\mathfrak{q}$. Hence $x_u=\varphi\left(\exp\frac{1}{2}X_n\right)\in\varphi(G)$. On the other hand, since $\mathrm{Ad}(x_s)X_n=X_n$,

$$x=x_sx_u=a\left(\exp\frac{1}{2}X_n\right)x_s.$$

Hence $x_s=a\left(\exp\left(-\frac{1}{2}X_n\right)\right)x\in\varphi(G)$.

LEMMA 12. *If $X\in N_\mathfrak{g}\cap\mathfrak{q}$, then there exists a $Y\in\mathfrak{h}$ such that $[Y, X]=2X$.*

PROOF. It is known that there is a $Y\in\mathfrak{g}$ such that $[Y, X]=2X$ ([**6**, p. 100]). Let $Y=Y_1+Y_2$ ($Y_1\in\mathfrak{h}$, $Y_2\in\mathfrak{q}$). Since $[Y_1, X]\in\mathfrak{q}$ and $[Y_2, X]\in\mathfrak{h}$, we have $[Y_2, X]=0$ and therefore $[Y_1, X]=2X$. q. e. d.

PROPOSITION 2. (i) *Let $X\in\mathfrak{q}$ and $X=X_s+X_n$ be the Jordan decomposition of X. Then X_s, $X_n\in\mathfrak{q}$ and $X_s\in(\mathrm{Ad}(H_0)X)^{cl}$.*

(ii) *Let $x\in\varphi(G)$ and $x=x_sx_u$ be the Jordan decomposition of x.* Then *x_s, $x_u\in\varphi(G)$ and $x_s\in(a(H_0)x)^{cl}$.*

This proposition can be proved in the same way as in [**12**, p. 106 and p. 121] using Lemma 12.

§6. The main theorems.

THEOREM 1. *Let (G_C, H_C, σ) be a complex affine symmetric space and $(\mathfrak{g}_C, \mathfrak{h}_C, \sigma)$ the corresponding complex symmetric Lie algebra. Let $\mathfrak{a}_{\mathfrak{q}_C}$ be an A-subspace and $A_{\mathfrak{q}_C}$ the associated A-subset. Then*

(i) *The mapping $\zeta_C: H_C/Z_{H_C}(\mathfrak{a}_{\mathfrak{q}_C})\times\mathfrak{a}'_{\mathfrak{q}_C}\to\mathfrak{q}'_C$ defined by $\zeta_C(hZ_{H_C}(\mathfrak{a}_{\mathfrak{q}_C}), Y)=\mathrm{Ad}(h)Y$ $(h\in H_C,\ Y\in\mathfrak{a}'_{\mathfrak{q}_C})$ is an everywhere regular $|W(\mathfrak{a}_{\mathfrak{q}_C}, H_C)|$-to-one mapping onto $\mathfrak{q}'_C$.*

(ii) *The mapping $\eta_C: H_C/Z_{H_C}(\mathfrak{a}_{\mathfrak{q}_C})\times A'_{\mathfrak{q}_C}\to\varphi(G_C)'$ defined by $\eta_C(hZ_{H_C}(\mathfrak{a}_{\mathfrak{q}_C}), Y)$ $=hyh^{-1}$ $(h\in H_C,\ y\in A'_{\mathfrak{q}_C})$ is an everywhere regular $|W(\mathfrak{a}_{\mathfrak{q}_C}, H_C)|$-to-one mapping onto $\varphi(G_C)'$.*

PROOF. Seeing Lemma 4 and Lemma 10, we have only to prove the ontoness of ζ_C and η_C. Moreover since d_k and D_k are holomorphic functions on $\mathfrak{q}_C$ and $\varphi(G_C)$, respectively, $\mathfrak{q}'_C$ and $\varphi(G_C)'$ are connected. Since $\mathrm{Im}\,\zeta_C$ and $\mathrm{Im}\,\eta_C$ are open sets, it suffices to prove that $\mathrm{Im}\,\zeta_C$ and $\mathrm{Im}\,\eta_C$ are closed subsets of $\mathfrak{q}'_C$ and $\varphi(G_C)'$ respectively.

