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On the existence of optimal controls of the first boundary value problems for parabolic partial delay-differential equations in divergence form

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1. Introduction.

A number of results concerning the existence of optimal controls for systems governed by parabolic partial differential equations without delays has been reported in [1-5] and others. However, it appear that only few results are available for systems with time delayed arguments appearing in the coefficients of the differential equations ([3, p. 262], [6]).

In [6], Teo and Ahmed considered the existence of optimal controls for second order parabolic partial delay-differential equations with controls and delayed arguments appearing in the first and zero order coefficients and the forcing terms. Further, the solutions of that system satisfy the differential equations a. e.

In this paper, we consider the question on the existence of optimal controls which minimize a given cost functional subject to the system monitored by the following parabolic partial differential equations

$$\begin{cases} L(u)\phi(x, t)=\sum_{k=0}^{N}\left\{\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(F_{kj}(x, t-h_k, u(x, t-h_k))) \right. \\ \qquad\qquad \left. +f_k(x, t-h_k, u(x, t-h_k))\right\} & \text{on } Q=\Omega\times(0, T] \\ \phi(x, t)=\Phi(x, t) & \text{on } \Omega\times[-h_N, 0] \\ \phi(x, t)=0 & \text{on } \partial\Omega\times[0, T], \end{cases} \tag{1.1}$$

where $h_0, h_1, \cdots, h_N, T$ are constants so that

$$0=h_0<h_1<\cdots<h_N<T<\infty;\ N \text{ finite},\quad u\in D;$$

D is the class of controls to be defined later; and, for each $u\in D$, the operator $L(u)$ is given by

$$
\begin{aligned}
L(u)\phi(x,t) \triangleq \frac{\partial\phi(x,t)}{\partial t} - \sum_{k=0}^{N}\Big\{\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\Big(\sum_{i=1}^{n} a_{kij}(x,t-h_k)\cdot\frac{\partial\phi(x,t-h_k)}{\partial x_i} \\
+ a_{kj}(x,t-h_k,u(x,t-h_k))\cdot\phi(x,t-h_k)\Big)\Big\} \\
- \sum_{j=1}^{n} b_j(x,t,u(x,t))\cdot\phi_{x_j}(x,t) \\
- c(x,t,u(x,t))\cdot\phi(x,t).
\end{aligned}
\tag{1.2}
$$

Note that solutions of system (1.1) are weak solutions in the sense of Aronson [7]. The existence and uniqueness of weak solutions are established in Theorem 4.1 of § 4. Notations are given in § 2 and the statement of the control problem in § 3. In § 4, we also present some Lemmas which will be needed in proving the existence of optimal controls. The existence theorem of optimal controls is presented in Theorem 5.1 of § 5 and is proved by using Filippov's method. Note that the control restraint set U is taken as either a measurable multifunction or an upper semicontinuous multifunction.

2. Notations.

Let R^n denote the n-dimensional Euclidean space. For any $x\in R^n$, let $|x|=(\sum|x_i|^2)^{1/2}$. "a. e" means almost everywhere with respect to Lebesgue measure. $|E|$ denotes the Lebesgue measure of the measurable set E of any finite dimensional Euclidean space. ∂B denotes the boundary of the set B and $\bar{B}$ its closure.

A function $f: X\times Y\to R^m$ is said to be a Carathéodory function if $f(\cdot, b)$ is measurable on X for every $b\in Y$ and $f(a, \cdot)$ is continuous on Y for almost all $a\in X$.

Let G be any bounded connected subset of R^n, and denote by $C^l(G)$ the class of all continuous l-times differentiable real-valued functions on G, where l is a positive integer or equal to ∞. $C_0^l(G)$ denotes the class of all functions in $C^l(G)$ with compact support in G.

$H_0^{1,2}(G)$ is the completion of $C_0^\infty(G)$ in the norm

$$\|z\|\triangleq\|z\|_{2,G}+\|z_x\|_{2,G},$$

where

$$\|z\|_{2,G}\triangleq\Big(\int_G|z(x)|^2dx\Big)^{1/2}; \quad \text{and}$$

$$\|z_x\|_{2,G}\triangleq\Big(\int_G\sum_{i=1}^{n}|z_{x_i}(x)|^2dx\Big)^{1/2}.$$

$W^{1,2}(G)$ is the class of all measurable functions $z: G\to R^1$ having a generalized derivative z_x and satisfying

$$\|z\|_{2,G}+\|z_x\|_{2,G}<\infty,$$

where $\|z\|_{2,G}$ and $\|z_x\|_{2,G}$ are as defined above.

$L^q(G)$, $(q\geqq 1)$, is the Banach space of all measurable functions $z: G\to R^1$ that are q^{th} power integrable on G. Its norm is defined by

$$\|z\|_{q,G}\triangleq\left(\int_G |z(x)|^q dx\right)^{1/q}, \qquad \text{for } 1\leqq q<\infty; \text{ and}$$

$$\|z\|_{\infty,G}\triangleq\operatorname*{ess\,sup}_{x\in G} |z(x)| \qquad \text{for } q=\infty.$$

$L^{q,r}(G\times I)$, $(1\leqq q, r\leqq\infty)$, is the Banach space of all measurable functions $z: I\to L^q(G)$ such that $\|z\|_{q,r,G\times I}<\infty$, where

$$\|z\|_{q,r,G\times I}\triangleq\left(\int_I\left(\int_G |z(x, t)|^q dx\right)^{r/q} dt\right)^{1/r}, \qquad \text{for } 1\leqq q,\ r<\infty;$$

$$\|z\|_{q,\infty,G\times I}\triangleq\operatorname*{ess\,sup}_{t\in I} \|z(\cdot, t)\|_{q,G} \qquad \text{for } 1\leqq q<\infty,\ r=\infty;$$

$$\|z\|_{\infty,r,G\times I}\triangleq\left(\int_I (\|z(\cdot, t)\|_{\infty,G})^r dt\right)^{1/r} \qquad \text{for } q=\infty,\ 1\leqq r<\infty;$$

and

$$\|z\|_{\infty,\infty,G\times I}\triangleq\operatorname*{ess\,sup}_{(x,t)\in G\times I} |z(x, t)| \qquad \text{for } q=r=\infty.$$

$L^r(I, H_0^{1,2}(G))$ $(L^r(I, W^{1,2}(G)))$, $(r\geqq 1)$, is the Banach space of all measurable functions $z: I\to H_0^{1,2}(G)$ $(z: I\to W^{1,2}(G))$ having a finite norm $|||z|||_r$, where

$$|||z|||_r\triangleq\left\{\int_I (\|z(\cdot, t)\|_{2,G}^r+\|z_x(\cdot, t)\|_{2,G}^r) dt\right\}^{1/r} \qquad \text{for } 1\leqq r<\infty$$

and

$$|||z|||_\infty\triangleq\operatorname*{ess\,sup}_{t\in I} (\|z(\cdot, t)\|_{2,G}+\|z_x(\cdot, t)\|_{2,G}) \qquad \text{for } r=\infty.$$

$$\psi_t\triangleq\frac{\partial\psi}{\partial t}, \qquad \psi_{x_i}\triangleq\frac{\partial\psi}{\partial x_i}; \qquad (\ \ )_{x_j}\triangleq\frac{\partial}{\partial x_j}(\ \ ).$$

When there is no confusion, any subsequence will be denoted by its original sequence.

3. Definitions, basic assumptions and the statement of the problem.

Let Ω be a bounded connected open set in R^n and let N be a positive integer. Let h_k $(k=0, 1, \cdots, N)$, and T be fixed constants so that $0=h_0<h_1<\cdots<h_N<T<\infty$. Let $I_0\triangleq[-h_N, 0]$, $I_1\triangleq(0, T]$, $I_2\triangleq[-h_N, T]$, $Q_0\triangleq\Omega\times I_0$, $Q\triangleq\Omega\times I_1$, and $Q_2\triangleq\Omega\times I_2$.

DEFINITION 3.1. *A multifunction* $U: \bar{Q} \rightarrow R^m$ is a function from $\bar{Q}$ into the set of nonempty subsets of R^m. A multifunction $U: \bar{Q} \rightarrow R^m$ has *complete values* if and only if $U(x, t)$ is complete for all $(x, t) \in \bar{Q}$. A multifunction $U: \bar{Q} \rightarrow R^m$ is *measurable* if and only if $U^{-1}(B) = \{(x, t) \in \bar{Q} : U(x, t) \cap B \neq \emptyset\}$ is measurable for every closed subset B of R^m.

