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Continuation of A^{∞}-functions from submanifolds to strictly pseudoconvex domains

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§1. Introduction.

Let D be an open set in C^{n}. Let $H^{\infty}(D)$ be the space of all bounded holomorphic functions in D. By $A^{k}(D)$, $k\geqq 0$, we denote the algebra of C^{k}-functions on $\bar{D}$ which are holomorphic in D, where $v\in C^{k}(\bar{D})$ means that all derivatives of order$\leqq k$ of v admit a continuous extension to $\bar{D}$. $A^{k}(D)$ is a Banach space with respect to the norm

$$\|f\|_{k}=\sup_{z\in\partial D}\sup_{|\alpha|\leqq k}|D^{\alpha}f(z)|,$$

where α is an n-tuple of nonnegative integers $\alpha=(\alpha_{1}, \cdots, \alpha_{n})$, $|\alpha|=\alpha_{1}+\cdots+\alpha_{n}$ and

$$D^{\alpha}=\frac{\partial^{|\alpha|}}{\partial z_{1}^{\alpha_{1}}\cdots\partial z_{n}^{\alpha_{n}}}.$$

Let D be a strictly pseudoconvex domain in C^{n} with C^{∞}-boundary and let M be a complex submanifold of D which intersects ∂D transversally. G. M. Henkin [4] proved that there exists a bounded operator $E: H^{\infty}(M)\rightarrow H^{\infty}(D)$, which continues bounded holomorphic functions on M to bounded holomorphic functions on D and moreover $Ef\in A^{0}(D)$ if $f\in A^{0}(M)$. The related results have been given by K. Adachi [1] and J. E. Fornaess [3]. In this paper we prove that the extended function Ef belongs to $A^{\infty}(D)$ if $f\in A^{\infty}(M)$. In the second section we prove this theorem for a strictly convex domain D with C^{∞}-boundary and $M=\{z_{k+1}=\cdots=z_{n}=0\}\cap D$. The proof of this theorem is based on the method of Y. T. Siu [5] used to obtain the estimates for derivatives of the solutions in the $\bar{\partial}$-problem. In the third section, by applying the method of G. M. Henkin [4], we prove this theorem for strictly pseudoconvex domains with C^{∞}-boundaries. In the fourth section we prove that the Ramírez-Henkin kernel is considered as an operator which maps $C^{k}(\partial D)$ to $A^{m}(D)$, provided $k\geqq 2m+4$. Also, we prove an approximation theorem: $O(\bar{D})$ is dense in $A^{k}(D)$ in the $\|\ \|_{m}$-norm, provided $k\geqq 2m+4$. In the final section we prove that the multiplicative Cousin problem for $A^{\infty}(D)$

is solvable, provided that D is a strictly convex domain with C^∞-boundary.

§ 2. The case of strictly convex domains.

Let D be a strictly convex domain in C^n with C^∞-boundary, i.e., $D=\{z\in\tilde{D}: \rho(z)<0\}$, where $\tilde{D}$ is a domain, $\bar{D}\subset\tilde{D}$, ρ is a strictly convex C^∞-function on $\tilde{D}$. Then by G. M. Henkin [4], $d\rho\neq 0$ on ∂D. Let $\tilde{M}=\{z_{k+1}=\cdots=z_n=0\}\cap\tilde{D}$ and $M=\tilde{M}\cap D$. We set $\eta=(\eta_1, \cdots, \eta_n)$ and $\zeta=(\zeta_1, \cdots, \zeta_n)$, and

$$\omega_k'(\eta)\wedge\omega_k(\zeta)=\Big(\sum_{\nu=1}^{k}(-1)^{\nu-1}\eta_\nu d\eta_1\wedge\cdots\wedge\widehat{d\eta_\nu}\wedge\cdots\wedge d\eta_k\Big)\wedge(d\zeta_1\wedge\cdots\wedge d\zeta_k), \tag{2.1}$$

where $\wedge$ means that $d\eta_\nu$ is omitted. Let $\Phi(\zeta, z)=\sum_{i=1}^{n}\frac{\partial\rho}{\partial\zeta_i}(\zeta)(\zeta_i-z_i)$, $\zeta^0\in\partial M$ and $\tilde{U}$ be some neighborhood of ζ^0 in $\tilde{D}$. Now we need the following lemmata.