(i) $\mathrm{Im}\,\zeta_C$ is closed in $\mathfrak{q}'_C$. Suppose $\mathrm{Im}\,\zeta_C$ is not closed in $\mathfrak{q}'_C$. Then there exists an $X\in(\mathrm{Im}\,\zeta_C)^{cl}\cap\mathfrak{q}'_C$ such that $X\notin\mathrm{Im}\,\zeta_C$. Let $X=X_s+X_n$ be the Jordan decomposition of X. Then it follows from Proposition 2 that $X_s\in(\mathrm{Ad}(H_C)_0X)^{cl}$ and therefore $X_s\in((\mathrm{Im}\,\zeta_C)^{cl}\cap\mathfrak{q}'_C)-\mathrm{Im}\,\zeta_C$.

Let ξ be the mapping of $\mathfrak{h}_C\times\mathfrak{z}_{\mathfrak{q}_C}(X_s)$ into $\mathfrak{q}_C$ defined by $\xi(Y, Y_1)=e^{\mathrm{ad}\,Y}(X_s+Y_1)$ $(Y\in\mathfrak{h}_C,\ Y_1\in\mathfrak{z}_{\mathfrak{q}_C}(X_s))$. Then the differential of ξ at $(0, 0)$ is given by the linear map $(Y, Y_1)\to Y_1+[Y, X_s]$ of $\mathfrak{h}_C\times\mathfrak{z}_{\mathfrak{q}_C}(X_s)$ onto $\mathfrak{q}_C$. Hence $\mathrm{Im}\,\xi$ contains a neighborhood of X_s in $\mathfrak{q}_C$ and therefore $\mathrm{Im}\,\xi\cap\mathrm{Im}\,\zeta_C\neq\emptyset$. Thus $\mathfrak{z}_{\mathfrak{q}_C}(X_s)\cap\mathrm{Im}\,\zeta_C\neq\emptyset$. Let Y_2 be an element of $\mathfrak{z}_{\mathfrak{q}_C}(X_s)\cap\mathrm{Im}\,\zeta_C$. Then $Y_2=\mathrm{Ad}(h)Y_3$ for some $h\in H_C$ and $Y_3\in\mathfrak{a}'_{\mathfrak{q}_C}$. Therefore $\mathfrak{z}_{\mathfrak{q}_C}(Y_2)=\mathrm{Ad}(h)\mathfrak{z}_{\mathfrak{q}_C}(Y_3)=\mathrm{Ad}(h)\mathfrak{a}_{\mathfrak{q}_C}$. Since $X_s\in\mathfrak{z}_{\mathfrak{q}_C}(Y_2)$, we have $X_s\in\mathrm{Ad}(h)\mathfrak{a}'_{\mathfrak{q}_C}\subset\mathrm{Im}\,\zeta_C$ a contradiction.

(ii) $\mathrm{Im}\,\eta_C$ is closed in $\varphi(G_C)'$. Suppose that $\mathrm{Im}\,\eta_C$ is not closed in $\varphi(G_C)'$. Then we have a semi-simple element x contained in $((\mathrm{Im}\,\eta_C)^{cl}\cap\varphi(G_C)')-\mathrm{Im}\,\eta_C$ by a similar argument given in (i). Let μ be the mapping of $H_C\times\exp\mathfrak{z}_{\mathfrak{q}_C}(x)$ into $\varphi(G_C)$ defined by $\mu(y, y_1)=yxy_1y^{-1}$ ($y\in H_C$, $y_1\in\exp\mathfrak{z}_{\mathfrak{q}_C}(x)$). Then it follows from Lemma 11 that $\mathrm{Im}\,\mu$ contains a neighborhood of x in $\varphi(G)$.

