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A finite difference approach to the number of peaks of solutions for semilinear parabolic problems

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Introduction.

In this paper we study the number of peaks of solutions for one-dimensional semilinear parabolic problems by a finite difference method. As a model problem let us consider the equation in $u=u(x, t)$.

$$(0.1)\qquad \begin{cases} u_t=u_{xx}+f(u), & 0<x<1 \quad \text{and} \quad t>0, \\ u_x(0, t)=u_x(1, t)=0, & t>0, \\ u(x, 0)=u^0(x), & 0<x<1, \end{cases}$$

where f is a smooth function. We have two purposes; one is to know how the number of peaks of $u(\cdot, t)$ changes as t passes, the other is to present a finite difference scheme for (0.1) whose solution has the same behavior as the exact one concerning the number of peaks. Our result for the former is that the number of peaks is monotonically decreasing. We do not prove it independently of the latter. But we first attack the latter and present a finite difference scheme whose solution has the monotonically decreasing property with regard to the number of peaks. After that we prove the above result by the limit process.

Thus our main effort is devoted to constructing a finite difference scheme whose solution has the property mentioned above under appropriate conditions. Of course, it should also be shown that the finite difference solutions converge to the exact one as h and τ (space mesh and time mesh) tend to zero. In our scheme for (0.1), roughly speaking, the condition $\tau/h^2 \leqq 1/2$ yields the convergence, while the condition $\tau/h^2<1/4$ leads to the property in question (Remark 2.7).

As a simple application of our result let us show a consequence relating to the stability of equilibrium solution. Chafee [1] and Matano [5] showed that every nonconstant equilibrium solution of (0.1) is unstable, while Ito [3] proved that for each (unstable) equilibrium solution there exists a stable manifold such that the solution of (0.1) starting from any function on the manifold converges to the equilibrium solution. The monotonically decreasing property of the number of peaks leads to a characterization of the stable manifold: The number of peaks for the function on a stable manifold is not less than that of the corresponding equilibrium solution.

For systems of semilinear equations it does not in general hold that the number of peaks is monotonically decreasing. This fact may be observed in the result of numerical experiments performed by Mimura [6]. He proposed a system of diffusion equations with autonomous nonlinear couplings as a planktonic prey and predator model in order to obtain a nonconstant stable equilibrium solution with large amplitude. His numerical experiments for this system show that the number of peaks may increase for each component of solutions in some cases.

The plan of this paper is as follows. In § 1 we define the number of peaks of a function. Then we state the result for semilinear parabolic problems with homogeneous Neumann boundary conditions. The proof is given in the latter part of the next section. In § 2 we present a finite difference scheme and show that, under appropriate conditions, the difference solution possesses the property that the number of peaks is monotonically decreasing. In that proof we use a theorem concerning a property of a class of linear operators in the finite dimensional space, which plays a key role there. This theorem is proved in § 3. In § 4 we consider the case of homogeneous Dirichlet boundary conditions. After imposing a restriction to the inhomogeneous term, we show that the same property still holds. The case of the third boundary conditions is also noted. In § 5 we discuss the case when solutions blow up.

We use the following notation throughout this paper:

$$\|a\|_{\bar{Q}}=\max\{|a(x, t)|\,;\,(x, t)\in\bar{Q}\},$$

$$\|a\|_{[0,1]}=\max\{|a(x)|\,;\,x\in[0, 1]\},$$

$$\|a\|_{[0,T]}=\max\{|a(t)|\,;\,t\in[0, T]\},$$

for continuous functions a on each closed set. The same symbol $\|a\|$ is used if there is no fear of confusion. We also use

$$\langle i, j\rangle=\{i, i+1, \cdots, j\} \quad \text{for integers} \quad i<j.$$

§ 1. The number of peaks of solutions.

Let T be a positive number. Consider the following semilinear parabolic equation in $u=u(x, t)$,

$$(1.1)\quad \begin{cases} u_t=a(x,t)u_{xx}+b(x,t)u_x+f(t,u) & \text{in } Q=(0,1)\times(0,T), \\ u_x(0,t)=u_x(1,t)=0, & t\in(0,T), \\ u(x,0)=u^0(x), & x\in(0,1), \end{cases}$$

where $u^0(x)$ is a given continuous function.

ASSUMPTION 1. a, b and f satisfy the following conditions:

(i) a is continuous in $\bar{Q}$ together with its first derivatives with respect to x and t. There exists a positive number a_0 such that $a\geqq a_0$ in $\bar{Q}$.

(ii) b is continuous in $\bar{Q}$ together with its first derivative with respect to x.

(iii) f is continuous in $[0,T]\times\boldsymbol{R}$ together with its first and second derivatives with respect to u. There exists a real number M_0 such that $\partial f/\partial u\leqq M_0$ in $[0,T]\times\boldsymbol{R}$.

REMARK 1.1. We divide f into two parts as follows:

$$f(t,u)=f_0(t)+f_1(t,u)u\,,$$

where $f_0(t)=f(t,0)$ and $f_1(t,u)=\{f(t,u)-f(t,0)\}/u$. By virtue of Assumption 1 f_0, f_1 and $\partial f_1/\partial u$ are continuous and it holds that $f_1(t,u)\leqq M_0$ in $[0,T]\times\boldsymbol{R}$.

Let $u(x,t)$ be the solution of (1.1). Considering $u=\{u(t)\}$, $t\in[0,T]$, as a one-parameter family of t, we observe how "the number of peaks" of $u(t)$ changes as t passes. Our result is that the number is monotonically decreasing. Before stating the theorem, we give the definition of the number of peaks for a function belonging to $C^1[0,1]$.

At first we assign two integers $N_{\pm}(p)$ to each continuous function p; $N_+(p)$ (resp. $N_-(p)$) is the number of those zeroes and zero-intervals of p where p changes its sign from positive (resp. negative) to negative (resp. positive). The detailed definition of $N_{\pm}(p)$ is as follows. Let $p(x)$ be a continuous function defined on $[0,1]$. When p is nonnegative or nonpositive on $[0,1]$, we define $N_+(p)=N_-(p)=0$. Otherwise, put $A_+=\{x\,;x\in[0,1],\ p(x)>0\}$, $A_{+,0}=\{x\,;x\in[0,1],\ p(x)\geqq 0\}$. Let m be the number of those connected components I of $A_{+,0}$ such that $I\cap A_+\neq\emptyset$ and let $\tilde{A}_+$ be the union of those I. When m is not a finite number, we define $N_+(p)=N_-(p)=+\infty$. When m is finite, we define

$$N_+(p)=\begin{cases} m-1 & \text{if } 1\in\tilde{A}_+, \\ m & \text{otherwise}, \end{cases} \quad\text{and}$$

$$N_-(p)=\begin{cases} m-1 & \text{if } 0\in\tilde{A}_+, \\ m & \text{otherwise}. \end{cases}$$

For $u(x)\in C^1[0,1]$ the number of peaks, $\sharp_p(u)$, is defined by $\sharp_p(u)=N_+(du/dx)$.

Similarly the number of valleys, $\sharp_v(u)$, is defined by $\sharp_v(u)=N_-(du/dx)$.

THEOREM 1.2. *Suppose Assumption* 1 *and* $u^0\in C^1[0, 1]$. *Then, equation* (1.1) *has a unique solution* $u(t)\in C^1[0, 1]$, $t\in[0, T]$, *and it holds that*

$$(1.2)\qquad \sharp_p(u(t))\leqq\sharp_p(u^0),\quad \sharp_v(u(t))\leqq\sharp_v(u^0)\qquad \text{for}\quad t\in[0, T].$$

COROLLARY 1.3. *Under the same assumptions as Theorem* 1.2, $\sharp_p(u(t))$ *and* $\sharp_v(u(t))$ *are monotonically decreasing.*

Corollary 1.3 is a direct consequence of Theorem 1.2. As stated in the introduction, we shall prove Theorem 1.2 after constructing a finite difference scheme for (1.1) whose solution satisfies the same property as (1.2). The complete proof is given in the following section.

Here we note that (1.2) fails if the term f in (1.1) depends on x. For example, consider the heat equation (i.e., $a\equiv1$, $b\equiv0$) with an inhomogeneous term $f(x, t)=1/2-(1/2+2\pi^2t)\cos 2\pi x$. Then $u(x, t)=t\sin^2\pi x$ is the solution corresponding to the initial value $u^0=0$. Therefore we have $\sharp_p(u^0)=0$ and $\sharp_p(u(t))=1$ for $t>0$, so that (1.2) fails.

§ 2. A finite difference approximation.

In this section we approximate (1.1) by a finite difference scheme and prove that the approximate solutions converge to the exact one. Furthermore we show that under appropriate conditions the difference solution possesses the property that the number of peaks is monotonically decreasing. After that the proof of Theorem 1.2 is given.

