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On foliations with the structure group of automorphisms of a geometric structure

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Introduction.

In this paper we shall study foliations with the structure pseudogroup Γ of local automorphisms of a certain 2nd order G-structure. Our purpose is to prove a vanishing theorem for certain characteristic classes of such Γ-foliations and to give a geometric construction of examples of these Γ-foliations.

Let G/U be a semi-simple flat homogeneous space of $\dim G/U=q$ in the sense of Ochiai [**12**]. It is a connected homogeneous space, on which a semi-simple Lie group G acts transitively and effectively, and $\mathfrak{g}=\mathrm{Lie}\, G$, the Lie algebra of G, has a graded Lie algebra structure:

$$\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1\,, \qquad \dim \mathfrak{g}_{-1}=q\,,$$

with $\mathfrak{u}=\mathrm{Lie}\, U=\mathfrak{g}_0+\mathfrak{g}_1$. We identify $\mathfrak{g}_{-1}$ with $\boldsymbol{R}^q$ by a basis for $\mathfrak{g}_{-1}$, and then $\boldsymbol{R}^q$ with an open neighbourhood of the origin U in G/U by the imbedding $\mathfrak{g}_{-1}\ni x\mapsto(\exp x)U\in G/U$. Then we can define an imbedding ι of G into the 2nd order frame bundle $P^2(G/U)$ of G/U by

$$\iota(a)=j_0^2(a) \qquad \text{for} \quad a\in G\,.$$

In particular, ι identifies U with a Lie subgroup of the structure group $G^2(q)$ of $P^2(G/U)$. Let B be a smooth manifold of $\dim B=q$. A U-subbundle Q of the 2nd order frame bundle $P^2(B)$ of B is called a *2nd order structure of type* G/U over B. For instance, the image $Q_G=\iota(G)$ of ι is a 2nd order structure of type G/U over G/U. Let $\Gamma=\Gamma(Q)$ denote the pseudogroup of all local diffeomorphisms φ of B such that the 2nd prolongation $\varphi^{(2)}$ leaves Q invariant. We shall study Γ-foliations for these pseudogroups Γ.

For example, the pseudogroup Γ of local projective or conformal transformations for a Riemannian metric on a smooth manifold B is obtained in this way from a certain semi-simple flat homogeneous space (cf. §4). The Γ-foliations for these Γ are the so-called projective and conformal foliations.

In general, for a Lie group L and a Lie subalgebra $\mathfrak{h}$ of $\mathrm{Lie}\, L$, we define

$$I_L(\mathfrak{h})=\{f|\mathfrak{h}\,;f \text{ is an } L\text{-invariant polynomial on Lie } L\}\,.$$

In particular, for a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(q)=\mathrm{Lie}\,GL(q)$, we write $\mathrm{Char}(\mathfrak{h})=I_{GL(q)}(\mathfrak{h})$. We define a closed subgroup G_0 of U by

$$G_0=\{a\in U\,;\,\mathrm{Ad}\,a\mathfrak{g}_0=\mathfrak{g}_0\}\,.$$

Then $\mathrm{Lie}\,G_0=\mathfrak{g}_0$ and the linear isotropy representation $\rho: U\to GL(\mathfrak{g}_{-1})=GL(q)$ of G/U identifies G_0 with the linear isotropy subgroup in $GL(q)$. Identifying $\mathfrak{g}_0$ with a Lie subalgebra of $\mathfrak{gl}(q)$ by ρ, we set

$$\mathfrak{g}_0'=\{x\in\mathfrak{g}_0\,;\,\mathrm{Tr}\,x=0\}\,.$$

Let $\{[\bar{u}_I\bar{c}_J]\}$ be the Vey basis for $H(WO_q)$ (cf. §3), and for a foliation $\mathcal{F}$ of codimension q, let $[\bar{u}_I\bar{c}_J](\mathcal{F})$ denote the corresponding characteristic class of $\mathcal{F}$. With these notations, we have the following vanishing theorem:

Suppose

(1) *Spencer cohomology* $H^{2,1}(\mathfrak{g})=\{0\}$;

(2) $\mathrm{Char}(\mathfrak{g}_0')\subset I_G(\mathfrak{g}_0')$.

