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On the stochastic differential equation for a two-dimensional Brownian motion with boundary condition

By Masaaki TSUCHIYA

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§ 1. Introduction.

The stochastic differential equations for multi-dimensional diffusion processes with boundary conditions were formulated by Ikeda [5] in the two-dimensional case and Watanabe [17] in the case of general dimensions. They solved the equations with Lipschitz continuous coefficients and showed the strong Markov property of the solutions in the sense of the measure on the path space. Furthermore, Nakao and Shiga [11], [12] solved the equations under more general conditions on the coefficients.

In this paper, we shall consider the stochastic differential equation for a Brownian motion on the upper half plane $\bar{D}=\{(x, y): x\in R^1, y\geqq 0\}$ with the boundary condition: $\frac{\partial u}{\partial y}+\alpha(x)\frac{\partial u}{\partial x}+\frac{1}{2}\beta^2(x)\frac{\partial^2 u}{\partial x^2}=0$. Let $(B(t), b(t), g(t))$ be a three-dimensional Brownian motion starting from the origin. Then the equation is written in the form:

$$(1.1)\qquad \begin{cases} dy(t)=db(t)+d\phi(t)\,, \\ dx(t)=dB(t)+\alpha(x(t))d\phi(t)+\beta(x(t))dg(\phi(t))\,. \end{cases}$$

Here the first equation of (1.1) is the so-called Skorohod equation, which determines the reflecting Brownian motion $y(t)$ on $[0, \infty)$ and its local time $\phi(t)$ at 0, that is,

$$(1.2)\qquad y(t)=y(0)+b(t)+\phi(t) \qquad (y(0)\geqq 0)\,,$$

$$(1.3)\qquad \phi(t)=-\min_{0\leqq s\leqq t}\{(y(0)+b(s))\wedge 0\}\,,$$

where we set $a\wedge b=\min(a, b)$.

We consider the case where $\alpha(x)$ is bounded Borel measurable, and $\beta(x)$ is bounded and satisfies the continuity condition of Yamada and Watanabe (see [19], Theorem 1, (i)). The equation (1.1) is a special type of the equations treated in the papers [5], [17], [11], [12]; however, the existence of solutions of (1.1) is unknown in our case, in general. So, we will generalize the concept of the solution by introducing the concept of relaxed solutions. The concept of relaxed solutions was first introduced by Conway [1] for Itô's stochastic differential equations, which describe Markov processes without boundary. If $\alpha(x)$ is continuous, the relaxed solutions of (1.1) coincide with the ordinary solutions which are considered in the papers [5], [17], [11], [12].

Our relaxed solution is constructed in the following way. We consider a stochastic differential equation on the boundary $\partial\bar{D}=R^1$, which is obtained from (1.1) through time change by $\phi(t)$. Then we show the solutions of (1.1) can be represented by the underlying Brownian motion and the solutions of the equation on the boundary (see Lemma 3.2). Using this, we then prove that if $\alpha(x)$ satisfies the Osgood type condition and $\beta(x)$ satisfies the condition mentioned above, the existence and the uniqueness of solutions hold for (1.1) (see Proposition 4.1). Furthermore, we get a comparison theorem (see Theorem 4.1) analogous to the one obtained by Ikeda and Watanabe [6] for Itô's stochastic differential equations. For a general $\alpha(x)$, we take the minimum upper semi-continuous majorant $\bar{\alpha}(x)$ and the maximum lower semi-continuous minorant $\underline{\alpha}(x)$ of $\alpha(x)$. Here the majorants and the minorants are defined in the almost everywhere sense with respect to the Lebesgue measure on R^1. Approximate $\bar{\alpha}(x)$ or $\underline{\alpha}(x)$ by continuous functions, monotonically. A relaxed solution of (1.1) will then be obtained on a given probability space. So, we can construct a strong solution with shift property (see Theorem 2.1).

In Section 2, we will give the definition of the relaxed solutions of (1.1) and state the main result (Theorem 2.1). We will prove Lemma 3.2 in Section 3. In Section 4, a comparison theorem (Theorem 4.1) will be stated and construct the solutions described in Theorem 2.1. Finally, in Section 5, we will give a condition for the uniqueness of solutions, and under this condition we prove that the relaxed solution coincides with the ordinary solution. In relation to this condition, some correction of a result in my paper [15] is made.

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§ 2. The main result.

For a real function f on R^1, define functions $\bar{f}$ and $\underline{f}$ by

$$\bar{f}(x)=\lim_{\delta\downarrow 0}\operatorname{ess\,sup}_{|y-x|\leq\delta} f(y), \qquad \underline{f}(x)=\lim_{\delta\downarrow 0}\operatorname{ess\,inf}_{|y-x|\leq\delta} f(y),$$

where "ess sup" and "ess inf" mean the essential supremum and the essential infimum with respect to the Lebesgue measure on R^1, respectively. Then $\underline{f}(x) \leqq \bar{f}(x)$ for all $x \in R^1$. Furthermore, the functions $\bar{f}$ and $\underline{f}$ have the following properties.

LEMMA 2.1. *Let f be essentially bounded. Then $\bar{f}$ is bounded and upper semi-continuous, and $\underline{f}$ is bounded and lower semi-continuous. Furthermore, $\underline{f}(x) \leqq f(x) \leqq \bar{f}(x)$ a.e. x. If g is lower semi-continuous and $g(x) \leqq f(x)$ a.e. x, then $g(x) \leqq \underline{f}(x)$ for all $x \in R^1$. If h is upper semi-continuous and $f(x) \leqq h(x)$ a.e. x, then $\bar{f}(x) \leqq h(x)$ for all $x \in R^1$.*

PROOF. The boundedness and the semi-continuity of $\bar{f}$ and $\underline{f}$ follow from the definition. Moreover, it is easy to see that for every Lebesgue point x of f, $\underline{f}(x) \leqq f(x) \leqq \bar{f}(x)$ and it is well-known that almost every point is a Lebesgue point of a bounded function. So, this proves the second assertion. The last assertion for $\underline{f}$ and $\bar{f}$ is obvious. Q.E.D.

In the same way as in the proof of Dini's theorem, we have

LEMMA 2.2. *Let $\{f_n\}$ be a decreasing sequence of continuous functions on R^1 and let $f(x) = \lim_{n \to \infty} f_n(x)$ be finite for each $x \in R^1$. Then, for a real sequence $\{x_n\}$ with $x_n \to x_0$ $(n \to \infty)$, we have*

$$\limsup_{n \to \infty} f_n(x_n) \leqq f(x_0).$$

PROOF. Since the function f is upper semi-continuous, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(x) < f(x_0) + \varepsilon \qquad \text{for all } x \in [x_0 - \delta, x_0 + \delta].$$

Let $G_n = \{x : f_n(x) < f(x_0) + \varepsilon\}$ $(n = 1, 2, \cdots)$. Then G_n are open sets. By the assumption that $f_n(x) \downarrow f(x)$ as $n \to \infty$ for each $x \in R^1$, it follows that $[x_0 - \delta, x_0 + \delta] \subset \bigcup_{m=1}^{\infty} G_m$. So, by the compactness of the interval $[x_0 - \delta, x_0 + \delta]$, there is a finite subcovering $\{G_{m(j)} : j = 1, 2, \cdots, k\}$ of $\{G_m\}$. Let $N_1 = \max\{m(j) : j = 1, 2, \cdots, k\}$. Since $x_n \to x_0$ as $n \to \infty$, for the above $\delta > 0$ there exists a positive integer N_2 such that $x_n \in [x_0 - \delta, x_0 + \delta]$ for all $n \geqq N_2$. Therefore, for $n \geqq N_2$,

$$x_n \in G_{m(j)} \qquad \text{for some } j : 1 \leqq j \leqq k.$$

Hence, for $n \geqq \max(N_1, N_2)$,

$$f_n(x_n) < f(x_0) + \varepsilon,$$

so the proof is finished. Q.E.D.