We discretize $\bar{Q}$ by a (h, τ)-rectangular net, where $h=1/N$ (N is a natural number) is a space mesh and $\tau>0$ is a time mesh. Put $J=\{1/2, 3/2, \cdots, N-1/2\}$. Our grid points consist of $(x_j, k\tau)$, $x_j=jh$, $j\in J$, $k=0, \cdots, N_T$ $(=[T/\tau])$. We seek a net function $u_h(x_j, k\tau)=u_h^k(x_j)$ satisfying

$$(2.1)\qquad \begin{cases} \{u_h^{k+1}(x_j)-u_h^k(x_j)\}/\tau=a_j^k\Delta_h u_h^k(x_j)+b_j^kD_hu_h^k(x_j)+f_0(k\tau) \\ \qquad\qquad\qquad\qquad\qquad +f_1(k\tau, u_h^k(x_j))u_h^{k+1}(x_j), \\ u_h^k(-h/2)=u_h^k(h/2), \qquad u_h^k(1+h/2)=u_h^k(1-h/2), \\ u_h^0(x_j)=u^0(x_j) \qquad \text{for}\quad j\in J \quad\text{and}\quad k\in\langle 0, N_T-1\rangle, \end{cases}$$

where $a_j^k=a(x_j, k\tau)$, $b_j^k=b(x_j, k\tau)$, Δ_h and D_h are difference operators defined by

$$\Delta_hv(x_j)=\{v(x_j+h)-2v(x_j)+v(x_j-h)\}/h^2,$$

$$D_hv(x_j)=\{v(x_j+h)-v(x_j-h)\}/(2h).$$

THEOREM 2.1. *Under Assumption* 1 *and the conditions*

$$(2.2)\qquad \tau\leqq h^2/(2\|a\|),\qquad h\leqq 2a_0/\|b\|,$$

(2.3) $$\tau<1/M_0 \quad \text{if} \quad M_0>0,$$

the difference scheme (2.1) *is L^∞-stable in the following sense,*

(2.4) $$\max_{j\in J,\, k\in\langle 0,\, N_T\rangle} |u_h^k(x_j)| \leqq U_0(\tau),$$

where

$$U_0(\tau)=\begin{cases} \|u^0\| \exp\{TM_0/(1-\tau M_0)\}+\|f_0\|\{\exp(TM_0/(1-\tau M_0))-1\}/M_0 & \text{if } M_0>0, \\ \|u^0\|+\|f_0\|T & \text{if } M_0\leqq 0. \end{cases}$$

Furthermore, if the exact solution u of (1.1) *is smooth* (*see Remark* 2.2), *u_h converge to u uniformly in $\bar{Q}$ as h tends to zero.*

REMARK 2.2. (i) More precise statement of Theorem 2.1 (also Theorem 4.4) is that interpolating functions of u_h, for example bilinear on each rectangular net, converge to u uniformly in $\bar{Q}$ as h tends to zero. This is true if u_t and u_{xx} are Hölder continuous in $\bar{Q}$. If u_t and u_{xxx} are Lipschitz continuous in $\bar{Q}$, then the rate of convergence is h^2.
(ii) Conditions (2.2) and (2.3) can be replaced by

$$\tau\leqq h^2/(2\|a\|+h\|b\|),$$

if we apply the upwind finite difference technique,

$$\{u_h^{k+1}(x_j)-u_h^k(x_j)\}/\tau=a_j^k\Delta_h u_h^k(x_j)+b_j^k\tilde{D}_h u_h^k(x_j)+f_0(k\tau) + f_1(k\tau,\, u_h^k(x_j))\tilde{I}_h^k u_h(x_j),$$

where

$$\tilde{D}_h u_h^k(x_j)=\begin{cases} \{u_h^k(x_j+h)-u_h^k(x_j)\}/h & \text{if } b_j^k\geqq 0, \\ \{u_h^k(x_j)-u_h^k(x_j-h)\}/h & \text{if } b_j^k<0, \end{cases}$$

$$\tilde{I}_h^k u_h(x_j)=\begin{cases} u_h^k(x_j) & \text{if } f_1(k\tau,\, u_h^k(x_j))\geqq 0, \\ u_h^{k+1}(x_j) & \text{if } f_1(k\tau,\, u_h^k(x_j))<0. \end{cases}$$

PROOF OF THEOREM 2.1. We show (2.4) by the comparison theorem for the difference equation. Consider the (ordinary) difference equation with respect to τ subject to (2.3),

(2.5) $$\begin{cases} (v^{k+1}-v^k)/\tau=\|f_0\|+M_0 v^{k+1}, & k\in\langle 0,\, N_T-1\rangle, \\ v^0=\|u^0\|. \end{cases}$$

The solution of (2.5) can be solved explicitly,

$$v^k=\begin{cases} \{1/(1-\tau M_0)\}^k(v^0+\|f_0\|/M_0)-\|f_0\|/M_0 & \text{if } M_0\neq 0, \\ v^0+\tau k\|f_0\| & \text{if } M_0=0. \end{cases}$$

An easy computation leads to

$$0 \leqq v^k \leqq U_0(\tau) \quad \text{for} \quad k \in \langle 0, N_T \rangle . \tag{2.6}$$

To show $u_h^k(x_j) \leqq v^k$, we set $w_j^k = v^k - u_h^k(x_j)$ and obtain

$$\begin{cases} (w_j^{k+1} - w_j^k)/\tau = a_j^k \Delta_h w_j^k + b_j^k D_h w_j^k + (f_1)_j^k w_j^{k+1} + \{M_0 - (f_1)_j^k\} v^{k+1} \\ \qquad\qquad + \{\|f_0\| - f_0^k\} \qquad \text{for} \quad j \in J,\ k \in \langle 0, N_T - 1 \rangle , \\ w_j^k = w_{j+1}^k \qquad\qquad \text{for} \quad j = -1/2,\ N - 1/2 , \end{cases} \tag{2.7}$$

where $(f_1)_j^k = f_1(k\tau, u_h^k(x_j))$ and $f_0^k = f_0(k\tau)$. Transforming (2.7), we have

$$\{1 - \tau (f_1)_j^k\} w_j^{k+1} = \tau \{(2a_j^k + hb_j^k)/(2h^2)\} w_{j+1}^k + \{1 - 2\tau a_j^k/h^2\} w_j^k$$
$$+ \tau \{(2a_j^k - hb_j^k)/(2h^2)\} w_{j-1}^k + \tau \{(M_0 - (f_1)_j^k) v^{k+1} + (\|f_0\| - f_0^k)\} .$$

The last term of the right is nonnegative in virtue of (2.6). By (2.2), (2.3) and the nonnegativity of w_j^0, we obtain $w_j^k \geqq 0$, $j \in J$, $k \in \langle 0, N_T \rangle$, which leads to $u_h^k(x_j) \leqq v^k$. Similarly we obtain $u_h^k(x_j) \geqq -v^k$. Hence we get (2.4) by (2.6).

Since $U_0(\tau)$ is bounded as $\tau \downarrow 0$ and since the difference scheme (2.1) is a consistent approximation for (1.1), we can easily show the uniform convergence of u_h to u. q. e. d.

We now define the number of peaks $\#_p^h$ for a net function. Let u_h be a net function defined on $x_j = jh$, $j \in J$. Considering the first difference of u_h, we construct a broken line $p_h \in C[0, 1]$ connecting $(jh, p_h(jh))$, $j = 0, \cdots, N$, where

$$p_h(jh) = \{u_h(jh + h/2) - u_h(jh - h/2)\}/h \qquad \text{for} \quad j \in \langle 0, N \rangle , \tag{2.8}$$
$$u_h(-h/2) = u_h(h/2) , \qquad u_h(1 + h/2) = u_h(1 - h/2) .$$

We define $\#_p^h(u_h)$ by $\#_p^h(u_h) = N_+(p_h)$, where N_+ is the one defined in the previous section. Similarly the number of valleys of u_h is defined by $\#_v^h(u_h) = N_-(p_h)$.

THEOREM 2.3. *Suppose Assumption* 1. *Then, for every* $\varepsilon > 0$ *there exists a number* $h_0 > 0$ *such that under the condition*

$$h \leqq h_0 \quad \textit{and} \quad \tau \leqq h^2/\{4\|a\| + (2\|a_x\| + \varepsilon)h\} \tag{2.9}$$

the solution u_h *of* (2.1) *satisfies*

$$\#_p^h(u_h^k) \leqq \#_p^h(u_h^0) , \qquad \#_v^h(u_h^k) \leqq \#_v^h(u_h^0) \qquad \textit{for} \quad k \in \langle 0, N_T \rangle . \tag{2.10}$$

REMARK 2.4. h_0 depends only on ε, a, b, u^0 and f. One could obtain a lower bound of h_0 if one computed some quantities without using order estimates in the subsequent proof.

For the proof of Theorem 2.3 we need the following theorem concerning a property of a class of linear mappings in $\boldsymbol{R}^{N+1}$. Let

$$R_0^{N+1}=\{p=(p_0, p_1, \cdots, p_N); p_j\in R, j\in\langle 0, N\rangle, p_0=p_N=0\}$$

and Π be an operator in R_0^{N+1} such that

$$(\Pi p)_j=\begin{cases} 0 & \text{for } j=0, N, \\ \lambda_{j,j-1}p_{j-1}+\lambda_{j,j}p_j+\lambda_{j,j+1}p_{j+1} & \text{for } j\in\langle 1, N-1\rangle, \end{cases}$$

where $\lambda_{j,i}$, $i=j$, $j\pm 1$, are given real numbers. For $p\in R^{N+1}$ we regard $N_\pm(p)$ as $N_\pm(p)=N_\pm(p_h)$, where p_h is the broken line connecting (jh, p_j), $j=0, 1, \cdots, N$.

THEOREM 2.5. *Let Π be a linear operator in R_0^{N+1} as above. Suppose $\lambda_{j,i}$ satisfy*

$$\text{(2.11)} \qquad \begin{cases} \lambda_{j,i}\geqq 0 & \textit{for } j\in\langle 1, N-1\rangle, i=j, j\pm 1, \\ \lambda_{j,j}\geqq\lambda_{j+1,j}+\lambda_{j-1,j} & \textit{for } j\in\langle 1, N-1\rangle, \end{cases}$$

where $\lambda_{0,1}=\lambda_{N,N-1}=0$. Then it holds that

$$\text{(2.12)} \qquad N_\pm(\Pi p)\leqq N_\pm(p) \quad \textit{for} \quad p\in R_0^{N+1},$$

where the same sign should be taken in both sides.