Then, for each Γ-foliation $\mathcal{F}$, we have

$$[\bar{u}_I\bar{c}_J](\mathcal{F})=0,$$

if $I\neq\emptyset$, $i_1+|J|\geqq q+2$ and $2|J|\geqq q+1$, where i_1 is the smallest element in $I=(i_1,\cdots,i_k)$ and $|J|=j_1+\cdots+j_l$ for $J=(j_1,\cdots,j_l)$.

The conditions (1), (2) are satisfied for projective foliations ($q\geqq 2$) and conformal foliations ($q\geqq 3$). Thus our theorem includes vanishing theorems for rigid classes of Yamato [**16**], Morita [**10**].

Kamber-Tondeur [**6**] constructed examples of such Γ-foliations as follows. Assume that G is connected and with the trivial center. Let K_0 be a maximal compact subgroup of G_0 and D a uniform discrete subgroup of G acting on G/K_0 properly discontinuously and without fix points. Let $\tilde{\mathcal{F}}$ be the G-invariant foliation of codimension q on G/K_0 characterized by that the leaf passing through the origin K_0 coincides with U/K_0. This induces a foliation $\mathcal{F}$ on the compact quotient $M=D\backslash G/K_0$ in a natural manner. It is a $\Gamma(Q_G)$-foliation and called a *locally homogeneous foliation* of type G/U. Actually Kamber-Tondeur [**7**] and Yamato [**15**] studied locally homogeneous projective resp. conformal foliations to prove the linear independence of certain characteristic classes.

In particular, locally homogeneous conformal foliations are related with Anosov flows in the following way. In this case, G/K_0 is identified with the unit tangent bundle T^1H^{q+1} of the hyperbolic space H^{q+1} of dimension $q+1$. The geodesic flow and the canonical Riemannian metric on T^1H^{q+1} induce a flow ϕ_t and a Riemannian metric g on M respectively. Then ϕ_t is an *Anosov flow* on the Riemannian manifold (M, g) in the following sense: There exist ϕ_{t*}-invariant subbundles F_0, F_+, F_- of the tangent bundle TM of M with

$$TM=F_0\oplus F_+\oplus F_- \qquad \text{(Whitney sum)},$$

such that

(1) $\|\phi_{t*}x\|=\|x\|$ for each $x\in F_0$, $t\in\boldsymbol{R}$;

(2) There exist positive constants C_1, C_2 such that

$$\|\phi_{t*}x\|\leqq C_2e^{-C_1t}\|x\| \qquad \text{for each} \quad x\in F_+,\ t\geqq 0,$$

$$\|\phi_{t*}x\|\geqq C_2e^{C_1t}\|x\| \qquad \text{for each} \quad x\in F_-,\ t\geqq 0.$$

Here $\|x\|$ denotes the length $\sqrt{g(x,x)}$ of $x\in TM$. Subbundles F_0, F_+, F_- are called the *invariant bundle, contracting bundle, expanding bundle* respectively. Moreover the tangent bundle $\tau(\mathcal{F})$ for $\mathcal{F}$ coincides with the Whitney sum $F_0\oplus F_+$.

In the second half of this note, we shall show that these hold also for a general locally homogeneous foliation $\mathcal{F}$ of type G/U. More precisely, we shall give a geometric construction of a such foliation $\mathcal{F}$ on $M=D\backslash G/K_0$, identifying G/K_0 with a closed submanifold of the unit tangent bundle T^1S of a symmetric space S of non-compact type. Furthermore we shall show:

The geodesic flow on T^1S induces an Anosov flow ϕ_t on M such that $\tau(\mathcal{F})=F_0\oplus F_+$.

§1. Normal Cartan connections.

In this section we shall recall the notion of a normal Cartan connection $\boldsymbol{\omega}$ on a 2nd order structure Q of type G/U, and then define a connection form ω on the "prolongation" of the normal frame bundle of a $\Gamma(Q)$-foliation by pulling back $\boldsymbol{\omega}$. The form ω plays an important role in defining characteristic homomorphisms for $\Gamma(Q)$-foliations.

