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Global and local equivariant characteristic numbers of G-manifolds

By Katsuo KAWAKUBO

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§1. Introduction and statement of results.

Let G be a compact Lie group and $h_G(\ )$ be an equivariant multiplicative cohomology theory. Let M and N be closed G-manifolds of class C^3. Then for a G-map $f: M \to N$, we defined an "equivariant Gysin homomorphism"

$$f_!: h_G(M) \longrightarrow h_G(N)$$

under certain conditions and obtained equivariant Riemann-Roch type theorems in general [**13**], [**14**]. When N is a point, $f_!$ is called an "index homomorphism" and is denoted by Ind. On the other hand, we got a localization theorem. Consequently by virtue of the functorial property of our equivariant Gysin homomorphism, we have many equations between invariants of a G-manifold and fixed point data.

In the present paper, we shall confine ourselves to two special cases. Let $G \to EG \to BG$ be the universal principal G-bundle.

Case 1. $G = T^n$ (torus), $h_G(M) = H^*(EG \times_G M : R)$ where R is the real number field, manifolds are oriented G-manifolds of class C^3.

Case 2. $G = (Z_2)^n$, $h_G(M) = H^*(EG \times_G M; Z_2)$, manifolds are non oriented G-manifolds of class C^3,

The greater part of the results in Case 1 will be those in [**12**]. The results in Case 2 will be analogous to those in Case 1 and include the main theorems of [**17**], [**18**].

First we shall show that our $f_!$ has the functorial property and is an $h_G(*)$-module homomorphism where $*$ stands for a point. Now we consider the set $S \subset h_G(*)$ of Euler classes of the vector bundles $EG \times_\phi R^m \to BG$ where G acts on R^m by representations $\phi: G \to O(m)$ without trivial direct summand. Then S is a multiplicative set of $h_G(*)$. It follows that we get a localization $S^{-1}h_G(M)$ and an induced homomorphism

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$$S^{-1}f_! : S^{-1}h_G(M) \longrightarrow S^{-1}h_G(N)$$

for a G-map $f: M \to N$ (see Bourbaki [6] for notion and notation).

Let F_μ be a component of the fixed point set of a G-manifold M and $i_\mu : F_\mu \to M$ be the inclusion. As in the case of the equivariant K-theory [2], there exists the following isomorphism,

$$\sum_\mu S^{-1}i_\mu^* : S^{-1}h_G(M) \longrightarrow \sum_\mu S^{-1}h_G(F_\mu)$$

where the summation is taken over all the components F_μ of the fixed point set (Lemma 3.1). Denote by N_μ the normal bundle of F_μ in M. In Case 1, N_μ has a complex vector bundle structure such that T^n acts on N_μ as complex vector bundle automorphism. It follows that a fiber of N_μ has the orientation induced by the complex structure. We then orient F_μ so that the orientation of a fiber followed by that of F_μ yields the orientation of N_μ, where N_μ has the orientation of a tubular neighborhood of F_μ in M.

Then we have

$$i_\mu^* i_{\mu !}(x) = \chi_G(N_\mu) \cdot x \qquad \text{for } x \in h_G(F_\mu)$$

where $\chi_G(N_\mu)$ denotes the Euler class of the bundle $EG \underset{G}{\times} N_\mu \to BG \times F_\mu$ (Lemma 2.2). One verifies that $\chi_G(N_\mu)$ is a unit in $S^{-1}h_G(F_\mu)$ (Lemma 3.5). By the functorial property of our $f_!$, we shall have the following commutative diagram (Lemma 3.6):

$$(*) \qquad \begin{tikzcd} S^{-1}h_G(M) \arrow[dr, "S^{-1}\,\mathrm{Ind}"] \arrow[d, "\sum_\mu \frac{S^{-1}i_\mu^*}{\chi_G(N_\mu)}"'] & \\ \sum_\mu S^{-1}h_G(F_\mu) \arrow[r, "S^{-1}\,\mathrm{Ind}"'] & S^{-1}h_G(*) . \end{tikzcd}$$

It will be shown that the local index is given by the generalized slant products $/[F_\mu]$ by the orientation classes $[F_\mu]$ chosen above. Thus we shall obtain the following theorem on which our results in the present paper are based.

THEOREM 1.1. *For any $x \in S^{-1}h_G(M)$, we have*

$$S^{-1}\,\mathrm{Ind}\, x = \sum_\mu \frac{S^{-1}i_\mu^*(x)}{\chi_G(N_\mu)} / [F_\mu] .$$

In the following, we shall apply the equation to suitable elements $x \in S^{-1}h_G(M)$ and have many relations between global invariants and local invariants.

We shall first deal with Case 1. Let M be an oriented T^n-manifold of class C^3. Denote by ET_m^n the product $S^{2m+1}\times\cdots\times S^{2m+1}$ of n-copies of the $(2m+1)$-sphere. Then the torus T^n acts on ET_m^n naturally and the orbit space $BT_m^n=ET_m^n/T^n$ is the product $CP^m\times\cdots\times CP^m$ of n-copies of the m-dimensional complex projective space. Hereafter we abbreviate the coefficients of equivariant cohomology theories. Consider the fiber bundle

$$M \longrightarrow ET_m^n \underset{T^n}{\times} M \overset{P}{\longrightarrow} BT_m^n$$

and the usual Gysin map (the Poincaré dual of the homology homomorphism)

$$\bar{P}_!: H^*(ET_m^n \underset{T^n}{\times} M) \longrightarrow H^*(BT_m^n).$$

Then we shall show that our Ind and the Gysin map $\bar{P}_!$ are related by the following commutative diagram (Lemma 4.1)

$$(**)\qquad \begin{array}{ccc} H^*(ET^n \underset{T^n}{\times} M) & \xrightarrow{\quad\text{Ind}\quad} & H^*(BT^n) \\ \Big\downarrow j_1^* & & \Big\downarrow j_4^* \\ H^*(ET_m^n \underset{T^n}{\times} M) & \xrightarrow{\quad\bar{P}_!\quad} & H^*(BT_m^n) \end{array}$$

where j_1^* and j_4^* are induced by the natural inclusions. Denote by TM the tangent bundle of M and by $L(\ )$ the Hirzebruch L-genus [**11**]. Then the strictly multiplicative property of the L-genus (see Borel-Hirzebruch [**5**]) implies that

$$\bar{P}_!(L(ET_m^n \underset{T^n}{\times} TM))\in H^0(BT_m^n).$$

It follows from the diagram (**) above that

$$\text{Ind}\ L(ET^n \underset{T^n}{\times} TM)\in H^0(BT^n).$$

Thus we shall show (Theorem 4.3)

$$\text{Ind}\ L(ET^n \underset{T^n}{\times} TM)=\text{Index of } M$$

where "Index of M" denotes the Thom-Hirzebruch index of an oriented manifold M.

Remark that this information corresponds to that of the analytic index of the Atiyah-Singer theory [4].

On the other hand, the local index can be expressed in terms of the L-genus of the fixed point set, of the Chern class of the normal bundle of the fixed point set and of the weights of the normal representations.

By combining the global index and the local index, we obtain the Atiyah-Singer G-signature theorem [4] in C^3 category.