We will state some conditions for the coefficients of (1.1).

(A) $\alpha(x)$ is a real Borel measurable function on R^1 and is bounded.

(A′) $\alpha(x)$ is a real function on R^1 and satisfies the conditions:

(i) $\alpha(x)$ is bounded;

(ii) there exists a positive increasing concave function $\kappa(u)$ on $[0, \infty)$ such that

$$\kappa(0)=0, \qquad \int_{0+}\kappa^{-1}(u)du=+\infty,$$

$$|\alpha(x)-\alpha(y)|\leqq\kappa(|x-y|) \qquad \text{for all } x, y\in R^1.$$

(B) $\beta(x)$ is a real function on R^1 and satisfies the conditions:

(i) $\beta(x)$ is bounded;

(ii) there exists a positive increasing function $\rho(u)$ on $[0, \infty)$ such that

$$\rho(0)=0, \qquad \int_{0+}\rho^{-2}(u)du=+\infty,$$

$$|\beta(x)-\beta(y)|\leqq\rho(|x-y|) \qquad \text{for all } x, y\in R^1.$$

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a probability space with an increasing family $(\mathcal{F}_t)$ of sub-σ-fields of a σ-field $\mathcal{F}$ on a set Ω and with a probability measure P. A k-dimensional process $\boldsymbol{B}(t)=(B_1(t), B_2(t), \cdots, B_k(t))$ defined on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ is said to be an $(\mathcal{F}_t)$-Brownian motion, if $B_j(t)$ $(j=1, 2, \cdots, k)$ are $(\mathcal{F}_t)$-adapted real continuous processes and, for $s<t$ and $\lambda_1, \lambda_2, \cdots, \lambda_k\in R^1$,

$$E\left[\exp\left\{i\sum_{j=1}^{k}\lambda_j(B_j(t)-B_j(s))\right\}\Big|\mathcal{F}_s\right]=\exp\left\{-\frac{1}{2}(t-s)\sum_{j=1}^{k}\lambda_j^2\right\}.$$

DEFINITION 2.1. By a *relaxed solution* of (1.1), we mean a $\bar{D}$-valued process $z(t)=(x(t), y(t))$ over a probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ satisfying the following conditions (i), (ii), (iii) and (iv).

(i) There exists a three-dimensional Brownian motion $(B(t), b(t), g(t))$ defined on the probability space such that $B(0)=b(0)=g(0)=0$, $(B(t), b(t))$ is a two-dimensional $(\mathcal{F}_t)$-Brownian motion and $g(\phi(t))$ is a continuous martingale with respect to $(\mathcal{F}_t)$ satisfying $E[\{g(\phi(t))-g(\phi(s))\}^2|\mathcal{F}_s]=E[\phi(t)-\phi(s)|\mathcal{F}_s]$ $(s<t)$, where $\phi(t)$ is the process defined from $b(t)$ as in (1.3).

(ii) $y(t)$ is the process defined from $b(t)$ as in (1.2).

(iii) $x(t)$ is an $(\mathcal{F}_t)$-adapted real continuous process.

(iv) With probability one, for $0\leqq s<t$,

$$x(t)-x(s)\leqq B(t)-B(s)+\int_s^t\bar{\alpha}(x(u))d\phi(u)+\int_s^t\beta(x(u))dg(\phi(u)),$$

$$x(t)-x(s)\geqq B(t)-B(s)+\int_s^t\underline{\alpha}(x(u))d\phi(u)+\int_s^t\beta(x(u))dg(\phi(u)).$$

The process $(B(t), b(t), g(t))$ is called the *underlying Brownian motion* for the solution $z(t)$.

REMARK 2.1. Let $A[x]$ denote the interval $[\underline{\alpha}(x), \bar{\alpha}(x)]$. Then, in the above definition, we can replace the condition (iv) by the following condition:

(iv′) There is a well measurable process $\alpha(t)$ such that with probability one,

$$x(t)=x(0)+B(t)+\int_0^t \alpha(s)d\phi(s)+\int_0^t \beta(x(s))dg(\phi(s)) \qquad (t\geqq 0)$$

and

$$\alpha(s)\in A[x(s)] \qquad \text{for all } s\geqq 0\,.$$

REMARK 2.2. If $\alpha(x)$ is continuous, $\underline{\alpha}(x)=\bar{\alpha}(x)$ for all $x\in R^1$. Hence, in this case, the relaxed solutions are the ordinary solutions, that is, the solutions considered by Ikeda, Watanabe, Nakao and Shiga.

DEFINITION 2.2. We say that the *uniqueness* of solutions holds for (1.1) if any two solutions $z_1(t)$ and $z_2(t)$, defined on a common probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ with $z_1(0)=z_2(0)$ and with a common underlying Brownian motion, are equal with probability one, that is, $P[z_1(t)=z_2(t)$ for all $t\geqq 0]=1$.

A relaxed solution $z(t)=(x(t), y(t))$ is said to be *maximum* if $x(t)$ is the maximum solution of the second equation of (1.1). The *minimum solution* of (1.1) is defined similarly.

We will introduce some notations. Hereafter, $\boldsymbol{B}(A)$ denotes the σ-field consisting of Borel sets in a topological space A. Let $\boldsymbol{C}$ be the space of real continuous functions on $[0, \infty)$ and let $\boldsymbol{C}_0$ be the subset of $\boldsymbol{C}$ consisting of functions vanishing at 0. We consider the compact uniform topology on $\boldsymbol{C}$. For $w\in\boldsymbol{C}$, let

$$\|w\|_T=\max_{0\leq t\leq T}|w(t)| \qquad (T>0)\,.$$

Let $\boldsymbol{W}=\boldsymbol{C}_0\times\boldsymbol{C}_0\times\boldsymbol{C}_0$. For $\boldsymbol{w}=(u, v, w)\in\boldsymbol{W}$ and $y\geqq 0$, we set

$$\phi(t, y, \boldsymbol{w})\equiv\phi(t, y, v)=-\min_{0\leq s\leq t}\{(y+v(s))\wedge 0\}\,, \tag{2.1}$$

$$y(t, y, \boldsymbol{w})\equiv y(t, y, v)=y+v(t)+\phi(t, y, v)\,. \tag{2.2}$$

Define mappings $\phi_y: \boldsymbol{W}\to\boldsymbol{W}$, $\rho_s: \boldsymbol{W}\to\boldsymbol{W}$ and $\theta_s: \boldsymbol{W}\to\boldsymbol{W}$ by

$$\phi_y\boldsymbol{w}(t)=(u(t), v(t), w(\phi(t, y, v))) \qquad \text{for } \boldsymbol{w}=(u, v, w)\,; \tag{2.3}$$