Let B be a smooth manifold of dimension q. Let $G^r(q)\to P^r(B)\xrightarrow{\boldsymbol{\pi}_r} B$ be the r-th frame bundle of B (cf. Kobayashi [9]). The general linear group $GL(q)=G^1(q)$ will be identified with a subgroup of $G^r(q)$ in the canonical way. Then the natural projection $\boldsymbol{\pi}_r^s: P^r(B)\to P^s(B)$ for $r>s$ is $GL(q)$-equivariant and satisfies $\boldsymbol{\pi}_s\circ\boldsymbol{\pi}_r^s=\boldsymbol{\pi}_r$. Let $\Gamma(B)$ denote the pseudogroup of all local diffeomorphisms of B. For $\varphi\in\Gamma(B)$, $\varphi^{(r)}$ denotes the r-th prolongation of φ. It is a local $G^r(q)$-bundle map of $P^r(B)$ satisfying $\boldsymbol{\pi}_r\circ\varphi^{(r)}=\varphi\circ\boldsymbol{\pi}_r$.

Now we recall the definition of the r-th canonical form $\boldsymbol{\theta}^{(r)}$ on $P^r(B)$. We define first the distinguished element $e^r\in P^r(\boldsymbol{R}^q)$ by $e^r=j_0^r(id)$ and set $\mathfrak{p}^r(q)=T_{e^r}(P^r(\boldsymbol{R}^q))$. The natural action of $G^r(q)$ on $\mathfrak{p}^{r-1}(q)$ will be denoted by Ad. We denote the differential of $\boldsymbol{\pi}_r^s: P^r(\boldsymbol{R}^q)\to P^s(\boldsymbol{R}^q)$ for $r>s$ at e^r by $p_r^s:\mathfrak{p}^r(q)\to\mathfrak{p}^s(q)$. For example, $\mathfrak{p}^0(q)=\boldsymbol{R}^q$, $\mathfrak{p}^1(q)=\boldsymbol{R}^q+\mathfrak{gl}(q)$, which may be identified with the Lie algebra of the group of affine automorphisms of $\boldsymbol{R}^q$, and $p_1^0:\mathfrak{p}^1(q)\to\mathfrak{p}^0(q)$ is the projection to the first factor. The *r-th canonical form* $\boldsymbol{\theta}^{(r)}$ is a $\mathfrak{p}^{r-1}(q)$-

valued 1-form on $P^r(B)$ defined as follows. Let $u=j_0^r(f)\in P^r(B)$, where $f:\boldsymbol{R}^q\to B$ is a local diffeomorphism defined around 0. Then the correspondence $j_0^{r-1}(\varphi)\mapsto j_0^{r-1}(f\circ\varphi)$ defines a local diffeomorphism $\bar{f}:P^{r-1}(\boldsymbol{R}^q)\to P^{r-1}(B)$ defined around e^{r-1} such that $\bar{f}(e^{r-1})=u'=\boldsymbol{\pi}_r^{r-1}(u)$, and the differential $\bar{u}:\mathfrak{p}^{r-1}(q)\to T_{u'}(P^{r-1}(B))$ of $\bar{f}$ at e^{r-1} is independent of the choice of f. Now $\boldsymbol{\theta}^{(r)}$ is defined by

$$\boldsymbol{\theta}^{(r)}(X)=\bar{u}^{-1}(\boldsymbol{\pi}_r^{r-1})_*X \qquad \text{for} \quad X\in T_u(P^r(B)).$$

It satisfies

$$R_a^*\boldsymbol{\theta}^{(r)}=\mathrm{Ad}\, a^{-1}\boldsymbol{\theta}^{(r)} \qquad \text{for} \quad a\in G^r(q), \tag{1.1}$$

$$(\boldsymbol{\pi}_r^s)^*\boldsymbol{\theta}^{(s)}=p_{r-1}^{s-1}\boldsymbol{\theta}^{(r)} \qquad \text{for} \quad r>s, \tag{1.2}$$

where R_a means the right translation of $P^r(B)$ by $a\in G^r(q)$. We are mainly interested in the 2nd canonical form $\boldsymbol{\theta}^{(2)}$, which is an $\boldsymbol{R}^q+\mathfrak{gl}(q)$-valued 1-form on $P^2(B)$. Let $\boldsymbol{\theta}_{-1}$, $\boldsymbol{\theta}_0$ denote the $\boldsymbol{R}^q$-component and $\mathfrak{gl}(q)$-component of $\boldsymbol{\theta}^{(2)}$ respectively, so that $\boldsymbol{\theta}^{(2)}=\boldsymbol{\theta}_{-1}+\boldsymbol{\theta}_0$. Then we have

$$d\boldsymbol{\theta}_{-1}+[\boldsymbol{\theta}_0, \boldsymbol{\theta}_{-1}]=0. \tag{1.3}$$