THEOREM 1.2. *Let M be an oriented T^n-manifold of class C^3. Denote by F_μ each component of the fixed point set and by N_μ the normal bundle of F_μ in M. Let*

$$N_\mu = \sum_\lambda N_\mu(\lambda)$$

be the decomposition of N_μ determined by the normal representation of T^n where $\lambda=\lambda_1 t_1+\cdots+\lambda_n t_n$, $\lambda_i \in Z$ and t_i correspond to the canonical generators of the representation ring $R(T^n)$. We regard an irreducible representation $\lambda=\lambda_1 t_1+\cdots+\lambda_n t_n$ also as an element of $H^2(BT^n)$. Finally let $L(\ )$ be the Hirzebruch L-genus. Then we can orient each F_μ so that we have the equation:

$$\text{Index of } M = \sum_\mu L(TF_\mu) \prod_{\lambda, i} \left(\frac{e^{(\lambda + x^i_{\mu\lambda})} + e^{-(\lambda + x^i_{\mu\lambda})}}{e^{(\lambda + x^i_{\mu\lambda})} - e^{-(\lambda + x^i_{\mu\lambda})}} \right) / [F_\mu] \,.$$

Here the summation is taken over all the components of the fixed point set and the total Chern class of the bundle $N_\mu(\lambda)$ is written formally as $\prod_i (1+x^i_{\mu\lambda})$ and $/[F_\mu]$ denotes the slant product.

COROLLARY 1.3 [10], [12], [15], [16]. *We can orient each F_μ so that we get*

$$\text{Index of } M = \sum_\mu \text{Index of } F_\mu \,.$$

Denote by $\dim F$ the maximum of $\{\dim F_\mu\}$. We can replace λ by a suitably chosen complex number in Theorem 1.2 and have

COROLLARY 1.4. *Let M be a semi-free S^1-manifold of class C^3. If Index of M is non zero, then* $\dim F \geqq \dim M/2$.

More generally we can apply our Theorem 1.1 as follows. Let $\xi \to M$ be a T^n-vector bundle of dimension $2k$ (resp. $2k+1$). Let $f(x_1, \cdots, x_k)$ be a symmetric formal power series of $(x_1)^2, \cdots, (x_k)^2$, $x_1 \cdots x_k$ (resp. $(x_1)^2, \cdots, (x_k)^2$) over R. Set $\xi_\mu = \xi | F_\mu$. Let

$$\xi_\mu = \sum_\rho \xi_\mu(\rho)$$

be the decomposition of ξ_μ determined by the representation of T^n where ρ run through the irreducible representations of T^n. Denote by ρ_0 the trivial real irreducible representation of T^n which corresponds to the zero of $H^2(BT^n)$. Then for $\rho \neq \rho_0$, $\xi_\mu(\rho)$ has a complex vector bundle structure. We express the total Pontrjagin class of $ET^n \times_{T^n} \xi$ (resp. $\xi_\mu(\rho_0)$) formally as $\prod_i (1+(x_i)^2)$ (resp. $\prod_i (1+(x^i_{\mu\rho_0})^2)$). When $\dim \xi = 2k$, we denote by $x_1 \cdots x_k$ (resp. $x^1_{\mu\rho_0} \cdots x^a_{\mu\rho_0}$) the Euler class of $ET^n \times_{T^n} \xi$ (resp. $\xi_\mu(\rho_0)$) where $a = \dim \xi_\mu(\rho_0)/2$. Similarly for $\rho \neq \rho_0$, we express the total Chern class of $\xi_\mu(\rho)$ formally as $\prod_i (1+x^i_{\mu\rho})$. Then we have

THEOREM 1.5.

$$\operatorname{Ind} f(x_1, \cdots, x_k) = \sum_\mu \left(\frac{f(\cdots, \rho + x^i_{\mu\rho}, \cdots)}{\prod_{\lambda, i} (\lambda + x^i_{\mu\lambda})} \right) / [F_\mu]$$

where $f(\cdots, \rho + x^i_{\rho\mu}, \cdots)$ *means that we replace* $\{x_i \mid i=1, \cdots, k\}$ *by* $\{\rho + x^i_{\mu\rho} \mid \rho, i\}$ *in* $f(x_1, \cdots, x_k)$ *and* $x^i_{\mu\lambda}$ *are those given in Theorem* 1.2. *The constant term of* $\operatorname{Ind} f(x_1, \cdots, x_k)$ *is the evaluation* $f(x'_1, \cdots, x'_k)[M]$ *where the* (*non equivariant*) *total Pontrjagin class* (*resp. the Euler class*) *of* $\xi \to M$ *is written formally as* $\prod_i (1+(x'_i)^2)$ (*resp.* $x'_1 \cdots x'_k$ *when* $\dim \xi = 2k$).

Let $f(t)$ be a formal power series of t^2 over R with leading term 1 and $K(\ )$ be the multiplicative sequence belonging to $f(t)$ [**11**]. Then as a special case of Theorem 1.5 we have

THEOREM 1.6.

$$\operatorname{Ind} K(ET^n \times_{T^n} TM) = \sum_\mu \left(\frac{\prod_i f(z^i_\mu) \prod_{\lambda, i} f(\lambda + x^i_{\mu\lambda})}{\prod_{\lambda, i} (\lambda + x^i_{\mu\lambda})} \right) / [F_\mu]$$

where the total Pontrjagin class of TF_μ *is written formally as* $\prod_i (1+(z^i_\mu)^2)$ *and* $x^i_{\mu\lambda}$ *are those given in Theorem* 1.2.

Let ω be a partition $(i_1, \cdots, i_r)$ of k and s_ω be the characteristic class defined by using Pontrjagin classes [**19**]. Let M be an oriented T^n-manifold of class C^3 and of dimension $4k$. Then we have

PROPOSITION 1.7.

$$s_\omega[M] = \sum_\mu \sum_{\omega_1 \omega_2 = \omega} \frac{s_{\omega_1}(\prod_i (1+(z^i_\mu)^2) s_{\omega_2}(\prod_{\lambda, i} (1+(\lambda + x^i_{\mu\lambda})^2))}{\prod_{\lambda, i} (\lambda + x^i_{\mu\lambda})} / [F_\mu] .$$

REMARK 1.8. Quite similar formulae hold for Stiefel-Whitney classes instead of Pontrjagin classes. Hence Proposition 1.7 gives an explicit way to compute the bordism class $[M]$ of the oriented bordism group from the fixed point data.

In particular, we have

PROPOSITION 1.9. When an action is non-trivial,

$$s_k[M] = \sum_\mu \left(\frac{\sum_{\lambda, i} (\lambda + x^i_{\mu\lambda})^{2k}}{\prod_{\lambda, i} (\lambda + x^i_{\mu\lambda})} \right) / [F_\mu] .$$

REMARK 1.10. It is pointed out by D. Zagier that there is an interesting relation between Proposition 1.9 and a residue formula when $M=CP^m$ and $T^n=S^1$.

Let M be an oriented T^n-manifold of dimension $4k$ and u be the number of the subgroups H of T^n satisfying:

(1) $T^n/H\cong S^1$

(2) H is an isotropy group at some point of M.

Let $k=(2u+1)a+b$, $0\leqq b\leqq 2u$. When $0\leqq b\leqq u$, we set $v=4a$. When $u<b\leqq 2u$, we set $v=4a+2$.

PROPOSITION 1.11. *If* $s_k[M]\neq 0$, *then* $\dim F\geqq v$.

PROPOSITION 1.12. *Let M be an oriented semi-free S^1-manifold of class C^3. Suppose that M satisfies one of the following conditions:*

(a) $\dim M$: *odd,* $\dim F<(2/5)\dim M$,

(b) $\dim M$: *even,* $\dim F<(1/4)\dim M$.

Then M bounds as S^1-manifold.

