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On the hyperplane section principle of Lefschetz

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The purpose of this article is to describe various versions of the following principle:

Let A be an ample divisor on a manifold M. Then the structure of M is closely related to that of A.

This was the philosophy of the study of Lefschetz on the hyperplane sections. Recently Sommese [**24**] developed various techniques in the above spirit and he found many examples of manifolds A which cannot be ample divisors in any manifold M because the above 'relation' implies so severe conditions on M that they cannot be compatible with each other. Here, inspired also by the work [**17**] of Mori, we develop further the methods of Sommese, improve some of his results, answer to questions and conjectures raised by him, and give several new examples. We shall find the tools thus obtained to be very powerful in the study of polarized varieties (see [**5**], [**6**]).

In this paper we work in the category of algebraic spaces defined over an algebraically closed field K of any characteristic (However, in the statements indicated by $/\boldsymbol{C}$, K is assumed to be the complex number field $\boldsymbol{C}$. For example, $K=\boldsymbol{C}$ in § 1, but (2.1) is valid in positive characteristic cases too.). In § 1 we review the classical Lefschetz theory. § 2 is devoted to the study of various types of extension theorems from A to M, for cohomologies, line bundles, linear systems, morphisms and so on. In § 3 we consider the case in which A is isomorphic to a complete intersection in a projective space. In § 4 we study the case in which A is a fiber bundle over a manifold. § 5 is for the case where A is a blowing up of another manifold. Two appendixes about a couple of techniques in this paper are added for the convenience of the reader.

Our results in the case char $K>0$ are far from satisfactory because of the lack of a vanishing theorem of Kodaira type. However, the author expects that our principle itself is of great importance in positive characteristic cases too.

NOTATION, CONVENTION and TERMINOLOGY.

Variety means an irreducible reduced algebraic space which is assumed to be proper over K unless otherwise stated explicitly. Smooth means nonsingular. Manifold means a smooth variety. Vector bundles are occasionally confused with the locally free sheaves of their sections. Tensor products of line bundles are indicated by additive notation.

We show examples of our notation which is similar to that in [8], [11], [3], [4] etc. Abbreviated forms are sometimes used when there is no danger of confusion.

$\mathrm{Pic}(S)$: The group of line bundles on an algebraic space S.

$\mathrm{Pic}_0(S)$: The group scheme of line bundles on S which are algebraically equivalent to zero.

I_S: The trivial line bundle on S.

$\mathcal{O}_S$: The structure sheaf of S.

$h^p(S, \mathcal{F}) := \dim H^p(S, \mathcal{F})$, where $\mathcal{F}$ is a coherent sheaf on S.

$\chi(S, \mathcal{F}) := \sum_{p=0}^{\infty}(-1)^p h^p(S, \mathcal{F})$.

$E^{\vee}$: The dual bundle of a vector bundle E on S.

$S^k E$: The k-th symmetric product of E.

$\det E$: The determinant line bundle of E.

$c_j(E)$: The j-th Chern class of E.

$\mathcal{F}[E] := \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{E}$, where $\mathcal{E}$ is the locally free sheaf of sections of E.

$\boldsymbol{P}(E)$: The projective space bundle $E^{\vee} - \{\text{zero section}\}/K^{\times}$ over S.

H^E: The relatively ample tautological line bundle on $\boldsymbol{P}(E)$ (see (4.6)).

$|L|$: The complete linear system of effective Cartier divisors associated with a line bundle L.

$G(S, L)$: The graded K-algebra $\oplus_{t=0}^{\infty} \Gamma(S, tL)$.

$[\Lambda]$: The line bundle defined by a linear system Λ on S.

$Bs\Lambda$: The intersection of all the members of Λ.

ρ_Λ: The rational mapping $S \to \boldsymbol{P}^{\dim \Lambda}$ induced by Λ.

L_T, Λ_T: The pull-backs of L, Λ by a given morphism $T \to S$.

$\mathcal{O}_W(1)$: The invertible sheaf defined by hyperplane sections on $W \subset \boldsymbol{P}^N$.

$\{X\}$: The algebraic cycle represented by a subspace X.

ω_V: The dualizing sheaf of a locally Macaulay variety V.

K^M: The canonical bundle of a manifold M.

Ω_M^p: The sheaf of holomorphic p-forms on M. Note that $\omega_M = \mathcal{O}_M[K^M] = \Omega_M^{\dim M}$.

$H^{p,q}(M, E) := H^q(M, \Omega_M^p[E])$.

$\mathrm{Alb}(M)$: The Albanese variety of M.

$Q_Z(M)$: The blowing up of M with center Z.

$g(C)$: The genus of a curve C.

$\pi_i(Y)$: The i-th homotopy group of an analytic space Y.

§ 1. Lefschetz Theorem.

In this section we summarize various versions of Lefschetz Theorem.

THEOREM/$\boldsymbol{C}$. *Let A be an ample divisor on a manifold M with* $\dim M=n$. *Let ι be the embedding $A\subset M$. Then*

I_i: $\pi_i(\iota):\pi_i(A)\to\pi_i(M)$ *is bijective for $i<n-1$.*

I_i^*: $\pi_i(\iota)$ *is surjective for $i=n-1$.*

II_i: $H_i(\iota):H_i(A;\boldsymbol{Z})\to H_i(M;\boldsymbol{Z})$ *is bijective for $i<n-1$.*

II_i^*: $H_i(\iota)$ *is surjective for $i=n-1$.*

III_i: $H^i(\iota):H^i(M;\boldsymbol{Z})\to H^i(A;\boldsymbol{Z})$ *is bijective for $i<n-1$.*

III_i^*: $H^i(\iota)$ *is injective and* $\mathrm{Coker}(H^i(\iota))$ *is torsion free for $i=n-1$.*

$\mathrm{IV}_{p,q}$: $H^{p,q}(\iota):H^q(M,\Omega_M^p)\to H^q(A,\Omega_A^p)$ *is bijective for $p+q<n-1$ if A is smooth or if $p=0$.*

$\mathrm{IV}_{p,q}^*$: $H^{p,q}(\iota)$ *is injective for $p+q=n-1$ if A is smooth or if $p=0$.*

V: $\mathrm{Pic}(\iota):\mathrm{Pic}(M)\to\mathrm{Pic}(A)$ *is bijective for $n>3$.*

V^*: $\mathrm{Pic}(\iota)$ *is injective and* $\mathrm{Coker}(\mathrm{Pic}(\iota))$ *is torsion free for $n=3$.*

In the following statements, A is assumed to be smooth.