Let U_0 be a given nonempty compact subset of R^m and let $U: \bar{Q} \rightarrow R^m$ be a multifunction such that $U(x, t) \subset U_0$ for all $(x, t) \in \bar{Q}$. Let $\hat{u}: \bar{Q}_0 \rightarrow U_0$ be a given measurable function. Let D be the set of measurable functions $u: \bar{Q}_2 \rightarrow U_0$ such that $u(x, t) = \hat{u}(x, t)$ a.e. on Q_0. We say that u is a control if $u \in D$.

For any pair of functions $\Psi \in L^2(I_1 H_0^{1,2}(\Omega))$ and $Z \in C_0^1(Q)$ defined on $\Omega \times [-h_N, T]$ and $\Omega \times [0, T]$ respectively, let the following abbreviations be defined by

$$\langle L(u)\Psi, Z \rangle_{\Omega \times (\tau, \sigma)} \triangleq \iint_{\Omega \times (\tau, \sigma)} \Big[-\Psi(x, t) \cdot Z_t(x, t) \tag{3.1}$$

$$+ \sum_{k=0}^{N} \Big\{ \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} a_{kij}(x, t-h_k) \cdot \Psi_{x_i}(x, t-h_k)$$

$$+ a_{kj}(x, t-h_k, u(x, t-h_k)) \cdot \Psi(x, t-h_k) \Big) \cdot Z_{x_j}(x, t) \Big\}$$

$$- \sum_{j=1}^{n} b_j(x, t, u(x, t)) \cdot \Psi_{x_j}(x, t) \cdot Z(x, t)$$

$$- c(x, t, u(x, t)) \cdot \Psi(x, t) \cdot Z(x, t) \Big] dx dt$$

and

$$\langle \mathcal{F}(u), Z \rangle_{\Omega \times (\tau, \sigma)} \triangleq \iint_{\Omega \times (\tau, \sigma)} \Big[\sum_{k=0}^{N} \Big\{ \sum_{j=1}^{n} F_{kj}(x, t-h_k, u(x, t-h_k)) \tag{3.2}$$

$$\times Z_{x_j}(x, t) - f_k(x, t-h_k, u(x, t-h_k)) \cdot Z(x, t) \Big\} \Big] dx dt,$$

where $0 \leqq \tau < \sigma \leqq T$, $L(u)$ is as defined in (1.2) and

$$\mathcal{F}(u)(x, t) \triangleq \sum_{k=0}^{N} \Big\{ \sum_{j=1}^{n} (F_{kj}(x, t-h_k, u(x, t-h_k)))_{x_j}$$

$$+ f_k(x, t-h_k, u(x, t-h_k)) \Big\}.$$

Corresponding to system (1.1), we need the following definition.

DEFINITION 3.2. For each $u \in D$, the function $\phi(u): Q_2 \rightarrow R^1$ is said to be a *weak solution* of system (1.1) in the sense of Aronson [7, p. 633] if

(i) $\phi(u)|_Q \in L^{2,\infty}(Q) \cap L^2(I_1, H_0^{1,2}(\Omega))$;

(ii) $\phi(u)(x, t) = \Phi(x, t)$ on Q_0;

(iii) $\langle L(u)\phi(u) + \mathfrak{F}(u), \eta \rangle_Q = 0$ for any $\eta \in C_0^1(Q)$;

and

(iv) $\lim_{t\to 0+} \int_\Omega \phi(u)(x, t) \cdot z(x) dx = \int_\Omega \Phi(x, 0) \cdot z(x) dx$

for any $z \in C_0^1(\Omega)$, where $\phi(u)|_Q$ denotes the restriction of $\phi(u)$ on Q.

For ease in future references, the following assumptions will be referred to as *assumptions* (A):

(i) For each $k \in \{0, 1, \cdots, N\}$, the functions a_{kij}, $(i, j=1, \cdots, n)$, are measurable on $\bar{\Omega} \times [-h_k, T-h_k]$ and a_{kj}, F_{kj}, f_k, $(j=1, \cdots, n)$, are Carathéodory functions in $(\bar{\Omega} \times [-h_k, T-h_k]) \times U_0$; and b_j, $(j=1, \cdots, n)$, c are Carathéodory functions on $(\bar{\Omega} \times [0, T]) \times U_0$;

(ii) There exist constants α, $\beta > 0$ such that $\sum_{i,j=1}^{n} a_{0ij}(x, t) \cdot \zeta_i \cdot \zeta_j \geqq \alpha |\zeta|^2$ a.e. on $\bar{Q}$ for all $\zeta \in R^n$; for each $k \in \{0, 1, \cdots, N\}$, $|a_{kij}| \leqq \beta$ a.e. on $\bar{\Omega} \times [-h_k, T-h_k]$ and $|a_{kj}| \leqq \beta$ a.e. on $\bar{\Omega} \times [-h_k, T-h_k] \times U_0$ for all $i, j=1, \cdots, n$; and $|b_j|$, $(j=1, \cdots, n)$, $|c| \leqq \beta$ a.e. on $\bar{\Omega} \times [0, T] \times U_0$;

(iii) There exists a constant $\gamma > 0$ such that for all $k \in \{0, 1, \cdots, N\}$ and $u \in D$,

$$\|F_{kj}(\cdot, \cdot, u(\cdot, \cdot))\|_{2,2,\Omega\times(-h_k, T-h_k]}, \qquad (j=1, \cdots, n),$$

$$\|f_k(\cdot, \cdot, u(\cdot, \cdot))\|_{q,r,\Omega\times(-h_k, T-h_k]} \leqq \gamma,$$

where q and r satisfy $1 < q, r \leqq \infty$ and $\frac{n}{2q} + \frac{1}{r} < 1$; and

(iv) $\Phi \in L^2(I_0, W^{1,2}(\Omega))$ and $\Phi(\cdot, 0) \in L^2(\Omega)$.

A control $u \in D$ is called an *admissible control* if

(i) $u(x, t) \in U(x, t)$ a.e. in Q; and

(ii) there exists a unique weak solution $\phi(u)$ of system (1.1) corresponding to u.

Let Δ denote the set of all admissible controls. Let the cost functional J be defined on Δ by

$$J[u] = \iint_Q d(x, t, u(x, t)) \cdot g(x, t, \phi(u)(x, t)) dx dt, \tag{3.3}$$

where $\phi(u)$ is the weak solution of system (1.1) corresponding to u and the functions $d: \bar{Q} \times U_0 \to R^1$ and $g: \bar{Q} \times R^1 \to R^1$ satisfy the following assumptions which will be referred to as *assumptions* (H):

(i) d and g are Carathéodory functions on $\bar{Q}\times U_0$ and $\bar{Q}\times R^1$, respectively; and

(ii) there exist a non-negative continuous function $h:[0, \infty)\rightarrow R^1$ and non-negative measurable functions $p_1\in L^{\lambda,\sigma}(Q)$, $p_2\in L^{\lambda',\sigma'}(Q)$, where λ, σ, λ', $\sigma'>1$, $\dfrac{1}{\lambda}+\dfrac{1}{\lambda'}=1$ and $\dfrac{1}{\sigma}+\dfrac{1}{\sigma'}=1$ such that for all $u\in U_0$ and $\phi\in L^{2,2}(Q)$

$$|d(x, t, u)|\leqq p_1(x, t); \quad \text{and}$$

$$|g(x, t, \phi(x, t))|\leqq p_2(x, t)\cdot h(\|\phi\|_{2,2,Q}) \quad \text{on } Q.$$

We now state our control problem "P" as: subject to system (1.1), find an admissible control $u_0\in\Delta$ such that

$$J[u_0]\leqq J[u] \tag{3.4}$$

for all $u\in\Delta$. Such an admissible control will be referred to as an *optimal control.*

4. Preliminary results.

We first consider the existence and uniqueness of weak solutions of system (1.1) for each $u\in D$.

THEOREM 4.1. *Consider system* (1.1). *Suppose that the assumptions* (A) *are satisfied. Then, for each* $u\in D$, *system* (1.1) *admits a unique weak solution* $\phi(u)$ *satisfying the estimate*

$$\|\phi(u)\|_{2,\infty,Q}^2+\|\phi_x(u)\|_{2,2,Q}^2\leqq M, \tag{4.1}$$

where the constant $M>0$ *depends only on* α, β, γ, n, T, N, q, r, h_1, $|\Omega|$, $\|\Phi(\cdot, 0)\|_{2,\Omega}^2$, $\|\Phi\|_{2,2,Q_0}^2$ *and* $\|\Phi_x\|_{2,2,Q_0}^2$.