The proof of Theorem 2.5 is rather complicated. Therefore we shall prove it in the next section. Here we prove Theorem 2.3 by using Theorem 2.5.

PROOF OF THEOREM 2.3. Let u_h^k, $k=0, \cdots, N_T$, be the solution of (2.1). For the proof of (2.10) it is sufficient to show that

$$\text{(2.13)} \qquad N_\pm(p_h^{k+1})\leqq N_\pm(p_h^k) \quad \text{for} \quad k\in\langle 0, N_T-1\rangle,$$

where p_h^k is the first differences of u_h^k defined by (2.8). Fix k arbitrarily. From (2.1) we have p_h^k, $p_h^{k+1}\in R_0^{N+1}$ and

$$\text{(2.14)} \qquad \begin{aligned} (p_j^{k+1}-p_j^k)/\tau= & a_{j+1/2}^k(p_{j+1}^k-2p_j^k+p_{j-1}^k)/h^2+(a_x)_j^k(p_j^k-p_{j-1}^k)/h \\ & +b_{j+1/2}^k(p_{j+1}^k-p_{j-1}^k)/(2h)+(b_x)_j^k(p_j^k+p_{j-1}^k)/2 \\ & +(f_1)_j^k p_j^{k+1}+(f_{1,u}u_h)_j^k p_j^k \quad \text{for} \quad j\in\langle 1, N-1\rangle, \end{aligned}$$

where $p_j^k=p_h^k(x_j)$,

$$(a_x)_j^k=a_x(\xi_{1j}^k, k\tau)=\{a((j+1/2)h, k\tau)-a((j-1/2)h, k\tau)\}/h,$$

$$(b_x)_j^k=b_x(\xi_{2j}^k, k\tau)=\{b((j+1/2)h, k\tau)-b((j-1/2)h, k\tau)\}/h,$$

$$(f_1)_j^k=f_1(k\tau, u_h^k((j+1/2)h)),$$

$$(f_{1,u}u_h)_j^k=\partial f_1/\partial u(k\tau, \eta_j^k)u_h^{k+1}((j-1/2)h),$$

ξ_{ij}^k, $i=1, 2$, are intermediate values between $jh\pm h/2$,

η_j^k is an intermediate value between $u_h^k((j\pm 1/2)h)$.

Hence we have

$$p_j^{k+1}=\lambda_{j,j-1}^k p_{j-1}^k+\lambda_{j,j}^k p_j^k+\lambda_{j,j+1}^k p_{j+1}^k \qquad \text{for} \quad j\in\langle 1, N-1\rangle,$$

where

$$\text{(2.15)} \qquad \begin{cases} \lambda_{j,j-1}^k=\tau\{2a_{j+1/2}^k-2h(a_x)_j^k-hb_{j+1/2}^k+h^2(b_x)_j^k\}/(2h^2\gamma_j^k), \\ \lambda_{j,j}^k=\{1-\tau(4a_{j+1/2}^k-2h(a_x)_j^k-h^2(b_x)_j^k-2h^2(f_{1,u}u_h)_j^k)\}/(2h^2\gamma_j^k), \\ \lambda_{j,j+1}^k=\tau\{2a_{j+1/2}^k+hb_{j+1/2}^k\}/(2h^2\gamma_j^k), \\ \gamma_j^k=1-\tau(f_1)_j^k. \end{cases}$$

Thus p_h^{k+1} can be regarded as the image of p_h^k by an operator which belongs to the class considered in Theorem 2.5. We show that $\lambda_{j,i}^k$ defined in (2.15) satisfy condition (2.11) if h_0 is chosen suitably small. Since condition (2.9) yields (2.2) and (2.3), the solutions u_h are bounded, which implies γ_j^k, $\gamma_{j\pm1}^k=1+O(h^2)$. This fact enables us to write

$$\begin{aligned} \lambda_{j+1,j}^k&=\tau\{2a_{j+1/2}^k-hb_{j+1/2}^k\}/(2h^2\gamma_{j+1}^k) \\ &=a_{j+1/2}^k\tau/h^2-b_{j+1/2}^k\tau/(2h)+O(h^2), \\ \lambda_{j,j}^k&=1-2a_{j+1/2}^k\tau/h^2+(a_x)_j^k\tau/h+O(h^2), \\ \lambda_{j-1,j}^k&=\tau\{2a_{j+1/2}^k-2h(a_x)_j^k+hb_{j+1/2}^k-h^2(b_x)_j^k\}/(2h^2\gamma_{j-1}^k) \\ &=a_{j+1/2}^k\tau/h^2-(a_x)_j^k\tau/h+b_{j+1/2}^k\tau/(2h)+O(h^2). \end{aligned}$$

Hence we have

$$\begin{aligned} \lambda_{j,j}^k-\lambda_{j+1,j}^k-\lambda_{j-1,j}^k&=1-4a_{j+1/2}^k\tau/h^2+2(a_x)_j^k\tau/h+O(h^2) \\ &\geqq \varepsilon h/\{4\|a\|+(2\|a_x\|+\varepsilon)h\}+O(h^2) \\ &>0 \qquad \text{for} \quad j\in\langle 2, N-2\rangle, \end{aligned}$$

$$\begin{aligned} \lambda_{1,1}^k-\lambda_{2,1}^k&=1-3a_{1+1/2}^k\tau/h^2+O(h) \\ &\geqq \{\|a\|+(2\|a_x\|+\varepsilon)h\}/\{4\|a\|+(2\|a_x\|+\varepsilon)h\}+O(h) \\ &>0. \end{aligned}$$

In a similar line we can show that the other conditions of (2.11) are also satisfied. Applying Theorem 2.5, we get (2.13). q. e. d.

PROOF OF THEOREM 1.2. We first take the case when the solution u is so smooth (cf. Remark 2.2) that Theorem 2.1 can be applied. We prove only the former of (1.2), because the latter is shown similarly. We may assume that $\sharp_p(u^0)<+\infty$, otherwise (1.2) is satisfied as a trivial relation. Set $m=\sharp_p(u^0)$. For the purpose of an indirect proof, assume that $\sharp_p(u(t_0'))>m$ for some $t_0'\in[0, T]$. Fix $\varepsilon>0$ and let h_0 be the one stated in Theorem 2.3. ($1/h_0$ may be supposed to be an integer.) Put $h(n)$, $\tau(n)$, $n=0, 1, \cdots$, and S as follows:

$$h(n)=h_0/2^n, \qquad \tau(n)=1/2^{n'},$$

$$S=\bigcup_{n=0}^{+\infty}\{k\tau(n)\,;\ k=0,\ 1,\ \cdots,\ [T/\tau(n)]\},$$

where n' is the smallest integer such that $\tau=1/2^{n'}$ satisfies (2.9) with $h=h(n)$. We consider a family of the difference equations (2.1) with $(h, \tau)=(h(n), \tau(n))$. Since S is dense in $[0, T]$, we can find a time $t_0\in S$ sufficiently near t_0' such that $\#_p(u(t_0))>m$. By the definition of $\tau(n)$, for every n greater than some integer there exists a positive integer $k(n)$ satisfying $t_0=k(n)\tau(n)$. Choose $m+1$ peaks of $u(t_0)$ arbitrarily. Then, depending on the $m+1$ peaks, there exists a small positive number η such that for every net function v_h, if the broken line connecting $(jh, v_h(jh))$, $j=0, \cdots, 1/h$, lies in the domain $G_\eta=\{(x, y)\,;\ |y-u(x, t_0)|<\eta,\ x\in[0, 1]\}$, it holds that $\#_p^h(v_h)\geqq m+1$. Since $u_{h(n)}^{k(n)}$ converge uniformly to $u(t_0)$ as $n\to+\infty$ by Theorem 2.1, we can find a positive integer n_1 such that the broken lines made from $u_{h(n)}^{k(n)}$, $n\geqq n_1$, lie in G_η. Therefore we have

$$\#_p^h(u_{h(n)}^{k(n)})\geqq m+1 \qquad \text{for} \quad n\geqq n_1. \tag{2.16}$$

Since m is a finite number, it is easy to see that

$$\#_p^h(u_{h(n)}^0)=m \qquad \text{for} \quad n\geqq n_2, \tag{2.17}$$

where n_2 is some positive integer. However, (2.16) and (2.17) contradict the result of Theorem 2.3 with $(h(n), \tau(n))$, $n=\max(n_1, n_2)$. Hence we have $\#_p(u(t))\leqq m$ for $t\in[0, T]$.

We next take the general case. Let u_n, $n=1, 2, \cdots$, be a sequence of smooth functions such that each u_n satisfies (1.2), that u_n converge to u uniformly in $\bar{Q}$ and that $\#_p(u_n(0))=\#_p(u^0)$ $(=m<+\infty)$. Such a sequence can be made by assigning to u_n the solutions of (1.1) subject to smooth initial values u_n^0 with m peaks, which satisfy compatibility conditions and converge to u^0. (If necessary, a, b, and f are also approximated by smooth functions.) That each u_n satisfies (1.2) is the consequence of the first case. Assume that there exists a time $t_0>0$ such that $m<\#_p(u(t_0))$. Since u_n converge uniformly to u, we have for sufficiently large n $\#_p(u_n(t_0))>m=\#_p(u_n(0))$, which is a contradiction. Thus we obtain (1.2). q. e. d.

