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On covariant representations of continuous C^*-dynamical systems

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Introduction.

The study of C^*-dynamical systems plays an important role in the theory of C^*-algebras. This paper is devoted to a study of covariant representations of continuous C^*-dynamical systems. A C^*-dynamical system is a pair (A, G), where A is a C^*-algebra and G is a locally compact Hausdorff group acting on A by *-automorphisms. The action of $g\in G$ on $a\in A$ is denoted by $g\cdot a$ or ga. If, for all $a\in A$, the map $g\mapsto g\cdot a$ of G into A is continuous for the norm topology of A, we say that the C^*-dynamical system (A, G) is continuous. From a continuous dynamical system (A, G), one can construct the crossed product $C^*(G, A)$, the covariance algebra in the sense of [6]. For a closed subgroup G_0 of G, there is a method to construct representations of $C^*(G, A)$ from covariant representations of (A, G_0), which are called the induced representations ([**10**], § 3). On the other hand, in [8], W. Krieger showed the construction of a von Neumann algebra from a commutative dynamical system $((M, \mathfrak{B}, m), G)$, where $(M, \mathfrak{B}, m)$ is a measure space and G is a countable discrete group. This construction coincides with that of the crossed product when the action of G is free.

In this paper, to study the continuous C^*-dynamical system (A, G), we try to apply the idea of Krieger's to the covariant representations of (A, G). For this purpose, in Section 1, we show the construction of covariant representations of (A, G) from representations of A, which is an analogue of the Krieger's construction, and then we construct a representation Cent ρ of $C^*(G, A)$ from a representation ρ of A. If the action of G on the quasi-dual $\hat{A}$ of A is free, Cent ρ coincides with the induced representation of ρ. Using the representation Cent ρ, we show the construction of a C^*-algebra G^*A from (A, G), which is different from that of the crossed product. In Section 2, we show that, if representations ρ_1 and ρ_2 of A are quasi-equivalent, then Cent ρ_1 and Cent ρ_2 are quasi-equivalent.

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§ 1. Centrally extended representations.

In the following, A denotes a separable C^*-algebra. By Rep A we mean the set of all non degenerate representations ρ of A on some separable Hilbert space $\mathfrak{H}_\rho$, and by Fac A we mean the set of all factor representations. For representations ρ_1 and ρ_2 of a C^*-algebra, $\rho_1 \cong \rho_2$ means that ρ_1 and ρ_2 are equivalent, and $\rho_1 \approx \rho_2$ means that ρ_1 and ρ_2 are quasi-equivalent (c.f. [**5**], 2.2.1. and 5.3.2.). Let $\hat{A}$ denotes the quasi-dual (called the quasi-spectrum in [**5**], 7.2.2.) of A endowed with the Mackey Borel structure, that is, $\hat{A}$ is the set of quasi-equivalence classes of non-trivial factor representations of A, and the Mackey Borel structure on $\hat{A}$ is the quotient structure of that of Fac A for the canonical mapping Fac $A \to \hat{A}$ ([**5**], 7.2.2). Let G be a second countable locally compact Hausdorff topological group, and (A, G) be a continuous C^*-dynamical system. Then G acts on the quasi-dual $\hat{A}$ of A as follows; for $\zeta \in \hat{A}$, $g \cdot \zeta$ denotes the quasi-equivalence class of $g \cdot \pi$, where π is a representation belonging to the quasi-equivalence class ζ and $(g \cdot \pi)(a) = \pi(g^{-1}a)$ for $a \in A$. For $\zeta \in \hat{A}$ we denote by G_ζ the stabilizer of ζ under G, i.e. $G_\zeta = \{g \in G;\ g \cdot \zeta = \zeta\}$. The stabilizer G_ζ is a subgroup of G. *We assume, throughout this paper, that G_ζ is closed in G for all $\zeta \in \hat{A}$.* This is true when A is a GCR-algebra ([**10**], Theorem 2.4, p. 280). The couple (π, U) is said to be a covariant representation of (A, G), if π is a representation of A on a Hilbert space $\mathfrak{H}$ and U is a unitary representation of G on the same space $\mathfrak{H}$ such that $U(g)\pi(x)U(g^{-1}) = \pi(gx)$ for $x \in A$, $g \in G$.

Let ν be a left invariant Haar measure on G, and ν_ζ be a left invariant Haar measure on G_ζ. As G and G_ζ are second countable locally compact Hausdorff groups, they are Polish spaces and ν and ν_ζ are standard measures.

For $\rho \in \text{Rep}\, A$, let the central decomposition of ρ be as follows;

$$\mathfrak{H}_\rho = \int_{\hat{A}}^{\oplus} \mathfrak{H}_{\rho(\zeta)} d\mu_\rho(\zeta), \qquad \rho = \int_{\hat{A}}^{\oplus} \rho(\zeta) d\mu_\rho(\zeta),$$

where the quasi-equivalence class of $\rho(\zeta)$ is ζ and μ_ρ is a standard measure on $\hat{A}$ which is uniquely determined up to equivalence. For every $\zeta \in \hat{A}$, we define $\mathfrak{H}(\zeta)$ and $G_\zeta \cdot \rho(\zeta)$ by the following:

$$\mathfrak{H}(\zeta) = \int_{G_\zeta}^{\oplus} \mathfrak{H}_{g \cdot \rho(\zeta)} d\nu_\zeta(g) = L^2(G_\zeta, \nu_\zeta) \otimes \mathfrak{H}_{\rho(\zeta)};$$

$$G_\zeta \cdot \rho(\zeta)(a) = \int_{G_\zeta}^{\oplus} g \cdot \rho(\zeta)(a) d\nu_\zeta(g) \qquad \text{for all} \quad a \in A.$$

PROPOSITION 1.1. *For every* $\zeta\in\hat{A}$, $G_\zeta\cdot\rho(\zeta)$ *belongs to* Fac A *and the quasi-equivalence class of* $G_\zeta\cdot\rho(\zeta)$ *is* ζ.