PROOF. Let $u\in D$ be arbitrary but fixed. Let σ be a positive integer such that $\sigma h_1<T\leqq(\sigma+1)h_1$. We now consider system (1.1) on $\Omega\times[(i-1)h_1, ih_1]$ successively in the order of $i=1, 2, \cdots, \sigma$ and on $\Omega\times[\sigma h_1, T]$. Clearly, system (1.1) on $\Omega\times[0, h_1]$ reduces to the system without time delayed arguments given by

$$\begin{cases} L_0(u)\phi(x, t)=\sum_{j=1}^{n}(F_j^1(u)(x, t))_{x_j}+\tilde{f}(u)(x, t) & \text{on } \Omega\times(0, h_1] \\ \phi(x, t)=0 \quad \text{on } \partial\Omega\times[0, h_1] \\ \phi(x, 0)=\Phi(x, 0) \quad x\in\Omega, \end{cases} \tag{4.2}$$

where the operator $L_0(u)$ is given by

$$L_0(u)\phi(x, t)=\phi_t(x, t)-\sum_{j=1}^{n}\Big\{\Big(\sum_{i=1}^{n} a_{0ij}(x, t)\cdot\phi_{x_i}(x, t) \tag{4.3}$$

$$+a_{0j}(x, t, u(x, t))\cdot\phi(x, t)\Big)_{x_j}+b_j(x, t, u(x, t))\cdot\phi_{x_j}(x, t)\Big\}$$

$$-c(x, t, u(x, t))\cdot\phi(x, t);$$

$$F_j^1(u)(x, t)=\sum_{k=1}^{N}\Big\{\sum_{i=1}^{n} a_{kij}(x, t-h_k)\cdot\Phi_{x_i}(x, t-h_k) \tag{4.4}$$

$$+a_{kj}(x, t-h_k, u(x, t-h_k))\cdot\Phi(x, t-h_k)\Big\}$$

$$+\sum_{k=0}^{N} F_{kj}(x, t-h_k, u(x, t-h_k)); \text{ and}$$

$$\tilde{f}(u)(x, t)=\sum_{k=0}^{N} f_k(x, t-h_k, u(x, t-h_k)). \tag{4.5}$$

Applying Minkowski's inequality to (4.4) and then using the assumptions A(ii), A(iv) and A(iii) and Cauchy's inequality, it can be shown that

$$\|F_j^1(u)\|_{2,2,\Omega\times(0,h_1]}^2 \tag{4.6}$$

$$\leqq(3N+1)\{Nn\beta^2\|\Phi_x\|_{2,2,Q_0}^2+N\beta^2\|\Phi\|_{2,2,Q_0}^2+(N+1)\gamma^2\}; \quad j=1, \cdots n.$$

Further, by applying Minkowski's inequality to (4.5), it follows from the assumption (Aiii) that

$$\|\tilde{f}(u)\|_{q,r,Q}^2\leqq(N+1)\gamma^2. \tag{4.7}$$

Since $\Phi(\cdot, 0)\in L^2(\Omega)$, we observe from Theorem 1 of [7, p. 634] that system (4.2) admits a unique weak solution $\phi^1(u)\in L^{2,\infty}(\Omega\times 0, h_1])\cap L^2((0, h_1], H_0^{1,2}(\Omega))$. By Lemma 1 of [7, p. 623] with $\zeta=1$, $s=0$ and $\mu=\infty$, we obtain the estimate

$$\|\phi^1(u)\|_{2,\infty,\Omega\times(0,h_1]}^2+\|\phi_x^1(u)\|_{2,2,\Omega\times(0,h_1]}^2 \tag{4.8}$$

$$\leqq d_0\Big(\|\Phi(\cdot, 0)\|_{2,\Omega}^2+\sum_{j=1}^{n}\|F_j^1(u)\|_{2,2,\Omega\times(0,h_1]}^2$$

$$+\|\tilde{f}(u)\|_{q,r,\Omega\times(0,h_1]}^2\Big),$$

where the constant $d_0>0$ depends only on α, β, n, q, r, $|\Omega|$ and h_1. Thus, it can be easily deduced from estimates (4.8), (4.6) and (4.7) that

$$\|\phi^1(u)\|_{2,\infty,\Omega\times(0,h_1]}^2+\|\phi_x^1(u)\|_{2,2,\Omega\times(0,h_1]}^2\leqq M_1, \tag{4.9}$$

where the constant $M_1>0$ depends only on $\alpha, \beta, \gamma, n, N, q, |\Omega|, r, h_1$, $\|\Phi(\cdot, 0)\|^2_{2,\Omega}$, $\|\Phi\|^2_{2,2,Q_0}$ and $\|\Phi_x\|^2_{2,2,Q_0}$.

Consider system (1.1) on $\Omega\times[h_1, 2h_1]$. Clearly, system (1.1) on $\Omega\times[h_1, 2h_1]$ reduces to the following system

$$\begin{cases} L_0(u)\psi(x, t)=\sum_{j=1}^{n}(F_j^2(u)(x, t))_{x_j}+\tilde{f}(u)(x, t) & \text{on } \Omega\times(h_1, 2h_1] \\ \psi(x, t)=0 \quad \text{on } \partial\Omega\times[h_1, 2h_1] \\ \psi(x, h_1)=\phi^1(u)(x, h_1) \quad x\in\Omega, \end{cases} \tag{4.10}$$

where $L_0(u)$ and $\tilde{f}(u)$ are as defined in (4.3) and (4.5), respectively, and $F_j^2(u)$ is as defined in (4.4) with Φ replaced by $\tilde{\phi}^1(u)$ in which $\tilde{\phi}^1(u)$ is defined by

$$\tilde{\phi}^1(u)(x, t)=\begin{cases} \Phi(x, t) & \text{on } Q_0 \\ \Phi^1(u)(x, t) & \text{on } \Omega\times(0, h_1]. \end{cases}$$

Since $\phi^1(u)$ is the weak solution of system (4.2), it follows from expression (2.3) of [7, p. 622] that $\phi^1(u)(\cdot, h_1)\in L^2(\Omega)$. Moreover,

$$\|\phi^1(u)(\cdot, h_1)\|_{2,\Omega}\leqq\|\phi^1(u)\|_{2,\infty,\Omega\times(0,h_1]}. \tag{4.11}$$

Note that for each $r\geqq 1$ and $\phi\in L^{2,\infty}(\Omega\times I)$,

$$\begin{aligned} \|\phi\|_{2,r,\Omega\times I}&=\left(\int_I\left(\int_\Omega|\phi(x, t)|^2dx\right)^{r/2}dt\right)^{1/r} \\ &\leqq\left(\int_I\left\{\operatorname*{ess\,sup}_{t\in I}\left(\int_\Omega|\phi(x, t)|^2dx\right)^{1/2}\right\}^r dt\right)^{1/r} \\ &\triangleq|I|^{1/r}\cdot\|\phi\|_{2,\infty,\Omega\times I}. \end{aligned} \tag{4.12}$$

As before, by applying Minkowski's inequality to $F_j^2(u)$ and using the assumptions A(ii)-A(iii) and Cauchy's inequality, it can be easily shown that

$$\begin{aligned} \|F_j^2(u)\|^2_{2,2,\Omega\times(h_1,2h_1]}\leqq(3N+1)\{Nn\beta^2\|\tilde{\phi}_x^1(u)\|^2_{2,2,\Omega\times(-h_N,h_1]} \\ +N\beta^2\|\tilde{\phi}^1(u)\|^2_{2,2,\Omega\times(-h_N,h_1]}+(N+1)\gamma^2\} \end{aligned} \tag{4.13}$$

for $j=1, \cdots, n$. Thus, it follows again from Theorem 1 and Lemma 2 of [7, p. 634, p. 623] that system (4.10) admits a unique weak solution $\phi^2(u)\in L^{2,\infty}(\Omega\times(h_1, 2h_1])\cap L^2((h_1, 2h_1], H_0^{1,2}(\Omega))$ satisfying the estimate

$$\begin{aligned} &\|\phi^2(u)\|^2_{2,\infty,\Omega\times(h_1,2h_1]}+\|\phi_x^2(u)\|^2_{2,2,\Omega\times(h_1,2h_1]} \\ &\leqq d_0(\|\phi^1(u)(\cdot, h_1)\|^2_{2,\Omega}+\sum_{j=1}^{n}\|F_j^2(u)\|^2_{2,2,\Omega\times(h_1,2h_1]} \\ &\quad+\|\tilde{f}(u)\|^2_{q,r,\Omega\times(h_1,2h_1]}), \end{aligned} \tag{4.14}$$

where the constant $d_0>0$ is as defined for (4.8).

Using (4.11), (4.13), (4.12), (4.9) and (4.7), it can be easily deduced from (4.14), that

$$\|\phi^2(u)\|^2_{2,\infty,\Omega\times(h_1,2h_1]}+\|\phi^2_x(u)\|^2_{2,2,\Omega\times(h_1,2h_1]}\leq M_2, \tag{4.15}$$

where the constant $M_2>0$ depends only on $\alpha, \beta, \gamma, n, q, |\Omega|, r, N, h_1$, $\|\Phi(\cdot, 0)\|^2_{2,\Omega}$, $\|\Phi\|^2_{2,2,Q_0}$ and $\|\Phi_x\|^2_{2,2,Q_0}$.