We put $W=N_{G_C}(\mathfrak{z}_{\mathfrak{g}_C}(\mathfrak{a}_{\mathfrak{q}_C}))/Z_{G_C}(\mathfrak{a}_{\mathfrak{q}_C})$. Then W is finite as is stated in the beginning of §4 and for every $w\in W$ the natural mappings $w: \mathfrak{a}_{\mathfrak{q}_C}\to\mathfrak{z}_{\mathfrak{g}_C}(\mathfrak{a}_{\mathfrak{q}_C})$ and $w: A_{\mathfrak{q}_C}\to(Z_{G_C}(\mathfrak{a}_{\mathfrak{q}_C}))_0$ are well-defined. Let $W'=\{w\in W\mid w(Y)\neq Y$ for some $Y\in\mathfrak{a}_{\mathfrak{q}_C}\}$ and

$$A''_{\mathfrak{q}_C}=\{y\in A'_{\mathfrak{q}_C}\mid w(y)\neq y \text{ for all } w\in W'\}\,. \tag{6.1}$$

Then $A''_{\mathfrak{q}_C}$ is a dense open subset of $A_{\mathfrak{q}_C}$ since W' is finite.

Thus $\mathrm{Im}\,\mu\cap\eta_C(H_C/Z_{H_C}(\mathfrak{a}_{\mathfrak{q}_C})\times A''_{\mathfrak{q}_C})\neq\emptyset$, and therefore there exist $h\in H_C$ and $y_1\in A''_{\mathfrak{q}_C}$ such that hy_1h^{-1} is commutative with x. Put $x_1=h^{-1}xh$. Then x_1 is commutative with y_1 and $x_1\notin\mathrm{Im}\,\eta_C$. Since

$$\mathrm{Ad}(x_1)\mathfrak{z}_{\mathfrak{g}_C}(y_1)=\mathfrak{z}_{\mathfrak{g}_C}(y_1)\,,$$

we have $\mathrm{Ad}(x_1)\mathfrak{z}_{\mathfrak{g}_C}(\mathfrak{a}_{\mathfrak{q}_C})=\mathfrak{z}_{\mathfrak{g}_C}(\mathfrak{a}_{\mathfrak{q}_C})$. Hence $x_1\in N_{G_C}(\mathfrak{z}_{\mathfrak{g}_C}(\mathfrak{a}_{\mathfrak{q}_C}))$. Since $y_1\in A''_{\mathfrak{q}_C}$, it follows from (6.1) that $x_1\in Z_{G_C}(\mathfrak{a}_{\mathfrak{q}_C})$ and therefore $x_1\in A'_{\mathfrak{q}_C}$ which is a contradiction. q.e.d.

THEOREM 2. *Let (G, H, σ) be an affine symmetric space and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ the corresponding symmetric Lie algebra. Let $\{\mathfrak{a}_{\mathfrak{q}i}\mid i\in I\}$ be a set of representatives of H-conjugacy classes of A-subspaces and $A_{\mathfrak{q}i}$ the A-subset associated with $\mathfrak{a}_{\mathfrak{q}i}$. Then*

(i) *$\mathfrak{q}'=\bigcup_{i\in I}\mathrm{Ad}(H)\mathfrak{a}'_{\mathfrak{q}i}$ (disjoint union) and the mapping $\zeta_i: H/Z_H(\mathfrak{a}_{\mathfrak{q}i})\times\mathfrak{a}'_{\mathfrak{q}i}\to\mathfrak{q}'$ defined by $\zeta_i(hZ_H(\mathfrak{a}_{\mathfrak{q}i}), Y)=\mathrm{Ad}(h)Y$ ($h\in H$, $Y\in\mathfrak{a}'_{\mathfrak{q}i}$) is an everywhere regular $|W(\mathfrak{a}_{\mathfrak{q}i}, H)|$-to-one mapping.*