REMARK 2.6. One could prove Theorem 1.2 without using the finite difference by considering the equation of p $(=u_x)$. The discussion is suggested in Matano [**3**; Lemma 2].

REMARK 2.7. Consider a special case when $a\equiv 1$ and $b\equiv 0$. Then Theorem 2.1 guarantees that, under the conditions $\tau/h^2\leqq 1/2$ and (2.3), the finite difference solutions of (2.1) converge to the exact one, while Theorem 2.3 ensures that, under the conditions $\tau \leqq h^2/(4+\varepsilon h)$ and $h \leqq h_0$, the difference solution holds the property (2.10).

REMARK 2.8. Theorems 2.1 and 2.3 are valid for the following difference schemes:

$$\begin{cases} (u_j^{k+1}-u_j^k)/\tau = a_j^k \Delta_h u_j^k + b_j^k D_h u_j^k + f_0^k + f_1(k\tau, u_j^k) u_j^{k+1}, \\ u_{-1}^k = u_1^k,\ u_{N+1}^k = u_{N-1}^k \qquad \text{for} \quad j \in \langle 0, N \rangle,\ k \in \langle 0, N_T - 1 \rangle, \end{cases}$$

$$\begin{cases} (u_j^{k+1}-u_j^k)/\tau = a_j^k \Delta_h u_j^k + b_j^k D_h u_j^k + f_0^k + f_1(k\tau, u_j^k) u_j^{k+1}, \\ u_0^k = u_1^k,\ u_N^k = u_{N-1}^k \qquad \text{for} \quad j \in \langle 1, N-1 \rangle,\ k \in \langle 0, N_T - 1 \rangle, \end{cases}$$

where $u_j^k = u_h(jh, k\tau)$, $h = 1/N$.

We conclude this section by showing two typical examples where (2.10) holds no longer without (2.9).

EXAMPLE 2.9. Consider the heat equation (i.e., $a \equiv 1$, $b \equiv f \equiv 0$). Fix $\tau/h^2 = 1/2$ and take

$$u^0(x) = \sum_{n=1}^{+\infty} \phi(2^n x - 1)/4^n ,$$

where $\phi(x)$ is a C^1-function defined in $\boldsymbol{R}^1$ such that (i) $\phi(3/8+x) = \phi(3/8-x)$, (ii) $\phi(x) = 0$ for $x \leqq 1/4$ and $\phi(3/8) = 1$, (iii) $d\phi/dx \geqq 0$ on $[1/4, 3/8]$. Obviously u^0 belongs to $C^1[0, 1]$. Then, for $N = 2^m$ ($m \geqq 3$) we have $\#_p^h(u_h^0) = m-2$ and $\#_p^h(u_h^1) = m-1$.

EXAMPLE 2.10. Consider the same equation as Example 2.9. Fix $\tau/h^2 = 1/4 + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. Let N be an odd number greater than $(8\varepsilon+1)/(4\varepsilon)$. Take $u_h^0(x_j)$, $j \in J = \{1/2, 3/2, \cdots, N-1/2\}$, as follows: $u_h^0(x_j) = u_h^0(1-x_j) = (-1)^{j'} j'$, $j \in \{1/2, 3/2, \cdots, N/2\}$, where $j' = j - 1/2$. Then we have

$$u_h^1(x_j) = u_h^1(1-x_j) = \begin{cases} -1/4 - \varepsilon , & j = 1/2 , \\ 4\varepsilon(-1)^{j'+1} j' , & j \in \{3/2, 5/2, \cdots, N/2-1\} , \\ 2\varepsilon(-1)^{j'+1}\{N - (8\varepsilon+1)/(4\varepsilon)\} , & j = N/2 . \end{cases}$$

Hence we have $\#_p^h(u_h^0) = (N-3)/2$ and $\#_p^h(u_h^1) = (N-1)/2$.

§ 3. Proof of Theorem 2.5.

Here we prove Theorem 2.5. Throughout this section $\hat{p}$ means the image Πp of $p \in \boldsymbol{R}_0^{N+1}$ by an operator Π which belongs to the class of linear operators in $\boldsymbol{R}^{N+1}$ considered in the preceding section. $\hat{p}_j$ means its j-th component. We often use double signs $\pm$ and $\mp$. In an expression including them the same order should be taken in both sides. For typographical reasons, we use no double subscripts but express them with parentheses, for example p_i with

$i=j_r$ is denoted by $p_{j(r)}$.

We extend the definition of $N_\pm$ to a part of $\boldsymbol{p}\in\boldsymbol{R}^{N+1}$. Define $N_\pm(\boldsymbol{p}\,;j_1, j_2)$, $0\leqq j_1<j_2\leqq N$, by $N_\pm(\tilde{\boldsymbol{p}})$, where $\tilde{\boldsymbol{p}}$ is an element in $\boldsymbol{R}^{N+1}$ taken as follows,

$$\tilde{p}_j=\begin{cases} p_{j(1)} & (0\leqq j\leqq j_1), \\ p_j & (j_1<j<j_2), \\ p_{j(2)} & (j_2\leqq j\leqq N). \end{cases}$$

The following proposition is trivial.

PROPOSITION 3.1. *Let* $\boldsymbol{p}\in\boldsymbol{R}_0^{N+1}$ *satisfy* $p_j\neq 0$, $j=1, \cdots, N-1$. *Then it holds that*

$$N_\pm(\boldsymbol{p})=\sum_{i=0}^{s-1} N_\pm(\boldsymbol{p}\,;j_i, j_{i+1}),$$

for any j_i, $i=0, 1, \cdots, s$, *satisfying* $0=j_0<j_1<\cdots<j_{s-1}<j_s=N$.

If $\boldsymbol{p}$ or $\hat{\boldsymbol{p}}$ includes zero-components except both edges, the analysis of $N_\pm$ is somewhat complicated. The following proposition shows that we can get rid of such circumstances if $\lambda_{j,i}$ satisfy the condition

$$\lambda_{j,i}>0 \quad \text{for} \quad i=j,\ j\pm 1,\ j\in\langle 1, N-1\rangle. \tag{3.1}$$

Define $\boldsymbol{R}_{0*}^{N+1}(\Pi)$ by

$$\boldsymbol{R}_{0*}^{N+1}(\Pi)=\{\boldsymbol{p}\ ;\ \boldsymbol{p}, \hat{\boldsymbol{p}}\in\boldsymbol{R}_0^{N+1},\ p_j,\ \hat{p}_j\neq 0 \text{ for } j\in\langle 1, N-1\rangle\}.$$

PROPOSITION 3.2. *Suppose condition* (3.1). *Then, for every non-zero element* $\boldsymbol{p}\in\boldsymbol{R}_0^{N+1}$ *there exists an element* $\boldsymbol{q}\in\boldsymbol{R}_{0*}^{N+1}(\Pi)$ *such that*

$$N_\pm(\boldsymbol{q})=N_\pm(\boldsymbol{p}) \quad \text{and} \quad N_\pm(\hat{\boldsymbol{q}})=N_\pm(\hat{\boldsymbol{p}}). \tag{3.2}$$

PROOF. We divide the proof into 3 steps. Throughout the following proof, σ, σ_1 and σ_2 indicate $+1$ or -1 and ε indicates a sufficiently small positive number, which may differ at each occurrence. We call $\boldsymbol{p}$ and $\boldsymbol{q}$ are equivalent to each other if (3.2) is satisfied.

1st step. To find $\boldsymbol{p}^1\in\boldsymbol{R}_0^{N+1}$ equivalent to $\boldsymbol{p}$ such that $p_i^1=0$ for some $i\in\langle 2, N-2\rangle$ implies $\operatorname{sgn} p_{i-1}^1\cdot\operatorname{sgn} p_{i+1}^1=-1$ and that $p_1^1, p_{N-1}^1\neq 0$. We can find such an element $\boldsymbol{p}^1$ by repeating the following two transformations (from $\boldsymbol{p}$ to $\boldsymbol{q}$):

(i) If there exists $j\in\langle 1, N-1\rangle$ such that $\operatorname{sgn}(p_{j-1}, p_j, p_{j+1})=(\sigma, 0, 0)$ or $(0, 0, \sigma)$, then $\boldsymbol{q}$ is defined by

$$q_i=\begin{cases} p_i+\sigma\varepsilon & \text{for } i=j, \\ p_i & \text{otherwise}. \end{cases} \tag{3.3}$$

(ii) If there exists $j\in\langle 2, N-2\rangle$ such that $\operatorname{sgn}(p_{j-1}, p_j, p_{j+1})=(\sigma, 0, \sigma)$, then $\boldsymbol{q}$ is defined by (3.3).

We show that $\boldsymbol{p}$ and $\boldsymbol{q}$ are equivalent to each other under the transformation (i) with a suitable choice of ε. Suppose $j\neq 1, N-1$. From the structure of Π and by (3.1), we know that $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}$ differ only at $i=j, j\pm 1$ and that $\operatorname{sgn}\hat{p}_j=\operatorname{sgn}\hat{q}_j=\sigma$. Four cases are considered: (a) $\operatorname{sgn}(\hat{p}_{j-1}, \hat{p}_j, \hat{p}_{j+1})=(\sigma_1, \sigma, \sigma_2)$, (b) $(0, \sigma, \sigma_1)$, (c) $(\sigma_1, \sigma, 0)$, (d) $(0, \sigma, 0)$. By choosing a sufficiently small ε (>0) we have (a) $\operatorname{sgn}(\hat{q}_{j-1}, \hat{q}_j, \hat{q}_{j+1})=(\sigma_1, \sigma, \sigma_2)$, (b) $(\sigma, \sigma, \sigma_1)$, (c) $(\sigma_1, \sigma, \sigma)$, (d) (σ, σ, σ) for each case. Hence we obtain $N_\pm(\hat{\boldsymbol{p}})=N_\pm(\hat{\boldsymbol{q}})$ in any case. A similar argument shows that $\boldsymbol{p}$ and $\boldsymbol{q}$ are equivalent to each other when $j=1, N-1$ and when the transformation (ii) is executed.