VI: $\mathrm{Pic}_0(\iota):\mathrm{Pic}_0(M)\to\mathrm{Pic}_0(A)$ *is an isomorphism for $n>2$.*

VI^*: $\mathrm{Pic}_0(\iota)$ *is injective for $n=2$.*

VII: $\mathrm{Alb}(\iota):\mathrm{Alb}(A)\to\mathrm{Alb}(M)$ *is an isomorphism for $n>2$.*

VII^*: $\mathrm{Alb}(\iota)$ *is surjective and a general fiber of it is connected if $n=2$.*

PROOF. Using the Morse theory, we prove $\mathrm{I}^{(*)}$ and $\mathrm{II}^{(*)}$ (see [**14**]). Dualizing $\mathrm{II}^{(*)}$, we obtain $\mathrm{III}^{(*)}$. Considering the Hodge decomposition for $\boldsymbol{C}$-cohomology, we prove $\mathrm{IV}_{p,q}^{(*)}$ if A is smooth. $\mathrm{IV}_{0,q}^{(*)}$ follows from the vanishing theorem of Kodaira. $\mathrm{V}^{(*)}$ follows from III_1, $\mathrm{III}_2^{(*)}$, $\mathrm{IV}_{0,1}$ and $\mathrm{IV}_{0,2}^{(*)}$ since we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} H^1(M;\boldsymbol{Z}) & \longrightarrow & H^1(M,\mathcal{O}_M) & \dashrightarrow & H^1(M,\mathcal{O}_M^\times) & \longrightarrow & H^2(M;\boldsymbol{Z}) & \longrightarrow & H^2(M,\mathcal{O}_M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(A;\boldsymbol{Z}) & \longrightarrow & H^1(A,\mathcal{O}_A) & \longrightarrow & H^1(A,\mathcal{O}_A^\times) & \longrightarrow & H^2(A;\boldsymbol{Z}) & \longrightarrow & H^2(A,\mathcal{O}_A). \end{array}$$

Similarly $\mathrm{VI}^{(*)}$ (resp. $\mathrm{VII}^{(*)}$) follows from $\mathrm{III}_1^{(*)}$ and $\mathrm{IV}_{0,1}^{(*)}$ (resp. $\mathrm{II}_1^{(*)}$ and $\mathrm{IV}_{1,0}^{(*)}$).

§ 2. Extension theorems.

(2.1) LEMMA (compare [**24**] Lemma I-B). *Let A be an ample divisor on a scheme S and let $\mathcal{F}$ be a coherent sheaf on S. Suppose that* $\mathrm{Ker}(\mathcal{F}\to\mathcal{F}_A)\cong\mathcal{F}[-A]$ *and that the restriction $H^i(S,\mathcal{F}[tA])\to H^i(A,\mathcal{F}[tA]_A)$ is surjective for a fixed i and for any $t>0$. Then $H^{i+1}(S,\mathcal{F})=0$.*

PROOF. We have a natural exact sequence $H^i(S,\mathcal{F}[tA])\to H^i(A,\mathcal{F}[tA]_A)\to H^{i+1}(S,\mathcal{F}[(t-1)A])\to H^{i+1}(S,\mathcal{F}[tA])$. So the second assumption implies

$h^{i+1}(S, \mathcal{F}[(t-1)A])\leqq h^{i+1}(S, \mathcal{F}[tA])$ for any $t\geqq 1$. On the other hand, $H^{i+1}(S, \mathcal{F}[tA])=0$ for $t\gg 0$ since A is ample. Combining them, we infer that $h^{i+1}(S, \mathcal{F})\leqq h^{i+1}(S, \mathcal{F}[tA])=0$.

(2.2) LEMMA. *Let A be an ample divisor on a variety V and let $\mathcal{E}$ be a locally free sheaf on V. Suppose that $H^i(A, \mathcal{E}[tA]_A)=0$ for a fixed $i<\dim V$ and for every $t\leqq 0$. Then $H^i(V, \mathcal{E})=0$ either a) if V is locally Macaulay, or b) if $i=1$ and V is normal.*

PROOF. In both cases a) and b) we have $H^i(V, \mathcal{E}[tA])=0$ for $t\ll 0$ (see [**23**]). So this lemma is proved by the same argument as in [**24**], Lemma I-B.

(2.3) PROPOSITION. *Let A be an ample divisor on a normal variety V with $\dim V\geqq 2$ and let F be a line bundle on V. Suppose that $H^1(A, [F-tA]_A)=0$ for any $t>0$. Then $\Gamma(V, F)\to\Gamma(A, F_A)$ is surjective and $|F|_A=|F_A|$.*

PROOF. Put $\mathcal{E}=\mathcal{O}_V(F-A)$ and apply (2.2) to obtain $H^1(V, F-A)=0$. Our assertion follows from this.

(2.4) COROLLARY. *Let A and V be as above. Let E be an effective divisor on A such that $H^1(A, [E-tA])=0$ for any $t>0$. Suppose that $[E]=F_A$ for some $F\in\mathrm{Pic}(V)$. Then there exists an effective divisor $D\in|F|$ such that $D_A=E$.*

(2.5) COROLLARY. *Let A and V be as above. Then $\mathrm{Pic}(V)\to\mathrm{Pic}(A)$ is injective if $H^1(A, [-tA])=0$ for any $t>0$.*

PROOF. Suppose that $F_A=0$ for $F\in\mathrm{Pic}(V)$. Applying (2.4) to the trivial divisor $\in|F_A|$, we obtain an effective divisor $D\in|F|$ such that $D\cap A=\emptyset$. We infer that $D=0$ since A is ample. Consequently $F=0$.

(2.6) COROLLARY/$\boldsymbol{C}$. *Let A be an ample divisor on a normal variety V with $\dim V\geqq 3$. Then $\mathrm{Pic}(V)\to\mathrm{Pic}(A)$ is injective if A is normal.*

For a proof, use the Theorem 2 in [**18**].

REMARK. Coker$(\mathrm{Pic}(V)\to\mathrm{Pic}(A))$ is not always torsion free if V is not smooth.

(2.7) PROPOSITION. *Let A be an ample divisor on a variety V. Let Λ be a linear system on V such that $Bs\Lambda_A=\emptyset$ and $\dim\rho_\Lambda(A)<n-1=\dim A$. Then $Bs\Lambda=\emptyset$ and ρ_Λ is a morphism. Moreover, $\rho_\Lambda(V)=\rho_\Lambda(A)$.*

PROOF. Let $D_1, \cdots, D_{n-1}$ be general members of Λ. Put $B=\cap_{i=1}^{n-1}D_i$. Clearly codim $B\leqq n-1$ unless $B=\emptyset$. On the other hand, $B\cap A=\emptyset$ since $A\cap Bs\Lambda=\emptyset$ and $\dim\rho_\Lambda(A)<n-1$. Hence $\dim B\leqq 0$ since A is ample. Consequently $B=\emptyset$, which proves $Bs\Lambda=\emptyset$. In order to show $\rho_\Lambda(V)=\rho_\Lambda(A)$, we put $[\Lambda]=L$. Recall that $\rho_\Lambda(V)$ (resp. $\rho_\Lambda(A)$) corresponds to the graded subalgebra of $G(V, L)=\bigoplus_{t=0}^{\infty}\Gamma(V, tL)$ (resp. $G(A, L_A)$) generated by the linear subspace of $\Gamma(V, L)$ (resp. $\Gamma(A, L_A)$) defined by Λ (resp. Λ_A). Therefore, it suffices to show that $\Gamma(V, tL)\to\Gamma(A, tL)$ is injective for any $t>0$. If this were not true, $|tL-A|$ would contain an effective divisor E. For $r=\dim\rho_\Lambda(A)$, we would have $L^rA^{n-r-1}\{E\}=tL^{r+1}A^{n-r-1}\{V\}-L^rA^{n-r}\{V\}=tL_A^{r+1}A_A^{n-r-2}\{A\}-L_A^rA_A^{n-r-1}\{A\}<0$

since $L_A^{r+1}=0$. This would contradict Lemma B5 in Appendix.

(2.8) COROLLARY. *Let A be an ample divisor on a normal variety V with* $\dim V=n\geqq 3$. *Let F be a line bundle on V such that $Bs|F_A|=\emptyset$, $F^{n-1}\{A\}=0$ and $H^1(A, [F-tA]_A)=0$ for any $t>0$. Then $Bs|F|=\emptyset$ and $\rho_{|F|}(V)=\rho_{|F_A|}(A)$.*

For a proof, combine (2.3) and (2.7).