LEMMA 1. *Let $\phi_i\in C^1(\tilde{U})$ such that for $\zeta\in\tilde{U}\cap\partial D$, $\phi_i(\zeta)=\frac{\partial\rho}{\partial\zeta_i}(\zeta)$. Let $\tilde{\Phi}(\zeta, z)=\sum_{i=1}^{n}\phi_i(\zeta)(\zeta_i-z_i)$. Then there exist $c>0$ and an open neighborhood U of ζ^0 in $\tilde{U}$ such that $|\tilde{\Phi}(\zeta, z)|\geqq c|\zeta-z|^2$ for $\zeta, z\in U$ and $\rho(z)\leqq 0\leqq\rho(\zeta)$.*

PROOF. There exist $A>0$ and an open neighborhood U_1 of ζ^0 in $\tilde{U}$ with diameter$\leqq\frac{1}{2}$ such that

$$\Big|\phi_i(\zeta)-\frac{\partial\rho}{\partial\zeta_i}(\zeta)\Big|\leqq A\rho(\zeta), \qquad (1\leqq i\leqq n)$$

for $\zeta\in U_1$ and $\rho(\zeta)\geqq 0$. It follows that

$$|\tilde{\Phi}(\zeta, z)-\Phi(\zeta, z)|\leqq nA\rho(\zeta)|\zeta-z|$$

for $\zeta, z\in U_1$ and $\rho(\zeta)\geqq 0$. By Taylor's formula and the strict convexity of ρ, there exist an open neighborhood U_2 of ζ^0 in U_1 and $\lambda>0$ such that

$$\rho(z)-\rho(\zeta)\geqq 2\mathrm{Re}\sum_{i=1}^{n}(z_i-\zeta_i)\frac{\partial\rho}{\partial\zeta_i}(\zeta)+\lambda|\zeta-z|^2$$

for $\zeta, z\in U_2$. Therefore

$$2|\Phi(\zeta, z)|\geqq\rho(\zeta)+\lambda|\zeta-z|^2$$

for $\zeta, z\in U_2$ and $\rho(z)\leqq 0\leqq\rho(\zeta)$. Let U be an open neighborhood of ζ^0 in U_2 with diameter$\leqq(4nA)^{-1}$. Then

$$|\tilde{\Phi}(\zeta, z)|\geqq|\Phi(\zeta, z)|-|\tilde{\Phi}(\zeta, z)-\Phi(\zeta, z)|\geqq\frac{1}{2}\lambda|\zeta-z|^2$$

for ζ, $z\in U$ and $\rho(z)\leqq 0\leqq\rho(\zeta)$. Q.E.D.

The next lemma is a consequence of Y. T. Siu [5].

LEMMA 2. *Let m be a positive integer and let G be an open subset of C^n. Suppose $1\leqq i\leqq n$ and $\frac{\partial\rho}{\partial\zeta_i}\neq 0$ for $\zeta\in\tilde{U}$. Then every C^{∞}-function $h(\zeta, z)$ on $(\tilde{U}\cap\partial D)\times G$ can be extended to a C^{∞}-function $\tilde{h}(\zeta, z)$ on $\tilde{U}\times G$ such that $\frac{\partial\tilde{h}}{\partial\bar{\zeta}_j}(\zeta, z)=\gamma(\zeta, z)\rho(\zeta)^m$ for some C^{∞}-function $\gamma(\zeta, z)$ on $\tilde{U}\times G$.*

G.M. Henkin [4] proved the following.

THEOREM 1. *Let $f(z)$ be a bounded holomorphic function on M. The formula*

$$(Lf)(z)=c_k\int_{\zeta\in\partial M}f(\zeta)\omega_k'(\eta)\wedge\omega_k(\zeta)\,, \tag{2.2}$$

where

$$\eta_i=\frac{\frac{\partial\rho}{\partial\zeta_i}(\zeta)}{\sum_{\nu=1}^{n}\frac{\partial\rho}{\partial\zeta_\nu}(\zeta)(\zeta_\nu-z_\nu)}\quad\text{and}\quad c_k=\frac{(k-1)!}{(2\pi i)^k}$$

defines a function $F(z)$ which is bounded and holomorphic in D, and is such that $F(z)=f(z)$ for any $z\in M$. Here $F(z)\in A^0(D)$ if $f(z)\in A^0(M)$.

Using lemmata 1, 2, we have the following theorem.

THEOREM 2. *Let $F(z)$ be the function obtained in theorem 1. If $f(z)\in A^{\infty}(M)$, then $F(z)\in A^{\infty}(D)$.*

PROOF. Since $F(z)$ can be written as

$$F(z)=\int_{\partial M}\frac{f(\zeta)\omega_1(\zeta)}{\Phi(\zeta, z)^k}$$

where $\omega_1(\zeta)$ is a $C^{\infty}(k, k-1)$-form in D, we have

$$D^mF(z)=\int_{\partial M}\frac{f(\zeta)\omega_2(\zeta, z)}{\Phi(\zeta, z)^{k+m}}$$

where $\omega_2(\zeta, z)$ is a $C^{\infty}(k, k-1)$-form in $\tilde{D}\times\tilde{D}$. In order to prove that $F(z)\in A^{\infty}(D)$, it suffices to show that

$$\int_{\partial M}\frac{\omega(\zeta, z)}{\Phi(\zeta, z)^{k+m}}$$

is uniformly bounded in $z\in D$, where

$$\omega(\zeta, z)=\sum_{\nu=1}^{k}a_\nu(\zeta, z)d\bar{\zeta}_1\wedge\cdots\wedge\widehat{d\bar{\zeta}_\nu}\wedge\cdots\wedge d\bar{\zeta}_k\wedge d\zeta_1\wedge\cdots\wedge d\zeta_k$$