2nd step. To find $\boldsymbol{p}^2\in\boldsymbol{R}_0^{N+1}$ equivalent to $\boldsymbol{p}^1$ such that $p_j^2=0$ for some $j\in\langle 2, N-2\rangle$ implies $\operatorname{sgn} p_{j-1}^2\cdot\operatorname{sgn} p_{j+1}^2=-1$, that $p_1^2, p_{N-1}^2\neq 0$, and that $\hat{p}_j^2\neq 0$ for all $j\in\langle 1, N-1\rangle$. We can find such an element $\boldsymbol{p}^2$ by repeating the following two transformations (from $\boldsymbol{p}$ to $\boldsymbol{q}$):

(i) If there exists $j\in\langle 1, N-1\rangle$ such that $\operatorname{sgn}(\hat{p}_{j-1}, \hat{p}_j, \hat{p}_{j+1})=(\sigma, 0, 0)$, $(0, 0, \sigma)$ or $(\sigma, 0, \sigma)$, then $\boldsymbol{q}$ is defined by (3.3).

(ii) If there exists $j\in\langle 2, N-2\rangle$ such that $\operatorname{sgn}(\hat{p}_{j-1}, \hat{p}_j, \hat{p}_{j+1})=(\sigma, 0, -\sigma)$, then $\boldsymbol{q}$ is defined by (3.3).

By (i) we have $\operatorname{sgn}(\hat{q}_{j-1}, \hat{q}_j, \hat{q}_{j+1})=(\sigma, \sigma, \sigma)$ and by (ii) $\operatorname{sgn}(\hat{q}_{j-1}, \hat{q}_j, \hat{q}_{j+1})=(\sigma, \sigma, -\sigma)$. Since ε is sufficiently small, the transformations (i) and (ii) preserve the property which $\boldsymbol{p}^1$ has. (It may happen that $p_j=0$ and $q_j=\sigma\varepsilon\neq 0$ for some j.) It is easy to see that $\boldsymbol{p}$ and $\boldsymbol{q}$ (therefore $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$) are equivalent to each other.

3rd step. To find $\boldsymbol{q}\in\boldsymbol{R}_{0*}^{N+1}(\Pi)$ equivalent to $\boldsymbol{p}^2$. We can find such an element $\boldsymbol{q}$ by repeating the following transformation (from $\boldsymbol{p}$ to $\boldsymbol{q}$): If there exists $j\in\langle 2, N-2\rangle$ such that $p_j=0$, then $\boldsymbol{q}$ is defined by (3.3), where σ is arbitrary (1 or -1). It is easy to see $\boldsymbol{q}\in\boldsymbol{R}_{0*}^{N+1}(\Pi)$ and that $\boldsymbol{p}$ and $\boldsymbol{q}$ are equivalent to each other. q. e. d.

Before stating the following lemmas, we introduce the definition of "chain". A part $\boldsymbol{c}$ of $\boldsymbol{p}\in\boldsymbol{R}_{0*}^{N+1}(\Pi)$ is called a chain if $\boldsymbol{c}=(p_{j(1)}, p_{j(1)+1}, \cdots, p_{j(2)})$, $1\leqq j_1<j_2\leqq N-1$, satisfies the following conditions: (i) $\operatorname{sgn} p_{j+1}=-\operatorname{sgn} p_j$ for $j\in\langle j_1, j_2-1\rangle$, (ii) $\operatorname{sgn} p_{j+1}\neq-\operatorname{sgn} p_j$ for $j=j_1-1, j_2$. We denote a chain of $\boldsymbol{p}$ by $\boldsymbol{c}(\boldsymbol{p}; j_1, j_2)$. A chain $\boldsymbol{c}(\boldsymbol{p}; j_1, j_2)$ is called an active chain (or a-chain) if there exists some $j\in\langle j_1, j_2\rangle$ such that $\operatorname{sgn}\hat{p}_j\neq\operatorname{sgn} p_j$. Put

$$N_0(\boldsymbol{p}; j_1, j_2)=N_+(\boldsymbol{p}; j_1, j_2)+N_-(\boldsymbol{p}; j_1, j_2).$$

LEMMA 3.3. *Let* $\boldsymbol{c}(\boldsymbol{p}; j, j+r)$, $1\leqq j<j+r\leqq N-1$, *be an a-chain of* $\boldsymbol{p}\in\boldsymbol{R}_{0*}^{N+1}(\Pi)$. *Then, under condition* (2.11) *it holds that*

(i) $N_0(\hat{\boldsymbol{p}}; j, j+r)\leqq r-1$,

(ii) *If* sgn $\hat{p}_i=-$sgn p_i *for* $i=j,\ j+r$, *then* $N_0(\hat{\boldsymbol{p}}\,;\,j,\ j+r)\leqq r-2$.

PROOF. From the definition of chain, $N_0(\boldsymbol{p}\,;\,j,\ j+r)=r$. The results (i) and (ii) are obtained at one stroke if we prove that it is impossible that sgn $\hat{p}_i=-$sgn p_i for all $i\in\langle j,\ j+r\rangle$. Without loss of generality we may assume that sgn $p_i=(-1)^i$, $i\in\langle j,\ j+r\rangle$. We have

$$(-1)^i\hat{p}_i=-\lambda_{i,i-1}(-1)^{i-1}p_{i-1}+\lambda_{i,i}(-1)^i p_i-\lambda_{i,i+1}(-1)^{i+1}p_{i+1}\,. \tag{3.4}$$

Summing up (3.4) from $i=j$ to $j+r$, we obtain

$$\begin{aligned}\sum_{i=j}^{j+r}(-1)^i\hat{p}_i=&-\lambda_{j,j-1}(-1)^{j-1}p_{j-1}+(\lambda_{j,j}-\lambda_{j+1,j})(-1)^j p_j\\&+\sum_{i=j+1}^{j+r-1}(\lambda_{i,i}-\lambda_{i-1,i}-\lambda_{i+1,i})(-1)^i p_i\\&+(\lambda_{j+r,j+r}-\lambda_{j+r-1,j+r})(-1)^{j+r}p_{j+r}\\&-\lambda_{j+r,j+r+1}(-1)^{j+r+1}p_{j+r+1}\,.\end{aligned} \tag{3.5}$$

By the definition of chain it holds that sgn $p_{j-1}=$sgn p_j or $=0$ (if $j-1=0$) and that sgn $p_{j+r+1}=$sgn p_{j+r} or $=0$ (if $j+r+1=N$). By virtue of (2.11) we have

$$\begin{aligned}\text{the right of (3.5)}\geqq&(\lambda_{j,j}-\lambda_{j+1,j})(-1)^j p_j\\&+(\lambda_{j+r,j+r}-\lambda_{j+r-1,j+r})(-1)^{j+r}p_{j+r}\\\geqq&\,0\,.\end{aligned}$$

Hence there exists at least one $i\in\langle j,\ j+r\rangle$ such that sgn $\hat{p}_i\neq(-1)^{i+1}$, which implies sgn $\hat{p}_i\neq-$sgn p_i. q. e. d.

LEMMA 3.4. *Suppose condition* (2.11). *Let* $\boldsymbol{t}(p\,;\,j_0,\ j_0+r)=(p_{j(0)},\ p_{j(0)+1},\ \cdots,\ p_{j(0)+r})$, $1\leqq j_0<j_0+r\leqq N-1$, *be a train of a-chains included in* $\boldsymbol{p}\in\boldsymbol{R}_{0*}^{N+1}(\Pi)$, *i.e., there exist* j_k, $k=1,\cdots,s$ $(\geqq 1)$, *such that* $j_s=j_0+r+1$, $j_{k+1}-j_k\geqq 2$ *for* $k=0,1,\cdots,s-1$ *and that* $\boldsymbol{c}(\boldsymbol{p}\,;\,j_k,\ j_{k+1}-1)$, $k=0,1,\cdots,s-1$, *are a-chains. If* sgn $\hat{p}_i$ $=$sgn p_i *for* $i=j_0-1$ *and* j_0+r+1, *then*

$$N_\pm(\hat{\boldsymbol{p}}\,;\,j_0-1,\ j_0+r+1)\leqq N_\pm(\boldsymbol{p}\,;\,j_0-1,\ j_0+r+1)\,.$$

PROOF. We divide the proof into two steps.