$$\rho_s\boldsymbol{w}(t)=\boldsymbol{w}(t\wedge s)\,; \tag{2.4}$$

$$\theta_s\boldsymbol{w}(t)=\boldsymbol{w}(t+s)-\boldsymbol{w}(s)\,. \tag{2.5}$$

Let $\boldsymbol{P}$ be the Wiener measure on $(\boldsymbol{W}, \boldsymbol{B}(\boldsymbol{W}))$. Denote by $\mathcal{M}$ the $\boldsymbol{P}$-completion of $\boldsymbol{B}(\boldsymbol{W})$. For $y\geqq 0$ and $t\geqq 0$, $\mathcal{M}_t^y$ denotes the $\boldsymbol{P}$-completion of $(\rho_t\circ\phi_y)^{-1}\boldsymbol{B}(\boldsymbol{W})$

in $\mathcal{M}$. Next, denote by $\boldsymbol{V}$ the space of $\bar{D}$-valued continuous functions on $[0, \infty)$ endowed with the compact uniform topology and let $\boldsymbol{B}_t(\boldsymbol{V})=\rho_t^{-1}\boldsymbol{B}(\boldsymbol{V})$, where the stopped operator $\rho_t: \boldsymbol{V}\to\boldsymbol{V}$ is defined as in (2.4). Furthermore, in what follows, for $z=(x, y)\in\bar{D}$, we set

$$\phi_z=\phi_y\,, \qquad \mathcal{M}_t^z=\mathcal{M}_t^y\,.$$

Following Watanabe ([**18**], [**19**]), who has investigated the concept of strong solutions for Itô's stochastic differential equations, we define the following concept.

DEFINITION 2.3. A function $\bar{z}: [0, \infty)\times\bar{D}\times\boldsymbol{W}\ni(t, z, \boldsymbol{w})\to\bar{z}(t, z, \boldsymbol{w})\in\bar{D}$ is said to be the *maximum strong solution* of (1.1), if it satisfies the following conditions (i), (ii), (iii) and (iv).

(i) The mapping: $\bar{D}\times\boldsymbol{W}\ni(z, \boldsymbol{w})\to\bar{z}(\cdot, z, \boldsymbol{w})\in\boldsymbol{V}$ is $\boldsymbol{B}(\bar{D}\times\boldsymbol{W})/\boldsymbol{B}(\boldsymbol{V})$-measurable.

(ii) For each $z\in\bar{D}$, the mapping: $\boldsymbol{W}\ni\boldsymbol{w}\to\bar{z}(\cdot, z, \boldsymbol{w})\in\boldsymbol{V}$ is $\mathcal{M}_t^z/\boldsymbol{B}_t(\boldsymbol{V})$-measurable for every $t\geqq 0$.

(iii) For each $z=(x, y)\in\bar{D}$, $\bar{z}(t, z, \boldsymbol{w})=(\bar{x}(t\,; x, y\,; \boldsymbol{w}),\ y(t, y, \boldsymbol{w}))$ is the maximum relaxed solution of (1.1) over the probability space $(\boldsymbol{W}, \mathcal{M}, \boldsymbol{P}\,; \mathcal{M}_t^z)$ with initial value z and with underlying Brownian motion $\boldsymbol{w}(t)=(u(t), v(t), w(t))$.

(iv) Let $z(t)$ be the maximum relaxed solution of (1.1) over a probability space $(\Omega, \mathcal{F}, P\,; \mathcal{F}_t)$ with underlying Brownian motion $\boldsymbol{B}(t)=(B(t), b(t), g(t))$. Then

$$P[z(t)=\bar{z}(t, z(0), \boldsymbol{B}) \text{ for all } t\geqq 0]=1\,.$$

Similarly, we define the *minimum strong solution* $\underline{z}(t, z, \boldsymbol{w})$ of (1.1). If the maximum strong solution and the minimum strong solution exist and $\boldsymbol{P}[\bar{z}(\cdot, z, \boldsymbol{w})=\underline{z}(\cdot, z, \boldsymbol{w})]=1$ for each $z\in\bar{D}$, then we say that the equation (1.1) has a *unique strong solution.*

THEOREM 2.1. *Suppose that the coefficients $\alpha(x)$ and $\beta(x)$ of* (1.1) *satisfy the conditions* (A) *and* (B), *respectively. Then, there exist the maximum strong solution and the minimum strong solution, and they have the shift property, that is, for each finite $(\mathcal{M}_t^z)$-stopping time S, with probability one*

$$z(\cdot+S, z, \boldsymbol{w})=z(\cdot, z(S, z, \boldsymbol{w}), \theta_S\circ\phi_z\boldsymbol{w})\,,$$

provided $z(t, z, \boldsymbol{w})$ is $\bar{z}(t, z, \boldsymbol{w})$ or $\underline{z}(t, z, \boldsymbol{w})$.

REMARK 2.3. The system $\{\boldsymbol{W}, \bar{z}(t, z, \boldsymbol{w}), \mathcal{M}_t^z, \boldsymbol{P}\,; z\in\bar{D}\}$ is a diffusion process on $\bar{D}$, that is, it has continuous paths and satisfies the strong Markov property: Let $z\in\bar{D}$ and let S be a finite $(\mathcal{M}_t^z)$-stopping time. Then

$$\int_{\boldsymbol{W}} L(\boldsymbol{w})f(\bar{z}(t+S, z, \boldsymbol{w}))\boldsymbol{P}(d\boldsymbol{w})=\int_{\boldsymbol{W}} L(\boldsymbol{w})\boldsymbol{P}(d\boldsymbol{w})\int_{\boldsymbol{W}} f(\bar{z}(t, \bar{z}(S, z, \boldsymbol{w}), \boldsymbol{w}'))\boldsymbol{P}(d\boldsymbol{w}')$$

for every bounded $\mathcal{M}_S^z$-measurable function L and every bounded $\boldsymbol{B}(\bar{D})$-measurable function f.

The system consisting of the minimum strong solution has the same property.

§ 3. Stochastic differential equation on the boundary.

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a probability space with a Brownian motion $(B(t), b(t), g(t))$ satisfying the condition (i) of Definition 2.1. Suppose that the probability space satisfies the *usual condition*, that is, $(\Omega, \mathcal{F}, P)$ is complete, $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$ and $\mathcal{F}_t=\mathcal{F}_{t+}\equiv \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$ $(t\geqq 0)$. Let $\phi(t)$ be the process defined by (1.3).

LEMMA 3.1. *Let $\phi^{-1}(t)=\inf\{u:\phi(u)>t\}$ and let $Y(t)$ be a bounded predictable process on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$. Then we have, with probability one,*

$$\text{(i)} \qquad \int_0^t Y(s)d\phi(s)=\int_0^{\phi(t)} Y(\phi^{-1}(s))ds \qquad (t\geqq 0),$$

$$\text{(ii)} \qquad \int_0^t Y(s)dg(\phi(s))=\int_0^{\phi(t)} Y(\phi^{-1}(s))dg(s) \qquad (t\geqq 0).$$