Let now G/U be a semi-simple flat homogeneous space of $\dim G/U=q$ as in Introduction. We set $\rho_0=\rho\,|\,G_0$. Then we have a commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\ \iota\ } & G^2(q) \\ \uparrow & \searrow^{\rho} & \downarrow \boldsymbol{\pi}_2^1 \\ G_0 & \xrightarrow{\ \rho_0\ } & GL(q). \end{array}$$

Recall that ρ_0 is an injective homomorphism, which is used to identify G_0 with the Lie subgroup $\rho(U)$ of $GL(q)$. Let $U\to Q\to B$ be a 2nd order structure of type G/U over B and let $\Gamma=\Gamma(Q)$. We define a G_0-subbundle P of $P^1(B)$ by $P=\boldsymbol{\pi}_2^1(Q)$. It should be noted that for each $\varphi\in\Gamma(Q)$ the 1st prolongation $\varphi^{(1)}$ leaves P invariant, since $\varphi^{(1)}\circ\boldsymbol{\pi}_2^1=\boldsymbol{\pi}_2^1\circ\varphi^{(2)}$.

Let $\mathcal{F}$ be a *Γ-foliation* on a smooth manifold M. It is by definition a maximal family of local submersions $f_\alpha:U_\alpha\to B$ of M such that

(1) $\{U_\alpha\}_\alpha$ is an open covering of M;

(2) For each $x\in U_\alpha\cap U_\beta$ there exists $\gamma_{\alpha\beta}^x\in\Gamma$ such that $\gamma_{\alpha\beta}^x\circ f_\beta=f_\alpha$ around x.

Then the kernels of differentials $f_{\alpha*}$ constitute the *tangent bundle* $\tau(\mathcal{F})$ for $\mathcal{F}$, which is an integrable subbundle of TM. The quotient bundle $\nu(\mathcal{F})=TM/\tau(\mathcal{F})$ will be called the *normal bundle* for $\mathcal{F}$. We associate to $\mathcal{F}$ a smooth foliation $\bar{\mathcal{F}}$ such that $\bar{\mathcal{F}}\supset\mathcal{F}$. Here $\bar{\mathcal{F}}$ is defined in the same way as $\mathcal{F}$ by replacing Γ by

$\Gamma(B)$. We shall define the *r-th frame bundle* $P^r(\mathcal{F})$ for $\mathcal{F}$ and the *r-th canonical form* $\theta^{(r)}$ on $P^r(\mathcal{F})$.

Take a point $o\in B$ and fix it once for all. Choose a local diffeomorphism $f:\boldsymbol{R}^q\to B$ defined around 0 such that $f(0)=o$ and $j_0^2(f)\in Q$, and then identify a neighbourhood of 0 in $\boldsymbol{R}^q$ with a neighbourhood of o in B by means of f. Set

$$P^r(\mathcal{F})=\{j_x^r(f)\,;f\in\mathcal{F} \text{ defined around } x \text{ with } f(x)=o\}$$

and define the projection $\pi_r:P^r(\mathcal{F})\to M$ by $\pi_r(j_x^r(f))=x$. The group $G^r(q)$ acts on $P^r(\mathcal{F})$ from the right by

$$j_x^r(f)j_0^r(\varphi)=j_x(\varphi^{-1}\circ f) \qquad \text{for} \quad j_0^r(\varphi)\in G^r(q)\,.$$

Thus we get a smooth $G^r(q)$-bundle $G^r(q)\to P^r(\mathcal{F})\xrightarrow{\pi_r}M$. Note that $P^1(\mathcal{F})$ may be identified with the frame bundle of the normal bundle $\nu(\mathcal{F})$. The natural projection $\pi_r^s:P^r(\mathcal{F})\to P^s(\mathcal{F})$ for $r>s$ is also $GL(q)$-equivariant and satisfies $\pi_s\circ\pi_r^s=\pi_r$. Let $f:V\to B$ be a local submersion in $\mathcal{F}$. For each $j_x^r(\varphi)\in P^r(\mathcal{F})$ with $x\in V$, there exists a local diffeomorphism $\phi:\boldsymbol{R}^q\to B$ defined around 0 such that $\phi(0)=f(x)$ and $\phi\circ\varphi=f$ around x. Then the correspondence $j_x^r(\varphi)\mapsto j_0^r(\phi)$ defines a $G^r(q)$-bundle map of $P^r(\mathcal{F})|V\to P^r(B)$, which will be denoted by $f^{(r)}$. It satisfies $\boldsymbol{\pi}_r\circ f^{(r)}=f\circ\pi_r$.