PROOF. For $g\in G_\zeta$, we have $\rho(\zeta)\approx g\cdot\rho(\zeta)$. Let $\aleph_0\cdot\rho(\zeta)$ be the representation $\bigoplus_{i=1}^\infty\rho_i$ of A, where $\rho_i=\rho(\zeta)$ for every i (c.f. [5], 2.2.3.). Since $\mathfrak{H}_{\rho(\zeta)}$ is separable, $\aleph_0\cdot\rho(\zeta)\cong\aleph_0\cdot g\cdot\rho(\zeta)$ ([5], 5.3.8.). Thus we have

$$\int_{G_\zeta}^{\oplus}\aleph_0\cdot\rho(\zeta)d\nu_\zeta(g)\cong\int_{G_\zeta}^{\oplus}\aleph_0\cdot g\cdot\rho(\zeta)d\nu_\zeta(g).$$

As

$$\int_{G_\zeta}^{\oplus}\aleph_0\cdot\rho(\zeta)d\nu_\zeta(g)\cong\aleph_0\cdot(I_{L^2(G_\zeta,\nu_\zeta)}\otimes\rho(\zeta))$$

and

$$\int_{G_\zeta}^{\oplus}\aleph_0\cdot g\cdot\rho(\zeta)d\nu_\zeta(g)\cong\aleph_0\cdot G_\zeta\cdot\rho(\zeta),$$

we have $\aleph_0\cdot(I_{L^2(G_\zeta,\nu_\zeta)}\otimes\rho(\zeta))\cong\aleph_0\cdot G_\zeta\cdot\rho(\zeta)$. Therefore we get $\rho(\zeta)\approx G_\zeta\cdot\rho(\zeta)$.

Q. E. D.

Let G/G_ζ be the set of left cosets of G by G_ζ. As G_ζ is assumed to be closed in G, G/G_ζ is a locally compact Hausdorff space by the quotient topology, and it is second countable. The group G acts continuously on G/G_ζ by left multiplication. There exist a non-zero quasi-invariant measure λ_ζ on G/G_ζ which is uniquely determined up to equivalence, and a continuous function $\chi_\zeta>0$ on $G\times G/G_\zeta$ such that

$$\int_{G/G_\zeta}f(\dot{t})d(h\cdot\lambda_\zeta)(\dot{t})=\int_{G/G_\zeta}f(\dot{t})\chi_\zeta(h^{-1},\dot{t})d\lambda_\zeta(\dot{t})$$

for every continuous function f on G/G_ζ with compact support and for all $h\in G$, where $(h\cdot\lambda_\zeta)(E)=\lambda_\zeta(h^{-1}\cdot E)$ for every integrable subset E of G/G_ζ ([3], Chap. 7, § 2, n°5). For $g\in G$, put $\dot{g}=g\cdot G_\zeta\in G/G_\zeta$.

Let L_ζ be the unitary representation of G_ζ on $L^2(G_\zeta,\nu_\zeta)\otimes\mathfrak{H}_{\rho(\zeta)}$ which is determined by $(L_\zeta(h)f)(g)=f(h^{-1}g)$ for $f\in L^2(G_\zeta,\nu_\zeta)\otimes\mathfrak{H}_{\rho(\zeta)}$ and $h\in G_\zeta$. Then $(G_\zeta\cdot\rho(\zeta), L_\zeta)$ is a covariant representation of (A, G_ζ). Let $(\Pi^\rho(\zeta), U^\rho(\zeta))$ be the covariant representation of (A, G) induced by $(G_\zeta\cdot\rho(\zeta), L_\zeta)$ with respect to the measure λ_ζ on G/G_ζ (c.f. [10], § 3). That is, let H_ζ^ρ be the space of all $L^2(G_\zeta,\nu_\zeta)\otimes\mathfrak{H}_{\rho(\zeta)}$-valued measurable functions η on G satisfying the conditions;

(1) $$\eta(st)=L_\zeta(t^{-1})\eta(s)\qquad\text{for}\quad s\in G,\ t\in G_\zeta,$$

(2) $$\int_{G/G_\zeta}\|\eta(s)\|^2d\lambda_\zeta(\dot{s})<+\infty.$$

The integral in (2) is well-defined by $\|\eta(st)\|=\|L_\zeta(t^{-1})\eta(s)\|=\|\eta(s)\|$ for $t\in G_\zeta$. For all $a\in A$ and $h\in G$, $\Pi^\rho(\zeta)(a)$ and $U^\rho(\zeta)(h)$ are operators on the Hilbert space H_ζ^ρ defined by $(\Pi^\rho(\zeta)(a)\eta)(g)=g\cdot G_\zeta\cdot\rho(\zeta)(a)\eta(g)$ and

$$(U^\rho(\zeta)(h)\eta)(g)=\chi_\zeta(h,\ h^{-1}\dot{g})^{-1/2}\eta(h^{-1}g) \qquad \text{for all } \eta\in H_\zeta^\rho \text{ and } g\in G.$$