PROOF. The assertion (i) is well-known (cf. [9], Chap. VII, T.12). To prove the second assertion, it is enough to consider the case of step functions. For $0=u_0<u_1<\cdots<u_n=t$, let

$$Y(s)=\sum_{i=0}^{n-1} Y_i I_{[u_i, u_{i+1})}(s),$$

where Y_i are bounded $\mathcal{F}_{u_i}$-measurable functions. Then

$$\begin{aligned} Y(\phi^{-1}(s)) &= \sum_{i=0}^{n-1} Y_i I_{[u_i, u_{i+1})}(\phi^{-1}(s)) \\ &= \sum_{i=0}^{n-1} Y_i I_{[\phi(u_i), \phi(u_{i+1}))}(s), \end{aligned}$$

because the inequality $u_i\leqq\phi^{-1}(s)<u_{i+1}$ is equivalent to the inequality $\phi(u_i)\leqq s<\phi(u_{i+1})$. On the other hand, setting $\mathcal{B}_u=\mathcal{F}_{\phi^{-1}(u)}$, then $\mathcal{F}_u\subset\mathcal{B}_{\phi(u)}$, $g(t)$ is a $(\mathcal{B}_t)$-Brownian motion and $\phi(u)$ is a $(\mathcal{B}_t)$-stopping time. Hence,

$$\begin{aligned} \int_0^{\phi(t)} Y(\phi^{-1}(s))dg(s) &= \sum_{i=0}^{n-1} Y_i\{g(\phi(u_{i+1}))-g(\phi(u_i))\} \\ &= \int_0^t Y(s)dg(\phi(s)). \end{aligned}$$

Q. E. D.

A process $l(t)$ defined on a probability space $(\Omega, \mathcal{B}, P; \mathcal{B}_t)$ is said to be a one-dimensional $(\mathcal{B}_t)$-Cauchy process, if $l(t)$ is a Lévy process adapted to $(\mathcal{B}_t)$ and for $0\leqq s<t$, $\lambda\in R^1$

$$E[\exp\{i\lambda(l(t)-l(s))\}\,|\,\mathcal{B}_s]=\exp\{-|\lambda|(t-s)\}.$$

We consider a stochastic differential equation

$$d\xi(t)=dl(t)+\alpha(\xi(t))dt+\beta(\xi(t))dg(t), \tag{3.1}$$

where $l(t)$ is a Cauchy process and $g(t)$ is a Brownian motion.

DEFINITION 3.1. By a *relaxed solution* of (3.1), we mean a real process $\xi(t)$ over a probability space $(\Omega, \mathcal{B}, P; \mathcal{B}_t)$ satisfying the following conditions (i), (ii), (iii) and (iv).

(i) There are a one-dimensional $(\mathcal{B}_t)$-Cauchy process $l(t)$ $(l(0)=0)$ and a one-dimensional $(\mathcal{B}_t)$-Brownian motion $g(t)$ $(g(0)=0)$.

(ii) With probability one, the paths of $\xi(t)$ are right continuous and have left-hand limits.

(iii) $\xi(t)$ is adapted to $(\mathcal{B}_t)$.

(iv) With probability one, for $0\leqq s<t$,

$$\xi(t)-\xi(s)\leqq l(t)-l(s)+\int_s^t\bar{\alpha}(\xi(u))du+\int_s^t\beta(\xi(u))dg(u),$$

$$\xi(t)-\xi(s)\geqq l(t)-l(s)+\int_s^t\underline{\alpha}(\xi(u))du+\int_s^t\beta(\xi(u))dg(u).$$

The process $(l(t), g(t))$ is called the *underlying process* for the solution $\xi(t)$.

REMARK 3.1. In the above definition, we can replace the condition (iv) by the condition:

(iv′) There is a well measurable process $\alpha(t)$ such that with probability one,

$$\xi(t)=\xi(0)+l(t)+\int_0^t\alpha(s)ds+\int_0^t\beta(\xi(s))dg(s) \qquad (t\geqq 0)$$

and

$$\alpha(s)\in A[\xi(s)] \qquad \text{for all } s\geqq 0.$$

DEFINITION 3.2. We say that the *uniqueness* of solutions holds for (3.1) if any two solutions $\xi_1(t)$ and $\xi_2(t)$, defined on a common probability space with $\xi_1(0)=\xi_2(0)$ and with a common underlying process, are equal with probability one.

The strong solution of (3.1) is defined in the analogous way to the case of (1.1). Denote by $\boldsymbol{D}$ the space of real right continuous functions on $[0, \infty)$ having limits from the left and by $\boldsymbol{D}_0$ the subset of $\boldsymbol{D}$ consisting of functions vanishing at 0. We endow the Skorohod topology on the space $\boldsymbol{D}$. Then $\boldsymbol{D}_0$

is a closed subset of $\boldsymbol{D}$. Set $\boldsymbol{U}=\boldsymbol{D}_0\times\boldsymbol{C}_0$ and define the stopped operators $\rho_t: \boldsymbol{U}\to\boldsymbol{U}$ and $\rho_t: \boldsymbol{D}\to\boldsymbol{D}$ as in (2.4). Let $\boldsymbol{R}$ be the measure on $(\boldsymbol{U}, \boldsymbol{B}(\boldsymbol{U}))$ induced by a process $(l(t), g(t))$, where $l(t)$ and $g(t)$ are a Cauchy process and a Brownian motion defined on a common probability space. Denote by $\mathcal{G}$ the $\boldsymbol{R}$-completion of $\boldsymbol{B}(U)$ and by $\mathcal{G}_t$ the $\boldsymbol{R}$-completion of $\boldsymbol{B}_t(U)\equiv\rho_t^{-1}\boldsymbol{B}(\boldsymbol{U})$ in $\mathcal{G}$.

DEFINITION 3.3. By the *maximum strong solution* of (3.1), we mean a function $\bar{\xi}: [0, \infty)\times R^1\times\boldsymbol{U}\ni(t, x, \boldsymbol{u})\to\bar{\xi}(t, x, \boldsymbol{u})\in R^1$ satisfying the following conditions (i), (ii), (iii) and (iv).

(i) The mapping: $R^1\times\boldsymbol{U}\ni(x, \boldsymbol{u})\to\bar{\xi}(\cdot, x, \boldsymbol{u})\in\boldsymbol{D}$ is $\boldsymbol{B}(R^1\times\boldsymbol{U})/\boldsymbol{B}(\boldsymbol{D})$-measurable.

(ii) For $x\in R^1$, the mapping: $\boldsymbol{U}\ni\boldsymbol{u}\to\bar{\xi}(\cdot, x, \boldsymbol{u})\in\boldsymbol{D}$ is $\mathcal{G}_t/\boldsymbol{B}_t(\boldsymbol{D})$-measurable for each $t\geqq 0$, where $\boldsymbol{B}_t(\boldsymbol{D})=\rho_t^{-1}\boldsymbol{B}(\boldsymbol{D})$.

(iii) For each $x\in R^1$, $\bar{\xi}(t, x, \boldsymbol{u})$ is the maximum relaxed solution of (3.1) over the probability space $(\boldsymbol{U}, \mathcal{G}, \boldsymbol{R}; \mathcal{G}_t)$ with initial value x and with underlying process $\boldsymbol{u}(t)=(p(t), q(t))$.