By the same token, we can show successively in the order of $i=3, \cdots, \sigma$ that system (1.1) on $\Omega\times[(i-1)h_1, ih_1]$ and on $\Omega\times[\sigma h_1, T]$ admits unique weak solutions $\phi^i(u)\in L^{2,\infty}(\Omega\times((i-1)h_1, ih_1])\cap L^2(((i-1)h_1, ih_1], H_0^{1,2}(\Omega))$, $(i=3, \cdots, \sigma)$, and $\phi^{\sigma+1}(u)\in L^{2,\infty}(\Omega\times(\sigma h_1, T])\cap L^2((\sigma h_1, T], H_0^{1,2}(\Omega))$. Further, $\phi^i(u)$, $(i=3, \cdots, \sigma+1)$, satisfy the corresponding estimates as for inequality (4.15) for some constants $M_i>0$ $(i=3, \cdots, \sigma+1)$, where M_i, $(i=3, \cdots, \sigma)$ depend only on $\alpha, \beta, \gamma, n, q, r, |\Omega|, N, h_1$, $\|\Phi(\cdot, 0)\|^2_{2,\Omega}$, $\|\Phi\|^2_{2,2,Q_0}$, and $\|\Phi_x\|^2_{2,2,Q_0}$, while $M_{\sigma+1}$ depends only on T and the above quantities.

Let $\phi(u)$ be defined on Q_2 by

$$\phi(u)(x, t)=\begin{cases} \Phi(x, t) & \text{on } Q_0 \\ \phi^i(u)(x, t) & \text{on } \Omega\times((i-1)h_1, ih_1], \quad i=1, \cdots, \sigma \\ \phi^{\sigma+1}(u)(x, t) & \text{on } \Omega\times(\sigma h_1, T]. \end{cases}$$

Then, by taking $M=\sum_{i=1}^{\sigma+1} M_i$, it follows readily that $\phi(u)$ satisfies estimate (4.1). We now show that $\phi(u)$ is the unique weak solution of system (1.1). Clearly, $\phi(u)$ satisfies the conditions (i), (ii) and (iv) of Definition 3.2. It remains to show that $\phi(u)$ satisfies the condition (iii) of Definition 3.2. Let $\eta\in C^1(\bar{Q})$ be arbitrary with compact support in Ω and vanishing at $t=T$. Let η_l, $(l=1, \cdots, \sigma)$, and $\eta_{\sigma+1}$ denote, respectively, the restrictions of η on $\Omega\times[(l-1)h_1, lh_1]$, $(l=1, \cdots, \sigma)$, and $\Omega\times[\sigma h_1, T]$. Clearly, $\eta_l\in C^1(\bar{\Omega}\times[(l-1)h_1, lh_1])$, $(l=1, \cdots, \sigma)$, $\eta_{\sigma+1}\in C^1(\bar{\Omega}\times[\sigma h_1, T])$ with compact support in Ω and $\eta_{\sigma+1}$ vanishes at $t=T$. Since, $\phi^l(u)$, $(l=1, \cdots, \sigma)$, and $\phi^{\sigma+1}(u)$ are weak solutions of system (1.1) on $\Omega\times[(l-1)h_1, lh_1]$, $(l=1, \cdots, \sigma)$, and $\Omega\times[\sigma h_1, T]$, respectively, it follows from expression (1.4) of [7, p. 620] that for each $l=1, \cdots, \sigma$,

$$\int_\Omega \phi^l(u)(x, lh_1)\cdot\eta_l(x, lh_1)dx+\langle L(u)\phi^l(u)+\mathcal{F}(u), \eta_l\rangle_{\Omega\times((l-1)h_1, lh_1]} \tag{4.16}$$
$$=\int_\Omega \phi^l(u)(x, (l-1)h_1)\cdot\eta_l(x, (l-1)h_1)dx;$$

and

$$\langle L(u)\phi^{\sigma+1}(u)+\mathcal{F}(u), \eta_{\sigma+1}\rangle_{\Omega\times(\sigma h_1, T]} \tag{4.17}$$
$$=\int_\Omega \phi^{\sigma+1}(u)(x, \sigma h_1)\cdot\eta_{\sigma+1}(x, \sigma h_1)dx.$$

Further, since $\phi^1(u)(x, 0)=\Phi(x, 0)$ on Ω, it follows from (4.16), (4.17) and the definitions of $\phi(u)$ and η that

$$\langle L(u)\phi(u)+\mathcal{F}(u),\ \eta\rangle_Q=\int_\Omega \Phi(x, 0)\cdot\eta(x, 0)dx\,. \tag{4.18}$$

In particular, if $\eta\in C_0^1(Q)$, then (4.18) holds with zero on its right hand side. Hence, $\phi(u)$ is a weak solution of system (1.1). The uniqueness of $\phi(u)$ follows from the uniqueness of $\phi^l(u)$, $(l=1, \cdots, \sigma+1)$. This completes the proof.

Let $0<\rho<\min\{1, h_1\}$ be an arbitrary but fixed constant. Let Ω_ρ denote a subset of Ω having a distance ρ from $\partial\Omega$; and let $K_\rho\triangleq\Omega_\rho\times(\rho, T]$.

LEMMA 4.2. *The set of weak solutions* $\{\phi(u)\colon u\in D\}$ *of system* (1.1) *is equicontinuous and uniformly bounded on* K_ρ.

PROOF. Let σ be a positive integer so that $\sigma h_1<T\leqq(\sigma+1)h_1$. Then, system (1.1) on $\Omega\times[0, h_1]$ reduces to system (4.2), which is a system without time delayed arguments. Let $K_\rho^1\triangleq K_\rho\cap\Omega\times[0, h_1]=\Omega_\rho\times(\rho, h_1]$ and let $\rho_0=\frac{1}{3}\rho$. For an arbitrary $(\bar{x}, \bar{t})\in K_\rho^1$, let $G(\rho_0)\triangleq\{x\in R^n\colon |x-\bar{x}|<\frac{1}{2}\rho_0\}\times(\bar{t}-\rho_0^2, \bar{t}]$. Let $\phi^1(u)=\phi(u)|_{\Omega\times(0,h_1]}$. Since $\phi^1(u)$ is the weak solution of system (4.2) and $G(3\rho_0)\subset\Omega\times(0, h_1]$, it follows from Theorem B of [7, p. 616] that for all $(x, t)\in\bar{G}(\rho_0)$,

$$|\phi^1(u)(x, t)|\leqq d_1(\rho_0^{-(n+1)/2}\|\phi^1(u)\|_{2,2,G(3\rho_0)}+\rho_0^\theta k_1)\,,$$

where the positive constant d_1 depends on n, ρ_0, α, β, $|\Omega|$, h_1, q and r, θ is a positive constant which is determined by the values of p and q occurring in assumptions (A), and $k_1=\left(\sum_{j=1}^n \|F_j^1(u)\|_{2,2,\Omega\times(0,h_1]}^2+\|\tilde{f}(u)\|_{q,r,\Omega\times(0,h_1]}^2\right)$. In particular,

$$|\phi^1(u)(\bar{x}, \bar{t})|\leqq d_1(\rho_0^{-(n+1)/2}\|\phi^1(u)\|_{2,2,G(3\rho_0)}^2+\rho_0^\theta k_1)\,. \tag{4.19}$$

Since $(\bar{x}, \bar{t})$ is an arbitrary element in K_ρ^1 and the constant d_1 is independent of $(\bar{x}, \bar{t})$, it follows that (4.19) holds for all $(\bar{x}, \bar{t})\in K_\rho^1$. In view of (4.1), (4.6) and (4.7), it can be easily shown that

$$|\phi^1(u)(x, t)|\leqq m_1 \tag{4.20}$$

for each $(x, t)\in K_\rho^1$, where the constant $m_1>0$ depends only on α, β, n, θ, $|\Omega|$, h_1, q, r, N, γ, T, ρ_0, $\|\Phi(\cdot, 0)\|_{2,\Omega}^2$, $\|\Phi\|_{2,2,Q_0}^2$ and $\|\Phi_x\|_{2,2,Q_0}^2$. Since m_1 is independent of $u\in D$, $\{\phi(u)\colon u\in D\}$ is uniformly bounded on K_ρ^1. The equicontinuity of $\{\phi(u)\colon u\in D\}$ on K_ρ^1 follows from Theorem C of [7, p. 616].