Since G is a Polish topological group and G_ζ is a closed subgroup, there exists a Borel cross section Φ_ζ of G/G_ζ in G ([1], Proposition 3.2, p. 14). Then, there exists an isomorphism Ψ_ζ of the Hilbert space $L^2(G/G_\zeta,\ \lambda_\zeta)\otimes L^2(G_\zeta,\ \nu_\zeta)\otimes\mathfrak{H}_{\rho(\zeta)}$ onto the Hilbert space H_ζ^ρ defined by $\Psi_\zeta(f)(s)=L_\zeta(s^{-1}\Phi_\zeta(\dot{s}))f(\dot{s})$ for $f\in L^2(G/G_\zeta,\ \lambda_\zeta)\otimes L^2(G_\zeta,\nu_\zeta)\otimes\mathfrak{H}_{\rho(\zeta)}$ and $s\in G$ ([7], p. 110). If fields $\zeta\mapsto L^2(G/G_\zeta,\ \lambda_\zeta)$ and $\zeta\mapsto L^2(G_\zeta,\ \nu_\zeta)$ are μ_ρ-measurable fields of Hilbert spaces on $\hat{A}$, the field $\zeta\mapsto L^2(G/G_\zeta,\ \lambda_\zeta)\otimes L^2(G_\zeta,\ \nu_\zeta)\otimes\mathfrak{H}_{\rho(\zeta)}$ is μ_ρ-measurable. Then, by the isomorphisms (Ψ_ζ), the field $\zeta\mapsto H_\zeta^\rho$ is a μ_ρ-measurable field of Hilbert spaces.

DEFINITION 1.2. A C^*-dynamical system $(A,\ G)$ is called centrally measurable, if there exist, for every $\zeta\in\hat{A}$, measures λ_ζ and ν_ζ which have the following properties;

1°. λ_ζ is a non-zero quasi-invariant measure on G/G_ζ, and ν_ζ is a left invariant Haar measure on G_ζ;

2°. $\zeta\mapsto L^2(G/G_\zeta,\ \lambda_\zeta)$ and $\zeta\mapsto L^2(G_\zeta,\ \nu_\zeta)$ are μ_ρ-measurable fields of Hilbert spaces on $\hat{A}$;

3°. for every $\rho\in\operatorname{Rep} A$, $\zeta\mapsto\Pi^\rho(\zeta)$ and $\zeta\mapsto U^\rho(\zeta)$ are μ_ρ-measurable fields of operators on $\hat{A}$ with respect to the structure of the μ_ρ-measurable field $\zeta\mapsto H_\zeta^\rho$ whose construction is described just before.

This notion makes sense as we have the following proposition.

PROPOSITION 1.3. *For a C^*-dynamical system $(A,\ G)$, if there exists a closed subgroup H of G such that the stabilizer G_ζ is H for every $\zeta\in\hat{A}$, then $(A,\ G)$ is centrally measurable.*

PROOF. Let $\{V_n\}_{n=1}^\infty$ (resp. $\{W_m\}_{m=1}^\infty$) be a relatively compact open basis of H (resp. G/H), and χ_{V_n} (resp. χ_{W_n}) be the characteristic function of each set. Then $\{\zeta\mapsto\chi_{V_n}\}_{n=1}^\infty$ (resp. $\{\zeta\mapsto\chi_{W_m}\}_{m=1}^\infty$) forms a fundamental subset of the μ_ρ-measurable field $\zeta\mapsto L^2(H,\ \nu_H)$ (resp. $\zeta\mapsto L^2(G/H,\ \lambda_H)$) where ν_H is a Haar measure on H and λ_H is a quasi-invariant measure on G/H. Let $\{\zeta\mapsto x_l(\zeta)\}_{l=1}^\infty$ be a fundamental subset of μ_ρ-measurable field $\zeta\mapsto\mathfrak{H}_{\rho(\zeta)}$. Then $\{\zeta\mapsto\Psi_\zeta(\chi_{W_m}\otimes\chi_{V_n}\otimes x_l(\zeta))\}_{m,n,l=1}^\infty$ is a fundamental subset of the μ_ρ-measurable field $\zeta\mapsto H_\zeta^\rho$. For $a\in A$, put $f(\zeta,\ h)=(\rho(\zeta)(h^{-1}a)x_l(\zeta)\,|\,x_{l'}(\zeta))\chi_{V_n\cap V_{n'}}(h)$. Then the function $\zeta\mapsto f(\zeta,\ h)$ is μ_ρ-measurable on $\hat{A}$ and the function $h\mapsto f(\zeta,\ h)$ is continuous on $V_n\cap V_{n'}\subset H$. Therefore the function

$$\zeta\mapsto\int_H f(\zeta,\ h)d\nu_H(h)=(H\cdot\rho(\zeta)(a)(\chi_{V_n}\otimes x_l(\zeta))\,|\,\chi_{V_{n'}}\otimes x_{l'}(\zeta))$$

is μ_ρ-measurable on $\hat{A}$. We use the notation mentioned before Definition 1.2

omitting the index ζ. As $\dot{s}\mapsto\Phi(\dot{s})$ is a Borel cross section of G/H into G, there exist a set of λ_H-measure zero $N\subset\overline{W_m\cap W_{m'}}$ and a sequence of compact sets $\{K_i\}$ which is a partition of $\overline{W_m\cap W_{m'}}-N$ such that $\dot{s}\mapsto\Phi(\dot{s})$ is continuous on K_i for all i. The function $\dot{s}\mapsto(\Phi(\dot{s})\cdot H\cdot\rho(\zeta)(a)(\chi_{V_n}\otimes x_l(\zeta))|\chi_{V_{n'}}\otimes x_{l'}(\zeta))$ is then continuous on K_i for all i and for all $\zeta\in\hat{A}$. Therefore the function