(2.9) COROLLARY/$\boldsymbol{C}$. *Let A and V be as in* (2.8) *and assume A to be smooth. Let $f: A\to S$ be a surjective morphism onto a projective scheme S with* $\dim S\leqq n-3$. *Suppose that $F_A=f^*H$ for some $F\in \mathrm{Pic}(V)$ and an ample line bundle H on S. Then f can be extended to a morphism $g: V\to S$.*

PROOF. Taking a positive multiple if necessary, we may assume that H is very ample. Then f is given by $\rho_{|f^*H|}$. We apply Lemma A4 in Appendix to obtain $H^1(A, [F-tA]_A)=0$ for any $t>0$. Now, from (2.8), we infer that $\rho_{|F|}$ gives the desired extension of f.

(2.10) COROLLARY/$\boldsymbol{C}$ (same as [**24**] Proposition III). *Let A be a smooth ample divisor on a manifold M with $n=\dim M$. Let $f: A\to S$ be a surjective morphism onto a projective scheme S with* $\dim S\leqq n-3$. *Then f can be extended to a morphism $g: M\to S$.*

PROOF. Clearly we may assume $\dim S>0$. So $n>3$. Using Lefschetz Theorem V, we find a line bundle F on M such that $F_A=f^*H$ for an ample line bundle H on S. Hence (2.9) applies.

REMARK. If in addition f is everywhere of maximal rank, then g is of maximal rank except at finite points.

§ 3. Defining equations.

(3.1) THEOREM. *Let L be a line bundle on a variety V and let A be the divisor defined by a non-zero section $\delta\in\Gamma(V, aL)$. Let $\xi_1, \cdots, \xi_k$ be homogeneous elements of the graded algebra $G(V, L)=\bigoplus_{t=0}^{\infty}\Gamma(V, tL)$ and suppose that $\eta_\alpha=\xi_{\alpha|A}$ ($\alpha=1, \cdots, k$) generate the graded algebra $G(A, L_A)$. Then $G(V, L)$ is generated by $\delta, \xi_1, \cdots, \xi_k$.*

(3.2) THEOREM. *Let V, L, A, δ, ξ_α and η_α be as above. Let $g_1, \cdots, g_r$ be homogeneous polynomials in k variables $Y_1, \cdots, Y_k$ with* $\deg Y_\alpha=\deg\eta_\alpha$ *for any $\alpha=1, \cdots, k$ such that all the relations among $\{\eta_\alpha\}$ in $G(A, L_A)$ are derived from $g_1(\eta_1, \cdots, \eta_k)=\cdots=g_r(\eta_1, \cdots, \eta_k)=0$. Then there exist r homogeneous polynomials $f_1, \cdots, f_r$ in $(k+1)$ variables $X_0, X_1, \cdots, X_k$ with* $\deg X_0=a=\deg\delta$, $\deg X_\alpha=\deg\xi_\alpha$ *for $\alpha\geqq 1$ such that $f_j(0, Y_1, \cdots, Y_k)=g_j$ for $1\leqq j\leqq r$ and that all the relations among $\delta, \xi_1, \cdots, \xi_k$ in $G(V, L)$ is derived from $f_1(\delta, \xi_1, \cdots \xi_k)=\cdots=f_r(\delta, \xi_1, \cdots, \xi_k)=0$.*

These two theorems can be proved by a similar argument to that in [**17**]. One can refer also [**4**], § 2. Roughly speaking, these theorems say that the structure of $G(V, L)$ is not much more complicated than that of $G(A, L_A)$

provided that $G(V, L)\to G(A, L_A)$ is surjective.

REMARK. V need not be assumed to be a variety if the defining ideal of A is isomorphic to $\mathcal{O}_V[-A]$.

(3.3) DEFINITION. A polarized scheme (S, L) with $\dim S=n$ is called a *weighted complete intersection of type* $(a_1, \cdots, a_r)$ *with weight* $(d_0, \cdots d_{n+r})$ if the graded algebra $G(S, L)$ has a generator system of homogeneous elements $\xi_0, \xi_1, \cdots \xi_{n+r}$ with $\deg \xi_j=d_j$ $(j=0, \cdots, n+r)$ such that the relation ideal among $\{\xi_j\}$ in $G(S, L)$ is generated by r homogeneous polynomials $f_1, \cdots, f_r$ with $\deg f_i=a_i$ for any i.

When there is no danger of confusion, we say that S is a weighted complete intersection if (S, L) is so for an ample line bundle L on S. We denote often $\mathcal{O}_S[tL]$ by $\mathcal{O}_S(t)$.

(3.4) It is not difficult to see that the above definition is equivalent to that of Mori [**17**]. Here we recall several results of him.

PROPOSITION. *Let* (V, L) *be a weighted complete intersection of type* $(a_1, \cdots, a_r)$ *with weight* $(d_0, \cdots, d_{n+r})$. *Then*

a) $H^p(V, tL)=0$ *for any* $t\in \boldsymbol{Z}$, $0<p<n$.

b) $L^n\{V\}=a_1\cdots a_r/d_0\cdots d_{n+r}$.

c) $\omega_V=\mathcal{O}_V(a_1+\cdots+a_r-(d_0+\cdots+d_{n+r}))$.

d) $\operatorname{Pic}(V)$ *is generated by* L *if* $n\geqq 3$.

(3.5) PROPOSITION. *Let* (V, L) *be a normal polarized variety and let* A *be a member of* $|dL|$ *for some* $d>0$. *Suppose that* (A, L_A) *is a weighted complete intersection of type* $(a_1, \cdots, a_r)$ *with weight* $(d_1, \cdots, d_{n+r})$ *where* $n=\dim V\geqq 3$. *Then* (V, L) *is a weighted complete intersection of type* $(a_1, \cdots, a_r)$ *with weight* $(d, d_1, \cdots, d_{n+r})$.

For a proof, combine (2.3) and (3.2). Details can be found in [**17**], Theorem 3.6.

(3.6) COROLLARY. *Let* A *be an ample divisor on a normal variety* V *with* $n=\dim V>3$. *Suppose that* (A, L_A) *is a weighted complete intersection for some* $L\in\operatorname{Pic}(V)$. *Then* (V, L) *is a weighted complete intersection, too.*

For a proof, use (2.5) and (3.4) d) to obtain that $[A]=dL$ for some positive integer d.

(3.7) COROLLARY/$\boldsymbol{C}$. *Let* A *be an ample divisor on a manifold* M *with* $n=\dim M\geqq 4$. *If* A *is a weighted complete intersection, then so is* M.

For a proof, use Lefschetz Theorem V.

(3.8) Finally, we consider the case in which A is a complete intersection in $\boldsymbol{P}^{n+r-1}$.

PROPOSITION (compare [**24**], Proposition VI). *Let* V, L *and* A *be as in* (3.5). *Suppose further that* $d_1=d_2=\cdots=d_{n+r}=1$, *or equivalently,* A *is a complete intersection in the usual sense. If* $d=1$, *then* V *is also a complete intersection of* *the same type. If $d>1$, then the inclusion $A\subset \boldsymbol{P}^{n+r-1}$ can be extended to a morphsim $V\to \boldsymbol{P}^{n+r-1}$. Moreover d divides one of $a_1, \cdots, a_r$.*

PROOF. The first assertion is a special case of (3.5). So we consider the case in which $d>1$. Then we see easily that $\Gamma(V, tdL)\otimes\Gamma(V, L)\to\Gamma(V, (td+1)L)$ is surjective for any t. Letting $t\gg 0$, we infer $Bs|L|\subset Bs|(td+1)L|=\emptyset$. Hence $\rho_{|L|}$ gives the desired extension $V\to \boldsymbol{P}^{n+r-1}$. In order to prove the final assertion, we apply Theorem 3 in [**19**] p. 45 to infer that $(\Gamma(V, (td+1)L))^{\otimes d}\to\Gamma(V, d(td+1)L)$ is surjective for $t\gg 0$. Therefore δ^{td+1} can be represented as a polynomial of elements in $\Gamma(V, (td+1)L)$, where δ is as in (3.2). This gives rise to a relation among $(\delta, \xi_1, \cdots, \xi_{n+r})$ which is monic with respect to δ since $\Gamma(V, L)\otimes\Gamma(V, tdL)\to\Gamma(V, (td+1)L)$ is surjective. This implies that one of the fundamental relations $f_1, \cdots, f_r$ among $(\delta, \xi_1, \cdots, \xi_{n+r})$ in $G(V, L)$ must be monic with respect to δ. So $d=\deg\delta$ divides one of $\{\deg f_j=a_j\}$.