Let $v=j_x^r(f)\in P^r(\mathcal{F})$ and set $v'=\pi_r^{r-1}(v)$. For each $j_y^{r-1}(\varphi)\in P^{r-1}(\mathcal{F})$ near to v', there exists a local diffeomorphism $\phi:\boldsymbol{R}^q\to\boldsymbol{R}^q$ defined around 0 such that $\phi(0)=f(y)$ and $\phi\circ\varphi=f$ around x. The correspondence $j_y^{r-1}(\varphi)\mapsto j_0^{r-1}(\phi)$ defines a local smooth map $\bar{f}:P^{r-1}(\mathcal{F})\to P^{r-1}(\boldsymbol{R}^q)$ defined around v' with $\bar{f}(v')=e^{r-1}$. The differential $\bar{v}:T_{v'}(P^{r-1}(\mathcal{F}))\to\mathfrak{p}^{r-1}(q)$ of $\bar{f}$ at v' is independent of the choice of f. Now $\theta^{(r)}$ is defined by

$$\theta^{(r)}(X)=\bar{v}(\pi_r^{r-1})_*X \qquad \text{for} \quad X\in T_v(P^r(\mathcal{F}))\,.$$

It satisfies also

$$R_a^*\theta^{(r)}=\operatorname{Ad}a^{-1}\theta^{(r)} \qquad \text{for} \quad a\in G^r(q)\,, \tag{1.4}$$

$$(\pi_r^s)^*\theta^{(s)}=p_{r-1}^{s-1}\theta^{(r)} \qquad \text{for} \quad r>s\,. \tag{1.5}$$

It is related with $\boldsymbol{\theta}^{(r)}$ by

$$\theta^{(r)}=f^{(r)*}\boldsymbol{\theta}^{(r)} \qquad \text{on} \quad P^r(\mathcal{F})|V \tag{1.6}$$

for each local submersion $f:V\to B$ in $\mathcal{F}$. Actually $\theta^{(r)}$ is characterized by the property (1.6).

The following lemma follows from the invariance of Q and P under $\Gamma(Q)$.

LEMMA 1.1. (Nishikawa-Takeuchi [**11**]) *Let $\mathcal{F}=\{f_\alpha\}_\alpha$ be a $\Gamma(Q)$-foliation on M. Then:*

1) *There exists a unique U-subbundle $Q(\mathcal{F})$ of $P^2(\mathcal{F})$ such that $Q(\mathcal{F})|U_\alpha=$*

$(f_\alpha^{(2)})^{-1}Q$ *for each* $f_\alpha: U_\alpha \to B$ *in* $\mathcal{F}$;

2) *There exists a unique* G_0*-subbundle* $P(\mathcal{F})$ *of* $P^1(\mathcal{F})$ *such that* $P(\mathcal{F})|U_\alpha = (f_\alpha^{(1)})^{-1}P$ *for each* $f_\alpha: U_\alpha \to B$ *in* $\mathcal{F}$;

3) $\pi_2^1 Q(\mathcal{F}) = P(\mathcal{F})$.

LEMMA 1.2. *There exists a smooth* G_0*-equivariant section* $s: P(\mathcal{F}) \to Q(\mathcal{F})$ *of the bundle* $\pi_2^1: Q(\mathcal{F}) \to P(\mathcal{F})$.

PROOF. Recall that U/G_0 is contractible, since the map $(a, x) \mapsto a \exp x$ ($a \in G_0$, $x \in \mathfrak{g}_1$) defines a diffeomorphism: $G_0 \times \mathfrak{g}_1 \approx U$ (Ochiai [12]). Therefore, the U-bundle $Q(\mathcal{F})$ has a G_0-subbundle P_1. Then $\varpi = \pi_2^1 | P_1 : P_1 \to P(\mathcal{F})$ is a G_0-bundle isomorphism. Set $s = \varpi^{-1}: P(\mathcal{F}) \to P_1 \subset Q(\mathcal{F})$. It is a required section.

q. e. d.