$$\zeta\mapsto\int_{K_i}(\Phi(\dot{s})\cdot H\cdot\rho(\zeta)(a)(\chi_{V_n}\otimes x_l(\zeta))|\chi_{V_{n'}}\otimes x_{l'}(\zeta))d\lambda_H(\dot{s})$$

is μ_ρ-measurable on $\hat{A}$. Hence, the function

$$\zeta\mapsto\sum_{i=1}^{\infty}\int_{K_i}(\Phi(\dot{s})\cdot H\cdot\rho(\zeta)(a)(\chi_{V_n}\otimes x_l(\zeta))|\chi_{V_{n'}}\otimes x_{l'}(\zeta))d\lambda_H(\dot{s})$$

$$=(\Psi^{-1}(\Pi^\rho(\zeta)(a)\Psi(\chi_{W_m}\otimes\chi_{V_n}\otimes x_l(\zeta)))|\Psi^{-1}\circ\Psi(\chi_{W_{m'}}\otimes\chi_{V_{n'}}\otimes x_{l'}(\zeta)))$$

is μ_ρ-measurable. It follows that the field $\zeta\mapsto\Pi^\rho(\zeta)$ is μ_ρ-measurable on $\hat{A}$. It is clear that the field $\zeta\mapsto U^\rho(\zeta)$ is μ_ρ-measurable on $\hat{A}$, as $\nu_\zeta=\nu_H$ and $\lambda_\zeta=\lambda_H$ for all $\zeta\in\hat{A}$. Q.E.D.

From now on, we assume that (A, G) is centrally measurable. Then we can define $\mathfrak{R}_\rho$, Π^ρ and U^ρ as follows;

$$\mathfrak{R}_\rho=\int_{\hat{A}}^{\oplus}H_\zeta^\rho d\mu_\rho(\zeta);$$

$$\Pi^\rho=\int_{\hat{A}}^{\oplus}\Pi^\rho(\zeta)d\mu_\rho(\zeta);$$

$$U^\rho=\int_{\hat{A}}^{\oplus}U^\rho(\zeta)d\mu_\rho(\zeta).$$

Since $(\Pi^\rho(\zeta), U^\rho(\zeta))$ is a covariant representation of (A, G) on H_ζ^ρ for every $\zeta\in\hat{A}$, (Π^ρ, U^ρ) is a covariant representation of (A, G) on $\mathfrak{R}_\rho$. $L^1(G, A)$ denotes the set of all Bochner integrable A-valued measurable functions on G which is the Banach *-algebra with the product and the involution defined by

$$(x*y)(g)=\int_G x(h)h\cdot y(h^{-1}g)d\nu(h)$$

$$x^*(g)=\Delta(g)^{-1}g\cdot(x(g^{-1}))^*$$

for all x, $y\in L^1(G, A)$ and $g\in G$, where Δ is the modular function of G with respect to ν. The crossed product $C^*(G, A)$ of A by G is the enveloping C^*-algebra of $L^1(G, A)$ (c.f. [**10**], p. 273).

DEFINITION 1.4. Let Cent ρ be the unique representation of the crossed product $C^*(G, A)$ such that

$$\mathrm{Cent}\ \rho(x)=\int_G \Pi^{\rho}(x(g))U^{\rho}(g)d\nu(g)$$

for all $x\in L^1(G, A)$ (c. f. [6], Theorem 3). The representation Cent ρ is called the centrally extended representation of $C^*(G, A)$ for ρ.

For $x\in L^1(G, A)$, define $\|x\|_*=\sup_{\rho\in\mathrm{Rep}\,A}\|\mathrm{Cent}\ \rho(x)\|$.

PROPOSITION 1.5. *The above* $\|\cdot\|_*$ *is a norm on* $L^1(G, A)$.

PROOF. It is sufficient to show that $\|x\|_*=0$ implies $x=0$. Let $\{W_n\}$ be a countable decreasing fundamental system of relatively compact open neighborhoods of the unit $e\in G$. Let ϕ_n be a nonnegative real-valued continuous function on G such that $\mathrm{supp}\,\phi_n\subset\overline{W}_n$ and $\|\phi_n\|_1=1$, where $\|\cdot\|_1$ denotes the L^1-norm. Let $\{u_n\}$ be an approximate unit of A, and $u_n\phi_n$ be an element of $L^1(G, A)$ defined by $(u_n\phi_n)(g)=\phi_n(g)u_n$ for $g\in G$. The function $x*(u_n\phi_n)$ is continuous on G, and we have $\lim_{n\to\infty}\|x*(u_n\phi_n)-x\|_1=0$ for every $x\in L^1(G, A)$ Suppose now $\|x\|_*=0$. Then, for every $\rho\in\mathrm{Rep}\,A$, we get

$$\|\mathrm{Cent}\ \rho(x*(u_n\phi_n))\|\leqq\|\mathrm{Cent}\ \rho(x)\|\cdot\|\mathrm{Cent}\ \rho(u_n\phi_n)\|=0\,.$$

Put $y=x*(u_n\phi_n)$. Note that $y(g)$ is continuous. We shall show $y=0$.