(3.9) COROLLARY. *Let A be an ample divisor on a normal variety V with* $\dim V\geqq 3$. *Suppose that $(A, L_A)\cong(\boldsymbol{P}^{n-1}, \mathcal{O}(1))$ for some $L\in \mathrm{Pic}(V)$. Then $V\cong \boldsymbol{P}^n$ and A is a hyperplane on it.*

PROOF. Put $[A]_A=\mathcal{O}(d)$. (2.5) proves $[A]=dL$. So L is ample on V. Hence (3.8) implies $d=1$. Our assertion follows from this.

(3.10) COROLLARY/$\boldsymbol{C}$. *Let M be a manifold which contains $A\cong \boldsymbol{P}^{n-1}$ ($n\geqq 3$) as an ample divisor. Then $M\cong \boldsymbol{P}^n$ and A is a hyperplane on it.*

For a proof, use Lefschetz Theorem V$^{(*)}$.

(3.11) EXAMPLE/$\boldsymbol{C}$ (compare [**24**], Conjecture IV-A). Let G be the graded algebra generated by $(\delta, \xi_1, \cdots, \xi_{n+1})$ with relation $\delta^3+\delta\sum_{j=1}^{n+1}\xi_j^4+\sum_{j=1}^{n+1}\xi_j^6=0$, where $\deg\delta=2$ and $\deg\xi_j=1$ for $1\leqq j\leqq n+1$. It is easy to see that $M=Proj(G)$ is a smooth weighted complete intersection. Let A be the ample divisor on M defined by $\delta=0$. Clearly A is isomorphic to a smooth hypersurface of degree six in $\boldsymbol{P}^n$. M is a three-sheeted covering of $\boldsymbol{P}^n$, but is not cyclic.

(3.12) EXAMPLE/$\boldsymbol{C}$. Consider the graded algebra G generated by $(\delta, \xi_1, \cdots, \xi_{n+2})$ with relations $\delta^2+\sum_{j=1}^{n+2}\xi_j^4=\delta^3+\sum_{j=1}^{n+2}\xi_j^6=0$, where $\deg\delta=2$ and $\deg\xi_j=1$. It is easy to see that $M=Proj(G)$ is a smooth weighted complete intersection of type (4, 6) with weight $(2, 1, \cdots, 1)$. Let A be the ample divisor on M defined by $\delta=0$. Then A is a smooth complete intersection of type (4, 6) in $\boldsymbol{P}^{n+1}$. M is the normalization of the hypersurface defined by $(\sum_{j=1}^{n+2}\xi_j^4)^3+(\sum_{j=1}^{n+2}\xi_j^6)^2=0$ in $\boldsymbol{P}^{n+1}$ of degree 12, which has cusp singularities along A.

§4. Fiber bundles and vector bundles.

(4.1) PROPOSITION. *Let $f: V\to S$ be a fiber space and suppose that $H^2(S, \mathcal{O}_S^\times)=0$. Then $H^1(V, \mathcal{O}_V^\times)\to H^0(S, R^1f_*\mathcal{O}_V^\times)$ is surjective.*

For a proof, apply the theory of Leray spectral sequence and use the fact $f_*\mathcal{O}_V^\times=\mathcal{O}_S^\times$.

(4.2) REMARK/$\boldsymbol{C}$. $H^2(S, \mathcal{O}_S^\times)=0$ if $H^2(S, \mathcal{O}_S)=H^3(S; \boldsymbol{Z})=0$. In particular, $H^2(C, \mathcal{O}_C^\times)=0$ for any curve C.

(4.3) REMARK/$\boldsymbol{C}$. When f is a fiber bundle with fibers being isomorphic to F, we have $R^1f_*\mathcal{O}_V^\times \cong R^2f_*\boldsymbol{Z}_V$ if $H^1(F, \mathcal{O}_F)=H^2(F, \mathcal{O}_F)=0$, where $\boldsymbol{Z}_V$ denotes the constant sheaf on V with each stalk $\cong \boldsymbol{Z}$. In this case $H^0(S, R^1f_*\mathcal{O}_V^\times)=H^0(S, R^2f_*\boldsymbol{Z}_V)$ is nothing other than the subgroup of $H^2(F; \boldsymbol{Z})$ consisting of stabilized elements with respect to the monodromy action of $\pi_1(S)$.

(4.4) PROPOSITION/$\boldsymbol{C}$. *Let $f: M \to S$ be a fiber bundle over a manifold S with $H^2(S, \mathcal{O}_S^\times)=0$. Suppose that the fiber F of f is a weighted complete intersection with* $\dim F \geqq 3$. *Then there exists $L \in \mathrm{Pic}(M)$ such that $L_F=\mathcal{O}_F(1)$.*

PROOF. We have $H^1(F, \mathcal{O})=H^2(F, \mathcal{O})=0$ (see (3.4), a)). Moreover (3.4), d) implies that $c_1(\mathcal{O}_F(1))$ is stabilized by the monodromy action. Hence our assertion follows from (4.3) and (4.1).

(4.5) PROPOSITION/$\boldsymbol{C}$. *Let $f: M \to S$ be a $\boldsymbol{P}^n$-bundle over a manifold S with $H^2(S, \mathcal{O}_S^\times)=0$. Then there exists $L \in \mathrm{Pic}(M)$ such that $L_F=\mathcal{O}(1)$ for each fiber $F \cong \boldsymbol{P}^n$ of f.*

PROOF. The above argument works also for $n=1, 2$.

(4.6) For any vector bundle E on S with rank $E=r=n+1$, we consider the $\boldsymbol{P}^n$-bundle $\Pi: \boldsymbol{P}(E)=E^\vee-\{0\text{-}section\}/K^\times \to S$. Each point y on $\boldsymbol{P}(E)$ corresponds to a vector subspace of $E^\vee_{\Pi(y)}$ of dimension one. Hence Π^*E has a natural quotient bundle of rank one on $\boldsymbol{P}(E)$, which shall be denoted by H^E. We see easily that $H^E{}_F=\mathcal{O}(1)$ for each fiber $F \cong \boldsymbol{P}^n$ of Π. Moreover $\Pi_*(\mathcal{O}_{\boldsymbol{P}(E)}[H^E])=\mathcal{O}_S[E]$. On the other hand, when $f: M \to S$ is a $\boldsymbol{P}^n$-bundle and L is a line bundle on M such that $L_F=\mathcal{O}(1)$ for each fiber $F \cong \boldsymbol{P}^n$ of f, then $f_*(\mathcal{O}_M[L])$ is locally free on S. Moreover, letting E be the corresponding vector bundle, we have $(M, L) \cong (\boldsymbol{P}(E), H^E)$. Thus, in many cases, the study of $\boldsymbol{P}^n$-bundles can be reduced to the study of vector bundles.