PROPOSITION 1.13. *Let M be an oriented T^n-manifold of class C^3. Then* $\chi(M)=\sum_{\mu}\chi(F_\mu)$ *where $\chi(\ )$ denotes the ordinary Euler number.*

Next we deal with Case 2. Except for Theorem 1.2, quite analogous theorems hold in this case too. Hence we only describe some of them in the following.

Let M be an unoriented $(Z_2)^n$-manifold of class C^3. Denote by F_μ each component of the fixed point set of M and by N_μ the normal bundle of F_μ in M. Let $\xi\to M$ be a $(Z_2)^n$-vector bundle of dimension k. Set $\xi_\mu=\xi|F_\mu$. Let

$$\xi_\mu=\sum_{\rho}\xi_\mu(\rho)\qquad(\text{resp. } N_\mu=\sum_{\lambda}N_\mu(\lambda))$$

be the decomposition of ξ_μ (resp. N_μ) determined by the representation of $(Z_2)^n$. We express the total Stiefel-Whitney classes of

$$E(Z_2)^n\underset{(Z_2)^n}{\times}\xi,\qquad \xi_\mu(\rho)\qquad\text{and}\qquad N_\mu(\lambda)$$

formally as

$$\prod_i(1+x_i),\qquad \prod_i(1+x^i_{\mu\rho})\qquad\text{and}\qquad \prod_i(1+x^i_{\mu\lambda})$$

respectively. Let $f(x_1,\cdots,x_k)$ be a symmetric formal power series over Z_2. Regarding irreducible representations ρ and λ as elements of $H^1(B(Z_2)^n;Z_2)$, we have the main theorem of [18].

THEOREM 1.14.

$$\operatorname{Ind}f(x_1,\cdots,x_k)=\sum_{\mu}\left(\frac{f(\cdots,\rho+x^i_{\mu\rho},\cdots)}{\prod_{\lambda,i}(\lambda+x^i_{\mu\lambda})}\right)/[F_\mu].$$

REMARK 1.15. The constant term of the left hand side is $f(x'_1, \cdots, x'_k)[M]$ where the (non equivariant) total Stiefel-Whitney class of $\xi \to M$ is written formally as $\prod_i (1+x'_i)$.

Let $f(x_1, \cdots, x_k)$ be a symmetric polynomial over Z_2 of degree at most $\dim M$.

THEOREM 1.16. *If there exists a homomorphism* $A: H^1(B(Z_2)^n; Z_2) \to Z_2$ *such that* $A(\lambda)=1$ *for all* μ, λ *with* $N_\mu(\lambda) \neq 0$, *then*

$$f(x'_1, \cdots, x'_k)[M] = \sum_\mu \frac{f(\cdots, A(\rho)+x^i_{\mu\rho}, \cdots)}{\prod_{\lambda, i} (1+x^i_{\mu\lambda})} [F_\mu].$$

In particular, we have the main theorem of [17].

COROLLARY 1.17. *Let* M *be a* Z_2*-manifold of dimension* k *and* $f(x_1, \cdots, x_k)$ *be a symmetric polynomial over* Z_2 *of degree at most* k. *Then*

$$f(x'_1, \cdots, x'_k)[M] = \sum_\mu \frac{f(1+y^1_\mu, \cdots, z^1_\mu, \cdots)}{\prod_i (1+y^i_\mu)} [F_\mu]$$

where the total Stiefel-Whitney classes of TM, N_μ *and* TF_μ *are written formally as*

$$\prod_i (1+x'_i), \qquad \prod_i (1+y^i_\mu) \qquad \text{and} \qquad \prod_i (1+z^i_\mu)$$

respectively.

The present paper is organized as follows. In § 2 we define our equivariant Gysin homomorphism and investigate fundamental properties of it. Theorem 1.1 is proved in § 3. In § 4 we shall analyze the global index and show that

$$\mathrm{Ind}\, L(ET^n \times_{T^n} TM) = \text{Index of } M.$$

By combining the above, we shall give proofs of Theorem 1.2 and of Corollaries 1.3 and 1.4 in § 5. Propositions 1.11 and 1.12 are proved in § 6. Since the proofs of the rest of the results are analogous, they are omitted.

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§ 2. Equivariant Gysin homomorphism.

Let G, $h_G(\ )$, M, N be those of Case 1 or of Case 2 in § 1. Then for a G-map $f: M \to N$, we define our equivariant Gysin homomorphism

$$f_!: h_G(M) \longrightarrow h_G(N)$$

as follows. Since f is G-homotopic to a differentiable G-map f' of class C^3, we first define our Gysin homomorphism $f'_!$ and then define $f_!$ to be $f'_!$. The forthcoming Lemma 2.2 will assure that $f_!$ is independent of the choice of f'. Therefore we may assume that f itself is differentiable of class C^3. As is well-known, there is a G-embedding e of M in some G-vector space V. For the proof, see Palais [20]. Choose a G-invariant Riemannian metric on $N\times V$ and let ν be an invariant open tubular neighborhood of $(f\times e)(M)$ in $N\times V$. Here we need the assumption C^3. Then ν may be identified with the normal G-vector bundle of $(f\times e)(M)$ in $N\times V$. For a G-vector bundle ξ, we denote by $D(\xi)$ (resp. $S(\xi)$) the disk bundle (resp. sphere bundle) associated with ξ. Denote by $D(V)$ (resp. $S(V)$) the unit disk (resp. unit sphere) in V. Here we may assume without loss of generality that $D(\nu)$ is in $N\times \operatorname{Int} D(V)$. Then the homomorphism $f_!$ is defined by the composition of the following three homomorphisms which we explain in a moment:

$$\phi_1:\ h_G(M)\longrightarrow \tilde{h}_G(D(\nu)/S(\nu))$$

$$\phi_2:\ \tilde{h}_G(D(\nu)/S(\nu))\longrightarrow \tilde{h}_G((N\times D(V))/(N\times S(V)))$$

$$\phi_3:\ \tilde{h}_G((N\times D(V))/(N\times S(V)))\longrightarrow h_G(N)\,.$$

Explanation: Here $\tilde{h}_G(\ )$ denotes the reduced cohomology ring as usual. Let $t(M)\in\tilde{h}_G(D(TM)/S(TM))$ (resp. $t(N)\in\tilde{h}_G(D(TN)/S(TN))$) be the orientation class of the manifold M (resp. N) where TM and TN denote the tangent G-vector bundles. Fix an orientation class $t(V)\in\tilde{h}_G(D(TV)/S(TV))$ of V. It is easy to see that we can choose a canonical orientation class $t(\nu)\in\tilde{h}_G(D(\nu)/S(\nu))$ such that

$$t(M)\times t(\nu)=(f\times e)^*(t(N)\times t(V))\,.$$

Then the homomorphism ϕ_1 is defined to be the Thom isomorphism by making use of the Thom class $t(\nu)$. The homomorphism ϕ_2 is the induced homomorphism by the natural collapsing map

$$(N\times D(V))/(N\times S(V))\longrightarrow D(\nu)/S(\nu)\,.$$

The homomorphism ϕ_3 is again defined by the Thom isomorphism using $t(V)$ in the manner of the definition of ϕ_1.