(iv) If $\xi(t)$ is the maximum relaxed solution of (3.1) over a probability space $(\Omega, \mathcal{B}, P; \mathcal{B}_t)$ with underlying process $\boldsymbol{L}(t)=(l(t), g(t))$,

$$P[\xi(t)=\bar{\xi}(t, \xi(0), \boldsymbol{L}) \text{ for all } t\geqq 0]=1\,.$$

The *minimum strong solution* $\underline{\xi}(t, x, \boldsymbol{u})$ of (3.1) is defined similarly. If the maximum strong solution and the minimum strong solution exist and $\boldsymbol{R}[\bar{\xi}(\cdot, x, \boldsymbol{u})=\underline{\xi}(\cdot, x, \boldsymbol{u})]=1$ for all $x\in R^1$, then we say that the equation (3.1) has a *unique strong solution.*

LEMMA 3.2. *Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a probability space with a Brownian motion $(B(t), b(t), g(t))$ satisfying the condition* (i) *of Definition* 2.1. *Let $y(t)$ and $\phi(t)$ be the processes defined from $b(t)$ as in* (1.2) *and* (1.3), *respectively. Suppose that the probability space satisfies the usual condition. Set*

$$l(t)=B(\phi^{-1}(t))-B(\phi^{-1}(0))\,, \qquad \mathcal{B}_t=\mathcal{F}_{\phi^{-1}(t)}\,.$$

Then we have:

(i) *Let $z(t)=(x(t), y(t))$ be a relaxed solution of* (1.1) *over the probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ with underlying Brownian motion $(B(t), b(t), g(t))$. If we set $\xi(t)=x(\phi^{-1}(t))$, then $\xi(t)$ is a relaxed solution of* (3.1) *over the probability space $(\Omega, \mathcal{F}, P; \mathcal{B}_t)$ with $\xi(0)=x(0)+B(\phi^{-1}(0))$ and with underlying process $(l(t), g(t))$. Moreover, with probability one,*

$$x(t)=B(t)-B(\phi^{-1}(0))+\xi(\phi(t))-l(\phi(t))\,, \qquad t\geqq 0\,.$$

(ii) *Take an $\mathcal{F}_0$-measurable real function x_0. Let $\xi(t)$ be a relaxed solution of* (3.1) *over the probability space $(\Omega, \mathcal{F}, P; \mathcal{B}_t)$ with $\xi(0)=x_0+B(\phi^{-1}(0))$ and with underlying process $(l(t), g(t))$. If we set*

$$x(t)=B(t)-B(\phi^{-1}(0))+\xi(\phi(t))-l(\phi(t)) \qquad (t\geqq 0),$$

then $z(t)=(x(t), y(t))$ *is a relaxed solution of* (1.1) *over the probability space* $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ *with* $x(0)=x_0$ *and with underlying Brownian motion* $(B(t), b(t), g(t))$. *Furthermore, with probability one,*

$$x(\phi^{-1}(t))=\xi(t) \qquad (t\geqq 0).$$

PROOF. Using Lemma 3.1, we have the assertion (i). So, we prove the assertion (ii). Obviously we have by the definition of $x(t)$ that, with probability one,

$$x(\phi^{-1}(t))=\xi(t) \qquad \text{for all } t\geqq 0,$$

because $\phi\circ\phi^{-1}(t)=t$ $(t\geqq 0)$. The paths of $x(t)$ are continuous. Next, we show that $x(t)$ is adapted to $(\mathcal{F}_t)$. For a $(\mathcal{B}_t)$-stopping time T, denote by $\mathcal{B}_{T-}$ the σ-field generated by $\mathcal{B}_0\cup\{A\cap(t<T): A\in\mathcal{B}_t,\ t\geqq 0\}$. Note that for each $t\geqq 0$, $\phi(t)$ is a $(\mathcal{B}_s)$-stopping time. We see that

$$\mathcal{B}_{\phi(t)-}|_{\{\phi^{-1}(0)\leqq t\}}\subset\mathcal{F}_t \qquad \text{for all } t\geqq 0, \tag{3.2}$$

that is,

$$B\cap\{\phi^{-1}(0)\leqq t\}\in\mathcal{F}_t \qquad \text{for all } B\in\mathcal{B}_{\phi(t)-}.$$

In fact, if $B\in\mathcal{B}_0$, then $B\cap\{\phi^{-1}(0)\leqq t\}\in\mathcal{F}_t$, because $\phi^{-1}(0)$ is an $(\mathcal{F}_s)$-stopping time. Let $A\in\mathcal{B}_u$. Then $A\cap\{\phi^{-1}(u)<v\}\in\mathcal{F}_v$ for all v, because $\phi^{-1}(u)$ is an $(\mathcal{F}_s)$-stopping time and $\mathcal{F}_v=\mathcal{F}_{v+}$. If $v>t$, then $\{u<\phi(t)\}\subset\{\phi^{-1}(u)<v\}$. Hence, for $v>t$,

$$A\cap\{u<\phi(t)\}=A\cap\{\phi^{-1}(u)<v\}\cap\{u<\phi(t)\}\in\mathcal{F}_v.$$

Therefore, using that $\mathcal{F}_t=\mathcal{F}_{t+}$, we have $A\cap\{u<\phi(t)\}\in\mathcal{F}_t$, so that

$$A\cap\{u<\phi(t)\}\cap\{\phi^{-1}(0)\leqq t\}\in\mathcal{F}_t.$$

This shows that (3.2) holds. Since the process $\xi(s)-l(s)$ is adapted to $(\mathcal{B}_s)$ and has continuous paths, it is predictable (see [3], T22). So $\xi(\phi(t))-l(\phi(t))$ is $\mathcal{B}_{\phi(t)-}$-measurable (see [3], T20). Therefore $x(t)I_{\{\phi^{-1}(0)\leqq t\}}$ is $\mathcal{F}_t$-measurable. On the other hand, $x(t)I_{\{\phi^{-1}(0)>t\}}=(x_0+B(t))I_{\{\phi^{-1}(0)>t\}}$ is also $\mathcal{F}_t$-measurable. This proves that $x(t)$ is $\mathcal{F}_t$-measurable. It is easy to see that $x(t)$ satisfies the condition (iv) of Definition 2.1. Hence $x(t)$ is a relaxed solution of (1.1) over the probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ with initial value x_0 and with underlying Brownian motion $(B(t), b(t), g(t))$. Q. E. D.

§4. Construction of the solution.

Let $\phi(t, y, v)$ and $y(t, y, v)$ be the processes defined by (2.1) and (2.2), respectively. If the shift operator $\theta_s: \boldsymbol{C}_0\to\boldsymbol{C}_0$ is defined as in (2.5), then, by elementary fashion, we have

LEMMA 4.1. (i) *For* $y, y'\geqq 0$; $v, v'\in \boldsymbol{C}_0$; $T>0$,

$$\|\phi(\cdot, y, v)-\phi(\cdot, y', v')\|_T\leqq |y-y'|+\|v-v'\|_T .$$

(ii) *For* $t, s\geqq 0$, $y\geqq 0$, $v\in \boldsymbol{C}_0$,

$$\phi(t+s, y, v)=\phi(s, y, v)+\phi(t, y(s, y, v), \theta_s v) .$$

(iii) *For* $t, s\geqq 0$, $y\geqq 0$, $v\in \boldsymbol{C}_0$,

$$y(t+s, y, v)=y(t, y(s, y, v), \theta_s v) .$$

We first prove Theorem 2.1 under stronger assumptions on the coefficients.