(ii) *Put $\varphi(G)_i=\bigcup_{h\in H}hA'_{\mathfrak{q}i}h^{-1}$. Then $\varphi(G)'=\bigcup_{i\in I}\varphi(G)_i$ (disjoint union) and the mapping $\eta_i: H/Z_H(A_{\mathfrak{q}i})\times A'_{\mathfrak{q}i}\to\varphi(G)'$ defined by $\eta_i(hZ_H(A_{\mathfrak{q}i}), y)=hyh^{-1}$ ($h\in H$, $y\in A'_{\mathfrak{q}i}$) is an everywhere regular $|W(A_{\mathfrak{q}i}, H)|$-to-one mapping. Moreover if the affine symmetric space (G, H, σ) has a complexification (G_C, H_C, σ) such that $\iota: G\to G_c$ is injective, then $Z_H(A_{\mathfrak{q}i})=Z_H(\mathfrak{a}_{\mathfrak{q}i})$ and $W(A_{\mathfrak{q}i}, H)=W(\mathfrak{a}_{\mathfrak{q}i}, H)$.*

PROOF. (i) Disjointness is clear. In view of Lemma 4 it suffices to prove $\mathfrak{q}'\subset\bigcup_{i\in I}\mathrm{Ad}(H)\mathfrak{a}'_{\mathfrak{q}i}$. Let $X\in\mathfrak{q}'$. It follows from (i) of Theorem 1 that $\mathfrak{z}_{\mathfrak{q}_C}(X)$ is an A-subspace of $(\mathfrak{g}_C, \mathfrak{h}_C, \sigma)$. Since $\mathfrak{z}_{\mathfrak{q}_C}(X)$ is stable by the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$, $\mathfrak{z}_{\mathfrak{q}}(X)$ is an A-subspace of $(\mathfrak{g}, \mathfrak{h}, \sigma)$. Hence there exist an $i\in I$ and an $h\in H$ such that $\mathfrak{z}_{\mathfrak{q}}(X)=\mathrm{Ad}(h)\mathfrak{a}_{\mathfrak{q}i}$ and therefore $X\in \mathrm{Ad}(h)\mathfrak{a}'_{\mathfrak{q}i}$.

(ii) In view of Lemma 10 it suffices to prove that $\varphi(G)'\subset\bigcup_{i\in I}\varphi(G)_i$ and that $\bigcup_{i\in I}\varphi(G)_i$ is disjoint union. Let $x\in\varphi(G)'$ and let (G_C, H_C, σ) be a complexification of (G, H, σ). Then it follows from Theorem 1 that $\iota(x)$ is contained in the A-subset $A_{\mathfrak{q}_C}$ associated with some A-subspace $\mathfrak{a}_{\mathfrak{q}_C}$ of $(\mathfrak{g}_C, \mathfrak{h}_C, \sigma)$. Clearly $\mathfrak{a}_{\mathfrak{q}_C}=\mathfrak{z}_{\mathfrak{q}_C}(x)$ because x is $\varphi(G)$-regular. Hence $\mathfrak{z}_{\mathfrak{q}}(x)$ is an A-subspace of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ and therefore $x\in hA_{\mathfrak{q}i}h^{-1}$ for some $i\in I$ and $h\in H$. Disjointness of $\bigcup_{i\in I}\varphi(G)_i$ and the last statement is clear. q. e. d.