1st step. We prove that

$$N_0(\hat{\boldsymbol{p}}\,;\,j_0-1,\ j_0+r+1)\leqq N_0(\boldsymbol{p}\,;\,j_0-1,\ j_0+r+1)\,. \tag{3.6}$$

Set $n_k=j_{k+1}-j_k-1$. Obviously it holds that

$$N_0(\boldsymbol{p}\,;\, j_0-1,\, j_0+r+1)=\sum_{k=0}^{s-1} n_k\,. \tag{3.7}$$

Applying Proposition 3.1 and Lemma 3.3, we have

$$\begin{aligned} N_0(\hat{\boldsymbol{p}}\,;\, j_0-1,\, j_0+r+1) &= \sum_{k=0}^{s} N_0(\hat{\boldsymbol{p}}\,;\, j_k-1,\, j_k)+\sum_{k=0}^{s-1} N_0(\hat{\boldsymbol{p}}\,;\, j_k,\, j_k+n_k) \\ &\leqq s+1+\sum_{k=0}^{s-1}(n_k-1) \\ &= \sum_{k=0}^{s-1} n_k+1\,. \end{aligned}$$

For (3.6) it suffices to prove that all the following equalities do not hold at one time,

$$N_0(\hat{\boldsymbol{p}}\,;\, j_k-1,\, j_k)=1 \qquad \text{for} \quad k=0,\, 1,\, \cdots,\, s\,, \tag{3.8}$$

$$N_0(\hat{\boldsymbol{p}}\,;\, j_k,\, j_k+n_k)=n_k-1 \qquad \text{for} \quad k=0,\, 1,\, \cdots,\, s-1\,. \tag{3.9}$$

If $j_0=1$ (resp. $j_0+r=N-1$), (3.8) does not hold for $k=0$ (resp. $k=s$). Consider the case $1<j_0<j_0+r<N-1$. Then we have

$$\operatorname{sgn} p_{j(k)-1}=\operatorname{sgn} p_{j(k)} \qquad \text{for} \quad k=0,\, 1,\, \cdots,\, s\,. \tag{3.10}$$

We show that, if we assume (3.8) for $k=0, 1, \cdots, s-1$ and (3.9) for $k=0, 1, \cdots, s-1$, (3.8) for s does not hold. For the proof we observe that those assumptions lead to

$$\operatorname{sgn} \hat{p}_{j(k)}=-\operatorname{sgn} p_{j(k)}\,, \qquad \operatorname{sgn} \hat{p}_{j(k)+n(k)}=\operatorname{sgn} p_{j(k)+n(k)} \qquad \text{for} \quad k=0,\, 1,\, \cdots,\, s-1\,. \tag{3.11}$$

In the case $k=0$, by (3.8), the assumption of Lemma, and (3.10), we have

$$\operatorname{sgn} \hat{p}_{j(0)}=-\operatorname{sgn} \hat{p}_{j(0)-1}=-\operatorname{sgn} p_{j(0)-1}=-\operatorname{sgn} p_{j(0)}\,,$$

which implies $\operatorname{sgn} \hat{p}_{j(0)+n(0)}=\operatorname{sgn} p_{j(0)+n(0)}$ by Lemma 3.3 and (3.9). We now proceed inductively, assuming that (3.11) is true for $k=l$. By (3.8), this assumption, and (3.10), we have

$$\operatorname{sgn} \hat{p}_{j(l+1)}=-\operatorname{sgn} \hat{p}_{j(l)+n(l)}=-\operatorname{sgn} p_{j(l)+n(l)}=-\operatorname{sgn} p_{j(l+1)}\,,$$

which implies $\operatorname{sgn} \hat{p}_{j(l+1)+n(l+1)}=\operatorname{sgn} p_{j(l+1)+n(l+1)}$ by Lemma 3.3 and (3.9). Hence (3.11) is true for $k=l+1$, which completes the proof of (3.11). By setting $k=s-1$ in the latter of (3.11), and by using (3.10) for $k=s$ and the assumption of Lemma, we have

$$\operatorname{sgn} \hat{p}_{j(s)-1}=\operatorname{sgn} p_{j(s)-1}=\operatorname{sgn} p_{j(s)}=\operatorname{sgn} \hat{p}_{j(s)},$$

which implies (3.8) for $k=s$ is false.

2nd step. In (3.6) two cases are considered: (i) both sides are not equal, (ii) both sides are equal. In the first case we can easily conclude the results of Lemma since it holds generally that for $0\leqq i_1<i_2\leqq N$,

$$N_{\pm}(\boldsymbol{p}\,;i_1,i_2)=\begin{cases} N_0(\boldsymbol{p}\,;i_1,i_2)/2 & \text{if } N_0(\boldsymbol{p}\,;i_1,i_2) \text{ is even,} \\ (N_0(\boldsymbol{p}\,;i_1,i_2)\pm1)/2 & \\ \text{or} & \text{if } N_0(\boldsymbol{p}\,;i_1,i_2) \text{ is odd.} \\ (N_0(\boldsymbol{p}\,;i_1,i_2)\mp1)/2 & \end{cases}$$

In the second case we have $j_0\geqq2$ or $j_0+r+1\leqq N-1$. In fact if we assume that $j_0=1$ and $j_0+r+1=N$, (3.8) does not hold for $k=0$ and s, which implies

$$N_0(\hat{\boldsymbol{p}}\,;j_0-1,j_0+r+1)\leqq\sum_{k=1}^{s-1}n_k-1<N_0(\boldsymbol{p}\,;j_0-1,j_0+r+1).$$

Hence this is reduced to the first case. Without loss of generality we can assume $j_0\geqq2$. Then we have $\operatorname{sgn} \hat{p}_{j(0)-1}=\operatorname{sgn} p_{j(0)-1}\neq0$. Since (3.6) is satisfied with equality, we obtain $N_{\pm}(\hat{\boldsymbol{p}}\,;j_0-1,j_0+r+1)=N_{\pm}(\boldsymbol{p}\,;j_0-1,j_0+r+1)$.

q. e. d.

PROOF OF THEOREM 2.5. We divide the proof into 2 steps.

1st step (the case when $\lambda_{j,i}$ satisfy the additional condition (3.1)). When $\boldsymbol{p}$ is nonnegative (resp. nonpositive), we have $\hat{\boldsymbol{p}}$ is nonnegative (resp. nonpositive). Hence it holds $N_{\pm}(\hat{\boldsymbol{p}})=N(\boldsymbol{p})=0$.

When $\boldsymbol{p}$ has both positive and negative components, fix $\boldsymbol{q}\in\boldsymbol{R}_{0*}^{N+1}(\Pi)$ equivalent to $\boldsymbol{p}$ by Proposition 3.2. For (2.12) it suffices to show $N_{\pm}(\hat{\boldsymbol{q}})\leqq N_{\pm}(\boldsymbol{q})$. We first take out all the a-chains $\boldsymbol{c}_j$ included in $\boldsymbol{q}$. Connecting $\boldsymbol{c}_j$ if they are adjacent to each other, we make up a set of trains of a-chains, which is denoted by $\boldsymbol{t}_k(\boldsymbol{q}\,;j_k,j_k+n_k)$, $k=1,\cdots,s$, $0<j_1<j_1+n_1<j_2<\cdots<j_s+n_s<N$. We show that

$$\operatorname{sgn} \hat{q}_j=\operatorname{sgn} q_j \quad \text{for all} \quad q_j\in\boldsymbol{q}-\bigcup_{k=1}^{s}\boldsymbol{t}_k. \tag{3.12}$$

Three cases are considered: (i) $j=0$ or N, (ii) q_j belongs to a non-active chain, (iii) otherwise. In cases (i) and (ii), (3.12) is trivial. In case (iii), q_j belongs to no chain. Hence, by the definition of chain we have

$$\begin{aligned} &\operatorname{sgn} q_{j-1}=\operatorname{sgn} q_j=\operatorname{sgn} q_{j+1} && \text{when} \quad j\neq1,\ N-1, \\ &q_0=0,\ \operatorname{sgn} q_1=\operatorname{sgn} q_2 && \text{when} \quad j=1, \quad \text{or} \\ &\operatorname{sgn} q_{N-2}=\operatorname{sgn} q_{N-1},\ q_N=0 && \text{when} \quad j=N-1. \end{aligned}$$

In any case we have (3.12).

If $s=0$, our proof is complete. Suppose $s\geqq 1$. By Proposition 3.1, we have

$$N_{\pm}(\hat{q})=N_{\pm}(\hat{q}\,;0,\ j_1-1)+\sum_{k=1}^{s}N_{\pm}(\hat{q}\,;j_k-1,\ j_k+n_k+1) \tag{3.13}$$

$$+\sum_{k=1}^{s-1}N_{\pm}(\hat{q}\,;j_k+n_k+1,\ j_{k+1}-1)+N_{\pm}(\hat{q}\,;j_s+n_s+1,\ N),$$

where it may happen that $j_1-1=0$, $j_k+n_k+1=j_{k+1}-1$, $j_s+n_s+1=N$. By using (3.12) and Lemma 3.4, we obtain

$$\text{the right of (3.13)}\leqq N_{\pm}(q\,;0,\ j_1-1)+\sum_{k=1}^{s}N_{\pm}(q\,;j_k-1,\ j_k+n_k+1)$$

$$+\sum_{k=1}^{s-1}N_{\pm}(q\,;j_k+n_k+1,\ j_{k+1}-1)+N_{\pm}(q\,;j_s+n_s+1,\ N)$$

$$=N_{\pm}(q)\,.$$

2nd step (the general case). Let Π_{ε} be the same kind operator as Π in $\boldsymbol{R}_0^{N+1}$ with

$$\lambda_{j,i}^{\varepsilon}=\begin{cases}\lambda_{j,i}+2\varepsilon & \text{for}\quad i=j,\ j\in\langle 1,\ N-1\rangle\,,\\ \lambda_{j,i}+\varepsilon & \text{for}\quad i=j\pm 1,\ j\in\langle 1,\ N-1\rangle\,,\end{cases}$$

where ε is a positive number. Since $\lambda_{j,i}^{\varepsilon}$ satisfy (2.11) as well as (3.1), Π_{ε} lies within the scope of the first step. Therefore we have $N_{\pm}(\Pi_{\varepsilon}\boldsymbol{p})\leqq N_{\pm}(\boldsymbol{p})$ for every $\varepsilon>0$.