DEFINITION 2.1. When N is a point $*$, $f_!$ is called an index homomorphism and denoted by

$$\operatorname{Ind}:h_G(M)\longrightarrow h_G(*)\,.$$

LEMMA 2.2. *The equivariant Gysin homomorphism is independent of all choices made and has the following properties:*

i) *$f_!$ depends only on the G-homotopy class of f*

ii) $f_!$ *is an* $h_G(*)$*-module homomorphism*

iii) $(fg)_!=f_!\cdot g_!$

iv) $f_!(x\cdot f^*(y))=f_!(x)\cdot y$ *for* $x\in h_G(M)$, $y\in h_G(N)$

v) *if* f *is a G-embedding of class* C^3 *with a normal bundle* ν, *then* $f^*f_!(x)=\chi_G(\nu)\cdot x$ *for* $x\in h_G(M)$ *where* $\chi_G(\nu)$ *denotes the equivariant Euler class of* ν.

PROOF. Easy and omitted.

§ 3. Localization.

We consider the subset S of $h_G(*)$ consisting of Euler classes of G-vector bundles $EG\underset{\phi}{\times}R^m\to BG$ where G acts on R^m by representations $\phi: G\to O(m)$ without trivial direct summand. Then S is a multiplicative set of $h_G(*)$ and we get a localization $S^{-1}h_G(M)$.

For a G-manifold M, we denote by F_μ each component of the fixed point set of M and by $i_\mu: F_\mu\to M$ the inclusion map.

LEMMA 3.1 [9]. *The following homomorphism*

$$\sum_\mu S^{-1}i_\mu^*: S^{-1}h_G(M)\longrightarrow \sum_\mu S^{-1}h_G(F_\mu)$$

is an isomorphism where the summation is taken over all the components F_μ *of the fixed point set.*

In the following, we consider Case 1 first. Let $\phi: T^n\to O(m)$ be a representation without trivial direct summand. Then by representation theory (see Adams [1]), m is even, say $2k$, and ϕ comes from a unitary representation

$$\psi: T^n\longrightarrow U(k).$$

Let T^k be the maximal torus of $U(k)$ consisting of diagonal matrices. In view of the maximal tori theorem of E. Cartan (see Weil [22]), we may assume that $\psi(T^n)\subset T^k$. Then ψ induces the homomorphism ψ^* of $H^1(T^k)$ in $H^1(T^n)$. Let $\{t_i\,|\,i=1, 2, \cdots, n\}$ (resp. $\{t_i'\,|\,i=1, 2, \cdots, k\}$) be the canonical base of $H^1(T^n)$ (resp. $H^1(T^k)$). The elements $\omega_i=\psi^*(t_i')$ will be called the weights of ψ and can be written as

$$\omega_i=\sum_{j=1}^n a_{ij}t_j, \qquad a_{ij}\in Z.$$

According to Borel-Hirzebruch [5], the total Chern class $c(ET^n\underset{\psi}{\times}C^k)$ of the complex vector bundle

$$ET^n\underset{\psi}{\times}C^k\longrightarrow BT^n$$

is given by

$$c(ET^n\underset{\psi}{\times}C^k)=\prod_{i=1}^k(1+\omega_i)=\prod_{i=1}^k(1+\sum_{j=1}^n a_{ij}t_j).$$

It follows that the Euler class $\chi(ET^n \underset{\phi}{\times} \boldsymbol{C}^k)$ is given by

$$\chi(ET^n \underset{\phi}{\times} \boldsymbol{C}^k)=\prod_{i=1}^{k}(\sum_{j=1}^{n} a_{ij}t_j).$$

Since ϕ has no trivial direct summand, $\omega_i \neq 0$ for all i, that is,

$$\prod_{i=1}^{k}(\sum_{j=1}^{n} a_{ij}^2)\neq 0.$$

Conversely, for a $(k\times n)$-matrix (a_{ij}) satisfying

$$\prod_{i=1}^{k}(\sum_{j=1}^{n} a_{ij}^2)\neq 0, \qquad a_{ij}\in Z,$$

we can construct a homomorphism

$$\phi: T^n \longrightarrow T^k \subset U(k),$$

such that

$$\phi^*(t_i')=\sum_{j=1}^{n} a_{ij}t_j \qquad \text{for all } i.$$

The representation ϕ has no trivial direct summand. Therefore we have shown the following

LEMMA 3.2. *The set S consists of those elements*

$$\prod_{i=1}^{k}(\sum_{j=1}^{n} a_{ij}t_j)$$

where a_{ij} satisfy $\prod_{i=1}^{k}(\sum_{j=1}^{n} a_{ij}^2)\neq 0$, $a_{ij}\in Z$ and k may vary.

Since S does not contain the zero element by Lemma 3.2 and since $H^*(BT^n)$ is an integral domain, the localization map

$$H^*(BT^n) \longrightarrow S^{-1}H^*(BT^n)$$

is injective.

Next we study the fixed point set and its normal bundle. As introduced in §1, we denote by F_μ a component of the fixed point set and by N_μ its normal bundle. As is well-known the normal bundle N_μ has a complex vector bundle structure such that the group T^n acts on N_μ as complex vector bundle automorphism. It follows from [3] that N_μ has the following decomposition

$$N_\mu=\sum_{\lambda} N_\mu(\lambda), \qquad N_\mu(\lambda)=E_{\mu\lambda}\otimes V_\lambda$$

where λ run through the complex irreducible representations, V_λ denote their representation spaces and $E_{\mu\lambda}$ denote complex vector bundles. We now show the following

LEMMA 3.3.

$$ET^n \underset{T^n}{\times} N_\mu = \sum_\lambda (ET^n \underset{T^n}{\times} V_\lambda) \hat{\otimes} E_{\mu\lambda}$$

where $\hat{\otimes}$ *denotes the external tensor product and* $ET^n \underset{T^n}{\times} V_\lambda$ *denotes the* λ-*extension of the principal* T^n-*bundle* $ET^n \to BT^n$.

PROOF. Obviously we have

$$ET^n \underset{T^n}{\times} N_\mu = \sum_\lambda (ET^n \underset{T^n}{\times} N_\mu(\lambda)).$$

Hence it will suffice to show that

$$ET^n \underset{T^n}{\times} (E_{\mu\lambda} \otimes V_\lambda) = (ET^n \underset{T^n}{\times} V_\lambda) \hat{\otimes} E_{\mu\lambda}.$$

But this is easily seen by the following correspondence

$$x \times (y \otimes z) \longmapsto (x \times z) \hat{\otimes} y$$

for $x \in ET^n$, $y \in E_{\mu\lambda}$, $z \in V_\lambda$.

For a complex vector bundle ξ, we denote by $c(\xi)$ the total Chern class of ξ.

LEMMA 3.4. *The total Chern class and the Euler class of the bundle* $ET^n \underset{T^n}{\times} N_\mu \to BT^n \times F_\mu$ *are given by*

$$c(ET^n \underset{T^n}{\times} N_\mu) = \prod_{\lambda, i} (1 + \lambda_1 t_1 + \cdots + \lambda_n t_n + x^i_{\mu\lambda})$$

$$\chi_{T^n}(N_\mu) = \chi(ET^n \underset{T^n}{\times} N_\mu) = \prod_{\lambda, i} (\lambda_1 t_1 + \cdots + \lambda_n t_n + x^i_{\mu\lambda}).$$

Here we identified λ *with the element* $\lambda_1 t_1 + \cdots + \lambda_n t_n$ *of* $H^2(BT^n)$ *by the following translations*

$$\left\{\begin{matrix}\text{complex irreducible} \\ \text{representations}\end{matrix}\right\} \longleftrightarrow H^1(T^n) \xleftrightarrow{\text{transgression}} H^2(BT^n)$$

and the total Chern class of the complex vector bundle $E_{\mu\lambda}$ *is written formally as* $\prod_i (1 + x^i_{\mu\lambda})$.