Consider system (1.1) on $\Omega\times[h_1-\rho, 2h_1]$. Then, system (1.1) on $\bar{\Omega}\times[h_1-\rho, 2h_1]$ reduces to the system without time delayed arguments given by

$$\begin{cases} L_0(u)\phi(x, t)=\sum_{j=1}^{n}(F_j^2(u)(x, t))_{x_j}+\tilde{f}(u)(x, t) & \text{on } \Omega\times(h_1-\rho, 2h_1] \\ \phi(x, t)=0 \quad \text{on } \partial\Omega\times[h_1-\rho, 2h_1] \\ \phi(x, h_1-\rho)=\phi^1(u)(x, h_1-\rho) \quad \text{on } \Omega, \end{cases} \tag{4.21}$$

where $F_j^2(u)$ and $\tilde{f}(u)$ are as defined for system (4.10). Let $\phi^2(u)=\phi(u)|_{\Omega\times[h_1, 2h_1]}$ and let

$$\tilde{\phi}^2(u)(x, t)=\begin{cases} \phi^1(u)(x, t) & \text{on } \Omega\times[h_1-\rho, h_1] \\ \phi^2(u)(x, t) & \text{on } \Omega\times(h_1, 2h_1]. \end{cases}$$

Then $\tilde{\phi}^2(u)$ is the weak solution of system (4.21). Let $K_\rho^2\triangleq K_\rho\cap\Omega\times(h_1, 2h_1]$ $=\Omega_\rho\times(h_1, 2h_1]$ and let $\rho_0=\frac{1}{3}\rho$. Then, by using a similar argument as that given above, we can show that $\{\phi^2(u): u\in D\}$ is uniformly bounded and equicontinuous on K_ρ^2.

By the same token, we can show successively in the order of $l=3, \cdots, \sigma$, that the sets $\{\phi^l(u): u\in D\}$, $(l=3, \cdots, \sigma)$, and $\{\phi^{\sigma+1}(u): u\in D\}$ are uniformly bounded and equicontinuous on K_ρ^l, $(l=3, \cdots, \sigma)$, and $K_\rho^{\sigma+1}$, respectively, where

$$\phi^l(u)=\phi(u)|_{\Omega\times[(l-1)h_1, lh_1]}, \quad (l=3, \cdots, \sigma),$$

$$\phi^{\sigma+1}(u)=\phi(u)|_{\Omega\times[\sigma h_1, T]},$$

$$K_\rho^l\triangleq K_\rho\cap((l-1)h_1, lh_1]=\Omega_\rho\times((l-1)h_1, lh_1], \quad (l=3, \cdots, \sigma),$$

and

$$K_\rho^{\sigma+1}\triangleq K_\rho\cap(\sigma h_1, T]=\Omega_\rho\times(\sigma h_1, T].$$

Thus, the set $\{\phi(u): u\in D\}$ on K_ρ is uniformly bounded and equicontinuous. This completes the proof.

Using Lemma 4.2 and the Ascoli-Arzela theorem, we can easily obtain the following result.

LEMMA 4.3. *Let* $\{u_l\}$ *be a sequence in* D *and let* $\{\phi(u_l)\}$ *denote the corresponding sequence of weak solutions of system* (1.1). *Then, there exist a subsequence of* $\{\phi(u_l)\}$, *which is denoted by the original sequence, and a continuous function* $\phi: Q\to R^1$ *so that* $\phi(u_l)\to\phi$ *uniformly on any compact subset of* Q.

With the help of Lemma 4.3, we can prove the following lemma.

LEMMA 4.4. *Consider system* (1.1). *Let the assumptions* (A) *be satisfied. Let* $\{u_l\}$ *be a sequence in* D *such that for each* $k\in\{0, 1, \cdots, N\}$,

$$a_{kj}(\cdot, \cdot, u_l(\cdot, \cdot))\to\tilde{a}_{kj}(\cdot, \cdot), \qquad (j=1, \cdots, n),$$

*in the weak * topology of* $L^{\infty,\infty}(\Omega\times(-h_k, T-h_k])$;

$$F_{kj}(\cdot, \cdot, u_l(\cdot, \cdot)) \to \tilde{F}_{kj}(\cdot, \cdot), \qquad (j=1, \cdots, n),$$

weakly in $L^{2,2}(\Omega \times (-h_k, T-h_k])$; *and*

$$f_k(\cdot, \cdot, u_l(\cdot, \cdot)) \to \tilde{f}_k(\cdot, \cdot) \text{ weakly in } L^{q,r}(\Omega \times (-h_k, T-h_k]);$$

and that

$$b_j(\cdot, \cdot, u_l(\cdot, \cdot)) \to \tilde{b}_j(\cdot, \cdot), \qquad (j=1, \cdots, n), \text{ and}$$

$$c(\cdot, \cdot, u_l(\cdot, \cdot)) \to \tilde{c}(\cdot, \cdot)$$

in the weak $*$ *topology of* $L^{\infty,\infty}(\Omega \times (0, T])$. *Let* $\phi(u_l)$ *denote the weak solution of system* (1.1) *corresponding to* u_l. *Then, there exist a subsequence of* $\{\phi(u_l)\}$, *which is denoted by the original sequence, and a function* $\phi : \Omega \times I_2 \to R^1$ *such that*

(i) $\phi(u_l) \to \phi$ *weakly in* $L^{2,2}(Q)$, *uniformly on any compact subset of* Q; *and*

(ii) ϕ *is the weak solution of the following system*

$$\begin{cases} \tilde{L}\phi(x, t) = \sum_{k=0}^{N} \left\{ \sum_{j=1}^{n} (\tilde{F}_{kj}(x, t-h_k))_{x_j} + \tilde{f}_k(x, t-h_k) \right\} & \text{on } Q \\ \phi(x, t) = \Phi(x, t) \quad \text{on } Q_0 \\ \phi(x, t) = 0 \quad \text{on } \partial\Omega \times [0, T], \end{cases} \tag{4.22}$$

where the operator $\tilde{L}$ *is defined by*

$$\tilde{L}\phi(x, t) \triangleq \phi_t(x, t) - \sum_{k=0}^{N} \left\{ \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{kij}(x, t-h_k) \cdot \phi_{x_i}(x, t-h_k) \right.\right.$$
$$\left.\left. + \tilde{a}_{kj}(x, t-h_k) \cdot \phi(x, t-h_k) \right)_{x_j} \right\} - \sum_{j=1}^{n} \tilde{b}_j(x, t) \phi_{x_j}(x, t)$$
$$- \tilde{c}(x, t) \cdot \phi(x, t).$$

PROOF. Since $\phi(u_l)$ is the weak solution of system (1.1) corresponding to u_l, it follows from the estimate (4.1) of Theorem 4.1 that, for each u_l,

$$\|\phi(u_l)\|_{2,\infty,Q}^2 + \|\phi_x(n_l)\|_{2,2,Q}^2 \leqq M, \tag{4.23}$$

where the constant $M>0$, which is independent of l, is as defined for (4.1). This implies that

$$\|\phi(u_l)\|_{2,2,Q}^2 \leqq T \cdot M \tag{4.24}$$

and

$$\|\phi_x(u_l)\|_{2,2,Q}^2 \leqq M. \tag{4.25}$$

In view of the weak compactness of bounded sets in $L^2(I_1, H_0^{1,2}(\Omega))$, it follows from (4.24) and (4.25) that there exists a subsequence of $\{\phi(u_l)\}$ which converges to $\tilde{\phi}$ weakly in $L^2(I_1, H_0^{1,2}(\Omega))$. By Lemma 4.3, we observe that, corresponding to this sequence, there exist a subsequence of $\{\phi(u_l)\}$, which is again denoted by the original sequence, and a function $\tilde{\tilde{\phi}}: Q \to R^1$ such that

$$\phi(u_l) \to \tilde{\tilde{\phi}} \tag{4.26}$$

uniformly on any compact subset of Q. However, by virtue of (4.24) and (4.26), it follows from Theorem 13.44 of [8, p. 207] that $\phi(u_l)$ also converges to $\tilde{\tilde{\phi}}$ weakly in $L^{2,2}(Q')$, where Q' is any compact subset of Q. By the uniqueness of the weak limit, we obtain readily that $\tilde{\phi}(x, t)=\tilde{\tilde{\phi}}(x, t)$ a.e. on any compact subset of Q. We may modify $\tilde{\phi}$ if necessary, by taking $\tilde{\phi}(x, t) =\tilde{\tilde{\phi}}(x, t)$ on any compact subset of Q. Thus, it follows from (4.26) that

$$\phi(u_l) \to \tilde{\phi} \tag{4.27}$$

uniformly on any compact subset of Q.

By estimate (4.23) and Lemma 3 of [7, p. 633], it follows that $\tilde{\phi} \in L^{2,\infty}(Q)$. Hence, $\tilde{\phi} \in L^{2,\infty}(Q) \cap L^2(I_1, H_0^{1,2}(\Omega))$.