PROPOSITION 4.1. *Suppose that* $\alpha(x)$ *and* $\beta(x)$ *of the coefficients of* (1.1) *satisfy the conditions* (A') *and* (B), *respectively. Then* (1.1) *has a unique strong solution, and it has the shift property.*

PROOF. Consider the equation (3.1) with coefficients $\alpha(x)$ and $\beta(x)$. Then, applying the same method as in Yamada and Watanabe [**19**], it is easy to see that the uniqueness holds for (3.1). So, using Lemma 3.2, the uniqueness holds for (1.1). Applying Skorohod's method, we see that for any probability measure μ on $\boldsymbol{B}(\bar{D})$ there exists a solution of (1.1) with initial distribution μ on the Wiener space (cf. [**11**], [**15**]). Next, we will prove that (1.1) has a unique strong solution. For $z=(x, y)\in\bar{D}$, let

$$\boldsymbol{W}'=\{\boldsymbol{w}: \lim_{t\to\infty}\phi(t, y, \boldsymbol{w})=\infty\} .$$

Then $\boldsymbol{W}'$ is a Borel set of $\boldsymbol{W}$. Since ϕ_z is continuous and $\phi_{z|\boldsymbol{W}'}$ is an injection, it follows from Kuratowski's theorem that $\phi_z(\boldsymbol{W}')$ is a Borel set of $\boldsymbol{W}$. Let $z(t)$ be a solution on a probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ with initial value z and with underlying Brownian motion $\boldsymbol{B}(t)=(B(t), b(t), g(t))$. Denote by P_z the probability measure on $(\boldsymbol{V}\times \boldsymbol{W}, \boldsymbol{B}(\boldsymbol{V}\times \boldsymbol{W}))$ induced by the process $(z(t), \boldsymbol{B}(t))$. Take the projection $\pi: \boldsymbol{V}\times\boldsymbol{W}\to\boldsymbol{W}$. Then the image measure πP_z is the Wiener measure $\boldsymbol{P}$ on $(\boldsymbol{W}, \boldsymbol{B}(\boldsymbol{W}))$. Denote by $\boldsymbol{P}_z$ the image measure $\phi_z\circ\pi P_z$. Then $\boldsymbol{P}_z=\phi_z\boldsymbol{P}$. Since $\boldsymbol{V}\times\boldsymbol{W}$ is a Polish space, there exists a disintegration $Q_z^{w'}(d\boldsymbol{v}d\boldsymbol{w})$ of P_z by $\phi_z\circ\pi$ (see [4]). Let

$$Q_z(\boldsymbol{w}'; d\boldsymbol{v})=Q_z^{w'}(d\boldsymbol{v}\times\boldsymbol{W}) .$$

Since $\boldsymbol{P}(\boldsymbol{W}')=1$, $\phi_{z|\boldsymbol{W}'}$ is an injection and since $Q_z^{w'}(d\boldsymbol{v}d\boldsymbol{w})$ is concentrated on $(\phi_z\circ\pi)^{-1}(\{\boldsymbol{w}'\})$ ($\boldsymbol{P}_z$-a. s. $\boldsymbol{w}'$), $Q_z(\phi_z\boldsymbol{w}; d\boldsymbol{v})$ satisfies the following conditions:

(i) for fixed $\boldsymbol{w}\in\boldsymbol{W}$, $Q_z(\phi_z\boldsymbol{w}; d\boldsymbol{v})$ is a probability measure on $(\boldsymbol{V}, \boldsymbol{B}(\boldsymbol{V}))$;

(ii) for fixed $A\in\boldsymbol{B}(\boldsymbol{V})$, $Q_z(\phi_z\boldsymbol{w}; A)$ is $\phi_z^{-1}\boldsymbol{B}(\boldsymbol{W})$-measurable;

(iii) for $A\in\boldsymbol{B}(\boldsymbol{V})$ and $B\in\boldsymbol{B}(\boldsymbol{W})$,

$$P_z(A\times B)=\int_B Q_z(\phi_z \boldsymbol{w}\,;\,A)\boldsymbol{P}(d\boldsymbol{w})\,.$$

Take another solution $z'(t)$ of (1.1) over a probability space with initial value z and with underlying Brownian motion $\boldsymbol{B}'(t)$. Denote by P_z' the measure on $(\boldsymbol{V}\times \boldsymbol{W}, \boldsymbol{B}(\boldsymbol{V}\times \boldsymbol{W}))$ induced by the process $(z'(t), \boldsymbol{B}'(t))$ and define $Q_z'(\boldsymbol{w}'\,;\,d\boldsymbol{v})$ from P_z' as before. Moreover, define a probability measure Q on $(\boldsymbol{V}\times \boldsymbol{V}\times \boldsymbol{W}, \boldsymbol{B}(\boldsymbol{V}\times \boldsymbol{V}\times \boldsymbol{W}))$ by

$$Q(d\boldsymbol{v}d\boldsymbol{v}'d\boldsymbol{w})=Q_z(\phi_z\boldsymbol{w}\,;\,d\boldsymbol{v})Q_z'(\phi_z\boldsymbol{w}\,;\,d\boldsymbol{v}')\boldsymbol{P}(d\boldsymbol{w})\,.$$

Then, in the same way as in [18], [19], we have $Q(\boldsymbol{v}=\boldsymbol{v}')=1$ and hence there exists a mapping $F_z: \boldsymbol{W}\to \boldsymbol{V}$ such that

(i) $$Q[\boldsymbol{v}=\boldsymbol{v}'=F_z(\boldsymbol{w})]=1\,;$$

(ii) F_z is $\phi_z^{-1}\boldsymbol{B}(\boldsymbol{W})/\boldsymbol{B}(\boldsymbol{V})$-measurable and $\mathcal{M}_t^z/\boldsymbol{B}_t(\boldsymbol{V})$-measurable for each $t\geqq 0$;

(iii) F_z is a solution of (1.1) over the probability space $(\boldsymbol{W}, \mathcal{M}, \boldsymbol{P}\,;\,\mathcal{M}_t^z)$ with initial value z and with underlying Brownian motion $\boldsymbol{w}(t)$. Hence, to prove that $F_z(\boldsymbol{w})\equiv F(z, \boldsymbol{w})$ is the unique strong solution of (1.1), it remains to show that $F(z, \boldsymbol{w})$ is measurable in $(z, \boldsymbol{w})$. We can write $F(z, \boldsymbol{w})$ in component wise

$$F(z, \boldsymbol{w})=(x(\cdot, z, \boldsymbol{w}), y(\cdot, y, \boldsymbol{w}))\,,$$

where $y(t, y, \boldsymbol{w})$ is the process defined by (2.2). So, we will prove that $x(\cdot, z, \boldsymbol{w})$ is measurable. To do this, we construct the unique strong solution of (3.1). In the same way as in the above argument, we can get a mapping $G: R^1\times \boldsymbol{U}\ni(x, \boldsymbol{u})\to G(x, \boldsymbol{u})\in \boldsymbol{D}$ such that

(i) for each $x\in R^1$, the mapping: $\boldsymbol{U}\ni \boldsymbol{u}\to G(x, \boldsymbol{u})\in \boldsymbol{D}$ is $\boldsymbol{B}(\boldsymbol{U})/\boldsymbol{B}(\boldsymbol{D})$-measurable and $\mathcal{G}_t/\boldsymbol{B}_t(\boldsymbol{D})$-measurable for each $t\geqq 0$;

(ii) for each $x\in R^1$, $G(x, \boldsymbol{u})$ is a solution of (3.1) over the probability space $(\boldsymbol{U}, \mathcal{G}, \boldsymbol{R}\,;\,\mathcal{G}_t)$ with initial value x and with underlying process $\boldsymbol{u}(t)=(p(t), q(t))$;

(iii) rewriting $G(x, \boldsymbol{u})=G(x\,;\,p, q)$ for $\boldsymbol{u}=(p, q)\in \boldsymbol{U}$, we have

$$\boldsymbol{R}[G(x\,;\,p, q)-p\in \boldsymbol{C}]=1 \qquad \text{for all } x\in R^1\,.$$

Since $\boldsymbol{C}$ is a Borel set of $\boldsymbol{D}$, we may assume that

(iii′) $$H(x, \boldsymbol{u})\equiv H(x\,;\,p, q)=G(x\,;\,p, q)-p\in \boldsymbol{C}$$

for $\boldsymbol{u}=(p, q)\in \boldsymbol{U}$ and $x\in R^1$.