(4.7) DEFINITION. A vector bundle E is said to be *ample* if the line bundle H^E on $\boldsymbol{P}(E)$ is so.

(4.8) REMARK. Let $0 \to I \to E \to Q \to 0$ be an exact sequence of vector bundles on S, where I denotes the trivial line bundle. Then $\boldsymbol{P}(Q)$ is a divisor on $\boldsymbol{P}(E)$ defined by $\delta \in \Gamma(\boldsymbol{P}(E), H^E) \cong \Gamma(S, E)$ which comes from $\Gamma(S, I) \cong K$. Moreover $[\boldsymbol{P}(Q)]=H^E$ and $H^E{}_{\boldsymbol{P}(Q)}=H^Q$. Conversely, let D be a member of $|H^E|$ such that the fiber D_x of D over $x \in S$ is a hyperplane on $\boldsymbol{P}(E)_x \cong \boldsymbol{P}^{r-1}$ for every $x \in S$. Then, the exact sequence $0 \to \mathcal{O}_{\boldsymbol{P}(E)} \to \mathcal{O}_{\boldsymbol{P}(E)}[H^E] \to \mathcal{O}_D[H^E] \to 0$ goes down by Π_* to the following exact sequence: $0 \to \mathcal{O}_S \to \mathcal{O}_S[E] \to \Pi_*\mathcal{O}_D[H^E] \to 0$. Thus D defines a trivial sub-line bundle of E since $\Pi_*\mathcal{O}_D[H^E]$ is locally free.

(4.9) PROPOSITION. *Let $f: V \to X$ be a surjective morphism from a locally Macaulay variety V onto a manifold X. Let A be an effective divisor on V which is relatively ample with respect to f. Suppose that f makes A a $\boldsymbol{P}^{n-1}$-*

bundle over X. Let V_x and A_x denote the fibers of f and f_A over $x \in X$ respectively. Further assume that one of the following conditions: a) $[A]_{A_x} = \mathcal{O}(1)$ *on $A_x \cong \boldsymbol{P}^{n-1}$, or* b) *$n \geqq 3$ and $L_{A_x} = \mathcal{O}(1)$ for some $L \in \mathrm{Pic}(V)$. Then f is a $\boldsymbol{P}^n$-bundle over X and A_x is a hyperplane on $V_x \cong \boldsymbol{P}^n$ for every x.*

PROOF. For every x on X, V_x has only finite singular points since the smooth divisor A_x is ample on it. Hence V_x is normal. So the sufficiency of the condition a) follows from (3.8) and that of b) follows from (3.9).

(4.10) PROPOSITION/$\boldsymbol{C}$. *Let $\Pi : A \to S$ be a fiber bundle over a manifold S with fiber being a smooth hyperquadric Q in $\boldsymbol{P}^{r-1}$. Then A cannot be an ample divisor in any manifold if $r > 3$.*

PROOF. Assume that A is an ample divisor on a manifold M. We will derive a contradiction. Using (2.10), we extend Π to a morphism $f : M \to S$. Taking general hyperplane sections on S successively, we find a smooth curve C on S such that $f^{-1}(C) = M_C$ is smooth. The ample divisor A_{M_C} on M_C is a fiber bundle over C with fiber Q. Therefore it suffices to consider the case in which S is a curve C.

Let M_x denote the fiber of f over $x \in C$. $A_x = M_x \cap A$ is an ample divisor on M_x and $A_x \cong Q$. So M_x has only finite singular points and hence M_x is normal. We claim that there exists $H \in \mathrm{Pic}(M)$ such that $H_{A_x} = \mathcal{O}_Q(1)$. Lefschetz Theorem V says $\mathrm{Pic}(M) \cong \mathrm{Pic}(A)$. So the claim follows from (4.4) directly if $r > 4$. If $r = 4$, then $\mathcal{O}_Q(1) \in \mathrm{Pic}(Q) \cong H^2(Q; \boldsymbol{Z})$ is stabilized by the monodromy action of $\pi_1(C)$ since $\omega_Q = \mathcal{O}_Q(-2)$ is stabilized. So (4.1) and (4.3) prove our claim. Now we can apply (3.8) to infer that M_x is $\boldsymbol{P}^{r-1}$ or a hyperquadric in $\boldsymbol{P}^r$.

First we consider the case in which $M_x \cong \boldsymbol{P}^{r-1}$ for some $x \in C$. Using [3], Corollary 5.4, we find a vector bundle E on C such that $(M, H) \cong (\boldsymbol{P}(E), H^E)$. We may put $[A] = 2H - f^*F$ for some $F \in \mathrm{Pic}(C)$. Taking f_* of the exact sequence $0 \to \mathcal{O}_M[2H - A] \to \mathcal{O}_M[2H] \to \mathcal{O}_A[2H] \to 0$, we infer that F is a subbundle of $S^2 E$. At each point y on C, F_y corresponds to the symmetric bilinear form on $E_y^\vee$ which gives the defining equation of A_y in M_y. Taking the determinant of the corresponding symmetric matrix at each point on C, we obtain a section of $2 \det E - rF$ over C, which has a zero at y if and only if A_y is singular. So $2 \det E = rF$ since Π is smooth. Now we have $[A]^r\{M\} = (2H - f^*F)^r = 2^r H^r - r2^{r-1} H^{r-1} F = 2^r c_1(E) - r2^{r-1} \deg F = 0$. This contradicts the ampleness of A.

Second, we consider the case in which M_x is a hyperquadric in $\boldsymbol{P}^r$ and A_x is a hyperplane section on it. So we may assume $H = [A]$. Let E^M and E^A be the vector bundles on C which correspond to $f_* \mathcal{O}_M[H]$ and $f_* \mathcal{O}_A[H]$ respectively. The exact sequence $0 \to \mathcal{O}_M \to \mathcal{O}_M[H] \to \mathcal{O}_A[H] \to 0$ goes down by f_* to an exact sequence $0 \to I_C \to E^M \to E^A \to 0$ on C. So $\det E^M = \det E^A$. M and $P^A = \boldsymbol{P}(E^A)$ are divisors on $P^M = \boldsymbol{P}(E^M)$. H and H^{E^A} are restrictions of $[P^A] =$

$H^{E^M}\in \mathrm{Pic}(P^M)$, which is denoted by H by abuse of notation. We put $[M]=2H-F_{P^M}\in \mathrm{Pic}(P^M)$, where $F\in \mathrm{Pic}(C)$. Taking the determinant of the symmetric matrix defining M_x at each point $x\in C$ as before, we obtain a non-zero section of $2\det E^M-(r+1)F$ over C. Similarly we get $2\det E^A=rF$ since $[A]=2H-F$ in $\mathrm{Pic}(P^A)$. Combining them with $\det E^M=\det E^A$, we infer that $\deg F\leqq 0$. This contradicts $H^r\{M\}=(2H-F)H^r\{P^M\}=2c_1(E^M)-\deg F=(r-1)\deg F$, since $H=[A]$ is ample on M.

(4.11) PROPOSITION/$\boldsymbol{C}$. *Let A be an ample divisor on a manifold M. Let $f: M\to X$ be a surjective morphism onto a manifold X such that f_A makes A a fiber bundle over X. Then* $\dim M\geqq 2\dim X$.

For a proof, see [**24**], Proposition V.

(4.12) COROLLARY/$\boldsymbol{C}$. *Let E be an ample vector bundle on a manifold M. If E has a trivial sub-bundle of rank one, then* $\mathrm{rank}\, E>\dim M$.