Now, for each $\boldsymbol{p}\in\boldsymbol{R}_0^{N+1}$ there exists a small positive number ε such that $(\Pi\boldsymbol{p})(j)\neq 0$ for some $j\in\langle 1,\ N-1\rangle$ implies $\operatorname{sgn}((\Pi\boldsymbol{p})(j))=\operatorname{sgn}((\Pi_{\varepsilon}\boldsymbol{p})(j))$, where $(\Pi\boldsymbol{p})(j)$ (resp. $(\Pi_{\varepsilon}\boldsymbol{p})(j)$) is the j-th component of $\Pi\boldsymbol{p}$ (resp. $\Pi_{\varepsilon}\boldsymbol{p}$). Then obviously we have $N_{\pm}(\Pi\boldsymbol{p})\leqq N_{\pm}(\Pi_{\varepsilon}\boldsymbol{p})$. Therefore we obtain $N_{\pm}(\Pi\boldsymbol{p})\leqq N_{\pm}(\boldsymbol{p})$ for every $\boldsymbol{p}\in\boldsymbol{R}_0^{N+1}$. q. e. d.

§ 4. Homogeneous Dirichlet boundary conditions.

In this section we consider semilinear parabolic equations with homogeneous Dirichlet boundary conditions. In this case it may happen in general that the number of peaks of a solution increases. After giving such an example, we impose an additional restriction to the term $f(t,\ u)$ and show that the same results as Sections 1 and 2 hold under the restriction.

EXAMPLE 4.1. Consider the following equation,

$$\begin{cases}u_t=u_{xx}+t & \text{in}\quad Q\,,\\ u(0,\ t)=u(1,\ t)=0\,, & t\in(0,\ T),\\ u(x,\ 0)=0\,, & x\in(0,\ 1)\,.\end{cases} \tag{4.1}$$

Then equation (4.1) has a smooth solution which can be expressed as

$$u(x,\ t)=\int_{0}^{t}ds\int_{0}^{1}U(x,\ y,\ t-s)s\,dy\ ,$$

where $U(x, y, t)$ is the Green function of $L=\partial^{2}/\partial x^{2}-\partial/\partial t$ with homogeneous Dirichlet boundary conditions. Since $u(x, t)$ is positive for $x\in(0, 1)$ and $t>0$, we have $\sharp_{p}(u(t))\geqq 1$ for $t>0$. Therefore we have $0=\sharp_{p}(u^{0})<\sharp_{p}(u(t))$ for $t>0$.

Considering Example 4.1, we impose the following additional restriction to f,

$$f(t,\ 0)=0\ . \tag{4.2}$$

By noting Remark 1.1, the equation we consider is written as

$$\begin{cases} u_{t}=a(x,\ t)u_{xx}+b(x,\ t)u_{x}+f_{1}(t,\ u)u & \text{in } Q\ , \\ u(0,\ t)=u(1,\ t)=0\ , & t\in(0,\ T)\ , \\ u(x,\ 0)=u^{0}(x)\ , & x\in(0,\ 1)\ . \end{cases} \tag{4.3}$$

Corresponding to Theorem 1.2 and Corollary 1.3, the following results hold.

THEOREM 4.2. *Suppose Assumption* 1 *and* (4.2). *Let* $u^{0}\in C^{1}[0, 1]$ *satisfy* $u^{0}(0)=u^{0}(1)=0$. *Then, equation* (4.3) *has a unique solution* $u(t)\in C^{1}[0, 1]$, $t\in[0, T]$ *and it holds that*

$$\sharp_{p}(u(t))\leqq\sharp_{p}(u^{0})\ ,\quad \sharp_{v}(u(t))\leqq\sharp_{v}(u^{0})\qquad \text{for}\quad t\in[0,\ T]\ . \tag{4.4}$$

COROLLARY 4.3. *Under the same assumptions as Theorem* 4.2, $\sharp_{p}(u(t))$ *and* $\sharp_{v}(u(t))$ *are monotonically decreasing.*

Theorem 4.2 can be proved in a similar line to Theorem 1.2 if we obtain a finite difference scheme for (4.3) whose solution has the same property as (4.4).

Discretize $\bar{Q}$ by a (h, τ)-rectangular net. This time our grid points consist of $(x_{j}, k\tau)$, $x_{j}=jh$, $j\in\langle 0, N\rangle$, $k\in\langle 0, N_{T}\rangle$. We seek a net function $u_{h}(x_{j}, k\tau)=u_{h}^{k}(x_{j})$ satisfying

$$\begin{cases} \{u_{h}^{k+1}(x_{j})-u_{h}^{k}(x_{j})\}/\tau=a_{j}^{k}\Delta_{h}u_{h}^{k}(x_{j})+b_{j}^{k}D_{h}u_{h}^{k}(x_{j}) \\ \qquad\qquad +f_{1}(k\tau,\ u_{h}^{k}(x_{j}))u_{h}^{k+1}(x_{j})\ , \\ u_{h}^{k}(x_{0})=u_{h}^{k}(x_{N})=0\ , \\ u_{h}^{0}(x_{j})=u^{0}(x_{j}) \qquad \text{for}\quad j\in\langle 1,\ N-1\rangle,\ k\in\langle 0,\ N_{T}-1\rangle\ . \end{cases} \tag{4.5}$$

THEOREM 4.4. *Under Assumption* 1 *and the conditions*

$$\tau\leqq h^{2}/(2\|a\|),\ h\leqq 2a_{0}/\|b\| \quad \textit{and} \tag{4.6}$$

$$\tau<1/M_{0} \qquad \textit{if}\quad M_{0}>0\ , \tag{4.7}$$

the difference scheme (4.5) *is* L^{∞}*-stable in the sense,*

$$\max_{j\in\langle 0,N\rangle,\, k\in\langle 0,N_T\rangle} |u_h^k(x_j)| \leqq U_0(\tau), \tag{4.8}$$

where

$$U_0(\tau)=\begin{cases} \|u^0\| \exp\{TM_0/(1-\tau M_0)\}, & M_0>0, \\ \|u^0\|, & M_0\leqq 0. \end{cases}$$

Furthermore, if the exact solution u *of* (4.3) *is smooth* (*see Remark* 2.2), u_h *converge to* u *uniformly in* $\bar{Q}$.

Theorem 4.4 can be proved in a similar line to Theorem 2.1. So we omit the proof.

THEOREM 4.5. *Suppose Assumption* 1. *Then, for every* $\varepsilon>0$ *there exists a number* $h_0>0$ *such that under the condition*

$$h\leqq h_0 \quad \text{and} \quad \tau\leqq h^2/\{4\|a\|+(2\|a_x\|+\varepsilon)h\} \tag{4.9}$$

the solution u_h *of* (4.5) *satisfies*

$$\#_p^h(u_h^k)\leqq\#_p^h(u_h^0), \quad \#_v^h(u_h^k)\leqq\#_v^h(u_h^0) \qquad \text{for} \quad k\in\langle 0, N_T\rangle. \tag{4.10}$$

REMARK 4.6. $\#_p^h(u_h^k)$ and $\#_v^h(u_h^k)$ in (4.10) should be understood as follows. Let p_h^k be broken lines connecting $(x_j, p_h^k(x_j))$, $j\in J=\{1/2, 3/2, \cdots, N-1/2\}$, where $x_j=jh$,

$$p_h^k(x_j)=\{u_h^k(x_j+h/2)-u_h^k(x_j-h/2)\}/h.$$

Replacing $[0, 1]$ by $[h/2, 1-h/2]$ in the definition of $N_\pm$, we define $N_\pm$ for continuous functions defined on $[h/2, 1-h/2]$. Thus we understand $\#_p^h(u_h^k)=N_+(p_h^k)$ and $\#_v^h(u_h^k)=N_-(p_h^k)$.

PROOF OF THEOREM 4.5. Let p_h^k be as above. Fix k arbitrarily. We show that $N_\pm(p_h^{k+1})\leqq N_\pm(p_h^k)$. p_h^k and p_h^{k+1} satisfy (2.14) for $j\in J_0=\{3/2, 5/2, \cdots, N-3/2\}$. Substituting $u_h^k(x_1)=hp_{1/2}^k$ and $u_h^k(x_2)=h(p_{3/2}^k+p_{1/2}^k)$ into (4.5) with $j=1$, we have

$$(p_{1/2}^{k+1}-p_{1/2}^k)/\tau=a_1^k(p_{3/2}^k-p_{1/2}^k)/h^2+b_1^k(p_{3/2}^k+p_{1/2}^k)/(2h)+(f_1)_{1/2}^k p_{1/2}^{k+1}.$$

Similarly we obtain

$$\begin{aligned}(p_{N-1/2}^{k+1}-p_{N-1/2}^k)/\tau=a_{N-1}^k(p_{N-3/2}^k-p_{N-1/2}^k)/h^2 \\ -b_{N-1}^k(p_{N-1/2}^k+p_{N-3/2}^k)/(2h)+(f_1)_{N-3/2}^k p_{N-1/2}^{k+1}.\end{aligned}$$