is a $C^{\infty}(k, k-1)$-form in $\tilde{D}\times\tilde{D}$. After changing the coordinates system linearly, we can assume without loss of generality that there exists an open neighborhood $\tilde{U}$ of ζ^0 in $\tilde{D}$ such that $\frac{\partial\rho}{\partial\zeta_j}(\zeta)\neq 0$ for $\zeta\in\tilde{U}$ and $1\leqq j\leqq n$. Fix $1\leqq\nu\leqq k$. By lemma 2, we can find C^∞-functions $\phi_i(\zeta)$ on $\tilde{U}$ such that $\phi_i(\zeta)=\frac{\partial\rho}{\partial\zeta_i}(\zeta)$ for $\zeta\in\tilde{U}\cap\partial D$ and $\frac{\partial\phi_i}{\partial\bar{\zeta}_i}(\zeta)=\alpha_i(\zeta)\rho(\zeta)^{2m+2}$ on $\tilde{U}$ for some C^∞-functions $\alpha_i(\zeta)$ on $\tilde{U}$. Also, by lemma 2, we can find C^∞-functions $\tilde{a}_\nu(\zeta, z)$ on $\tilde{U}\times\tilde{D}$ such that $\tilde{a}_\nu(\zeta, z)=a_\nu(\zeta, z)$ on $(\tilde{U}\cap\partial D)\times\tilde{D}$ and $\frac{\partial\tilde{a}_\nu}{\partial\bar{\zeta}_\nu}(\zeta, z)=\gamma_\nu(\zeta, z)\rho(\zeta)^{2m+1}$ on $\tilde{U}\times\tilde{D}$ for some C^∞-function $\gamma_\nu(\zeta, z)$ on $\tilde{U}\times\tilde{D}$. Let $\tilde{\Phi}(\zeta, z)=\sum_{i=1}^{n}\phi_i(\zeta)(\zeta_i-z_i)$. By lemma 1, there exist $c>0$ and a relatively compact open neighborhood U_ν of ζ^0 in $\tilde{U}$ such that

$$|\tilde{\Phi}(\zeta, z)|\geqq c_\nu|\zeta-z|^2 \qquad \text{for} \quad \zeta, z\in U_\nu \quad \text{and} \quad \rho(z)\leqq 0\leqq\rho(\zeta).$$

Let

$$R_\nu(\zeta, z)=\frac{\frac{\partial\tilde{a}_\nu}{\partial\bar{\zeta}_\nu}(\zeta, z)}{\tilde{\Phi}(\zeta, z)^{k+m}}-(k+m)\frac{\tilde{a}_\nu(\zeta, z)\frac{\partial\tilde{\Phi}}{\partial\bar{\zeta}_\nu}(\zeta, z)}{\tilde{\Phi}(\zeta, z)^{k+m+1}}.$$

It follows that

$$|R_\nu(\zeta, z)|\leqq\text{const}\,|\zeta-z|^{-2k+1} \qquad \text{for} \quad \zeta\in U_\nu-\bar{D} \quad \text{and} \quad z\in U_\nu\cap D.$$

Let B_ν be a relatively compact open neighborhood of ζ^0 in U_ν such that ∂B_ν is C^1 and the normal vector of ∂B_ν and ∂D are independent at every point of $\partial B_\nu\cap\partial D$. For $z\in B_\nu\cap D$, by applying Stokes' theorem to

$$d_\zeta\Big(\frac{\tilde{a}_\nu(\zeta, z)}{\tilde{\Phi}(\zeta, z)^{k+m}}d\bar{\zeta}_1\wedge\cdots\wedge\widehat{d\bar{\zeta}_\nu}\wedge\cdots\wedge d\bar{\zeta}_k\wedge d\zeta_1\wedge\cdots\wedge d\zeta_k\Big)$$
$$=(-1)^{\nu-1}R_\nu(\zeta, z)d\bar{\zeta}_1\wedge\cdots\wedge d\bar{\zeta}_k\wedge d\zeta_1\wedge\cdots\wedge d\zeta_k$$

on $(B_\nu\cap\tilde{M})-\bar{M}$, we obtain

$$\int_{B_\nu\cap\partial M}\frac{a_\nu(\zeta, z)}{\Phi(\zeta, z)^{k+m}}d\bar{\zeta}_1\wedge\cdots\wedge\widehat{d\bar{\zeta}_\nu}\wedge\cdots\wedge d\bar{\zeta}_k\wedge d\zeta_1\wedge\cdots\wedge d\zeta_k$$
$$=-\int_{\partial(B_\nu\cap\tilde{M})-M}\frac{\tilde{a}_\nu(\zeta, z)}{\tilde{\Phi}(\zeta, z)^{m+k}}d\bar{\zeta}_1\wedge\cdots\wedge\widehat{d\bar{\zeta}_\nu}\wedge\cdots\wedge d\bar{\zeta}_k\wedge d\zeta_1\wedge\cdots\wedge d\zeta_k$$
$$+\int_{(B_\nu\cap\tilde{M})-\bar{M}}(-1)^{\nu-1}R_\nu(\zeta, z)d\bar{\zeta}_1\wedge\cdots\wedge d\bar{\zeta}_k\wedge d\zeta_1\wedge\cdots\wedge d\zeta_k.$$

It follows that if U is a relatively compact open neighborhood of ζ^0 in $\bigcap_{\nu=1}^{k}B_\nu$, then

$$\int_{\zeta\in\partial M}\frac{\omega(\zeta, z)}{\Phi(\zeta, z)^{k+m}}$$

is uniformly bounded for $z\in U\cap D$.