For a proof, combine (4.7), (4.8) and (4.11).

(4.13) There are many ample vector bundles with $\mathrm{rank}\, E>\dim M$ which have a trivial sub-bundle of rank one.

EXAMPLE. Let $H_0, \cdots, H_n$ be very ample line bundles on a manifold M with $\dim M=n$. Let $\{\varphi_j\}$ $(j=0, \cdots, n)$ be general members of $\Gamma(M, H_j)$. Then $\bigcap_{j=0}^{n}\{x\in M\mid \varphi_j(x)=0\}=\emptyset$. Therefore $\varphi=\bigoplus_{j=0}^{n}\varphi_j$ defines a trivial sub-bundle of rank one of $E=\bigoplus_{j=0}^{n}H_j$.

(4.14) EXAMPLE. Let L be a very ample line bundle on a manifold X with $\dim X=k$. Let $Y=X\times \boldsymbol{P}^1$ and let $H=L_Y+[1]_Y$, where $[1]$ denotes the line bundle on $\boldsymbol{P}^1$ of degree one. Then H is very ample on Y and the direct sum E of $(k+2)$ copies of H has a subbundle I_Y by (4.13). Let Q be the quotient bundle E/I_Y. Then $A=\boldsymbol{P}(Q)$ is an ample divisor on $M=\boldsymbol{P}(E)$. We see easily that every fiber F_x of the natural morphism $A\to X$ over $x\in X$ is smooth. Moreover, $F_x\cong \boldsymbol{P}(Q_x)$ where Q_x is the restriction of Q to the fiber $C_x\cong \boldsymbol{P}^1$ of $Y\to X$. We have an exact sequence $0\to \mathcal{O}_{C_x}\to \bigoplus_{(k+2)\text{copies}}\mathcal{O}(1)\to \mathcal{O}(Q_x)\to 0$ induced by $0\to I_Y\to E\to Q\to 0$. Therefore $\mathcal{O}(Q_x)\cong \mathcal{O}(2)\oplus \mathcal{O}(1)\oplus \cdots \oplus \mathcal{O}(1)$. Thus we infer that every fiber F_x is isomorphic to $\boldsymbol{P}((1)\oplus(0)\oplus \cdots \oplus(0))$, which is the blowing up of $\boldsymbol{P}^{k+1}$ with center being a linear subspace of codimension two. So A itself is a fiber bundle over X.

(4.15) The above example shows the best-possibility of the dimension condition in the following

PROPOSITION/$\boldsymbol{C}$. *Let A, M, f, X and f_A be as in* (4.11). *Suppose further that* $\dim M\leqq 2\dim X+1$. *Then $H^i(F;\boldsymbol{Z})=0$ for odd i and $\cong \boldsymbol{Z}$ for even i, where F is the fiber of f_A.*

For a proof, see [**24**], Proposition V.

Now, in view of (4.10), we make the following

QUESTION. Is it possible that F is not a projective space in the above situation?

(4.16) Now we study $\boldsymbol{P}^n$-bundles over curves more precisely. First we will prove the following

PROPOSITION/$\boldsymbol{C}$. *Let $0\to I\to E\to Q\to 0$ be a non-splitting exact sequence of vector bundles over a curve C, where $I=I_C$. Then E is ample if so is Q.*

(4.17) PROPOSITION/$\boldsymbol{C}$. *Let $f: M\to N$ be a surjective morphism between manifolds with $\dim M=\dim N=n$. Then the natural homomorphism $H^{p,q}(f, E)$: $H^{p,q}(N, E)\to H^{p,q}(M, f^*E)$ is injective for any vector bundle E on N.*

PROOF. Take $0\neq\varphi\in H^{p,q}(N, E)$. Using the Serre duality theory (see, for example, [**13**]), we infer that $0\neq\varphi\wedge\psi\in H^n(N, \Omega_N^n)$ for some $\psi\in H^{n-p,n-q}(N, E^\vee)$. $H^{n,n}(f, I_N)$ is injective since f is surjective. Therefore $0\neq H^{n,n}(f, I)(\varphi\wedge\psi)=H^{p,q}(f, E)(\varphi)\wedge H^{n-p,n-q}(f, E^\vee)(\psi)$. Hence $H^{p,q}(f, E)(\varphi)\neq 0$. Thus $H^{p,q}(f, E)$ is shown to be injective.

(4.18) COROLLARY/C. *Let $f: M\to N$ be as above. Let $0\to F\to E\to Q\to 0$ be an exact sequence of vector bundles on N. This sequence splits if and only if the pull-back of this sequence on M splits.*

This follows from the injectivity of $H^{0,1}(f, \mathcal{H}om(Q, F))$.

(4.19) Now we prove Proposition (4.16). We will apply Theorem B6 in the Appendix. Let D be the divisor on $P=\boldsymbol{P}(E)$ which corresponds to the subbundle I of E (see (4.8)). Note that $[D]=H^E$ and $H^E{}_D=H^Q$. Since Q is ample, it suffices to show $Y\cap D\neq\emptyset$ for any subvariety Y in P with $\dim Y>0$. Assume $Y\cap D=\emptyset$. Then the projection $P\to C$ gives a finite morphism $f: Y\to C$. Let $\tilde{Y}$ be the normalization of Y and consider the exact sequence $0\to I_{\tilde{Y}}\to E_{\tilde{Y}}\to Q_{\tilde{Y}}\to 0$. $\boldsymbol{P}(E_{\tilde{Y}})=P\times_C\tilde{Y}$ has a section σ over $\tilde{Y}$ induced by the inclusion $Y\to P$. Moreover $\sigma(\tilde{Y})\cap D_{\tilde{Y}}=\emptyset$ since $Y\cap D=\emptyset$. Therefore the above exact sequence on $\tilde{Y}$ splits. This contradicts (4.18).

(4.20) COROLLARY/$\boldsymbol{C}$. *Let Q be any ample vector bundle on a curve C with $g(C)>0$. Then there is an exact sequence $0\to I\to E\to Q\to 0$ of vector bundles on C where E is ample.*

PROOF. Using Riemann-Roch Theorem, we obtain $\chi(Q^\vee)=-\deg(\det Q)-(g-1)\operatorname{rank} Q<0$. So $H^1(C, Q^\vee)\neq 0$ and there exists a non-splitting exact sequence $0\to I\to E\to Q\to 0$ on C. Hence (4.16) applies.

(4.21) EXAMPLE/$\boldsymbol{C}$ (compare [**24**], Question III-A). Any $\boldsymbol{P}^n$-bundle A over a curve C can be an ample divisor on a manifold.

PROOF. Take a vector bundle Q on C such that $A\cong\boldsymbol{P}(Q)$ (use (4.2), (4.5) and (4.6)). Replacing Q by $Q\otimes F$ with F being a sufficiently ample line bundle on C if necessary, we may assume that Q is ample and $H^1(C, Q^\vee)\neq 0$. Take a non-splitting extension $0\to I\to E\to Q\to 0$ as in (4.20). Then E is ample and $\boldsymbol{P}(Q)$ is an ample divisor on $\boldsymbol{P}(E)$ (see (4.8)).

(4.22) EXAMPLE/$\boldsymbol{C}$ (compare the Conjecture in [**24**], p. 63). For any $\boldsymbol{P}^n$-

bundle A over a curve C with $g(C)>0$, there exists a chain $A=A_0\subset A_1\subset\cdots\subset A_j\subset\cdots$ of infinite number of manifolds $\{A_j\}$ such that 1) A_j is $\boldsymbol{P}^{n+j}$-bundle over C, 2) A_j is an ample divisor on A_{j+1} and 3) $[A_j]_{A_j}=[A_{j-1}]$ in $\mathrm{Pic}(A_j)$ for every j.