COROLLARY. (i) *Let $X\in\mathfrak{q}$. Then X is semi-simple if and only if X is contained in some A-subspace.*

(ii) *Let $x\in\varphi(G)$. Then x is semi-simple if and only if x is contained in some A-subset.*

PROOF (cf. [**12**, p. 105 and p. 120]). (i) Let X be a semi-simple element in $\mathfrak{q}$. Consider the map ξ of $\mathfrak{h}\times\mathfrak{z}_{\mathfrak{q}}(X)$ into $\mathfrak{q}$ given by $\xi(Y, Y_1)=e^{\mathrm{ad}\,Y}(X+Y_1)$ for $Y\in\mathfrak{h}$, $Y_1\in\mathfrak{z}_{\mathfrak{q}}(X)$ (cf. the proof of Theorem 1). Then the image of ξ contains a neighborhood of X in $\mathfrak{q}$. Thus there is a $\mathfrak{q}$-regular element Γ such that $\Gamma=e^{\mathrm{ad}\,Y}(X+Y_1)$ for some $Y\in\mathfrak{h}$ and $Y_1\in\mathfrak{z}_{\mathfrak{q}}(X)$. Since $X+Y_1$ is also $\mathfrak{q}$-regular, $\mathfrak{z}_{\mathfrak{q}}(X+Y_1)$ is an A-subspace (Theorem 2) containing X. The converse assertion follows from the definition of A-subspaces.

(ii) Every element of an A-subset is semi-simple (Lemma 9). Conversely let x be a semi-simple element in $\varphi(G)$. Define an analytic function δ on $\mathfrak{z}_{\mathfrak{q}}(x)$ by

$$\delta(Y)=\det((\mathrm{Ad}(x\exp Y)-1)|_{(\mathrm{Ad}(x)-1)\mathfrak{g}}).$$

Then $\delta(0)\neq 0$. Put $\mathfrak{z}_{\mathfrak{q}}(x)'=\{Y\in\mathfrak{z}_{\mathfrak{q}}(x)\,|\,\delta(Y)\neq 0\}$. Since $\bigcup_{h\in H}hx\exp\mathfrak{z}_{\mathfrak{q}}(x)'h^{-1}$ contains a neighborhood of x in $\varphi(G)$ (Lemma 11), there exists a $\varphi(G)$-regular element y such that $y\in x\exp\mathfrak{z}_{\mathfrak{q}}(x)'$. Then $\mathfrak{z}_{\mathfrak{q}}(y)$ is an A-subspace (Theorem 2). Let $X\in\mathfrak{z}_{\mathfrak{q}}(y)$. Since x is semi-simple, $\mathfrak{g}=\mathfrak{z}_{\mathfrak{g}}(x)+(\mathrm{Ad}(x)-1)\mathfrak{g}$ (direct sum). Therefore X can be written as $X=X_1+X_2$ for some $X_1\in\mathfrak{z}_{\mathfrak{g}}(x)$ and $X_2\in(\mathrm{Ad}(x)-1)\mathfrak{g}$. Since $\mathfrak{z}_{\mathfrak{g}}(x)$ and $(\mathrm{Ad}(x)-1)\mathfrak{g}$ are $\mathrm{Ad}(y)$-stable, we have $(\mathrm{Ad}(y)-1)X_2=0$. Then $X_2=0$ because $y\in x\exp\mathfrak{z}_{\mathfrak{q}}(x)'$. Hence $X=X_1\in\mathfrak{z}_{\mathfrak{q}}(x)$. Thus $\mathfrak{z}_{\mathfrak{q}}(y)\subset\mathfrak{z}_{\mathfrak{q}}(x)$. This implies that x is contained in the A-subset associated with the A-subspace $\mathfrak{z}_{\mathfrak{q}}(y)$. q. e. d.

§ 7. *H*-conjugacy classes of *A*-subspaces.

Let θ be a Cartan involution of $\mathfrak{g}$ commutative with σ. Then as is stated in the remark following Lemma 5, every A-subspace is H_0-conjugate to a θ-

stable A-subspace. Moreover two θ-stable A-subspaces are H-conjugate if and only if they are K_+-conjugate. Here we put $K_+=K\cap H$ and $K=G_\theta$. This is proved easily by using [**12**, Lemma 1.1.3.7]. Hence in order to determine the H-conjugacy classes of A-subspaces, we have only to consider the K_+-conjugacy classes of θ-stable A-subspaces.