LEMMA 1.3. *Define a subbundle* $F^{(1)}$ *of* $T(P(\mathcal{F}))$ *of codimension* q *by*

$$F^{(1)} = \{X \in T(P(\mathcal{F}))\,;\, (\pi_1)_* X \in \tau(\mathcal{F})\}\,.$$

Then, for $X \in T(P(\mathcal{F}))$, *one has*

$$X \in F^{(1)} \Leftrightarrow \theta^{(1)}(X) = 0\,.$$

PROOF. Let $X \in T_v(P(\mathcal{F}))$ and choose $f_\alpha: U_\alpha \to B$ in $\mathcal{F}$ with $\pi_1(v) \in U_\alpha$. It follows from (1.6) that $\theta^{(1)}(X) = \boldsymbol{\theta}^{(1)}(f_{\alpha *}^{(1)} X) = \bar{u}^{-1}(\boldsymbol{\pi}_{1*} f_{\alpha *}^{(1)} X) = \bar{u}^{-1}(f_{\alpha *}\pi_{1*}X)$ for $u = f_\alpha^{(1)}(v)$. It follows that $\theta^{(1)}(X) = 0 \Leftrightarrow f_{\alpha *}\pi_{1*}X = 0 \Leftrightarrow \pi_{1*}X \in \tau(\mathcal{F}) \Leftrightarrow X \in F^{(1)}$. q. e. d.

Now we recall the existence theorem of Tanaka-Ochiai for a Cartan connection on our 2nd order structure Q of type G/U.

THEOREM 1.1. *If the Spencer cohomology* $H^{2,1}(\mathfrak{g}) = \{0\}$, *then there exists a unique normal Cartan connection of type* G/U *on* Q.

See Ochiai [12] for definitions of Spencer cohomology and a normal Cartan connection. See Tanaka [14], Ochiai [12] for a proof of Theorem 1.1. We shall require following properties of a normal Cartan connection $\boldsymbol{\omega}$:

(i) $\boldsymbol{\omega}$ is a $\mathfrak{g}$-valued 1-form on Q.

(ii) $\boldsymbol{\omega}$ is invariant under $\Gamma(Q)$.

(iii) Let $G \to Q^G = Q \times_U G \to B$ be the group extension of Q by G. Then $\boldsymbol{\omega}$ is extended to a unique G-connection form on Q^G, which will be also denoted by $\boldsymbol{\omega}$.

(iv) For $i = -1, 0, 1$, let $\theta_i: \mathfrak{g} \to \mathfrak{g}_i$ be the projection with respect to the decomposition: $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ and let $\boldsymbol{\omega}_i = \theta_i \circ \boldsymbol{\omega}$ so that

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{-1} + \boldsymbol{\omega}_0 + \boldsymbol{\omega}_1\,.$$

We regard $\mathfrak{g}_{-1} + \mathfrak{g}_0$ as a Lie subalgebra of $\mathfrak{p}^1(q) = \boldsymbol{R}^q + \mathfrak{gl}(q)$ by the map $id \oplus \rho_0$, where the differential of $\rho_0: G_0 \to GL(q)$ is also denoted by ρ_0. Then

$$\boldsymbol{\theta}^{(2)} = \boldsymbol{\omega}_{-1} + \boldsymbol{\omega}_0$$

on Q, and hence $\boldsymbol{\omega}_{-1} = p_1^0 \boldsymbol{\theta}^{(2)}$ on Q.

Now (iv) and (1.3) imply

(v) $d\boldsymbol{\omega}_{-1}+[\boldsymbol{\omega}_0, \boldsymbol{\omega}_{-1}]=0$ on Q.