For the construction, use (4.20) subsequently.

(4.23) In view of [**2**] and [**12**], we make the following

CONJECTURE. Let $\{(M_j, L_j)\}_{j=0,1,\ldots}$ be an infinite chain of polarized manifolds such that 1) M_j is a divisor on M_{j+1}, 2) $[M_j]=L_{j+1}\in\mathrm{Pic}(M_{j+1})$, 3) $L_{j+1}|_{M_j}=L_j$ and 4) the restriction homomorphism $G(M_{j+1}, L_{j+1})\to G(M_j, L_j)$ is surjective for every j. Then each (M_j, L_j) is a weighted complete intersection.

§ 5. Blowing down.

(5.1) In this section we consider the following

PROBLEM. Let A be an ample divisor on a manifold M. Suppose that A is the blowing up of a manifold B with center C, where C is a submanifold of B. Then does there exist a manifold N such that B lies on N as a divisor and that M is the blowing up of N with center C?

(5.2) In order to find such a manifold N, it suffices to find a divisor D on M which satisfies the following conditions (see [**1**], [**16**]).

1) $D_A=D\cap A$ is the exceptional divisor E on A over C.
2) The natural projection $E\to C$ can be extended to a surjective morphism $\Pi: D\to C$.
3) Π makes D a $\boldsymbol{P}^r$-bundle over C, where $r=\dim B-\dim C$. Moreover, each fiber E_x of E over $x\in C$ is a hyperplane on $D_x=\Pi^{-1}(x)\cong\boldsymbol{P}^r$.
4) $[D]_{D_x}$ is the dual of $\mathcal{O}(1)$.

(5.3) LEMMA. *The above condition* 4) *follows from the other conditions* 1), 2) *and* 3).

PROOF. $[D]_{E_x}=[E]_{E_x}=\mathcal{O}_{E_x}(-1)$ on $E_x\cong\boldsymbol{P}^{r-1}$ because E is exceptional. Hence $[D]_{D_x}=\mathcal{O}_{D_x}(-1)$ on $D_x\cong\boldsymbol{P}^r$ since E_x is a hyperplane on D_x.

(5.4) LEMMA. *The condition* 3) *follows from the conditions* 1) *and* 2) *if one of the following conditions are satisfied.*

a) $[A]_{E_x}=\mathcal{O}(1)$ *on* $E_x\cong\boldsymbol{P}^{r-1}$.
b) $r\geqq 3$.

This lemma follows from (4.9), since $[-D]_{E_x}=\mathcal{O}(1)$.

(5.5) THEOREM/$\boldsymbol{C}$. *Let A, M, B and C be as in* (5.1). *Suppose that* $r=\dim B-\dim C>2$ *and that B is projective. Then there exists a manifold N as in* (5.1).

PROOF. The exceptional divisor E on A is a $\boldsymbol{P}^{r-1}$-bundle over C. So, Lemma A4 implies that $H^1(E, [E-tA]_E)=0$ for any $t\geqq 0$. Using the exact sequence $H^1(A, [-tA])\to H^1(A, [E-tA])\to H^1(E, [E-tA]_E)$ and the vanishing theorem of Kodaira, we infer $H^1(A, [E-tA])=0$ for any $t>0$. On the other hand, Lefschetz Theorem V says that $\mathrm{Pic}(M)\cong\mathrm{Pic}(A)$. Hence we can apply (2.4) to obtain a divisor D on M which satisfies the condition 1) in (5.2). We want to prove the condition 2) by (2.9). Clearly D is locally Macaulay. D has only finite singular points since E lies on it as an ample divisor. Therefore D is normal. Now, take an ample line bundle H on B. We have $F\in\mathrm{Pic}(M)$ such that $F_A=H_A$ since $\mathrm{Pic}(M)\cong\mathrm{Pic}(A)$. Therefore H_E comes from F_D and H_C is ample. So (2.9) applies. The conditions 3) and 4) are proved by (5.4) and (5.3).

REMARK. The first step of this proof is due to Sommese [**24**], p. 69.

(5.6) PROPOSITION. *Let A, M, B, C, D, E and N be as in* (5.1) *and* (5.2). *Suppose that the restriction of $[B]\in\mathrm{Pic}(N)$ to C is ample. Then B is ample on N.*

PROOF. We have $[B]_M=[A]+[D]$ since A is the proper transform of B in M. So it suffices to show the following

(5.7) LEMMA. *Let C be a submanifold in a manifold N and let M be the blowing up of N with center C. Let L be a line bundle on N such that L_C is ample on C. Suppose that $L_M-k[E]$ is ample on M for some $k>0$, where E denotes the exceptional divisor on M over C. Then L is ample on N.*

PROOF. It suffices to show that L_Y is strictly effective for any subvariety Y of N with $\dim Y>0$ (see Theorem B6 in the Appendix). Clearly we may assume $Y\not\subset C$. Hence the proper transform $\tilde{Y}$ of Y in M is not contained in E. So $[E]_{\tilde{Y}}$ is effective. Therefore $L_{\tilde{Y}}=[L-kE]_{\tilde{Y}}+k[E]_{\tilde{Y}}$ is strictly effective since $L-kE$ is ample on M. Hence L_Y is also strictly effective (see Lemma B2).

(5.8) Now we give a couple of applications.

PROPOSITION/$\boldsymbol{C}$. *Let C be a submanifold in $P\cong\boldsymbol{P}^n$ with* $\operatorname{codim} C=r\geqq 3$. *Then the blowing-up $A=Q_C(P)$ of P with center C cannot be an ample divisor in any manifold.*

PROOF. Let H denote the hyperplane section bundle on P and let E denote the exceptional divisor on A over C. Assume that A is an ample divisor on a manifold M. Using (5.5) we find a manifold N which contains P as a divisor such that $M\cong Q_C(N)$. For $[P]\in\mathrm{Pic}(N)$ we put $[P]_P=kH$. We have $[A]_A=kH_A-[E]$ since A is the proper transform of P in M. $k>0$ since $[A]_A$ is ample. This implies that $[P]_C$ is ample. So (5.6) proves that P is ample on N. Hence (3.10) applies and we obtain $N\cong\boldsymbol{P}^{n+1}$ and $k=1$. Namely P is a hyperplane on N. Take a line Y on P such that $Y\cap C\neq\emptyset$ and $Y\not\subset C$. Then $[E]_{\tilde{Y}}\geqq 1$ for the proper transform $\tilde{Y}$ of Y in A. So $[A]\tilde{Y}=H\tilde{Y}-E\tilde{Y}\leqq 0$. This contradicts the ampleness of A.

(5.9) COROLLARY/$\boldsymbol{C}$. *Let C be a disjoint union of submanifolds in $\boldsymbol{P}^n$ of codimensions $\geqq 3$, where the dimensions of the components of C may differ to each other. Then $A=Q_C(\boldsymbol{P}^n)$ cannot be an ample divisor in any manifold.*

Using (5.5) and (5.6), we prove this corollary by the induction on the number of components of C.