Let ϕ be defined on Q_2 by

$$\phi(x, t)=\begin{cases} \Phi(x, t) & \text{on } Q_0 \\ \tilde{\phi}(x, t) & \text{on } Q\,. \end{cases}$$

Next, we shall show that ϕ is the weak solution of system (4.22). It is clear that ϕ satisfies the conditions (i) and (ii) of Definition 3.2. Let η be an arbitrary element in $C_0^1(Q)$. Since $\phi(u_l)$ is the weak solution of system (1.1), it follows from the condition (iii) of Definition 3.2 that

$$\langle L(u_l)\phi(u)+\mathcal{F}(u_l),\ \eta\rangle_Q=0\,. \tag{4.28}$$

By virtue of the hypotheses, (4.27), Lemma 4.2, and the weak convergence of $\{\phi(u_l)\}$ to ϕ in $L^2(I_1, H_0^{1,2}(\Omega))$, we can deduce by using Lemma 4.2 of [9] that (4.28) in the limit with respect to l reduces to

$$\langle \tilde{L}\phi+\tilde{\mathcal{F}},\ \eta\rangle_Q=0\,, \tag{4.29}$$

where

$$\tilde{\mathcal{F}}(x, t) \triangleq \sum_{k=0}^{N} \left\{ \sum_{j=1}^{N} (\tilde{F}_{kj}(x, t-h_k))_{x_j}+\tilde{f}_k(x, t-h_k)\right\}.$$

It remains to show that

$$\lim_{\sigma\to 0+}\int_\Omega \phi(x, \sigma)\cdot z(x)dx=\int_\Omega \phi(x, 0)\cdot z(x)dx \tag{4.30}$$

for any $z \in C_0^1(\Omega)$.

Note that by using the same arguments as those given for expressions (1.3) and (1.4) of [7, p. 619-620], we obtain, respectively, the following results:

(i) If the function $\phi(u): \Omega \times I_2 \to R^1$ satisfies the conditions (i), (ii) and (iii) of Definition 3.2 then

$$\int_\Omega \phi(u)(x, \tau)\cdot \eta(x, \tau)dx + \langle L(u)\phi(u)+F(u), \eta\rangle_{\Omega\times(\sigma,\tau)} \tag{4.31}$$

$$= \int_\Omega \phi(u)(x, \sigma)\cdot \eta(x, \sigma)dx$$

for any $\eta \in C^1(\bar{Q})$ with compact support in Ω, where $0<\sigma<\tau\leqq T$.

(ii) If $\phi(u)$ is a weak solution of system (1.1), then relation (4.31) holds with $\sigma=0$ and its right hand side replaced by $\int_\Omega \Phi(x, 0)\cdot \eta(x, 0)dx$. Since ϕ satisfies the conditions (i), (ii) and (iii) of Definition 3.2, it follows from the results stated in (i) that

$$\langle \tilde{L}\phi+\tilde{F}, \eta\rangle_{\Omega\times(\sigma,T)} = \int_\Omega \phi(x, \sigma)\cdot \eta(x, \sigma)dx \tag{4.32}$$

for any $\eta \in C^1(\bar{Q})$ with compact support in Ω and vanishing near $t=T$, where $0<\sigma<T$. Further, since $\phi(u_l)$ is a weak solution of system (1.1), it follows from the results stated in (ii) that

$$\langle L(u_l)\phi(u_l)+F(u_l), \eta\rangle_Q = \int_\Omega \Phi(x, 0)\cdot \eta(x, 0)dx \tag{4.33}$$

for any $\eta \in C^1(\bar{Q})$ with compact support in Ω and vanishing near $t=T$. Letting $l\to\infty$ in (4.33) and using an argument similar to that used to obtain (4.29), it follows that the relation (4.33) reduces to

$$\langle \tilde{L}\phi+\tilde{F}, \eta\rangle_Q = \int_\Omega \Phi(x, 0)\cdot \eta(x, 0)dx \tag{4.34}$$

for any $\eta \in C^1(\bar{Q})$ with compact support in Ω and vanishing near $t=T$. Let $0<\sigma<T$ and let q be a C^1-function defined on $[0, T]$ such that $q(t)=1$ on $[0, \sigma]$ and vanishes near $t=T$. Then for any $z\in C_0^1(\Omega)$, $q(\cdot)z(\cdot)\in C^1(\bar{Q})$ with compact support in Ω and vanishing near $t=T$. Thus, replacing $\eta(\cdot, \cdot)$ by $q(\cdot)z(\cdot)$ in (4.32) and (4.34) and comparing their results, we obtain (4.30). Therefore ϕ is a weak solution of system (4.22). By virtue of Theorem 4.1, ϕ is the unique weak solution of system (4.22). This completes the proof.

5. Existence of optimal controls.

In this section, we establish the existence of optimal controls for the problem P using Filippov's method and Lemma 4.4.

For each $k=1, 2, \cdots, N$, let $\hat{\alpha}_k$ be defined on $\bar{Q}\times U_0$ by

$$\hat{\alpha}_k(x, t, v)=\begin{cases} \alpha_k(x, t, v) & \text{on } \bar{\Omega}\times[0, T-h_k]\times U_0 \\ \alpha_k(x, T-h_k, v) & \text{on } \bar{\Omega}\times(T-h_k, T]\times U_0, \end{cases}$$

where α_k denotes a_{kj}, F_{kj} or f_k, $(j=1, \cdots, n)$.

Let Γ: $\bar{Q}\times U_0\to R^{(N+1)(2n+1)+n+2}$ be defined by

$$\begin{aligned}\Gamma(x, t, u)=\text{col}\,(&a_{01}(x, t, u), \cdots, a_{0n}(x, t, u), \hat{a}_{11}(x, t, u), \cdots, \\ &\hat{a}_{1n}(x, t, u), \cdots, \hat{\alpha}_{N1}(x, t, u), \cdots, \hat{\alpha}_{Nn}(x, t, u), b_1(x, t, u), \cdots, \\ &b_n(x, t, u), c(x, t, u), F_{01}(x, t, u), \cdots, F_{0n}(x, t, u), \\ &\hat{F}_{11}(x, t, u), \cdots, \hat{F}_{1n}(x, t, u), \cdots, \hat{F}_{N1}(x, t, u), \cdots, \\ &\hat{F}_{Nn}(x, t, u), f_0(x, t, u), \hat{f}_1(x, t, u), \cdots, \hat{f}_N(x, t, u), \\ &d(x, t, u)),\end{aligned}$$

where $\text{col}\,(a_1, \cdots, a_s)\triangleq\begin{bmatrix} a_1 \\ \vdots \\ a_s \end{bmatrix}$. For each $(x, t)\in\bar{Q}$, let $R(x, t)$ be defined by

$$R(x, t)=\{\Gamma(x, t, u): u\in U(x, t)\}\,.$$

Clearly, $R(x, t)\subset R^s$ for each $(x, t)\in\bar{Q}$ and $R: \bar{Q}\to R^s$ is a multifunction, where $s=(N+1)(2n+1)+n+2$.

The following assumptions will be referred to collectively as *assumptions* (B):

(i) U is a measurable multifunction and for each $(x, t)\in\bar{Q}$, $U(x, t)$ is compact;

(ii) For each $(x, t)\in\bar{Q}$, the set $R(x, t)$ is convex and closed; and

(iii) The multifunction $R: \bar{Q}\to R^s$ is upper semicontinuous with respect to inclusion (u. s. c. i) for all $(x, t)\in\bar{Q}$, where $s=(N+1)(2n+1)+n+2$.