(vi) Let

$$\boldsymbol{\Omega}=d\boldsymbol{\omega}+\frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}]$$

be the curvature of $\boldsymbol{\omega}$ and decompose it as for $\boldsymbol{\omega}$:

$$\boldsymbol{\Omega}=\boldsymbol{\Omega}_{-1}+\boldsymbol{\Omega}_0+\boldsymbol{\Omega}_1, \qquad \boldsymbol{\Omega}_i=\theta_i\circ\boldsymbol{\Omega} \qquad (i=-1, 0, 1).$$

Then one has

$$[\boldsymbol{\omega}_{-1}, \boldsymbol{\Omega}_0]=0, \tag{1.7}$$

$$\mathrm{Tr}\,[\boldsymbol{\omega}_{-1}, \boldsymbol{\Omega}_1]=0, \tag{1.8}$$

$$\mathrm{Tr}\,\boldsymbol{\Omega}_0=0 \tag{1.9}$$

on Q, where $\mathrm{Tr}\,x$ means the trace of $x\in\mathfrak{g}_0\subset\mathfrak{gl}(q)$.

In what follows, we assume $H^{2,1}(\mathfrak{g})=\{0\}$. Let $G\rightarrow Q(\mathcal{F})^G=Q(\mathcal{F})\times_U G\rightarrow M$ be the group extension of $Q(\mathcal{F})$ by G. For a local submersion $f_\alpha: U_\alpha\rightarrow B$ in $\mathcal{F}$, the natural extension $Q(\mathcal{F})^G|U_\alpha\rightarrow Q^G$ of the bundle map $f_\alpha^{(2)}: Q(\mathcal{F})|U_\alpha\rightarrow Q$ will be also denoted by $f_\alpha^{(2)}$. Now the $\Gamma(Q)$-invariance of the normal Cartan connection $\boldsymbol{\omega}$ implies the following lemma.

LEMMA 1.4. (Nishikawa-Takeuchi [**11**]) *There exists a unique G-connection form ω on $Q(\mathcal{F})^G$ such that $f_\alpha^{(2)*}\boldsymbol{\omega}=\omega$ on $Q(\mathcal{F})^G|U_\alpha$ for each $f_\alpha: U_\alpha\rightarrow B$ in $\mathcal{F}$.*

Let

$$\Omega=d\omega+\frac{1}{2}[\omega, \omega] \tag{1.10}$$

be the curvature of ω and decompose ω and Ω as:

$$\omega=\omega_{-1}+\omega_0+\omega_1, \qquad \omega_i=\theta_i\circ\omega,$$

$$\Omega=\Omega_{-1}+\Omega_0+\Omega_1, \qquad \Omega_i=\theta_i\circ\Omega.$$

Then the properties (iv), (v) and (vi) imply the following equalities on $Q(\mathcal{F})$.

$$\omega_{-1}=p_1^0\theta^{(2)}, \tag{1.11}$$

$$\Omega_{-1}=d\omega_{-1}+[\omega_0, \omega_{-1}]=0, \tag{1.12}$$

$$[\omega_{-1}, \Omega_0]=0, \tag{1.13}$$

$$\mathrm{Tr}\,[\omega_{-1}, \Omega_1]=0, \tag{1.14}$$

$$\mathrm{Tr}\,\Omega_0=0. \tag{1.15}$$

Choose a smooth G_0-equivariant section $s: P(\mathcal{F})\rightarrow Q(\mathcal{F})$ by Lemma 1.2. Then the pull back $s^*\omega_0$ of ω_0 is a G_0-connection form on $P(\mathcal{F})$. We extend this to a $GL(q)$-connection form on $P^1(\mathcal{F})$ by the homomorphism $\rho_0: G_0\rightarrow GL(q)$ and denote it by

$$\bar{\omega}_0 = \rho_0(s^*\omega_0)\,.$$

The curvature of $\bar{\omega}_0$ will be denoted by $\bar{\Omega}_0$.

LEMMA 1.5. 1) *$s^*\omega_{-1} = \theta^{(1)}$ on $P(\mathcal{F})$.*

2) *$\bar{\omega}_0$ is torsion free in the sense that*

$$d\theta^{(1)} + [\bar{\omega}_0, \theta^{(1)}] = 0\,.$$

3) *$\bar{\Omega}_0(X, Y) = 0$ for each $X, Y \in T_v(P^1(\mathcal{F}))$ with $\pi_{1*}X, \pi_{1*}Y \in \tau(\mathcal{F})$.*

PROOF. 1) We have $(\pi_2^1)^*\theta^{(1)} = p_1^0\theta^{(2)}$ by (1.5). It follows from (1.11) that $\omega_{-1} = (\pi_2^1)^*\theta^{(1)}$ on $Q(\mathcal{F})$. This implies 1).