Let $\mathfrak{k}$, $\mathfrak{k}_+$, $\mathfrak{k}_-$, $\mathfrak{p}$, $\mathfrak{p}_+$ and $\mathfrak{p}_-$ be as in § 1 and extend σ and θ to the complex linear automorphisms of $\mathfrak{g}_C$. Put $\bar{\mathfrak{g}}=\mathfrak{k}_++\sqrt{-1}\mathfrak{k}_-+\sqrt{-1}\mathfrak{p}_++\mathfrak{p}_-$, $\bar{\mathfrak{k}}=\mathfrak{k}_++\sqrt{-1}\mathfrak{k}_-$, $\bar{\mathfrak{p}}=\sqrt{-1}\mathfrak{p}_++\mathfrak{p}_-$, $\bar{\theta}=\theta|_{\bar{\mathfrak{g}}}$, and $\bar{\sigma}=\sigma|_{\bar{\mathfrak{g}}}$. Then the triple $(\bar{\mathfrak{g}}, \bar{\mathfrak{k}}, \bar{\theta})$ is a symmetric Lie algebra dual to $(\mathfrak{g}, \mathfrak{h}, \sigma)$ and $\bar{\sigma}$ is a Cartan involution of $\bar{\mathfrak{g}}$ commutative with $\bar{\theta}$. For a θ-stable A-subspace $\mathfrak{a}_\mathfrak{q}=\mathfrak{a}_{\mathfrak{k}_-}+\mathfrak{a}_{\mathfrak{p}_-}$ of $(\mathfrak{g}, \mathfrak{h}, \sigma)$, $\bar{\mathfrak{a}}_\mathfrak{q}=\sqrt{-1}\mathfrak{a}_{\mathfrak{k}_-}+\mathfrak{a}_{\mathfrak{p}_-}$ is a $\bar{\theta}$-stable maximal abelian subspace of $\bar{\mathfrak{q}}=\sqrt{-1}\mathfrak{k}_-+\mathfrak{p}_-$. Hence the problem is reduced to the determination of the K_+-conjugacy classes of $\bar{\theta}$-stable maximal abelian subspace of $\bar{\mathfrak{q}}$, which is studied in [**9**, § 2].

Theorem 2 of [**9**] can be rewritten as the following Theorem 3. Let $\mathfrak{a}_{\mathfrak{k}_-}$ be a maximal abelian subspace of $\mathfrak{k}_-$ and let $\mathfrak{a}_\mathfrak{q}=\mathfrak{a}_{\mathfrak{k}_-}+\mathfrak{a}_{\mathfrak{p}_-}$ be a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}_{\mathfrak{k}_-}$. We fix this A-subspace $\mathfrak{a}_\mathfrak{q}$ in the following. Put $\bar{\mathfrak{a}}_\mathfrak{q}=\sqrt{-1}\mathfrak{a}_{\mathfrak{k}_-}+\mathfrak{a}_{\mathfrak{p}_-}$ and let $\Phi(\bar{\mathfrak{a}}_\mathfrak{q})$ denote the root system of the pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{a}}_\mathfrak{q})$. Put $\Phi(\mathfrak{a}_{\mathfrak{k}_-})=\{\lambda\in\Phi(\bar{\mathfrak{a}}_\mathfrak{q})\,|\,H_\lambda\in\sqrt{-1}\mathfrak{a}_{\mathfrak{k}_-}\}$ where H_λ is the unique element of $\bar{\mathfrak{a}}_\mathfrak{q}$ such that $\lambda(H)=B(H_\lambda, H)$ for all $H\in\bar{\mathfrak{a}}_\mathfrak{q}$ ($B(\ ,\ )$ is the Killing form of $\mathfrak{g}_C$).