THEOREM 5.1. *Consider the problem* P. *Suppose that the assumptions* (A), (H) *and* (B) *are satisfied. Then, there exists an optimal control.*

PROOF. By virtue of the assumption H(ii) and estimate (4.1), it can be easily deduced from Hölder's inequality that $|\inf\{J[u]: u\in\Delta\}|<\infty$. Let $\nu=\inf\{J[u]: u\in\Delta\}$ and let $\{u_l\}\subset\Delta$ be a minimizing sequence such that

$$\lim_{l\to\infty} J[u_l]=\nu\,. \tag{5.1}$$

Let us first consider the case when $1<q,\ r<\infty$. The cases when $1<q<\infty$, $r=\infty$ and $q=\infty$, $1<r<\infty$ will be considered later. In view of the assumptions A(ii), A(iii) and H(ii), there exist subsequences of $\{a_{0j}(\cdot,\cdot,u_l(\cdot,\cdot))\}$, $\{\hat{a}_{kj}(\cdot,\cdot,u_l(\cdot,\cdot))\}$, $\{b_j(\cdot,\cdot,u_l(\cdot,\cdot))\}$, $\{c(\cdot,\cdot,u_l(\cdot,\cdot))\}$, $\{F_{0j}(\cdot,\cdot,u_l(\cdot,\cdot))\}$, $\{\hat{F}_{kj}(\cdot,\cdot,u_l(\cdot,\cdot))\}$, $\{f_0(\cdot,\cdot,u_l(\cdot,\cdot))\}$, $\{\hat{f}_k(\cdot,\cdot,u_l(\cdot,\cdot))\}$, $(j=1,\cdots,n;\ k=1,\cdots,N)$, and $\{d(\cdot,\cdot,u_l(\cdot,\cdot))\}$, which will be denoted by their original sequences, and functions $\tilde{a}_{kj}$, $\tilde{b}_j$, $\tilde{c}\in L^{\infty,\infty}(Q)$, $\tilde{F}_{kj}\in L^{2,2}(Q)$, $\tilde{f}_k\in L^{q,r}(Q)$, $(j=1,\cdots,n;\ k=0,1,\cdots,N)$, and $\tilde{d}\in L^{\lambda,\sigma}(Q)$ such that for each $k\in\{1,\cdots,N\}$ and $j\in\{1,\cdots,n\}$,

$$a_{0j}(\cdot,\cdot,u_l(\cdot,\cdot))\to\tilde{a}_{0j}(\cdot,\cdot),\ \hat{a}_{kj}(\cdot,\cdot,u_l(\cdot,\cdot))\to\tilde{a}_{kj}(\cdot,\cdot),$$

$$b_j(\cdot,\cdot,u_l(\cdot,\cdot))\to\tilde{b}_j(\cdot,\cdot),\ c(\cdot,\cdot,u_l(\cdot,\cdot))\to\tilde{c}(\cdot,\cdot) \tag{5.2}$$

in the weak $*$ topology of $L^{\infty,\infty}(Q)$;

$$F_{0j}(\cdot,\cdot,u_l(\cdot,\cdot))\to\tilde{F}_{0j}(\cdot,\cdot),\ \hat{F}_{kj}(\cdot,\cdot,u_l(\cdot,\cdot))\to\tilde{F}_{kj}(\cdot,\cdot) \tag{5.3}$$

weakly in $L^{2,2}(Q)$;

$$f_0(\cdot,\cdot,u_l(\cdot,\cdot))\to\tilde{f}_0(\cdot,\cdot),\ \hat{f}_k(\cdot,\cdot,u_l(\cdot,\cdot))\to\tilde{f}_k(\cdot,\cdot) \tag{5.4}$$

weakly in $L^{q,r}(Q)$; and

$$d(\cdot,\cdot,u_l(\cdot,\cdot))\to\tilde{d}(\cdot,\cdot) \tag{5.5}$$

weakly in $L^{\lambda,\sigma}(Q)$.

Let

$$\begin{aligned} y(x,t)=\mathrm{col}\,(&\tilde{a}_{01}(x,t),\cdots,\tilde{a}_{0n}(x,t),\tilde{a}_{11}(x,t),\cdots,\tilde{a}_{Nn}(x,t),\tilde{b}_1(x,t),\cdots,\\ &\tilde{b}_n(x,t),\tilde{c}(x,t),\tilde{F}_{01}(x,t),\cdots,\tilde{F}_{0n}(x,t),\tilde{F}_{11}(x,t),\cdots,\\ &\tilde{F}_{Nn}(x,t),\tilde{f}_0(x,t),\cdots,\tilde{f}_N(x,t),\tilde{d}(x,t)).\end{aligned}$$

Then, y is a measurable function from Q to R^s, where $s=(N+1)(2n+1)+n+2$.

We shall show that $y(x,t)\in R(x,t)$ a.e. in Q. For each $l=1,2,\cdots$, let $\tilde{Q}_l\triangleq\{(x,t)\in Q:u_l(x,t)\notin U(x,t)\}$. Let $\tilde{Q}=\bigcup_{l=1}^{\infty}\tilde{Q}_l$. Then $|\tilde{Q}|=0$. Let $Q^1=Q\backslash\tilde{Q}$. Let $(x_0,t_0)\in Q^1$ be an arbitrary regular point of the function y. Since $R(x,t)$ is u.s.c.i on $\bar{Q}$, for any given $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that

$$R(x,t)\subset R^{\varepsilon}(x_0,t_0) \tag{5.6}$$

whenever $|(x,t)-(x_0,t_0)|<\delta$, where $R^{\varepsilon}(x,t)$ denotes the closed ε-neighbourhood of $R(x,t)$.

Let $Q_\delta=\{(x,t)\in Q:|(x,t)-(x_0,t_0)|<\delta\}$. Let $E\subset Q_\delta$ be any measurable set such that $(x_0,t_0)\in E$. Let

$$y_l(x,\ t)=\Gamma(x,\ t,\ u_l(x,\ t)).$$

Then, $y_l(x,\ t)\in R(x,\ t)$ a.e. on Q for $l=1,2,\cdots$. Thus, it follows from (5.6) that $y_l(x,\ t)\in R^{\varepsilon}(x_0,\ t_0)$ for almost all $(x,\ t)\in E$ and for all $l=1,2,\cdots$. Further, since $R^{\varepsilon}(x_0,\ t_0)$ is closed and convex, we obtain that for all $l=1,2,\cdots$,

$$\frac{1}{|E|}\iint_E y_l(x,\ t)dxdt\in R^{\varepsilon}(x_0,\ t_0). \tag{5.7}$$

Since $R^{\varepsilon}(x,\ t)$ is closed, it follows from (5.2), (5.3), (5.4), (5.5), (5.7) and the fact that $|E|<\infty$ that

$$\lim_{l\to\infty}\frac{1}{|E|}\iint_E y_l(x,\ t)dxdt=\frac{1}{|E|}\iint_E y(x,\ t)dxdt \tag{5.8}$$

$$\in R^{\varepsilon}(x_0,\ t_0),$$

and consequently

$$y(x_0,\ t_0)=\lim_{|E|\to 0}\left\{\frac{1}{|E|}\iint_E y(x,\ t)dxdt\right\}\in R^{\varepsilon}(x_0,\ t_0).$$

Since ε was arbitrary and $R(x_0,\ t_0)$ is closed, $y(x_0,\ t_0)\in R(x_0,\ t_0)$. Further, since almost all points $(x,\ t)\in Q$ are regular points of y, we have that $y(x,\ t)\in R(x,\ t)$ a.e. on Q. By a modification on a set of measure zero if necessary, we can assume that $y(x,\ t)\in R(x,\ t)$ for all $(x,\ t)\in\bar{Q}$.

Note that $\bar{Q}$ can be considered as a locally compact Hausdorff space in which each compact subspace is metrizable, U_0 as a separable metric space and $U:\bar{Q}\to U_0$ as a measurable multifunction with complete values. Further, Γ is a Carathéodory function from $\bar{Q}\times U_0$ into the Hausdorff space R^s, where $s=(N+1)(2n+1)+n+2$, and $y:\bar{Q}\to R^s$, where $s=(N+1)(2n+1)+n+2$, is a measurable function with $y(x,\ t)\in R(x,\ t)=\Gamma(x,\ t,\ U(x,\ t))$ for all $(x,\ t)\in\bar{Q}$. Thus, it follows from Theorem 3′ of [**10**, p. 281] that there exists a measurable function $u_0:\bar{Q}\to U_0$ such that $u_0(x,\ t)\in U(x,\ t)$ and $y(x,\ t)=\Gamma(x,\ t,\ u_0(x,\ t))$ for all $(x,\ t)\in\bar{Q}$. This, in turn, implies that $\tilde{a}_{0j}(x,\ t)=a_{0j}(x,\ t,\ u_0(x,\ t))$, $\tilde{a}_{kj}(x,\ t)=\hat{a}_{kj}(x,\ t,\ u_0(x,\ t))$, $\tilde{b}_j(x,\ t)=b_j(x,\ t,\ u_0(x,\ t))$, $\tilde{c}(x,\ t)=c(x,\ t,\ u_0(x,\ t))$, $\tilde{F}_{0j}(x,\ t)=F_{0j}(x,\ t,\ u_0(x,\ t))$, $\tilde{F}_{kj}(x,\ t)=\hat{F}_{kj}(x,\ t,\ u_0(x,\ t))$, $\tilde{f}_0(x,\ t)=f_0(x,\ t,\ u_0(x,\ t))$, $\tilde{f}_k(x,\ t)=\hat{f}_k(x,\ t,\ u_0(x,\ t))$, $(j=1,\cdots,n;\ k=1,\cdots,N)$, and $\tilde{d}(x,\ t)=d(x,\ t,\ u_0(x,\ t))$.