PROOF. Lemma 3.4 will follow from Lemma 3.3 by the arguments of Borel-Hirzebruch [**5**].

We are now ready to prove that the Euler class $\chi_{T^n}(N_\mu)$ is a unit in $S^{-1}H^*(BT^n \times F_\mu)$. It follows from Lemma 3.4 that $\chi_{T^n}(N_\mu)$ has the form $\prod_{\lambda, i} (\lambda + x^i_{\mu\lambda})$. Since the representations λ are non trivial, λ are in $\pi^*(S)$ where $\pi : BT^n \times F_\mu \to BT^n$ is the projection. Consider the formal equation

$$1=\prod_{\lambda, i}\left(1+\frac{x_{\mu\lambda}^{i}}{\lambda}\right)\cdot\left\{\sum_{j=0}^{\infty}\left(\frac{-x_{\mu\lambda}^{i}}{\lambda}\right)^{j}\right\}.$$

Since F_{μ} is of finite dimension, we have the equation

$$1=\prod_{\lambda, i}\left(1+\frac{x_{\mu\lambda}^{i}}{\lambda}\right)\cdot\left\{\sum_{j=0}^{m}\left(\frac{-x_{\mu\lambda}^{i}}{\lambda}\right)^{j}\right\}$$

in $S^{-1}H^{*}(BT^{n}\times F_{\mu})$ where $m=[\dim M/2]$. It follows that

$$1=\chi_{T^{n}}(N_{\mu})\prod_{\lambda}\frac{A(\mu, \lambda)}{\lambda^{n(\mu, \lambda)}}$$

in $S^{-1}H^{*}(BT^{n}\times F_{\mu})$ where $A(\mu, \lambda)$ is given by

$$A(\mu, \lambda)=\prod_{i}\left\{\sum_{j=0}^{m}\lambda^{m-j}(-x_{\mu\lambda}^{i})^{j}\right\}\in H^{*}(BT^{n}\times F_{\mu})$$

and $n(\mu, \lambda)=(m+1)\dim_{C}E_{\mu\lambda}$.

Thus we have shown the following

LEMMA 3.5. *Each equivariant Euler class $\chi_{T^{n}}(N_{\mu})$ is a unit in $S^{-1}H^{*}(BT^{n}\times F_{\mu})$ for any component F_{μ}.*

We are now ready to prove

LEMMA 3.6. *The following diagram*

$$\begin{array}{ccc}
S^{-1}H^{*}(ET^{n}\underset{T^{n}}{\times}M) & \xrightarrow{S^{-1}\mathrm{Ind}} & \\
{\scriptstyle \sum_{\mu}\frac{S^{-1}i_{\mu}^{*}}{\chi_{T^{n}}(N_{\mu})}}\Big\downarrow & & S^{-1}H^{*}(BT^{n}) \\
\sum_{\mu}S^{-1}H^{*}(BT^{n}\times F_{\mu}) & \xrightarrow{S^{-1}\mathrm{Ind}} &
\end{array}$$

commutes.

PROOF. It follows from (V) in Lemma 2.2 that for an element $x=\sum_{\mu}x_{\mu}$ of $\sum_{\mu}H^{*}(BT^{n}\times F_{\mu})$, we have

$$\left(\sum_{\mu}i_{\mu}^{*}\right)\left(\sum_{\mu}i_{\mu!}\right)\left(\sum_{\mu}x_{\mu}\right)=\sum_{\mu}\chi_{T^{n}}(N_{\mu})\cdot x_{\mu}.$$

Since $\chi_{T^{n}}(N_{\mu})$ is a unit in $S^{-1}H^{*}(BT^{n}\times F_{\mu})$ and since $\sum_{\mu}S^{-1}i_{\mu}^{*}$ is an isomorphism of the localized rings by Lemma 3.1, $\sum_{\mu}S^{-1}i_{\mu!}$ is also an isomorphism and its inverse is given by

$$(\sum_{\mu} S^{-1}i_{\mu!})^{-1}=\sum_{\mu}\frac{S^{-1}i_{\mu}^{*}}{\chi_{T^n}(N_{\mu})}\,.$$

Hence Lemma 3.6 follows from the functorial property (iii) of Lemma 2.2.

LEMMA 3.7. *Let F be an oriented manifold on which T^n acts trivially. Then our localized index homomorphism*

$$S^{-1}\operatorname{Ind}: S^{-1}H^*(BT^n\times F)\longrightarrow S^{-1}H^*(BT^n)$$

is given by the generalized slant product $/[F]$ where $[F]$ denotes the orientation class of F.

PROOF. Let $F\subset R^m$ be a T^n-embedding where T^n acts on R^m trivially. Denote by $t\times x$ $(\in H^*(BT^n\times F))$ the cross product of t $(\in H^*(BT^n))$ and x $(\in H^*(F))$. Let $f: F\to *$ be the constant map. In view of Lemma 2.2, we may use R^m as V in §2 and have easily that

$$\operatorname{Ind}(t\times x)=\bar{f}_!(x)t$$

where $\bar{f}_!$ denotes the classical Gysin map

$$\bar{f}_!: H^*(F)\longrightarrow H^*(*)\,.$$

It follows by definition that

$$\bar{f}_!(x)t=t\times x/[F]\,.$$

Since any element of $H^*(BT^n\times F)$ can be written as the sum of elements of the form $t\times x$, Ind is given by the slant product $/[F]$. Since the slant product $/[F]$ is an $H^*(BT^n)$-module homomorphism, it induces naturally the localized homomorphism

$$S^{-1}H^*(BT^n\times F)\longrightarrow S^{-1}H^*(BT^n)$$

which is denoted also by $/[F]$. Hence $S^{-1}\operatorname{Ind}$ is given by this generalized slant product $/[F]$.

PROOF OF THEOREM 1.1. Combining Lemmas 3.6 and 3.7, we have Theorem 1.1 in Case 1. The proof of Theorem 1.1 in Case 2 is quite similar.

§4. The global index.

In this section, we are concerned with toral actions on oriented manifolds and analyze the global index, which will give an information corresponding to that of the analytic index of the Atiyah-Singer G-signature theorem.

First, we show the following

LEMMA 4.1. *Let M be an oriented T^n-manifold, then the diagram*

$$\begin{tikzcd} H^*(ET^n \times_{T^n} M) \arrow[r, "\mathrm{Ind}"] \arrow[d, "j_1^*"] & H^*(BT^n) \arrow[d, "j_4^*"] \\ H^*(ET_m^n \times_{T^n} M) \arrow[r, "\bar{P}_!"] & H^*(BT_m^n) \end{tikzcd}$$

commutes, where j_1^* and j_4^* are induced by the natural inclusions and $\bar{P}_!$ is the classical Gysin map. Here BT_m^n $(=CP^m \times \cdots \times CP^m)$ has a canonical orientation class and the orientation class of $ET_m^n \times_{T^n} M$ is the induced one from BT_m^n and M.

PROOF. First we consider four homomorphisms:

$$H^*(ET^n \times_{T^n} M) \xrightarrow{\phi_1} H^*(ET^n \times_{T^n} D(\nu),\ ET^n \times_{T^n} S(\nu)) \xrightarrow{\phi_2'}$$

$$H^*(ET^n \times_{T^n} D(V),\ ET^n \times_{T^n} (D(V)-\mathrm{Int}\, D(\nu))) \xrightarrow{\phi_2''}$$

$$H^*(ET^n \times_{T^n} D(V),\ ET^n \times_{T^n} S(V)) \xrightarrow{\phi_3} H^*(BT^n),$$

where ϕ_1 and ϕ_3 are those in § 2 and ϕ_2' is an excision isomorphism and ϕ_2'' is induced by the natural inclusion. The composition $\phi_2'' \circ \phi_2'$ is nothing but ϕ_2 in § 2.