Hence, if we set $p_{-1/2}^k=p_{N+1/2}^k=0$, we have

$$p_j^{k+1}=\lambda_{j,j-1}^k p_{j-1}^k+\lambda_{j,j}^k p_j^k+\lambda_{j,j+1}^k p_{j+1}^k \qquad \text{for} \quad j\in J, \tag{4.11}$$

where $\lambda_{j,j}^k$ and $\lambda_{j,j\pm1}^k$, $j\in J_0$ are those defined in (2.15) and

$$\lambda_{1/2,1/2}^k=\{2h^2-\tau(2a_1^k-hb_1^k)\}/\{2h^2(1-\tau(f_1)_{1/2}^k)\},$$

$$\lambda_{1/2,3/2}^k=\tau(2a_1^k+hb_1^k)/\{2h^2(1-\tau(f_1)_{1/2}^k)\},$$

$$\lambda_{N-1/2,N-3/2}^k=\tau(2a_{N-1}^k-hb_{N-1}^k)/\{2h^2(1-\tau(f_1)_{N-3/2}^k\},$$

$$\lambda_{N-1/2,N-1/2}^k=\{2h^2-\tau(2a_{N-1}^k+hb_{N-1}^k)\}/\{2h^2(1-\tau(f_1)_{N-3/2}^k)\}.$$

Thus p_h^{k+1} can be regarded as the image of p_h^k by a linear operator in $\boldsymbol{R}_0^{N+2}=\{\boldsymbol{p}=(p_{-1/2}, p_{1/2}, \cdots, p_{N+1/2});\ p_j\in\boldsymbol{R},\ j\in J,\ p_{-1/2}=p_{N+1/2}=0\}$. It is not difficult to see that condition (4.9) implies (2.11). Applying Theorem 2.5, we get

$$N_\pm(p_h^{k+1};-1/2,\ N+1/2)\leqq N_\pm(p_h^k;\ -1/2,\ N+1/2).$$

Since $N_\pm(\boldsymbol{p};\ -1/2,\ N+1/2)=N_\pm(\boldsymbol{p};\ 1/2,\ N-1/2)$, we obtain $N_\pm(p_h^{k+1})\leqq N_\pm(p_h^k)$.

q. e. d.

REMARK 4.7. We can deal with the third boundary condition

$$\begin{cases}\alpha_0(t)u(0,\ t)-(1-\alpha_0(t))\partial u/\partial x(0,\ t)=0, \\ \alpha_1(t)u(1,\ t)+(1-\alpha_1(t))\partial u/\partial x(1,\ t)=0,\end{cases} \tag{4.12}$$

where $\alpha_i(t)$, $i=0$, 1, are smooth functions satisfying $0\leqq\alpha_i\leqq 1$. We approximate (4.12) by

$$\begin{cases}\alpha_0^k u_h^k(x_0)-(1-\alpha_0^k)(u_h^k(x_1)-u_h^k(x_0))/h=0, \\ \alpha_1^k u_h^k(x_N)+(1-\alpha_1^k)(u_h^k(x_N)-u_h^k(x_{N-1}))/h=0,\end{cases}$$

which lead to

$$\alpha_0^k u_h^k(x_0)=(1-\alpha_0^k)p_{1/2}^k, \qquad \alpha_0^k u_h^k(x_1)=\{h\alpha_0^k+(1-\alpha_0^k)\}\,p_{1/2}^k,$$

$$\alpha_1^k u_h^k(x_N)=-(1-\alpha_1^k)p_{N-1/2}^k, \qquad \alpha_1^k u_h^k(x_{N-1})=-\{h\alpha_1^k+(1-\alpha_1^k)\}\,p_{N-1/2}^k.$$

By using these equations, we can obtain the same results as those in the case of Dirichlet boundary conditions.

§ 5. The blowing-up case.

We have hitherto limited ourselves to non-blowing-up solutions by imposing (iii) of Assumption 1. Here we remove the assumption that $\partial f/\partial u$ is less than some real number in $[0, T]\times\boldsymbol{R}$ from Assumption 1 and denote by Assumption 1′ the remaining assumptions. Under Assumption 1′ solutions may blow up at a time $T_*\in(0, T]$ but the following results corresponding to Corollaries 1.3 and 4.3 are obtained.

COROLLARY 5.1. *Suppose Assumption* 1′ *and* $u^0\in C^1[0, 1]$. *Let* $[0, T_*)$ *be the interval where the solution of* (1.1) *exists. Then,* $u(t)\in C^1[0, 1]$ *for* $t\in[0, T_*)$ *and* $\#_p(u(t))$ *and* $\#_v(u(t))$ *are monotonically decreasing in* $[0, T_*)$.

COROLLARY 5.2. *Suppose Assumption* 1′ *and* (4.2). *Let* $u^0 \in C^1[0, 1]$ *satisfy* $u^0(0)=u^0(1)=0$ *and* $[0, T_*)$ *be the interval where the solution of* (4.3) *exists. Then,* $u(t) \in C^1[0, 1]$ *for* $t \in [0, T_*)$ *and* $\#_p(u(t))$ *and* $\#_v(u(t))$ *are monotonically decreasing in* $[0, T_*)$.

PROOFS OF COROLLARIES 5.1 AND 5.2. Both corollaries are proved in a similar line. So we show only the proof of the result of Corollary 5.1 concerning the number of peaks. It is sufficient to prove that $\#_p(u(t_2)) \leqq \#_p(u(t_1))$ for any fixed t_1, t_2, $0 \leqq t_1 < t_2 < T_*$. Since the solution u is bounded in $[0, 1] \times [0, t_2]$, we can modify f to obtain $\tilde{f}$ such that $\tilde{f}$ satisfies (iii) of Assumption 1 and that $\tilde{f}$ is equal to f in $[0, t_2] \times [U_1, U_2]$, where U_2 (resp. U_1) is the lowest upper (resp. largest lower) bound of u in $[0, 1] \times [0, t_2]$. The equation (1.1) with $\tilde{f}$ in place of f has the same solution u in $[0, 1] \times [0, t_2]$. Hence we have $\#_p(u(t_2)) \leqq \#_p(u(t_1))$ by Corollary 1.3. q.e.d.

We conclude by applying Theorem 2.5 to a finite difference scheme for a blowing-up problem considered by Nakagawa [7].

EXAMPLE 5.3. In (4.3) take $a \equiv 1$, $b \equiv 0$, $f_1 = u$ and $u^0 \geqq 0$. Nakagawa's scheme for this equation is the following one with variable time-steps τ_k:

$$
(5.1) \quad \begin{cases} (u_h^{k+1}(x_j) - u_h^k(x_j))/\tau_k = \Delta_h u_h^k(x_j) + (u_h^k(x_j))^2 , \\ u_h^k(x_0) = u_h^k(x_N) = 0 , \\ u_h^0(x_j) = u^0(x_j) , \\ \tau_k = \tau \times \min(1, 1/\{h \sum_{j=1}^{N-1} (u_h^k(x_j))^2\}^{1/2}) , \qquad \text{for} \quad j \in \langle 1, N-1 \rangle,\ k=0, 1, \cdots , \end{cases}
$$

where h $(=1/N)$ and τ are given positive numbers and $x_j = jh$. In his paper it is proved that when the exact solution blows up at a finite time T_*, the numerical blowing-up times $\left(\sum_{k=0}^{+\infty} \tau_k\right)(h)$ converge to T_* as $h \downarrow 0$ under the condition $\tau \leqq h^2/2$. In a similar line to the proof of Theorem 4.5, we observe that the first difference p_j^k satisfies (4.11) with

$$
\begin{aligned}
\lambda_{j,j+1}^k &= \tau_k/h^2 & & j \in J - \{N-1/2\} , \\
\lambda_{j,j}^k &= \begin{cases} 1 - \tau_k/h^2 + \tau_k u_h^k(x_1) , & j = 1/2 , \\ 1 - 2\tau_k/h^2 + \tau_k \{u_h^k(x_{j+1/2}) + u_h^k(x_{j-1/2})\} , & j \in J_0 , \\ 1 - \tau_k/h^2 + \tau_k u_h^k(x_{N-1}) , & j = N-1/2 , \end{cases} \\
\lambda_{j,j-1}^k &= \tau_k/h^2 , & & j \in J - \{1/2\} .
\end{aligned}
$$

From (5.1) we have $u_h^k(x_j) \geqq 0$ for $j \in \langle 0, N \rangle$, $k=0, 1, \cdots$ if $\tau \leqq h^2/2$. Hence the condition $\tau \leqq h^2/4$ leads to (2.11), which implies

$$
\#_p^h(u_h^{k+1}) \leqq \#_p^h(u_h^k) , \quad \#_v^h(u_h^{k+1}) \leqq \#_v^h(u_h^k) \qquad \text{for any} \quad k .
$$

Thus, the difference scheme (5.1) with $\tau \leqq h^2/4$ gives solutions such that the numerical blowing-up times converge to the exact one and that the number of peaks of each solution is monotonically decreasing.

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Added in proof.

After having written this article the author was kindly informed by Mr. Akira Mizutani of Gakushuin University that the proof of Theorem 2.5 could be shorten by applying the L-U decomposition to the matrix Π. The condition (2.11) and (3.1) imply that Π is decomposed into $\Pi_1\Pi_2$, where $\Pi_k=(\lambda^k_{i,j})$, $k=1, 2$, are matrices such that $\lambda^1_{i,i}=1$ and $\lambda^2_{i,i}>0$ $(i=1, \cdots, N-1)$, $\lambda^1_{i+1,i}$, $\lambda^2_{i,i+1}>0$ $(i=1, \cdots, N-2)$, and that the other elements are all zeros. Hence for Theorem 2.5 it is sufficient to show that $N_\pm(\Pi_k \boldsymbol{p}) \leqq N_\pm(\boldsymbol{p})$, $k=1, 2$, which is simpler than to show (2.12) directly since Π_k are bi-diagonal.

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