Note that for each $k\in\{1,\cdots,N\}$, $a_{kj}(x,\ t,\ u_l(x,\ t))=a_{kj}(x,\ t,\ \hat{u}(x,\ t))$, $F_{kj}(x,\ t,\ u_l(x,\ t))=F_{kj}(x,\ t,\ \hat{u}(x,\ t))$, $(j=1,\cdots,n)$, and $f_k(x,\ t,\ u_l(x,\ t))=f_k(x,\ t,\ \hat{u}(x,\ t))$ on $\Omega\times[-h_k,\ 0]$ for all $l=1,2,\cdots$. For each $k=1,\cdots,N$, extend the definitions of $\tilde{a}_{kj}$, $\tilde{F}_{kj}$, $(j=1,\cdots,n)$, $\tilde{f}_k$ on $\Omega\times[-h_k,\ 0]$ and the definition of u_0 on $\Omega\times[-h_N,\ 0]$ by defining

$$\tilde{a}_{kj}(x,t)=a_{kj}(x,t,\hat{u}(x,t)),\ \tilde{F}_{kj}(x,t)=F_{kj}(x,t,\hat{u}(x,t)),\ (j=1,\cdots,n),$$

$$\tilde{f}_k(x,t)=f_k(x,t,\hat{u}(x,t)) \quad \text{on} \quad \Omega\times[-h_N,0] \quad \text{and}$$

$$u_0(x,t)=\hat{u}(x,t) \quad \text{on} \quad \bar{\Omega}\times[-h_N,0].$$

Then, it follows from Lemma 4.4 that there exist a subsequence of $\{\phi(u_l)\}$, which is denoted by the original sequence, and a unique weak solution $\phi(u_0)$ of system (1.1) corresponding to u_0 such that

$$\phi(u_l)\to\phi(u_0) \tag{5.9}$$

uniformly on any compact subset of Q. Hence, u_0 is an admissible control.

We now show that u_0 is an optimal control. From (5.9), it follows that $(u_l, \phi(u_l))-(u_l, \phi(u_0))\to 0$ in measure on Q. Further, $(u_l, \phi(u_l)), (u_l, \phi(u_0)) \in L^{p,p}(Q)\times L^{2,2}(Q)$, $(p\geqq 1)$. Thus, it follows from the assumption H(i) and Lemma 3.1 of [**11**, p. 524] that

$$\Psi_l(x,t)\triangleq d(x,t,u_l(x,t))\cdot(g(x,t,\phi(u_l)(x,t))-g(x,t,\phi(u_0)(x,t)))\to 0$$

in measure on Q. This implies that there exists a subsequence of $\{\Psi_l\}$ which is denoted by the original sequence, such that

$$\Psi_l(x,t)\to 0 \text{ a.e. in } Q. \tag{5.10}$$

By virtue of the assumption H(ii) and estimate (4.1), Ψ_l is bounded by a $L^{1,1}(Q)$-function. Hence, it follows from (5.10) and the Lebesgue dominated convergence theorem that

$$\iint_Q d(x,t,u_l(x,t))\cdot(g(x,t,\phi(u_l)(x,t))-g(x,t,\phi(u_0)(x,t)))dxdt\to 0 \tag{5.11}$$

as $l\to\infty$. Recall that $d(\cdot,\cdot,u_l(\cdot,\cdot))\to d(\cdot,\cdot,u_0(\cdot,\cdot))$ weakly in $L^{\lambda,\sigma}(Q)$. Further, by assumption H(ii) and estimate (4.1), $g(\cdot,\cdot,u_0(\cdot,\cdot))\in L^{\lambda',\sigma'}(Q)$, where $\frac{1}{\lambda}+\frac{1}{\lambda'}=1$ and $\frac{1}{\sigma}+\frac{1}{\sigma'}=1$. Thus, it follows readily that

$$\iint_Q (d(x,t,u_l(x,t))-d(x,t,u_0(x,t)))\cdot g(x,t,u_0(x,t))dxdt\to 0 \tag{5.12}$$

as $l\to\infty$. Thus, it follows from (5.11) and (5.12) that

$$\lim_{l\to\infty} J[u_l]=J[u_0]=\nu.$$

Therefore, u_0 is an optimal control.

In the case when $1<q<\infty$ and $r=\infty$, it follows from (4.12) and the assumption A(iii) that $\|\hat{f}_k(\cdot,\cdot,u_l(\cdot,\cdot))\|_{q,q,Q}\leqq T^{1/q}\gamma$. Using this estimate and a similar argument to the one used above, we can show that an optimal control exists. The case when $q=\infty$ and $1<r<\infty$, can be similarly treated. This completes the proof.

REMARK 5.2. Let $U: \bar{Q}\rightarrow R^m$ be a multifunction such that U is u.s.c.i and $U(x, t)$ is compact for each $(x, t)\in\bar{Q}$. Then, it is clear that U is a measurable function. In addition, if we further assume that the functions $a_{kj}, b_j, c, F_{kj}, f_k$ and d, $(k=0, 1, \cdots, N;\ j=1, \cdots, n)$, are continuous, it follows that $R(x, t)$ is u.s.c.i and compact for each $(x, t)\in\bar{Q}$. Thus, Theorem 5.1 remains valid if the functions $a_{kj}, b_j, c, F_{kj}, f_k$, $(j=1, \cdots, n;\ k=0, 1, \cdots, N)$, and d are assumed continuous; and when the assumptions (B) are replaced by the following assumptions (B′):

(i) U is u.s.c.i on $\bar{Q}$ and for each $(x, t)\in\bar{Q}$, $U(x, t)$ is compact; and

(ii) For each $(x, t)\in\bar{Q}$, the set $R(x, t)$ is convex.

REMARK 5.3. Theorem 5.1 remains valid when the cost functional J is defined by

$$J[u]=\iint_Q H(x, t, \phi(u)(x, t), u(x, t))dxdt\,,$$

where $\phi(u)$ is the weak solution of system (1.1) corresponding to u and the function $H: \bar{Q}\times R^1\times U_0\rightarrow R^1$ satisfies the following assumptions (H′):

(i) $H(x, t, \psi, u)$ is continuous in (ψ, u) for all $(x, t)\in\bar{Q}$ and measurable in (x, t) for each $(\psi, u)\in R^1\times U_0$;

(ii) There exist a nonnegative continuous function $h: [0, \infty)\rightarrow R^1$ and a nonnegative measurable function $p\in L^{1,1}(Q)$ such that for all $u\in U_0$ and for all $\phi\in L^{2,2}(Q)$,

$$|H(x, t, \phi(x, t), u)|\leqq p(x, t)h(\|\phi\|_{2,2,Q}) \quad \text{a.e. on } \bar{Q}\text{; and}$$

(iii) There exist measurable functions $p_1, p_2\in L^{\lambda,\sigma}(Q)$ and Carathéodory functions $a_1, a_2: \bar{Q}\times U_0\rightarrow R^1$ such that

$$\|a_1(\cdot, \cdot, u(\cdot, \cdot))\|_{\lambda',\sigma',Q},\quad \|a_2(\cdot, \cdot, u(\cdot, \cdot))\|_{\lambda',\sigma',Q}\leqq\delta$$

for all $u\in D$ and for some constant $\delta>0$, where $\lambda, \lambda', \sigma, \sigma'>1$, $\dfrac{1}{\lambda}+\dfrac{1}{\lambda'}=1$ and $\dfrac{1}{\sigma}+\dfrac{1}{\sigma'}=1$: Further for all $u_1, u_2\in U_0$,

$$p_1(x, t)\cdot(a_1(x, t, u_1)-a_1(x, t, u_2))\leqq H(x, t, \psi, u_1)-H(x, t, \psi, u_2)$$

$$\leqq p_2(x, t)\cdot(a_2(x, t, u_1)-a_2(x, t, u_2))\,,$$

for all $(x, t, \psi)\in\bar{Q}\times R^1$.

For this case we let $\Gamma: \bar{Q}\times U_0\rightarrow R^s$, where $s=(N+1)(2n+1)+n+3$, be defined by

$$\Gamma(x, t, u)=\operatorname{col}(a_{01}(x, t, u), \cdots, a_{0n}(x, t, u), \hat{a}_{11}(x, t, u), \cdots, \hat{a}_{Nn}(x, t, u), b_1(x, t, u), \cdots, b_n(x, t, u), c(x, t, u), F_{01}(x, t, u), \cdots, F_{0n}(x, t, u), \hat{F}_{11}(x, t, u), \cdots, \hat{F}_{Nn}(x, t, u), f_0(x, t, u), \hat{f}_1(x, t, u), \cdots, \hat{f}_N(x, t, u), a_1(x, t, u), a_2(x, t, u)); \text{ and}$$

$$R(x, t)=\{\Gamma(x, t, u):\ u\in U(x, t)\}.$$

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