Similarly we define ψ_1, ψ_2', ψ_2'', ψ_3 using ET_m^n instead of ET^n. Let

$$j_2:\ ET_m^n \times_{T^n} D(\nu) \longrightarrow ET^n \times_{T^n} D(\nu)$$

and

$$j_3:\ ET_m^n \times_{T^n} D(V) \longrightarrow ET^n \times_{T^n} D(V)$$

be the natural inclusions. Then we have

$$j_{i+1}^* \circ \phi_i = \psi_i \circ j_i^* \qquad \text{for } i=1, 3,$$

and

$$j_3^* \circ \phi_2' = \psi_2' \circ j_2^*, \qquad j_3^* \circ \phi_2'' = \psi_2'' \circ j_3^*.$$

If we denote by Ind_m the composition $\psi_3 \circ \psi_2'' \circ \psi_2' \circ \psi_1$, then we have

$$\mathrm{Ind}_m \circ j_1^* = j_4^* \circ \mathrm{Ind}.$$

Next we show that Ind_m is equal to the Gysin map $\bar{P}_!$. For an oriented manifold X (with or without boundary), we denote by $[X]$ the orientation class. Let $p_1: ET^n_m \times_{T^n} D(\nu) \to ET^n_m \times_{T^n} M$ be the projection. Then we show that the following diagram is commutative

$$\text{(1)}\qquad \begin{tikzcd} H^*(ET^n_m \times_{T^n} M) \arrow[r, "\phi_1"] \arrow[d, "\cap[ET^n_m \times_{T^n} M]"] & H^*(ET^n_m \times_{T^n} D(\nu),\ ET^n_m \times_{T^n} S(\nu)) \arrow[d, "\cap[ET^n_m \times_{T^n} D(\nu)]"] \\ H_*(ET^n_m \times_{T^n} M) & H_*(ET^n_m \times_{T^n} D(\nu)) \arrow[l, "p_{1*}"'] \end{tikzcd}$$

where $\cap$ denote the cap products.

Denote by τ the Thom class of the Thom isomorphism ϕ_1. Notice that

$$p_{1*}(\tau \cap [ET^n_m \times_{T^n} D(\nu)]) = [ET^n_m \times_{T^n} M]$$

by Thom [21]. It follows that, for any $x \in H^*(ET^n_m \times_{T^n} M)$, we have

$$\begin{aligned} p_{1*}\{(p_1^* x \cup \tau) \cap [ET^n_m \times_{T^n} D(\nu)]\} &= p_{1*}\{p_1^* x \cap (\tau \cap [ET^n_m \times_{T^n} D(\nu)])\} \\ &= x \cap p_{1*}(\tau \cap [ET^n_m \times_{T^n} D(\nu)]) \\ &= x \cap [ET^n_m \times_{T^n} M], \end{aligned}$$

that is, the diagram (1) commutes where $\cup$ denotes the cup product. Next we show that the following diagram

$$\text{(2)}\qquad \begin{tikzcd} H^*(ET^n_m \times_{T^n} D(\nu),\ ET^n_m \times_{T^n} S(\nu)) \arrow[dr, "\phi_2'"] \arrow[ddd, "\cap[ET^n_m \times_{T^n} D(\nu)]"'] & \\ & H^*(ET^n_m \times_{T^n} D(V),\ ET^n_m \times_{T^n} (D(V) - \mathrm{Int}\, D(\nu))) \arrow[d, "\cap \bar{k}_*[ET^n_m \times_{T^n} D(V)]"] \\ & H_*(ET^n_m \times_{T^n} D(V)) \\ H_*(ET^n_m \times_{T^n} D(\nu)) \arrow[ur, "e_*"'] & \end{tikzcd}$$

commutes, where e_* is induced by the natural inclusion and $\bar{k}_*$ is induced by the natural inclusion:

$$\bar{k}: (ET^n_m \underset{T^n}{\times} D(V),\ ET^n_m \underset{T^n}{\times} S(V)) \longrightarrow (ET^n_m \underset{T^n}{\times} D(V),\ ET^n_m \underset{T^n}{\times} (D(V)-\mathrm{Int}\, D(\nu))).$$

Let

$$\bar{e}: (ET^n_m \underset{T^n}{\times} D(\nu),\ ET^n_m \underset{T^n}{\times} S(\nu)) \longrightarrow (ET^n_m \underset{T^n}{\times} D(V),\ ET^n_m \underset{T^n}{\times} (D(V)-\mathrm{Int}\, D(\nu)))$$

be the natural inclusion, then we have

$$\bar{k}_*[ET^n_m \underset{T^n}{\times} D(V)] = \bar{e}_*[ET^n_m \underset{T^n}{\times} D(\nu)].$$

Hence for any $x \in H^*(ET^n_m \underset{T^n}{\times} D(V),\ ET^n_m \underset{T^n}{\times} (D(V)-\mathrm{Int}\, D(\nu)))$, we have

$$\begin{aligned} e_*(\phi_2'^{-1} x \cap [ET^n_m \underset{T^n}{\times} D(\nu)]) &= e_*(\bar{e}^* x \cap [ET^n_m \underset{T^n}{\times} D(\nu)]) \\ &= x \cap \bar{e}_*[ET^n_m \underset{T^n}{\times} D(\nu)] \\ &= x \cap \bar{k}_*[ET^n_m \underset{T^n}{\times} D(V)], \end{aligned}$$

that is, the diagram (2) commutes.

Similar arguments prove that the following two diagrams commute:

$$(3)\qquad \begin{tikzcd} H^*(ET^n_m \underset{T^n}{\times} D(V),\ ET^n_m \underset{T^n}{\times} (D(V)-\mathrm{Int}\, D(\nu))) \arrow[dd, "\cap \bar{k}_*[ET^n_m \underset{T^n}{\times} D(V)]"] \arrow[dr, "\phi''_2"] & \\ & H^*(ET^n_m \underset{T^n}{\times} D(V),\ ET^n_m \underset{T^n}{\times} S(V)) \arrow[dl, "\cap[ET^n_m \underset{T^n}{\times} D(V)]"] \\ H_*(ET^n_m \underset{T^n}{\times} D(V)) & \end{tikzcd}$$

$$
(4)\qquad
\begin{array}{ccc}
H^*(ET^n_m \underset{T^n}{\times} D(V),\ ET^n_m \underset{T^n}{\times} S(V)) & \xrightarrow{\ \ \phi_3\ \ } & H^*(BT^n_m) \\
\Big\downarrow \cap[ET^n_m \underset{T^n}{\times} D(V)] & & \Big\downarrow \cap[BT^n_m] \\
H_*(ET^n_m \underset{T^n}{\times} D(V)) & \xrightarrow{\ \ p_{2*}\ \ } & H_*(BT^n_m)
\end{array}
$$

where p_2 is the projection.