§ 3. The case of strictly pseudoconvex domains.

Let D be a strictly pseudoconvex domain in C^n with C^∞-boundary, i. e., $D=\{z\in\tilde{D}: \rho(z)<0\}$, where $\tilde{D}$ is a domain, $\bar{D}\subset\tilde{D}$, and ρ is a strictly plurisubharmonic function in $\tilde{D}$ and $d\rho\neq 0$ on ∂D. Let $\tilde{M}$ be a k-dimensional complex submanifold in $\tilde{D}$ which intersects ∂D transversally. Let $M=\tilde{M}\cap D$. Let D' be a strictly pseudoconvex domain with $\tilde{D}\supset\bar{D}'\supset D'\supset\bar{D}$ and let $M'=\tilde{M}\cap D'$. Let

$$F(z, \zeta)=\sum_{i=1}^{n}\frac{\partial\rho}{\partial z_i}(\zeta)(z_i-\zeta_i)+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\rho}{\partial z_i\partial z_j}(\zeta)(z_i-\zeta_i)(z_j-\zeta_j)\,. \tag{3.1}$$

It follows from the Oka-Cartan theory that the ideal $I_{M'}$ of functions holomorphic in D' and equal to zero on M' has a finite set of generators $\{F_1(z), \cdots, F_q(z)\}$. Let $S_{\zeta,\sigma}=\{z: |\zeta-z|<\sigma\}$. G. M. Henkin [4] proved the following

LEMMA. 3. *There exist constants $\sigma>\delta>0$ such that for any $\zeta\in\partial M$ the following assertions are true.*

(a) *For certain numbers $q_1(\zeta), \cdots, q_{n-k}(\zeta)$ from the set $\{1, \cdots, q\}$ and for certain numbers $n_1(\zeta), \cdots, n_{k-1}(\zeta)$ from the set $\{1, \cdots, n\}$ the map*

$$z\to w(z, \zeta)=\{z_{n_1}-\zeta_{n_1}, \cdots, z_{n_{k-1}}-\zeta_{n_{k-1}}, F(z, \zeta), F_{q_1}(z), \cdots F_{q_{n-k}}(z)\}$$

is a biholomorphic map of the ball $S_{\zeta,\sigma}\subset D'$ onto a neighborhood of zero W_ζ in the space of the variables $(w_1, \cdots, w_n)=w$.

(b) *The preimage G_ζ of some strictly convex domain V_ζ of W_ζ contains the domain $D\cap S_{\zeta,\delta}$ and is contained in D, i. e.,*

$$D\cap S_{\zeta,\delta}\subset G_\zeta=\{z\in S_{\zeta,\delta}: w(z, \zeta)\in V_\zeta\subset\bar{V}_\zeta\subset W_\zeta\}\subset D\,,$$

where $V_\zeta=\{w\in W_\zeta: \rho_\zeta(w)<0\}$, and $\rho_\zeta(w)$ is a real valued C^∞-function in the domain W_ζ and is strictly convex in a neighborhood of $\bar{V}_\zeta$.

Let σ and δ be the constants from lemma 3. We may assume that $\varepsilon<\frac{\delta}{2}$. Let $\chi_i(z)$, $i=1, \cdots, N(\varepsilon)$, be C^∞-functions such that $\chi_i(z)\geqq 0$, $i=1, \cdots, N(\varepsilon)$, $\sum_i\chi_i(z)=1$ in a neighborhood of $\bar{M}'$ and for any $i=1, \cdots, N$ the diameter of $\operatorname{supp}\chi_i$ is less than $\frac{\varepsilon}{3}$. We set

$$\chi_\nu^1 = \sum_{i:\operatorname{supp}\chi_i \cap \operatorname{supp}\chi_\nu \neq \emptyset} \chi_i, \qquad \tilde{\chi}_\nu = \sum_{i:\operatorname{supp}\chi_i^1 \cap \operatorname{supp}\chi_\nu = \emptyset} \chi_i,$$

$$D_\nu = \left\{ z \in \tilde{D} : \rho(z) - \sum_{i=1}^{\nu} \lambda_i \chi_i(z) < 0 \right\}, \qquad \nu = 1, \cdots, N,$$

$$\tilde{D}_\nu = \left\{ z \in \tilde{D} : \rho(z) - \sum_{i=1}^{\nu-1} \lambda_i \chi_i(z) - \lambda_\nu \tilde{\chi}_\nu(z) < 0 \right\}, \qquad \nu = 1, \cdots, N.$$

$D_0 = D$, $M_\nu = \tilde{M} \cap D_\nu$, $\tilde{M}_\nu = \tilde{M} \cap \tilde{D}_\nu$, $\nu = 1, \cdots, N$. If λ_ν, $\nu = 1, \cdots, N$, are sufficiently small, then D_ν and $\tilde{D}_\nu$, $\nu = 1, \cdots, N$, are strictly pseudoconvex and the assertion of lemma 3 holds for any $\zeta \in \partial M$ with constants $\sigma_\nu > \delta_\nu \geqq \frac{3}{4}\delta$, where $\sigma_0 = \sigma$, $\delta_0 = \delta$.