(5.10) PROPOSITION/$\boldsymbol{C}$. *Let C be a disjoint union of submanifolds in B of codimensions $\geqq 3$, where B is a smooth hyperquadric in $P \cong \boldsymbol{P}^n$. Suppose that the blowing-up $A=Q_C(B)$ of B with center C is an ample divisor in a manifold M. Then C is a linear submanifold in P and $M=Q_C(P)$.*

PROOF. Using (5.5) and (5.6), we infer by induction on the number of components of C that B is an ample divisor on a manifold N such that $M=Q_C(N)$. In view of Lefschetz Theorem V we take $L \in \mathrm{Pic}(N)$ so that $L_B=\mathcal{O}_B(1)$. Put $[B]=eL$. There is a line Y in P which lies on B such that $Y \cap C \neq \emptyset$ and $Y \not\subset C$. Hence we infer $e>1$ as in (5.8). On the other hand, e divides 2 (see (3.8)). Therefore $e=2$ and $N=P$. Assume that C is not a linear submanifold in P. Then there is a line X on P such that $X \not\subset C$ and X meets C at more than one points. Then $E\tilde{X} \geqq 2$ for the proper transform $\tilde{X}$ of X in $M=Q_C(P)$. So $A\tilde{X}=(2L-E)\tilde{X} \leqq 0$, which contradicts the ampleness of A. Consequently C must be a linear submanifold in P.

(5.11) REMARK. If $\operatorname{codim} C=2$, then $Q_C(P)$ and $Q_C(B)$ can be ample divisors in various cases.

Appendix A. A vanishing theorem for $R^p f_*$ of Kodaira-Ramanujam type.

DEFINITION A1. A line bundle L on a variety V is said to be *semi-ample* if $Bs|tL|=\emptyset$ for some positive integer t. If so, $\kappa(L)$ denotes $\dim \rho_{|tL|}(V)$.

DEFINITION A2. Let $f: V \to S$ be a surjective morphism onto a projective scheme S. Then $L \in \mathrm{Pic}(V)$ is said to be *semi-ample with respect to f* if $L+f^*F$ is semi-ample for some $F \in \mathrm{Pic}(S)$. If so, $L+f^*H$ is semi-ample for any sufficiently ample line bundle H on S. Moreover, $\kappa(L+f^*H)=\dim S+\kappa(L_x)$, where L_x is the restriction of L to a general fiber of f over $x \in S$. By $\kappa_f(L)$ we denote $\kappa(L+f^*H)-\dim S=\kappa(L_x)$.

LEMMA A3. *Let $f: V \to S$ be as above and let L be a relatively semi-ample line bundle on V with respect to f. Let T be any subscheme of S and let $f_T: V_T=f^{-1}(T) \to T$ be the restriction of f. Then L_{V_T} is semi-ample with respect to f_T and $\kappa_{f_T}(L_{V_T}) \geqq \kappa_f(L)$.*

Proof is easy and is omitted.

LEMMA/$\boldsymbol{C}$ A4. *Let $f: M \to S$ be a surjective morphism onto a projective variety S from a manifold M. Let L be a relatively semi-ample line bundle on M with respect to f. Then $H^p(M, -L)=0$ for $p<\kappa_f(L)$.*

PROOF. We use the induction on $s=\dim S$. When $s=0$, the assertion can be reduced to the vanishing theorem due to Ramanujam [22]. So we consider the case $s \geqq 1$. Take a sufficiently ample general hypersurface section H on S. Note that $D=f^{-1}(H)$ is smooth and that L_D is relatively semi-ample with respect to f_H and $\kappa_{f_H}(L_D)\geqq\kappa_f(L)$. Consider the exact sequence: $H^p(M, -L-f^*H)\to H^p(M, -L)\to H^p(D, -L_D)$. The first term vanishes for any $p<\kappa_f(L)$, since $\kappa_f(L)<\kappa(L+f^*H)$. The third term also vanishes for $p<\kappa_f(L)$ by the induction hypothesis for $s-1$. Hence $H^p(M, -L)=0$ for $p<\kappa_f(L)$. Thus we prove the lemma.

THEOREM/**C** A5. *Let $f: M\to S$ be a surjective morphism onto a projective variety S from a manifold M. Let L be a relatively semi-ample line bundle on M with respect to f. Then $R^pf_*\mathcal{O}_M[-L]=0$ for $p<\kappa_f(L)$.*

PROOF. Put $\mathcal{H}^p=R^pf_*(\mathcal{O}_M[-L])$ for each p and take a sufficiently ample line bundle H on S such that $H^q(S, \mathcal{H}^p[H])=0$ for any $q>0$ and that $\mathcal{H}^p[H]$ is generated by its global sections for any p. Observing the Leray spectral sequence for $\mathcal{O}_M[f^*H-L]$ with respect to f, we infer that $H^p(M, f^*H-L)\cong H^0(S, \mathcal{H}^p[H])$. The left hand side vanishes for $p<\kappa_f(L)=\kappa_f(L-f^*H)$ by Lemma A4. Combining them we prove $\mathcal{H}^p=0$ since $\mathcal{H}^p[H]$ is generated by its global sections.

COROLLARY/**C** A6. *Let $f: M\to S$ and L be as above. Then $H^p(M, f^*E\otimes L^\vee)=0$ for any $p<\kappa_f(L)$ and any vector bundle E on S.*

For a proof, consider the Leray spectral sequence.

Appendix B. A reformed version of Nakai-Moĭšezon criterion.

DEFINITION B1. Let L be a line bundle on a variety V. L is said to be *c-effective* if there exists an effective divisor D on the normalization $\tilde{V}$ of V such that $c_1(D)=mc_1(L_{\tilde{V}})$ for some $m>0$. If in addition $D\neq 0$, then L is said to be *strictly c-effective.*

LEMMA B2. *Let $f: V\to W$ be a surjective morphism between varieties and let $L\in\mathrm{Pic}(W)$. Then f^*L is (strictly) c-effective if and only if so is L on W.*

Proof is easy since f induces $\tilde{f}: \tilde{V}\to\tilde{W}$.

DEFINITION B3. Let L be a line bundle on a variety V. L is said to be *c-semipositive* if L_W is c-effective for any subvariety W of V. L is said to be *c-positive* if in addition L_W is strictly c-effective for any W with $\dim W>0$.

LEMMA B4. *Let L be a line bundle on a variety V and let $f: W\to V$ be a morphism. Then f^*L is c-semipositive if so is L. If in addition f is finite and L is c-positive, then f^*L is c-positive.*

PROOF. Let Y be any subvariety in W. Put $X=f(Y)$. Then L_X is c-effective by Definition B3. So Lemma B2 proves that L_Y is c-effective. Hence L_W is c-semipositive by definition. A similar argument proves the second assertion.

LEMMA B5. *Let $L_1, L_2, \cdots, L_r$ be c-semipositive line bundles on a variety V with $\dim V=r$. Then $L_1L_2\cdots L_r\{V\}\geqq 0$. If in addition each L_j is c-positive, then $L_1L_2\cdots L_r\{V\}>0$.*

PROOF. We use the induction on r. Clearly it suffices to consider the case $r>1$. We may assume that V is normal since the morphism $\tilde{V}\to V$ is finite. So we find an effective divisor D on V such that $c_1(D)=mc_1(L_r)$ for some $m>0$. Then the induction hypothesis proves $mL_1L_2\cdots L_r\{V\}=L_1\cdots L_{r-1}\{D\}\geqq 0$ since $[L_j]_D$ is c-semipositive. In the latter case we have the inequality because $[L_j]_D$ is c-positive (Lemma B4). Thus we prove the lemma.

THEOREM B6. *Let L be a c-positive line bundle on an irreducible reduced algebraic space V. Then L is ample.*

PROOF. The above lemmata enable us to apply the criterion of Nakai-Moišezon (see [**11**], [**21**], [**15**] and [**1**]).

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Added note: After the typing, I found that the 'only if' part of Lemma B2 is not clear when char K divides $\deg f$. However, this lemma is not used in such a situation in any other part of this article.