Let ρ be a factor representation of A and ζ be the quasi-equivalence class of ρ. Then we have Cent $\rho(y)=0$. As the group is second countable, we have a countable family Υ of continuous functions with compact support on G_ζ in which we can find, corresponding to each pair of a compact set K and a relatively compact open set U containing K, a function ψ taking 1 on K, 0 outside U and the values between 0 and 1 everywhere. Let $\dot{\Upsilon}$ be a similar family with respect to G/G_ζ instead of G_ζ. In what follows, functions ϕ_1, ϕ_2 are chosen from $\dot{\Upsilon}$ and ψ_1, ψ_2 are chosen from Υ. We have now, for $\eta\in\mathfrak{H}_\rho$,

$$0=(\mathrm{Cent}\ \rho(y)\Psi_\zeta(\phi_1\otimes\psi_1\otimes\eta)\,|\,\Psi_\zeta(\phi_2\otimes\psi_2\otimes\eta))$$

$$=\int_G\int_{G/G_\zeta}\chi_\zeta(g,\ g^{-1}\dot{h})^{-1/2}(\Phi_\zeta(\dot{h})G_\zeta\cdot\rho(y(g))L_\zeta(\Phi_\zeta(\dot{h})^{-1}g\Phi_\zeta(g^{-1}\dot{h}))(\psi_1\otimes\eta)\,|\,\psi_2\otimes\eta)$$

$$\times\phi_1(g^{-1}\dot{h})\overline{\phi_2(\dot{h})}d\lambda_\zeta(\dot{h})d\nu(g)\,.$$

Then we have

$$(1)\quad 0=\int_G\chi_\zeta(g,\ g^{-1}\dot{h})^{-1/2}(\Phi_\zeta(\dot{h})G_\zeta\cdot\rho(y(g))L_\zeta(\Phi_\zeta(\dot{h})^{-1}g\Phi_\zeta(g^{-1}\dot{h}))(\psi_1\otimes\eta)\,|\,\psi_2\otimes\eta)$$

$$\times\phi_1(g^{-1}\dot{h})d\nu(g)$$

$$=\int_G\int_{G_\zeta}\chi_\zeta(g,\ g^{-1}\dot{h})^{-1/2}(\Phi_\zeta(\dot{h})s\cdot\rho(y(g))\eta\,|\,\eta)\phi_1(g^{-1}\dot{h})\psi_1(\Phi_\zeta(g^{-1}\dot{h})^{-1}g^{-1}\Phi_\zeta(\dot{h})s)$$

$$\times\overline{\psi_2(s)}d\nu_\zeta(s)d\nu(g),$$

for $\dot{h}\in G/G_\zeta$ except for a λ_ζ-null set which can be taken common for all $\phi_1\in\dot{\Upsilon}$, ψ_1, $\psi_2\in\Upsilon$. By [9], Lemma 1.1, we can assume that, for each compact subset K of G, $\Phi_\zeta(\dot{K})$ has a compact closure. Put $\operatorname{supp}\phi_1=K$, $\operatorname{supp}\psi_1=K_1$ and $\operatorname{supp}\psi_2=K_2$. Put

$$\alpha(g, h)=\chi_\zeta(g, g^{-1}\dot{h})^{-1/2},\ \beta(g, h, s)=(\Phi_\zeta(\dot{h})s\cdot\rho(y(g))\eta\,|\,\eta)$$

and

$$t(g, h, s)=\Phi_\zeta(g^{-1}\dot{h})^{-1}g^{-1}\Phi_\zeta(\dot{h})s.$$

Consider the following function on K_2;

$$s\mapsto\int_G\alpha(g, h)\beta(g, h, s)\phi_1(g^{-1}\dot{h})\psi_1(t(g, h, s))d\nu(g).$$

The integrand vanishes outside the compact set $\Phi_\zeta(\dot{h})K_2K_1^{-1}\overline{\Phi_\zeta(K)}^{-1}$. Since $\psi_1(ts)$ is an equi-continuous function of s for

$$t=\Phi_\zeta(g^{-1}\dot{h})^{-1}g^{-1}\Phi_\zeta(\dot{h})\in\overline{\Phi_\zeta(K)}^{-1}(\Phi_\zeta(\dot{h})K_2K_1^{-1}\overline{\Phi_\zeta(K)}^{-1})^{-1}\Phi_\zeta(\dot{h}),$$

the function we are considering is continuous on K_2. As this is true for any $\psi_2\in\Upsilon$, (1) implies that

$$\int_G\alpha(g, h)\beta(g, h, s)\phi_1(g^{-1}\dot{h})\psi_1(t(g, h, s))d\nu(g)=0 \tag{2}$$

for almost all $\dot{h}\in G/G_\zeta$ and all $s\in G_\zeta$.

Suppose that there exist $g_0\in G$, $\dot{h}\in G/G_\zeta$ and $s\in G_\zeta$ for which (2) holds such that $\beta(g_0, h, s)=(\Phi_\zeta(\dot{h})s\cdot\rho(y(g_0))\eta\,|\,\eta)\neq 0$. We may suppose, without loss of generality, that there exist $\delta>0$, and a relatively compact open neighborhood U of g_0 such that $\operatorname{Re}\beta(g, h, s)>\delta$ for all $g\in U$. Take a compact set $K\subset\{g^{-1}\dot{h}\in G/G_\zeta;\ g\in U\}$ such that Φ_ζ is continuous on K, and then an element $\dot{g}_1\in K$ such that, for every neighborhood O of $\dot{g}_1$ in G/G_ζ, $O\cap K$ is not of λ_ζ-measure zero. Let $g_2\in U$ be such that $g_2^{-1}\dot{h}=\dot{g}_1$ and $t_0\in G_\zeta$ be such that $g_2=\Phi_\zeta(\dot{h})st_0^{-1}\Phi_\zeta(\dot{g}_1)^{-1}$. Then we can choose a neighborhood V_1 of $\Phi_\zeta(\dot{g}_1)$ in G and a relatively compact neighborhood V_2 of t_0 in G_ζ, which satisfy $\Phi_\zeta(\dot{h})sV_2^{-1}V_1^{-1}\subset U$. Since Φ_ζ is continuous on K, there exists a neighborhood O of $\dot{g}_1$ in G/G_ζ such that $\Phi_\zeta(O\cap K)\subset V_1$ and that $\Phi_\zeta(O)$ has a compact closure in G. As $\lambda_\zeta(O\cap K)>0$ and $\nu_\zeta(V_2)>0$, we have $\nu(\Phi_\zeta(\dot{h})sV_2^{-1}\Phi_\zeta(O\cap K)^{-1})>0$. Let $\chi_{O\cap K}$ (resp. χ_{V_2}) be the characteristic function of $O\cap K$ (resp. V_2). We have