In this setting we have the following.

LEMMA 4. *There exist constants $\lambda_1, \cdots, \lambda_N > 0$ such that, for any $\nu = 1, \cdots, N$, bounded operators*

$$L_\nu^0 : H^\infty(M_{\nu-1}) \to H^\infty(M_\nu) \quad \text{and} \quad L_\nu^1 : H^\infty(M_{\nu-1}) \to H^\infty(\tilde{M}_\nu)$$

exist with the following properties.

(a^ν) $f(z) = (L^0 f)(z) + (L^1 f)(z)$ *for any* $f \in H^\infty(M_{\nu-1})$ *and any* $z \in M_{\nu-1}$.

(b^ν) $L_\nu^0 f \in A^\infty(M_\nu)$ *and* $L_\nu^1 f \in A^\infty(\tilde{M}_\nu)$ *if* $f \in A^\infty(M_{\nu-1})$.

PROOF. Now we follow the proof of lemma 12 of G. M. Henkin [4]. Suppose that constants $\lambda_1, \cdots, \lambda_{\nu-1}$ satisfying the conditions of the lemma have already been chosen. We set $U_\nu = \operatorname{supp} \chi_\nu^1$. In the case when $U_\nu \cap \partial M_{\nu-1} = \emptyset$, there is nothing to prove. Let $U_\nu \cap \partial M_{\nu-1} \neq \emptyset$. We fix a point $\zeta^0 \in U_\nu \cap \partial M_{\nu-1}$. By lemma 3, there exists a biholomorphic $w(z, \zeta^0) : S_{\zeta^0, 3\sigma/4} \to W_{\zeta^0}$, where W_{ζ^0} is some neighborhood of zero in the space of the variables $(w_1, \cdots, w_n) = w$. By the same lemma, it follows that

$$D_{\nu-1} \cap S_{\zeta^0, 3\delta/4} \subset G_{\zeta^0} = \{ z \in S_{\zeta^0, 3\sigma/4} : \rho_{\zeta^0}(w(z, \zeta^0)) < 0 \} \subset D_{\nu-1}$$

where $\rho_{\zeta^0}(w)$ is a strictly convex function in a neighborhood of the set $\bar{V}_{\zeta^0} = \{ w \in W_{\zeta^0} : \rho_{\zeta^0}(w) \leqq 0 \}$. Let $w \to z(w, \zeta^0)$ be the inverse map of the map $z \to w(z, \zeta^0)$. For any $z \in G_{\zeta^0} \cap M_{\nu-1}$, by the Cauchy-Fantappiè integral formula, we have

$$f(z) = c_k \int_{\partial V_{\zeta^0}} f(z(\zeta, \zeta^0)) \frac{\omega_k'\left(\frac{\partial \rho_{\zeta^0}}{\partial \zeta}(\zeta) \right) \wedge \omega_k(\zeta)}{\left[\sum_{i=1}^{k} \frac{\partial \rho_{\zeta^0}}{\partial \zeta_i}(\zeta)(\zeta_i - w_i(z, \zeta^0)) \right]^k}.$$

We set $M'' = \{ z \in \tilde{M} : \rho(z) - \sum_{i=1}^{\nu-1} \lambda_i \chi_i(z) < \lambda_0 \} = M_0'' \cup M_1''$, where $M_0'' = M'' \cap S_{\zeta^0, 3\delta/4}$, $M_1'' = M'' | S_{\zeta^0, \delta/2}$. If $\lambda_\nu < \lambda_0$, then $[\tilde{M}_\nu \cup M_\nu] \subset M''$. M'' is pseudoconvex for sufficiently small $\lambda_0 > 0$. Let $\chi_\nu^0 = 1 - \chi_\nu^1$. We define for $\alpha = 0, 1$,

$$R_\nu^\alpha f(z)=c_k\int_{\partial V_{\zeta^0}} f(z(\zeta,\zeta^0))\chi_\nu^\alpha(z(\zeta,\zeta^0))\frac{\omega_k'\left(\frac{\partial\rho_{\zeta^0}}{\partial\zeta}(\zeta)\right)\omega_k(\zeta)}{\left[\sum_{i=1}^k\frac{\partial\rho_{\zeta^0}}{\partial\zeta_i}(\zeta)(\zeta_i-w_i(z,\zeta^0))\right]^k}.$$