Let M be the set of all $\boldsymbol{B}(\boldsymbol{U})/\boldsymbol{B}(\boldsymbol{C})$-measurable mappings from $\boldsymbol{U}$ to $\boldsymbol{C}$, with two mappings identified if they are equal $\boldsymbol{R}$-almost everywhere. If f is a measurable mapping from $\boldsymbol{U}$ to $\boldsymbol{C}$, we denote by $[f]$ the equivalence class containing f. Define a metric ρ on M by

$$\rho([f], [g])=\int_U \frac{d(f(\boldsymbol{u}), g(\boldsymbol{u}))}{1+d(f(\boldsymbol{u}), g(\boldsymbol{u}))}\boldsymbol{R}(d\boldsymbol{u}),$$

where d is the metric on $\boldsymbol{C}$. By virtue of the uniqueness of solutions and the continuity of the coefficients of (3.1), the mapping $\hat{H}: R^1 \ni x \rightarrow [H(x, \cdot)] \in M$ is continuous. Therefore, it follows from Chung and Doob's result ([**2**], Proposition 32) that H has a measurable version (again denote by H), that is, H is $\boldsymbol{B}(R^1\times \boldsymbol{U})/\boldsymbol{B}(\boldsymbol{C})$-measurable. Furthermore, we find that $G(x, \boldsymbol{u})$ is the unique strong solution of (3.1). Therefore, by Lemma 3.2, we have $\boldsymbol{P}$-almost surely,

$$\begin{aligned} x(\cdot, z, \boldsymbol{w}) &= u(\cdot)-u(\phi^{-1}(0, y, v)) \\ &\quad +H(x+u(\phi^{-1}(0, y, v))\,;\, u(\phi^{-1}(\cdot, y, v))-u(\phi^{-1}(0, y, v)), w)(\phi(\cdot, y, v)). \end{aligned}$$

Hence $x(\cdot, z, \boldsymbol{w})$ has a measurable version, that is, we may assume that the mapping: $\bar{D}\times \boldsymbol{W} \ni (z, \boldsymbol{w}) \rightarrow x(\cdot, z, \boldsymbol{w}) \in \boldsymbol{C}$ is $\boldsymbol{B}(\bar{D}\times \boldsymbol{W})/\boldsymbol{B}(\boldsymbol{C})$-measurable. Moreover, setting $z(t, z, \boldsymbol{w})=(x(t, z, \boldsymbol{w}), y(t, y, \boldsymbol{w}))$, it is easily seen that for any solution $z(t)$ over a probability space with underlying Brownian motion $\boldsymbol{B}(t)$, with probability one,

$$z(t)=z(t, z(0), \boldsymbol{B}) \qquad \text{for all } t\geqq 0.$$

Finally, by virtue of the uniqueness, $z(t, z, \boldsymbol{w})$ has the shift property. Q. E. D.

Next, exactly in the same way as in Ikeda and Watanabe [**6**], we can obtain the following comparison theorem for solutions of (1.1). So, the proof will be omitted.

THEOREM 4.1. *Suppose that $\beta(x)$ satisfies the condition* (B). *Let $\alpha_1(x)$ and $\alpha_2(x)$ be real continuous functions on R^1 such that*

$$\alpha_1(x)<\alpha_2(x) \qquad \textit{for all } x\in R^1. \tag{4.1}$$

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a probability space satisfying the usual condition. Suppose that on this probability space, the following stochastic processes are given:

(i) *two real $\mathcal{F}_t$-adapted continuous processes $x_1(t)$ and $x_2(t)$;*

(ii) *a three-dimensional Brownian motion $(B(t), b(t), g(t))$ satisfying the condition* (i) *of Definition* 2.1;

(iii) *two real $(\mathcal{F}_t)$-adapted well measurable processes $\alpha_1(t)$ and $\alpha_2(t)$.*

Assume that the following conditions satisfy with probability one:

$$x_i(t)=x_i(0)+B(t)+\int_0^t \alpha_i(s)d\phi(s)+\int_0^t \beta(x_i(s))dg(\phi(s)) \qquad (i=1, 2),$$

$$x_1(0)\leqq x_2(0),$$

$$\alpha_1(t) \leqq \alpha_1(x_1(t)) \qquad \textit{for all } t \geqq 0,$$

$$\alpha_2(t) \geqq \alpha_2(x_2(t)) \qquad \textit{for all } t \geqq 0.$$

Then we have

$$P[x_1(t) \leqq x_2(t) \textit{ for all } t \geqq 0] = 1. \tag{4.2}$$

If, furthermore, the uniqueness of solutions holds for at least one of the stochastic differential equations (1.1) *with coefficients* $\alpha(x) = \alpha_i(x)$ *and* $\beta(x)$ $(i = 1, 2)$, *then we have the same conclusion* (4.2) *replacing* (4.1) *by the following weaker condition:*

$$\alpha_1(x) \leqq \alpha_2(x) \qquad \textit{for all } x \in R^1. \tag{4.1'}$$

PROOF OF THEOREM 2.1. Since $\bar{\alpha}(x)$ is bounded upper semi-continuous, it can be monotonically approximated by a decreasing sequence $\{\alpha_n(x)\}$ of uniformly bounded Lipschitz continuous functions. Let $z_n(t) = (x_n(t), y(t))$ $(n = 1, 2, \cdots)$ be the solutions of (1.1) with coefficients $\alpha(x) = \alpha_n(x)$ and $\beta(x)$ over a probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ with underlying Brownian motion $(B(t), b(t), g(t))$, that is, $y(t)$ and $\phi(t)$ are the processes defined from $b(t)$ as in (1.2) and (1.3), respectively, and with probability one

$$x_n(t) = x_n(0) + B(t) + \int_0^t \alpha_n(x_n(s)) d\phi(s) + \int_0^t \beta(x_n(s)) dg(\phi(s)).$$

Suppose that $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ satisfies the usual condition. Moreover, assume that $\{x_n(0)\}$ is a decreasing sequence of $\mathcal{F}_0$-measurable real functions and $x_n(0) \to \bar{x}(0)$ as $n \to \infty$, with probability one. Then, by Theorem 4.1, $x_n(t)$ converges to an $(\mathcal{F}_t)$-adapted process $\bar{x}(t)$. Hence we can choose a subsequence $\{n'\}$ of $\{n\}$ such that with probability one, $\int_0^t \beta(x_{n'}(s)) dg(\phi(s))$ converges to $\int_0^t \beta(\bar{x}(s)) dg(\phi(s))$ as $n' \to \infty$, uniformly on each compact time interval. For simplicity, we write $\{n'\} = \{n\}$ again. Then

$$\bar{x}(t) = \bar{x}(0) + B(t) + \lim_{n \to \infty} \int_0^t \alpha_n(x_n(s)) d\phi(s) + \int_0^t \beta(\bar{x}(s)) dg(\phi(s)).$$