$$\begin{aligned}(3)\quad &\int_G\alpha(g, h)\operatorname{Re}\beta(g, h, s)\chi_{O\cap K}(g^{-1}\dot{h})\chi_{V_2}(t(g, h, s))d\nu(g)\\ &\geqq\inf\{\alpha(g, h);\ g\in\Phi_\zeta(\dot{h})sV_2^{-1}\Phi_\zeta(O\cap K)^{-1}\}\,\delta\cdot\nu(\Phi_\zeta(\dot{h})sV_2^{-1}\Phi_\zeta(O\cap K)^{-1})\\ &>0.\end{aligned}$$

There exist then families (ϕ_n) of functions in $\dot{\Upsilon}$ and (ψ_n) in Υ such that, as $n\rightarrow\infty$,

$$\int_G \alpha(g, h)\,\mathrm{Re}\,\beta(g, h, s)\phi_n(g^{-1}\dot{h})\psi_n(t(g, h, s))d\nu(g)$$

$$\rightarrow\int_G \alpha(g, h)\,\mathrm{Re}\,\beta(g, h, s)\chi_{O\cap K}(g^{-1}\dot{h})\chi_{V_2}(t(g, h, s))d\nu(g).$$

This is absurd under the conditions (2) and (3). Therefore we have $(\Phi_\zeta(\dot{h})s\cdot\rho(y(g))\eta\,|\,\eta)=0$ for all $g\in G$, almost all $\dot{h}\in G/G_\zeta$ and all $s\in G_\zeta$.

Since the inverse image of a set of λ_ζ-measure zero under the canonical map of G onto G/G_ζ is of ν-measure zero ([3], Chap. 7, § 2, n°5, Theorem 1), we have $(t\cdot\rho(y(g))\eta\,|\,\eta)=0$ for almost all $t\in G$. As the complement of a set of ν-measure zero is dense in G, we have $(\rho(y(g))\eta\,|\,\eta)=0$. So we conclude that $\rho(y(g))=0$ for all $g\in G$ and $\rho\in\mathrm{Fac}\,A$, that is, $x*(u_n\phi_n)=y=0$.

We hence have seen that if $\|x\|_*=0$ then $x*(u_n\phi_n)=0$ for any n. As $\lim x*(u_n\phi_n)=x$, this implies $x=0$, which was to be established. Q. E. D.

By Proposition 1.5, we have the following result.

THEOREM 1.6. *Let $G*A$ be the completion of $L^1(G, A)$ by the norm $\|\cdot\|_*$. Then $G*A$ is a C^*-algebra.*

REMARK 1.7. The C^*-algebra $G*A$ is called the quasi-reduced crossed product of A by G, and the norm $\|\cdot\|_*$ is called the quasi-reduced norm.

For a representation ρ of A, $\mathrm{Ind}\,\rho$ denotes the representation of $C^*(G, A)$ induced from the covariant representation $(\rho, id.)$ of $(A, \{e\})$, where $id.$ is the trivial representation of the trivial group $\{e\}$. The reduced norm $\|\cdot\|_r$ is the norm on $L^1(G, A)$ defined by, for $x\in L^1(G, A)$, $\|x\|_r=\sup_{\rho\in\mathrm{Rep}\,A}\|\mathrm{Ind}\,\rho(x)\|$. The reduced crossed product $C_r^*(G, A)$ of A by G is the completion of $L^1(G, A)$ by the reduced norm (c. f. [11], p. 171). Suppose that G is freely acting on $\hat{A}$, that is, $G_\zeta=\{e\}$ for all $\zeta\in\hat{A}$. Then we have $\mathrm{Cent}\,\rho=\mathrm{Ind}\,\rho$ and $G*A=C_r^*(G, A)$.

§ 2. Some properties of centrally extended representations.

In this section, we study some properties of centrally extended representations. Especially we show that, if representations ρ_1 and ρ_2 of A are quasi-equivalent, then $\mathrm{Cent}\,\rho_1$ and $\mathrm{Cent}\,\rho_2$ are quasi-equivalent.

We assume throughout that (A, G) is centrally measurable. Note that, for $\eta\in L^2(G/G_\zeta, \lambda_\zeta)\otimes L^2(G_\zeta, \nu_\zeta)\otimes\mathfrak{H}_{\rho(\zeta)}$, we get

$$\Psi_\zeta^{-1}(\Pi^\rho(\zeta)(a)\Psi_\zeta(\eta))(\dot{s})=\Phi_\zeta(\dot{s})G_\zeta\cdot\rho(\zeta)(a)\eta(\dot{s})$$

and

$$\Psi_\zeta^{-1}(U^\rho(\zeta)(h)\Psi_\zeta(\eta))(\dot{s})=\chi_\zeta(h, h^{-1}\dot{s})^{-1/2}L_\zeta(\Phi_\zeta(\dot{s})^{-1}h\Phi_\zeta(h^{-1}\dot{s}))\eta(h^{-1}\dot{s}).$$

PROPOSITION 2.1. *For* ρ, $\pi\in\operatorname{Rep} A$, *if* $\rho\cong\pi$, $\operatorname{Cent}\rho$ *is equivalent to* $\operatorname{Cent}\pi$.