Let $i: ET^n_m \underset{T^n}{\times} M \to ET^n_m \underset{T^n}{\times} D(\nu)$ be the zero section map. Then $(p_{1*})^{-1}$ is given by i_* and the composition $p_2 \circ e \circ i$ is nothing but the projection

$$P:\ ET^n_m \underset{T^n}{\times} M \longrightarrow BT^n_m .$$

It follows that the composition

$$(\cap[BT^n_m])^{-1} \circ p_{2*} \circ e_* \circ i_*(\cap[ET^n_m \underset{T^n}{\times} M])$$

is the Gysin homomorphism $\bar{P}_!$.

Putting the commutative diagrams (1)-(4) together, we obtain the required equation

$$\bar{P}_! = \phi_3 \circ \phi_2'' \circ \phi_2' \circ \phi_1 .$$

Thus we have shown that

$$j_4^* \circ \mathrm{Ind} = \mathrm{Ind}_m \circ j_1^* = \bar{P}_! \circ j_1^* .$$

This makes the proof of Lemma 4.1 complete.

LEMMA 4.2. *Ind $L(ET^n \underset{T^n}{\times} TM)$ is in $H^0(BT^n)$.*

PROOF. According to Chern [7], the Gysin homomorphism $\bar{P}_!$ is equivalent to the *integration over the fiber* (see Borel-Hirzebruch [5]) of the fiber bundle

$$P:\ ET^n_m \underset{T^n}{\times} M \longrightarrow BT^n_m .$$

It is easy to see that the bundle

$$ET^n_m \underset{T^n}{\times} TM \longrightarrow ET^n_m \underset{T^n}{\times} M$$

is equivalent to the bundle along the fiber of the fiber bundle

$$P:\ ET^n_m \underset{T^n}{\times} M \longrightarrow BT^n_m .$$

It follows from Borel-Hirzebruch [5] that the L-genus is strictly multiplicative for the fiber bundle above, that is,

$$\bar{P}_! L(ET^n_m \underset{T^n}{\times} TM) \in H^0(BT^n_m).$$

Obviously there is a natural bundle map:

$$\begin{tikzcd} ET^n_m \underset{T^n}{\times} TM \arrow[r] \arrow[d] & ET^n \underset{T^n}{\times} TM \arrow[d] \\ ET^n_m \underset{T^n}{\times} M \arrow[r, "j_1"] & ET^n \underset{T^n}{\times} M. \end{tikzcd}$$

Hence, by Lemma 4.1, we have

$$\begin{aligned} j_4^* \operatorname{Ind} L(ET^n \underset{T^n}{\times} TM) &= \bar{P}_! j_1^* L(ET^n \underset{T^n}{\times} TM) \\ &= \bar{P}_! L(ET^n_m \underset{T^n}{\times} TM) \end{aligned}$$

which belongs to $H^0(BT^n_m)$. Since m is arbitrary, we may conclude that

$$\operatorname{Ind} L(ET^n \underset{T^n}{\times} TM) \in H^0(BT^n).$$

This completes the proof of Lemma 4.2.

THEOREM 4.3.

$$\operatorname{Ind} L(ET^n \underset{T^n}{\times} TM) = \textit{Index of } M.$$

PROOF. Taking 0 as m in Lemma 4.1, we have the following commutative diagram

$$\begin{tikzcd} H^*(ET^n \underset{T^n}{\times} M) \arrow[r, "\operatorname{Ind}"] \arrow[d, "j_1^*"] & H^*(BT^n) \arrow[d, "j_1^*"] \\ H^*(ET^n_0 \underset{T^n}{\times} M) \arrow[r, "\bar{P}_!"] & H^*(BT^n_0). \end{tikzcd}$$

Notice that $ET^n_0 \underset{T^n}{\times} M = M$, $BT^n_0 = *$ where $*$ denotes a point. Hence the Gysin homomorphism

$$\bar{P}_! : H^{\dim M}(ET^n_0 \underset{T^n}{\times} M) = Z \longrightarrow H^0(*) = Z$$

is the identity map and is trivial for other dimensions. Obviously the bundle

$$ET^n_0 \underset{T^n}{\times} TM \longrightarrow ET^n_0 \underset{T^n}{\times} M$$

is equivalent to the tangent bundle $TM \to M$, Hence we have:

$$
\begin{aligned}
j_4^* \operatorname{Ind} L(ET^n \underset{T^n}{\times} TM) &= \bar{P}_! j_1^* L(ET^n \underset{T^n}{\times} TM) \\
&= \bar{P}_! L(TM) \\
&= L(M)[M] \\
&= \text{Index of } M .
\end{aligned}
$$

Since $\operatorname{Ind} L(ET^n \underset{T^n}{\times} TM) \in H^0(BT^n)$ by Lemma 4.2 and since

$$j_4^* : H^0(BT^n) = Z \longrightarrow H^0(*) = Z$$

is the identity map, we may conclude that

$$\operatorname{Ind} L(ET^n \underset{T^n}{\times} TM) = \text{Index of } M,$$

by the natural identifications.

This makes the proof of Theorem 4.3 complete.

§5. *G*-signature theorem.

In this section we prove G-signature Theorem 1.2 and Corollaries 1.3 and 1.4. In view of Lemma 3.3, the total Pontrjagin class of the bundle

$$ET^n \underset{T^n}{\times} N_\mu \longrightarrow BT^n \times F_\mu$$

is given by

$$P(ET^n \underset{T^n}{\times} N_\mu) = \prod_{\lambda, i} (1 + (\lambda + x^i_{\mu\lambda})^2) .$$

Therefore, in the localized ring $S^{-1}H^*(BT^n)$, we have

$$
\begin{aligned}
\text{Index of } M &= \operatorname{Ind} L(ET^n \underset{T^n}{\times} TM) && \text{by Theorem 4.3,} \\
&= \sum_\mu \frac{i_\mu^* L(ET^n \underset{T^n}{\times} TM)}{\chi_{T^n}(N_\mu)} / [F_\mu] && \text{by Theorem 1.1,} \\
&= \sum_\mu \frac{L(TF_\mu) \cdot L(ET^n \underset{T^n}{\times} N_\mu)}{\chi_{T^n}(N_\mu)} / [F_\mu] \\
&= \sum_\mu L(TF_\mu) \cdot \prod_{\lambda, i} \left(\frac{e^{(\lambda + x^i_{\mu\lambda})} + e^{-(\lambda + x^i_{\mu\lambda})}}{e^{(\lambda + x^i_{\mu\lambda})} - e^{-(\lambda + x^i_{\mu\lambda})}} \right) / [F_\mu] .
\end{aligned}
$$

This makes the proof of Theorem 1.2 complete.

PROOF OF COROLLARY 1.3. Since the number of the irreducible representations λ which appear as normal representations are finite, we can find a sequence $a=(a_1, \cdots, a_n)$ of integers such that

$$(\lambda, a)=\lambda_1 a_1+\cdots+\lambda_n a_n \neq 0$$

for every irreducible representations λ which appear in normal representations. We can choose a complex structure of N_μ so that $(\lambda, a)>0$ for every λ. Accordingly F_μ is oriented. Note that the formula in Theorem 1.2 holds even if we regard t_i as real numbers satisfying $\lambda=\sum_i \lambda_i t_i \neq 0$. Set $t_i=a_i t$. Then we have

$$\lim_{t\to\infty} \frac{e^{((\lambda, a)t+x_{\mu\lambda}^i)}+e^{-((\lambda, a)t+x_{\mu\lambda}^i)}}{e^{((\lambda, a)t+x_{\mu\lambda}^i)}-e^{-((\lambda, a)t+x_{\mu\lambda}^i)}}=1 \qquad \text{in } H^*(F_\mu : R).$$

Therefore Corollary 1.3 is an immediate consequence of Theorem 1.2.