Since $\{\alpha_n\}$ is uniformly bounded, we find that, with probability one, $\lim_{n \to \infty} \int_0^t \alpha_n(x_n(s)) d\phi(s)$ is continuous in t. Therefore, $\bar{x}(t)$ is a continuous process. By Fubini's lemma and Lemma 2.2,

$$\lim_{n \to \infty} \int_s^t \alpha_n(x_n(u)) d\phi(u) \leqq \int_s^t \bar{\alpha}(\bar{x}(u)) d\phi(u) \qquad \text{for all } 0 \leqq s < t,$$

with probability one. Furthermore, with probability one

$$\lim_{n\to\infty}\int_s^t \alpha_n(x_n(u))d\phi(u)\geqq\int_s^t \underline{\alpha}(\bar{x}(u))d\phi(u) \qquad \text{for all } 0\leqq s<t\,,$$

because $\liminf_{n\to\infty}\alpha_n(x_n(u))\geqq\underline{\alpha}(\bar{x}(u))$. Therefore, $\bar{z}(t)=(\bar{x}(t),\ y(t))$ is a relaxed solution with coefficients $\alpha(x)$ and $\beta(x)$ and with initial value $\bar{z}(0)=(\bar{x}(0),\ y(0))$ over the probability space $(\Omega,\ \mathcal{F},\ P;\ \mathcal{F}_t)$. By Remark 2.1 and Theorem 4.1, it follows that $\bar{z}(t)$ is the maximum relaxed solution of (1.1) with initial value $\bar{z}(0)$. Hence for any probability measure μ on $\boldsymbol{B}(\bar{D})$, we can construct the maximum relaxed solution of (1.1) with initial distribution μ over a probability space. Denote by $z_n(t,\ z,\ \boldsymbol{w})$ the unique strong solution of (1.1) with coefficients $\alpha_n(x)$ and $\beta(x)$. Then for any probability measure μ on $\boldsymbol{B}(\bar{D})$, we have

$$\mu\times\boldsymbol{P}[z_n(\cdot,\ z,\ \boldsymbol{w}) \text{ converges in } \boldsymbol{V} \text{ as } n\to\infty]=1\,.$$

So if we set

$$\bar{z}(\cdot,\ z,\ \boldsymbol{w})=\lim_{n\to\infty} z_n(\cdot,\ z,\ \boldsymbol{w})\,,$$

then $\bar{z}(t,\ z,\ \boldsymbol{w})$ is the maximum strong solution of (1.1). Finally, we show the shift property of $\bar{z}(t,\ z,\ \boldsymbol{w})$. We can write $\bar{z}(t,\ z,\ \boldsymbol{w})$ in component wise

$$\bar{z}(t,\ z,\ \boldsymbol{w})=(\bar{x}(t\,;\ x,\ y\,;\ \boldsymbol{w}),\ y(t,\ y,\ \boldsymbol{w})) \qquad \text{for } z=(x,\ y)\,.$$

Let S be a finite $(\mathcal{M}_t^z)$-stopping time. By the shift property of $z_n(t,\ z,\ \boldsymbol{w})$, we have

$$\boldsymbol{P}[x_n(\cdot+S\,;\ x,\ y\,;\ \boldsymbol{w})=x_n(\cdot\,;\ x_n(S\,;\ x,\ y\,;\ \boldsymbol{w}),\ y(S,\ y,\ \boldsymbol{w})\,;\ \theta_S\circ\phi_z\boldsymbol{w})]=1\,.$$

Since $\boldsymbol{P}[x_n(S\,;\ x,\ y\,;\ \boldsymbol{w})\downarrow\bar{x}(S\,;\ x,\ y\,;\ \boldsymbol{w})$ as $n\to\infty]=1$, it follows from Theorem 4.1 that

$$\boldsymbol{P}[\bar{x}(\cdot+S\,;\ x,\ y\,;\ \boldsymbol{w})=\bar{x}(\cdot\,;\ \bar{x}(S\,;\ x,\ y\,;\ \boldsymbol{w}),\ y(S,\ y,\ \boldsymbol{w})\,;\ \theta_S\circ\phi_z\boldsymbol{w})]=1\,.$$

Hence $\bar{z}(t,\ z,\ \boldsymbol{w})$ has the shift property. Q. E. D.

§ 5. Final remarks.

First, we will give a condition for the relaxed solutions to coincide with the ordinary solutions.

THEOREM 5.1. *Suppose that* (A) *and* (B) *hold for the coefficients* $\alpha(x)$ *and* $\beta(x)$ *of* (1.1), *respectively. Furthermore, assume that*

$$\underline{\alpha}(x)=\bar{\alpha}(x) \qquad a.\,e.\ x\,,$$

$$\beta(x)>0 \qquad \text{for all } x\in R^1\,.$$

Then, the relaxed solutions of (1.1) *coincide with the ordinary solutions and the unieqness of solutions holds for* (1.1).

PROOF. Consider the equation (3.1) with coefficients $\alpha(x)$ and $\beta(x)$. Noting Remark 3.1, it follows from the Krylov type inequality (see [8], Théorème III_{15}, Corollaire III_{16}) that for any relaxed solution $\xi(t)$ of (3.1) over a probability space $(\Omega, \mathcal{B}, P; \mathcal{B}_t)$ with underlying process $(l(t), g(t))$,

$$E\left[\int_0^t I_N(\xi(s))ds\right]=0$$

for every $t\geqq 0$ and Lebesgue measure null set N of R^1. By the assumption and Lemma 2.1, $\underline{\alpha}(x)=\alpha(x)=\bar{\alpha}(x)$ *a. e.* x. Hence, with probability one,

$$\xi(t)=\xi(0)+l(t)+\int_0^t \alpha(\xi(s))ds+\int_0^t \beta(\xi(s))dg(s) \qquad (t\geqq 0),$$

that is, $\xi(t)$ is an ordinary solution of (3.1). On the other hand, it is well-known that the uniqueness holds for solutions of the martingale problem associated with (3.1) (see [7], [13]). Therefore, the maximum relaxed solution and the minimum relaxed solution of (3.1) over any probability space coincide, that is, the uniqueness holds for (3.1). This implies that the uniqueness of solutions holds for (1.1). Q. E. D.

REMARK 5.1. If $\alpha(x)$ is Riemann-integrable on each compact interval, then $\bar{\alpha}(x)=\underline{\alpha}(x)$ *a. e.* x. If $\beta(x)>0$ and if the set $\{x:\bar{\alpha}(x)>\underline{\alpha}(x)\}$ has positive Lebesgue measure, then the uniqueness of relaxed solutions of (1.1) does not generally hold. There exists a bounded semi-continuous function $\alpha(x)$ satisfying such a condition.

In the case where $\beta=0$, in [15], we discussed conditions for the uniqueness of solutions, and a relation between the relaxed solutions of (1.1) and the processes constructed by Motoo [10]. In this case, the maximum strong solution and the minimum strong solution can also be constructed by following Viktrovskiĭ [16] and the outline was given in [15]. However, the construction of the solutions in this paper lies in the same lines as in [14]. Some conditions for the uniqueness of solutions were given in Theorem 2.2 in [15]. It should be noticed that the statement (ii) of the theorem is incorrect. To correct it, in addition to the assumption of Theorem 2.2, (ii), we have to assume that $\underline{a}(x)=\bar{a}(x)$ *a. e.* x, here $a(x)$ indicates $\alpha(x)$ in this paper.

Finally, we note that our result of this paper remains valid when we replace the condition of the boundedness of $\alpha(x)$ and $\beta(x)$ by the condition:

$$|\alpha(x)|+|\beta(x)|\leqq K(1+|x|) \qquad \text{for all } x\in R^1$$

with a positive constant K.

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Masaaki Tsuchiya
Department of Mathematics
College of Liberal Arts
Kanazawa University
Kanazawa 920
Japan