PROOF. Let $\rho=\int_{\hat{A}}^{\oplus}\rho(\zeta)d\mu_{\rho}(\zeta)$ and $\pi=\int_{\hat{A}}^{\oplus}\pi(\zeta)d\mu_{\pi}(\zeta)$ be the central decomposition of ρ and π. Then μ_{ρ} and μ_{π} are equivalent, and $\rho(\zeta)\cong\pi(\zeta)$ for almost all $\zeta\in\hat{A}$. As A is separable, there exists a μ_{π}-measurable field of unitary operators $\zeta\mapsto v_0(\zeta)\in\mathfrak{L}(\mathfrak{H}_{\rho(\zeta)}, \mathfrak{H}_{\pi(\zeta)})$ such that $\rho(\zeta)(a)=v_0(\zeta)^{-1}\pi(\zeta)(a)v_0(\zeta)$ for $a\in A$ and almost all $\zeta\in\hat{A}$ ([5], 8.4.2.). If we put $v(\zeta)=I_{L^2(G_{\zeta},\nu_{\zeta})}\otimes v_0(\zeta)$, we have $G_{\zeta}\cdot\rho(\zeta)(a)=v(\zeta)^{-1}G_{\zeta}\cdot\pi(\zeta)(a)v(\zeta)$ for all $a\in A$. Put $V(\zeta)=(d\mu_{\rho}/d\mu_{\pi})^{1/2}(\zeta)(I_{L^2(G/G_{\zeta},\lambda_{\zeta})}\otimes v(\zeta))$. As the field $\zeta\mapsto V(\zeta)$ is measurable on $\hat{A}$, we can define V by $V=\int_{\hat{A}}^{\oplus}V(\zeta)d\mu_{\rho}(\zeta)$. Then V is a unitary operator of $\int_{\hat{A}}^{\oplus}L^2(G/G_{\zeta}, \lambda_{\zeta})\otimes L^2(G_{\zeta}, \nu_{\zeta})\otimes\mathfrak{H}_{\rho(\zeta)}d\mu_{\rho}(\zeta)$ onto $\int_{\hat{A}}^{\oplus}L^2(G/G_{\zeta}, \lambda_{\zeta})\otimes L^2(G_{\zeta}, \nu_{\zeta})\otimes\mathfrak{H}_{\pi(\zeta)}d\mu_{\pi}(\zeta)$. Let Ψ_{ζ}^{ρ} be the isomorphism of $L^2(G/G_{\zeta}, \lambda_{\zeta})\otimes L^2(G_{\zeta}, \nu_{\zeta})\otimes\mathfrak{H}_{\rho(\zeta)}$ onto H_{ζ}^{ρ} and Ψ_{ζ}^{π} be the similar isomorphism with respect to π. Then $V_0=\int_{\hat{A}}^{\oplus}\Psi_{\zeta}^{\pi}\circ V(\zeta)\circ\Psi_{\zeta}^{\rho^{-1}}d\mu_{\rho}(\zeta)$ is a unitary operator of $\mathfrak{K}_{\rho}$ onto $\mathfrak{K}_{\pi}$. We find by an easy computation that $\Pi^{\rho}(a)=V_0^{-1}\Pi^{\pi}(a)V_0$ for $a\in A$, and $U^{\rho}(h)=V_0^{-1}U^{\pi}(h)V_0$ for $h\in G$, so that $\operatorname{Cent}\rho(x)=V_0^{-1}\operatorname{Cent}\pi(x)V_0$ for $x\in G*A$. Q.E.D.

LEMMA 2.2. *For* $\rho\in\operatorname{Rep} A$, *let* E *be a projection of* $\rho(A)'$. *Then there exists a projection* $\tilde{E}$ *of* $(\operatorname{Cent}\rho(G*A))'$ *such that* $\operatorname{Cent}(\rho_E)\cong(\operatorname{Cent}\rho)_{\tilde{E}}$.

PROOF. Let $\rho=\int_{\hat{A}}^{\oplus}\rho(\zeta)d\mu_{\rho}(\zeta)$ be the central decomposition of ρ. Let $\mathfrak{D}_{\rho}$ be the algebra of diagonalizable operators of $\int_{\hat{A}}^{\oplus}\mathfrak{H}_{\rho(\zeta)}d\mu_{\rho}(\zeta)$, and $Z_{\rho(A)'}$ be the center of $\rho(A)'$. Since $\mathfrak{D}_{\rho}=Z_{\rho(A)'}\subset\rho(A)''$ and A is separable, we have