Let α_i $(i=1, \cdots, k)$ be elements of $\Phi(\mathfrak{a}_{\mathfrak{k}_-})$ and X_{α_i} $(i=1, \cdots, k)$ be non-zero elements of $\bar{\mathfrak{g}}_{\alpha_i}$ where $\bar{\mathfrak{g}}_{\alpha_i}=\{X\in\bar{\mathfrak{g}}\,|\,[Y, X]=\alpha_i(Y)X$ for all $Y\in\bar{\mathfrak{a}}_\mathfrak{q}\}$. Then $\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ is said to be a $\bar{\mathfrak{p}}$-*orthogonal system* of $\Phi(\mathfrak{a}_{\mathfrak{k}_-})$ ([**9**]) if the following two conditions are satisfied:

(i) $X_{\alpha_i}\in\bar{\mathfrak{p}}$ for $i=1, \cdots, k$,

(ii) $[X_{\alpha_i}, X_{\alpha_j}]=0$ and $[X_{\alpha_i}, \sigma(X_{\alpha_j})]=0$ for $i, j=1, \cdots, k, i\neq j$.

Two $\bar{\mathfrak{p}}$-orthogonal systems $\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ and $\{Y_{\beta_1}, \cdots, Y_{\beta_k}\}$ are said to be conjugate under $W(\mathfrak{a}_\mathfrak{q}, K_+)$ if there is a $w\in W(\mathfrak{a}_\mathfrak{q}, K_+)$ such that $w\left(\sum_{i=1}^k \boldsymbol{R}H_{\alpha_i}\right)=\sum_{i=1}^k \boldsymbol{R}H_{\beta_i}$.

THEOREM 3. *There is a one-to-one correspondence between the H-conjugacy classes of A-subspaces and the* $W(\mathfrak{a}_\mathfrak{q}, K_+)$*-conjugacy classes of* $\bar{\mathfrak{p}}$*-orthogonal systems of* $\Phi(\mathfrak{a}_{\mathfrak{k}_-})$. *The correspondence is given as follows. Let* $P=\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ *be a* $\bar{\mathfrak{p}}$*-orthogonal system of* $\Phi(\mathfrak{a}_{\mathfrak{k}_-})$. *Put* $\mathfrak{r}=\sqrt{-1}\sum_{i=1}^k \boldsymbol{R}H_{\alpha_i}$, $\mathfrak{a}^*_{\mathfrak{k}_-}=\{H\in\mathfrak{a}_{\mathfrak{k}_-}\,|\,B(H, \mathfrak{r})=0\}$, $\mathfrak{a}^*_{\mathfrak{p}_-}=\mathfrak{a}_{\mathfrak{p}_-}+\sum_{i=1}^k \boldsymbol{R}(X_{\alpha_i}-\sigma(X_{\alpha_i}))$ *and* $\mathfrak{a}^*_\mathfrak{q}=\mathfrak{a}^*_{\mathfrak{k}_-}+\mathfrak{a}^*_{\mathfrak{p}_-}$. *Then the* $W(\mathfrak{a}_\mathfrak{q}, K_+)$*-conjugacy class of* $\bar{\mathfrak{p}}$*-orthogonal systems of* $\Phi(\mathfrak{a}_{\mathfrak{k}_-})$ *containing P corresponds to the H-conjugacy class of A-subspaces containing* $\mathfrak{a}^*_\mathfrak{q}$. *Moreover if* X_{α_i} *is normalized such that* $2\alpha_i(H_{\alpha_i})B(X_{\alpha_i}, \sigma(X_{\alpha_i}))=-1$ *for* $i=1, \cdots, k$, *then*

$$(\mathfrak{a}_{\mathfrak{q}}^{*})_C = \mathrm{Ad}\,(\exp \frac{\pi}{2}(X_{\alpha_1}+\sigma(X_{\alpha_1})) \cdots \exp \frac{\pi}{2}(X_{\alpha_k}+\sigma(X_{\alpha_k}))(\mathfrak{a}_{\mathfrak{q}})_C$$

(in a complexification (G_C, H_C, σ) *of* (G, H, σ)*).*

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