If the constants $0<\lambda_\nu<\lambda_0$ are sufficiently small, by the same method as in the proof of theorem 2, it follows that $R_\nu^0 f\in A^\infty(M_\nu\cap S_{\zeta^0,3\delta/4})$, $R_\nu^1 f\in A^\infty(\tilde{M}_\nu\cap S_{\zeta^0,3\delta/4})$. Moreover R_ν^1 is a bounded operator from $H^\infty(M_{\nu-1})$ to $A^0(M_0''\cap M_1'')$. It follows that there exist bounded operators $T_\nu^\alpha: H^\infty(M_0''\cap M_1'')\to H(M_\alpha'')$, $\alpha=0, 1$, such that $f(z)=(T^0f)(z)+(T^1f)(z)$, where $f\in H^\infty(M_0''\cap M_1'')$ and $z\in M_0''\cap M_1''$. We set

$$(L_\nu^0 f)(z)=\begin{cases}(R_\nu^0 f)(z)+(T_\nu^0\circ R_\nu^1 f)(z) & \text{if } z\in M_\nu\cap S_{\zeta^0\,3\delta/4}\\ f(z)-(T_\nu^1\circ R_\nu^1 f)(z) & \text{if } z\in M_\nu\,|\,S_{\zeta^0,\delta/2}\end{cases}$$

$$(L_\nu^1 f)(z)=\begin{cases}(T_\nu^1\circ R_\nu^1 f)(z) & \text{if } z\in \tilde{M}_\nu\,|\,S_{\zeta^0,\delta/2}\\ (R_\nu^1 f)(z)-(T_\nu^0\circ R_\nu^1 f)(z) & \text{if } z\in \tilde{M}_\nu\cap S_{\zeta^0,3\delta/4}.\end{cases}$$

Then L_ν^0 and L_ν^1 are the operators satisfying the condition of lemma 4.

Q. E. D.

If we set $L_i=L_i^1\circ L_{i-1}^0\circ\cdots\circ L_1^0$, $i=1,\cdots,N-1$, and $L_N=L_{N-1}^0\circ L_{N-2}^0\circ\cdots\circ L_1^0$, and $S_i=\tilde{M}_i$, $i=1, 2,\cdots, N-1$, $S_N=M_{N-1}$. Then we have the following.

THEOREM 3. *For any $\varepsilon>0$, there exist a covering $\{S_i\}$ of the set ∂M by domains $S_i\supset M$, $i=1,\cdots, N(\varepsilon)$, and bounded operators $L_i: H^\infty(M)\to H^\infty(S_i)$ such that*

(1) $L_i: A^\infty(M)\to A^\infty(S_i)$ $(i=1,\cdots,N)$

(2) $f(z)=\sum_i (L_i f)(z)$ *for any function* $f\in H^\infty(M)$

(3) *the diameter of $\bar{M}|S_i$ is less than ε for any $i=1,\cdots,N$.*

From this theorem, by following the proof of the fundamental theorem of G. M. Henkin [4], we obtain the following theorem.

THEOREM 4. *There exists a bounded extension operator $E: H^\infty(M)\to H^\infty(D)$ and moreover $Ef\in A^\alpha(D)$ if $f\in A^\alpha(M)$, $\alpha=0,\infty$.*

§ 4. The Ramírez-Henkin kernel.

In this section we study properties about the Ramírez-Henkin kernels for strictly pseudoconvex domains with C^∞-boundaries and an approximation theorem for $A^k(D)$. Let D be a strictly pseudoconvex domain in C^n with C^∞-boundary and ρ be a defining function of D defined in $\tilde{D}$. Then we obtain a C^∞-function $\Psi(z,\zeta)$ on $\tilde{D}\times\tilde{D}$ holomorphic in z with the following properties:

(1) $\Psi(z,\zeta)\neq 0$ for $\zeta, z\in\tilde{D}$ with $\rho(\zeta)>\rho(z)$

(2) for $\zeta^0\in\partial D$ there exist an open neighborhood U of ζ^0 in $\tilde{D}$ and a

nowhere vanishing C^{∞}-function $G(z, \zeta)$ on $U\times U$ holomorphic in z such that $\Psi(z, \zeta)=G(z, \zeta)F(z, \zeta)$ on $U\times U$ where $F(z, \zeta)$ is the function defined in (3.1)

(3) there exist C^{∞}-functions $P_i(z, \zeta)$ on $\tilde{D}\times\tilde{D}$ holomorphic in z such that $\Psi(z, \zeta)=\sum_{i=1}^{n}(z_i-\zeta_i)P_i(z, \zeta)$.

Let $P(z, \zeta)=(P_1(z, \zeta), \cdots, P_n(z, \zeta))$. Then the Ramírez-Henkin kernel $H(z, \zeta)$ for D can be written as

$$H(z, \zeta)=\frac{\omega_n'(P(z, \zeta))\wedge\omega(\zeta)}{\Psi(z, \zeta)^n}$$

where $\omega_n'(P(z, \zeta))\wedge\omega(\zeta)$ is the differetial form with respect to ζ defined in (2.1).