$$\rho(A)''=\int_{\hat{A}}^{\oplus}\rho(\zeta)(A)''d\mu_{\rho}(\zeta)$$

([5], 8.4.1). As μ_{ρ} is standard on $\hat{A}$, $\rho(A)'=\int_{\hat{A}}^{\oplus}\rho(\zeta)(A)'d\mu_{\rho}(\zeta)$ ([4], Chap. II, § 3, Theorem 4). Therefore there exists a μ_{ρ}-measurable field $\zeta\mapsto E(\zeta)$ on $\hat{A}$ such that each $E(\zeta)$ is a projection of $\rho(\zeta)(A)'$ and $E=\int_{\hat{A}}^{\oplus}E(\zeta)d\mu_{\rho}(\zeta)$. Let $C(E)\in Z_{\rho(A)'}$ be the central support of E in $Z_{\rho(A)'}$. By $\mathfrak{D}_{\rho}=Z_{\rho(A)'}$, there exists a μ_{ρ}-measurable set F of $\hat{A}$ such that $C(E)=\int_{\hat{A}}^{\oplus}\chi_F(\zeta)I_{\mathfrak{H}_{\rho(\zeta)}}d\mu_{\rho}(\zeta)$. We may consider $\rho_E=\int_{\hat{A}}^{\oplus}\rho(\zeta)_{E(\zeta)}d(\chi_F\mu_{\rho})(\zeta)$ as the central decomposition of ρ_E. Then, we have $\Pi^{\rho_E}=\int_{\hat{A}}^{\oplus}\Pi^{\rho_E}(\zeta)d(\chi_F\mu_{\rho})(\zeta)$ and $U^{\rho_E}=\int_{\hat{A}}^{\oplus}U^{\rho_E}(\zeta)d(\chi_F\mu_{\rho})(\zeta)$.

Let next M_ζ be the left regular representation of G_ζ and denote by K_ζ the space of all $L^2(G_\zeta, \nu_\zeta)$-valued measurable function f on G satisfying the conditions;

(1) $$f(st)=M_\zeta(t^{-1})f(s) \quad \text{for } s\in G \text{ and } t\in G_\zeta;$$

(2) $$\int_{G/G_\zeta} \|f(s)\|^2 d\lambda_\zeta(\dot{s})<+\infty .$$

Then we have $H_\zeta^\rho=K_\zeta\otimes\mathfrak{H}_{\rho(\zeta)}$ and $L_\zeta=M_\zeta\otimes I_{\mathfrak{H}_{\rho(\zeta)}}$. By the definition, for $\eta\in H_\zeta^{\rho_E}=K_\zeta\otimes E(\zeta)\mathfrak{H}_{\rho(\zeta)}$, we have

$$\Pi^{\rho_E}(\zeta)(a)\eta=\Pi^\rho(\zeta)(a)(I_{K_\zeta}\otimes E(\zeta))\eta$$

and

$$U^{\rho_E}(\zeta)(h)\eta=U^\rho(\zeta)(h)(I_{K_\zeta}\otimes E(\zeta))\eta .$$

Now we define a projection $\tilde{E}$ on $\mathfrak{K}_\rho$ by $\tilde{E}=\int_{\hat{A}}^{\oplus}(I_{K_\zeta}\otimes E(\zeta))d(\chi_F\mu_\rho)(\zeta)$. Since $\chi_F(\zeta)E(\zeta)=E(\zeta)$, for almost all $\zeta\in\hat{A}$, we have $\tilde{E}=\int_{\hat{A}}^{\oplus}(I_{K_\zeta}\otimes E(\zeta))d\mu_\rho(\zeta)$. We also have that $\tilde{E}\in\text{Cent}\,\rho(G*A)'$. Since $\Pi^{\rho_E}=\Pi^\rho\tilde{E}$ and $U^{\rho_E}=U^\rho\tilde{E}$, we can conclude that $\text{Cent}\,(\rho_E)\cong(\text{Cent}\ \rho)_{\tilde{E}}$. Q.E.D.

PROPOSITION 2.3. *For $\rho_i\in\text{Rep}\,A$, let $\rho=\oplus_{i=1}^\infty\rho_i$. Then* Cent ρ *is equivalent to* $\oplus_{i=1}^\infty$ Cent ρ_i.

PROOF. There exists a family of projections $\{E_i\}_{i=1}^\infty$ of $\rho(A)'$ such that $E_iE_j=0$ for $i\neq j$ and $\sum_{i=1}^\infty E_i=I_{\mathfrak{H}_\rho}$, and such that $\rho_i\cong\rho_{E_i}$ for each i. By Proposition 2.1, we have Cent $\rho_i\cong\text{Cent}\,(\rho_{E_i})$. By Lemma 2.2, for each i, there exists a projection $\tilde{E}_i$ of Cent $\rho(G*A)'$ such that Cent $(\rho_{E_i})\cong(\text{Cent}\ \rho)_{\tilde{E}_i}$. Thus we get $\oplus_{i=1}^\infty$ Cent $\rho_i\cong\oplus_{i=1}^\infty(\text{Cent}\ \rho)_{\tilde{E}_i}$. Since $\sum_{i=1}^\infty\tilde{E}_i=I_{\mathfrak{K}_\rho}$, we have

$$\oplus_{i=1}^\infty\text{Cent}\ \rho_i\cong(\text{Cent}\ \rho)_{\Sigma\tilde{E}_i}=\text{Cent}\ \rho.$$

Q.E.D.

THEOREM 2.4. *For ρ_1, $\rho_2\in\text{Rep}\,A$, if $\rho_1\approx\rho_2$,* Cent ρ_1 *is quasi-equivalent to* Cent ρ_2.

PROOF. As $\mathfrak{H}_{\rho_1}$ and $\mathfrak{H}_{\rho_2}$ are separable, we have $\aleph_0\cdot\rho_1\cong\aleph_0\cdot\rho_2$ ([5], 5.3.8). By Propositions 2.1 and 2.3, this means that

$$\aleph_0\cdot(\text{Cent}\ \rho_1)\cong\text{Cent}\,(\aleph_0\cdot\rho_1)\cong\text{Cent}\,(\aleph_0\cdot\rho_2)\cong\aleph_0\cdot(\text{Cent}\ \rho_2)\,.$$

Thus we have Cent $\rho_1\approx$Cent ρ_2. Q.E.D.

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