PROOF OF COROLLARY 1.4. In this case, $\lambda=t$ and we extend the coefficients R to $\boldsymbol{C}$. As in the manner of the proof of Corollary 1.3, we can put $\lambda=\sqrt{-1}\pi/2$. Then we have

$$\text{Index of } M=\sum_\mu L(TF_\mu)\{(\prod_i x_\mu^i)+\text{higher term}\}/[F_\mu].$$

Hence if $\dim M-\dim F_\mu>\dim F_\mu$ for every component F_μ, Index of M must vanish. This completes the proof of Corollary 1.4.

§ 6. Dimension of fixed point set.

In this section, we prove Propositions 1.11 and 1.12.

PROOF OF PROPOSITION 1.11. Let B be the set of the irreducible representations ρ which appear as $N_\mu(\rho)$ for some N_μ. Denote by u' the order of the set B. Let $k=(2u'+1)a'+b'$, $0\leqq b'\leqq 2u'$. When $0\leqq b'\leqq u'$, we set $c=2a'$, $d=a'+b'$. When $u'<b'\leqq 2u'$, we set $c=2a'+1$, $d=a'+b'-u'$. In both cases, $k=u'\cdot c+d$ and $\min\{c, 2d\}=c$. Since $u'\leqq u$, $2c\geqq v$.

We now consider

$$\begin{aligned} f(\rho, x)&=\sum_{i=1}^{2k}(x_i)^{2d}\prod_{\rho\in B}(-\rho^2+(x_i)^2)^c \\ &=s_k(x)+\text{terms of lower degree in } x, \end{aligned}$$

which is inspired by [**17**]. As before, we regard each element ρ of B as an element of $H^2(BT^n)$ and express the total Pontrjagin class of $ET^n \times_{T^n} TM$ formally as $\prod(1+(x_i)^2)$. We also use the notations z_μ^i, $x_{\mu\lambda}^i$ as in § 1. We then have

$$\operatorname{Ind} f(\rho, x)=s_k[M]\neq 0.$$

Note that

$$i_\mu^* f(\rho, x) = \sum_{i=1}^{d(\mu)} (z_\mu^i)^{2d} \prod_{\rho \in B} (-\rho^2 + (z_\mu^i)^2)^c$$
$$+ \sum_{\lambda, i} (\lambda + x_{\mu\lambda}^i)^{2d} \prod_{\rho \in B} (-\rho^2 + (\lambda + x_{\mu\lambda}^i)^2)^c$$

where $d(\mu) = \dim F_\mu / 2$. Hence the term of least degree of $H^*(F_\mu)$ part in

$$\frac{i_\mu^* f(\rho, x)}{\prod_{\lambda, i} (\lambda + x_{\mu\lambda}^i)}$$

is of degree at least c. It follows that $\dim F \geqq 2c$.

This completes the proof of Proposition 1.11.

PROOF OF PROPOSITION 1.12. Let M^m be a semi-free S^1-manifold. Then M bounds as semi-free S^1-manifold if and only if $\sum_\mu (F_\mu, N_\mu)$ represents the zero element of

$$\sum_k \Omega_{m-2k}(BU(k)).$$

Since $H_*(BU(k))$ has no torsion, any element of $\sum_k \Omega_{m-2k}(BU(k))$ is characterized by bordism Stiefel-Whitney numbers and by bordism Pontrjagin numbers [8].

Now for any pair of partitions $\omega = (i_1, \cdots, i_r)$ and $\omega' = (j_1, \cdots, j_s)$, we consider

$$f_{\omega, \omega'}(x) = \{\sum (-t^2 + (x_1)^2)^{2i_1+1} \cdot (x_1)^{2i_1} \cdots (-t^2 + (x_r)^2)^{2i_r+1} \cdot (x_r)^{2i_r}\}$$
$$\times \{\sum (-t^2 + (x_1)^2)^{j_1} \cdot (x_1)^{2[j_1/2]+2} \cdots (-t^2 + (x_s)^2)^{j_s} \cdot (x_s)^{2[j_s/2]+2}\},$$

where the summations are taken as these are smallest symmetric polynomials containing the given terms.

As usual, the total Pontrjagin class of $ES^1 \underset{S^1}{\times} TM$ (resp. TF) is expressed formally as $\prod (1 + (x_i)^2)$ (resp. $\prod (1 + (z_\mu^i)^2)$). Similarly the total Chern class of N_μ is expressed formally as $\prod (1 + x_\mu^i)$. Denote by t the first Chern class of the bundle $S^1 \to ES^1 \to BS^1$ where we identify S^1 with $U(1)$ canonically. Then the term of least degree of $H^*(F_\mu)$ part in

$$\frac{i_\mu^* f_{\omega, \omega'}(x)}{\prod_i (t + x_\mu^i)}$$

is of degree at least $2|\omega| + |\omega'|$ and

$$\frac{i_\mu^* f_{\omega, \omega'}(x)}{\prod (t + x_\mu^i)} = c(t) s_\omega((z_\mu)^2) \cdot s_{\omega'}(x_\mu) + \text{terms of higher degree},$$

where $c(t)$ is a non zero rational function of t and

$$s_\omega((z_\mu)^2)=\sum(z_\mu^1)^{2i_1}\cdots(z_\mu^r)^{2i_r}$$

and

$$s_{\omega'}(x_\mu)=\sum(x_\mu^1)^{j_1}\cdots(x_\mu^s)^{j_s}.$$

Here the summations are taken as above.

The term of highest degree in $f_{\omega,\omega'}(x)$ is of degree

$$6|\omega|+2\left\{|\omega'|+r+s+\sum_{i=1}^{s}\left[\frac{j_i}{2}\right]\right\}.$$

Since $1+\left[\frac{j_i}{2}\right]\leqq j_i$, we have

$$6|\omega|+2\left\{|\omega'|+r+s+\sum_{i=1}^{s}\left[\frac{j_i}{2}\right]\right\}\leqq 8|\omega|+4|\omega'|.$$

Note that any bordism Pontrjagin numbers of $\sum_\mu(F_\mu, N_\mu)$ can be written as a linear combination of $s_\omega((z_\mu)^2)\cdot s_{\omega'}(x_\mu)[F_\mu]$. Therefore if $\dim F<\dim M/4$, any bordism Pontrjagin numbers must vanish.

A similar argument works for bordism Stiefel-Whitney numbers. Let $\omega=(i_1, \cdots, i_r)$ and $\omega'=(j_1, \cdots, j_s)$ be partitions such that none of the $i_j\in\omega$ is of the form 2^p-1 and all j_i are even. Then

$$2|\omega|+r+2|\omega'|+s\leqq\frac{5}{2}(|\omega|+|\omega'|)$$

and we make use of

$$f_{\omega,\omega'}(x)=\{\sum(1+x_1)^{i_1+1}\cdot x_1^{i_1}\cdots(1+x_r)^{i_r+1}\cdot x_r^{i_r}\}$$
$$\times\{\sum(1+x_1)^{j_1}\cdot x_1^{j_1+1}\cdots(1+x_s)^{j_s}\cdot x_s^{j_s+1}\}.$$

Notice that in our situation we have only to consider the partitions as above.

When $\dim M$ is odd, each $\dim F_\mu$ is also odd. Hence every bordism Pontrjagin number of (F_μ, N_μ) vanishes.

This makes the proof of Proposition 1.12 complete.

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Katsuo KAWAKUBO
Department of Mathematics
Faculty of Science
Osaka University
Toyonaka 560
Japan