From proposition 3.3 in Y. T. Siu [5], we have:

LEMMA 5. *If $\omega(z, \zeta)$ is a $C^{2m+2}(n, n-1)$-form on $\tilde{D}\times\tilde{D}$, then* $\int_{\zeta\in\partial D}\frac{\omega(z, \zeta)}{\Psi(z, \zeta)^{n+m}}$ *is uniformly bounded for $z\in D$.*

From this lemma, we obtain:

THEOREM 5. *Let s, m be nonnegative integers such that $s\geqq 2m+4$. Let $f(\zeta)$ be a C^s-function on ∂D. Then*

$$\int_{\zeta\in\partial D}f(\zeta)H(z, \zeta)\in A^m(D).$$

Next we can prove the following corollary as in the proof of corollary II. 3 in E. L. Stout [7].

COROLLARY. *Let s, m be nonnegative integers such that $s\geqq 2m+4$. If $f\in A^s(D)$ and if $\mathfrak{U}=\{U_1, \cdots, U_q\}$ is an open cover for ∂D, there exist functions $f_1, \cdots, f_q\in A^m(D)$, such that $f=f_1+\cdots+f_q$ and such that each function f_j is holomorphic on a neighborhood of the compact set $\partial D|U_j$.*

By applying the method used in the proof of theorem II. 4 of E. L. Stout [7], we obtain an approximation theorem for functions in $A^k(D)$.

THEOREM 6. *Let k, m be nonnegative integers such that $k\geqq 2m+4$. If $f\in A^k(D)$, then there exists a sequence $\{f_n\}$ in $O(\bar{D})$ that converges in $\|\ \|_m$-norm to f.*

PROOF. Choose a finite open covering $\mathfrak{U}=\{U_1, \cdots, U_q\}$ of ∂D, and, for each j, choose a point $P_j\in U_j$. Let Π_j be the real tangent plane to ∂D at P_j, and let ν_j be the unit outward normal to ∂D at P_j. Assume the U_j's and the P_j's have been chosen to satisfy these conditions:

A) The real linear orthogonal projection π_j that carries C^n onto Π_j carries a neighborhood U_j' of $\bar{U}_j$ diffeomorphically onto the open set $\pi_j(U_j)\subset\Pi_j$.

B) For all $z\in U_j$, the points $z-\varepsilon\nu_j$ approach z nontangentially through D as $\varepsilon\to 0^+$.

Choose $\gamma_1, \cdots, \gamma_q$ to constitute a C^{∞}-partition of unity on ∂D subordinate to $\mathfrak{U}$, and write $f=f_1+\cdots+f_q$ as in the corollary above. In order to approximate f, it suffices to approximate the functions f_j. For this purpose, define, for $\varepsilon>0$ but sufficiently small, a function $f_j^{(\varepsilon)}(z)$ by $f_j^{(\varepsilon)}(z)=f_j(z-\varepsilon\nu_j)$. It follows that $f_j^{(\varepsilon)}\in O(\bar{D})$ for all sufficiently small $\varepsilon>0$. We have

$$\|f_j^{(\varepsilon)}(z)-f_j(z)\|_m=\sup_{z\in\partial D}\sup_{|\alpha|\leq m}|D^\alpha(f_j^{(\varepsilon)}(z)-f_j(z))|\to 0 \quad \text{as} \quad \varepsilon\to 0\,.$$

Q. E. D.

§5. The multiplicative Cousin problem for $A^\infty(D)$.

Let S_n be the set of all strictly convex domains in C^n with C^∞-boundary. Let $D\in S_n$. Then D can be written in the form $D=\{z\in\tilde{D}: \rho(z)<1\}$, where $\tilde{D}$ is a domain, $\tilde{D}\supset\bar{D}$, ρ is a strictly convex function in $\tilde{D}$. Let

$$M=\max\{x_{2n}: \text{for some } z\in\bar{D},\ z=(z_1,\cdots,z_n),\ x_{2n}=\operatorname{Im} z_n\}\,,$$

and let m be the corresponding minimum. Let ε_0 satisfy $0<\varepsilon_0<(1/12)(M-m)$. Let η_i, $i=1, 2$, be real valued functions of a real variable such that

(1) η_i is of class C^∞, $i=1, 2$,

(2) $\eta_1(t)=0 \quad \text{if} \quad t\leqq\frac{1}{2}(M+m)+\frac{5}{2}\varepsilon_0$,

$\eta_2(t)=0 \quad \text{if} \quad t\geqq\frac{1}{2}(M+m)-\frac{5}{2}\varepsilon_0$,

(3) $\eta_1(t)\geqq 2 \quad \text{if} \quad t\geqq\frac{1}{2}(M+m)+3\varepsilon_0$,

$\eta_2(t)\geqq 2 \quad \text{if} \quad t\leqq\frac{1}{2}(M+m)-3\varepsilon_0$,

(4) $\eta_1''(t)>0 \quad \text{if} \quad t>\frac{1}{2}(M+m)+\frac{5}{2}\varepsilon_0$,

$\eta_2''(t)>0 \quad \text{if} \quad t<\frac{1}{2}(M+m)-\frac{5}{2}\varepsilon_0$.

Let $D_1=\{z: \rho(z)+\eta_1(x_{2n})<1\}$, $D_2=\{z: \rho(z)+\eta_2(x_{2n})<1\}$. Then it is easily verified that D_1, D_2 and $D_1\cap D_2$ are elements of S_n.

LEMMA 6. *Let D, D_1, D_2 be as above. If $f\in A^\infty(D_1\cap D_2)$, then we can write $f=f_1+f_2$, where $f_1\in A^\infty(D_1)$ and $f_2\in A^\infty(D_2)$.*

PROOF. Let ψ be a C^∞-function on C^n which has the properties that

$$\psi=0 \quad \text{on} \quad \left\{z\in\partial(D_1\cap D_2): x_{2n}<\frac{1}{2}(M+m)-\varepsilon_0\right\}$$

$$\psi=1 \quad \text{on} \quad \left\{z\in\partial(D_1\cap D_2):\ x_{2n}>\frac{1}{2}(M+m)+\varepsilon_0\right\}.$$

Let $\tilde{\rho}$ be a defining function of $D_1\cap D_2$. For $w\in D_1\cap D_2$, we can write

$$f(z)=c_n\int_{\partial(D_1\cap D_2)}\frac{f(\zeta)k(\zeta)}{\Phi(\zeta,\ z)^n}=f_1(z)+f_2(z)$$

$$f_1(z)=c_n\int_{\partial(D_1\cap D_2)}\frac{f(\zeta)(1-\psi(\zeta))k(\zeta)}{\Phi(\zeta,\ z)^n}$$

$$f_2(z)=c_n\int_{\partial(D_1\cap D_2)}\frac{f(\zeta)\psi(\zeta)k(\zeta)}{\Phi(\zeta,\ z)^n}$$

where $\Phi(z,\ z)=\sum_{i=1}^{n}\frac{\partial\tilde{\rho}}{\partial\zeta_i}(\zeta)(\zeta_i-z_i)$ and $k(z)$ is a $C^\infty(n,\ n-1)$-form on some neighborhood of $\overline{D_1\cap D_2}$. From the proof of theorem 2 and the properties of ψ it is easily seen that $F_1\in A^\infty(D_1)$, $f_2\in A^\infty(D_2)$. Q.E.D.

Next we prove that the multiplicative Cousin problem with A^∞ data is solvable on every domain of class S_n. The related results have been given by K. Adachi [2] and E. L. Stout [6].

THEOREM 7. *Let $D\in S_n$, and let $\{V_\alpha\}_{\alpha\in I}$ be an open cover for $\bar{D}$. If for each α, $f_\alpha\in A^\infty(V_\alpha\cap D)$, and if for all α, $\beta\in I$, $f_\alpha f_\beta^{-1}\in A^\infty(V_\alpha\cap V_\beta\cap D)$, then there exists $F\in A^\infty(D)$ such that for all α, Ff_α^{-1} is an invertible element of $A^\infty(D\cap V_\alpha)$.*

PROOF. Suppose that no F with the stated properties exists. Suppose there exist $F_1\in O(D_1)$ and $F_2\in O(D_2)$ such that for all α, $F_1f_\alpha^{-1}$ and $F_2f_\alpha^{-1}$ are invertible elements of $A^\infty(V_\alpha\cap D_1)$ and $A^\infty(V_\alpha\cap D_2)$, respectively. Then $f_0=F_1F_2^{-1}$ is an invertible element of $A^\infty(D_1\cap D_2)$. If $f_0=\exp(f)$ then $f\in A^\infty(D_1\cap D_2)$. By lemma 6 we can write $f=f_1+f_2$ where $f_1\in A^\infty(D_1)$ and $f_2\in A^\infty(D_2)$. Define G on D by $G=F_1\exp(-f_1)$ on D_1, $G=F_2\exp(f_2)$ on D_2. Then Gf_α^{-1} is an invertible element of $A^\infty(V_\alpha\cap D)$. We have supposed that no such function G exists, so either F_1 or F_2 does not exist. Say F_1. The x_{2n}-width of D_1, i.e., the number $\max|x'_{2n}-x''_{2n}|$, the maximum taken over all pairs of points z', z'' in D_1, is not more than three fourths of the x_{2n}-width of D. We now treat D_1 as we treated D, using the coordinate x_{2n-1} rather than x_{2n}, and we find a smaller set $D_{11}\subset D_1$ on which the problem is not solvable and which has the property that the x_{2n-1}-width of D_{11} is not more than three fourths that of D_1. We iterate this process, running cyclically through the real coordinate of C^n, and we obtain a shrinking sequence of sets on which our problem is not solvable. The sets we obtain eventually lie in some element V_α, and on V_α, the function f_α is a solution to the induced problem. Thus we have a contradiction. Q.E.D.

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