2) Follows from 1), (1.12) and (1.4).

3) Note first that we may assume that $X, Y \in F^{(1)}$. Taking $\mathfrak{g}_0$-components of (1.10), we get

$$\Omega_0 = d\omega_0 + \frac{1}{2}[\omega_0, \omega_0] + [\omega_{-1}, \omega_1]\,,$$

and hence

$$\bar{\Omega}_0 = s^*\Omega_0 - [s^*\omega_{-1}, s^*\omega_1]\,.$$

Let $X, Y \in F_v^{(1)}$ with $v \in P(\mathcal{F})$ and choose $f_\alpha : U_\alpha \to B$ in $\mathcal{F}$ with $\pi_1(v) \in U_\alpha$. Then $\Omega_0 = f_\alpha^{(2)*}\boldsymbol{\Omega}_0$ on $Q(\mathcal{F})|U_\alpha$ by Lemma 1.4, and hence $s^*\Omega_0 = s^*f_\alpha^{(2)*}\boldsymbol{\Omega}_0$ on $P(\mathcal{F})|U_\alpha$. Since $\boldsymbol{\Omega}_0$ is horizontal and

$$\boldsymbol{\pi}_{2*}(f_{\alpha}^{(2)}{}_*s_*X) = f_{\alpha*}\pi_{2*}s_*X = f_{\alpha*}\pi_{1*}X = 0\,,$$

we have $(s^*\Omega_0)(X, Y) = 0$. On the other hand, 1) and Lemma 1.3 imply $[s^*\omega_{-1}, s^*\omega_1](X, Y) = 0$. Thus we get $\bar{\Omega}_0(X, Y) = 0$. q. e. d.

LEMMA 1.6. *Define a subbundle $F^{(2)}$ of $T(Q(\mathcal{F}))$ of codimension q by*

$$F^{(2)} = \{X \in T(Q(\mathcal{F}))\,;\, \pi_{2*}X \in \tau(\mathcal{F})\}\,.$$

Then

1) *$\Omega(X, Y) = 0$ if $X \in F^{(2)}$.*

2) *$\omega_{-1}(X) = 0$ for each $X \in F^{(2)}$.*

PROOF. 1) Let $X, Y \in T_v(Q(\mathcal{F}))$ and choose $f_\alpha : U_\alpha \to B$ in $\mathcal{F}$ with $\pi_2(v) \in U_\alpha$. Then $\Omega = f_\alpha^{(2)*}\boldsymbol{\Omega}$ on $Q(\mathcal{F})|U_\alpha$ by Lemma 1.4. Since $\boldsymbol{\Omega}$ is horizontal and $\boldsymbol{\pi}_{2*}(f_\alpha^{(2)}{}_*X) = f_{\alpha*}\pi_{2*}X = 0$, we have $\Omega(X, Y) = 0$.

2) Let $X \in T_v(Q(\mathcal{F}))$ and set $v' = \pi_2^1(v) \in P(\mathcal{F})$. Note that then $(\pi_2^1)_*F_v^{(2)} = F_{v'}^{(1)}$. We have $\omega_{-1} = (\pi_2^1)^*\theta^{(1)}$, as we have seen in the proof of Lemma 1.5. Now Lemma 1.3 implies $\omega_{-1}(X) = 0$. q. e. d.

§ 2. Characteristic homomorphisms.

In this section we shall define a characteristic homomorphism for $\Gamma(Q)$-foliations by the methods of Kamber-Tondeur [6], Morita [10].

We shall first recall the notion of the canonical filtration of a G-DGA (cf. Kamber-Tondeur [**6**]). Let G be a Lie group and

$$E=\sum_{p\geq 0} E^p, \qquad \text{with the differential } d$$

be a G-differential graded algebra over $\boldsymbol{R}$, which will be abbreviated to a G-DGA. The action of G or $\mathfrak{g}=\mathrm{Lie}\, G$ will be denoted by $L(a)$ $(a\in G)$ or $L(x)$ $(x\in\mathfrak{g})$, and the contraction by $x\in\mathfrak{g}$ will be denoted by $i(x)$. For a Lie subgroup H of G and $\mathfrak{h}=\mathrm{Lie}\, H$, we denote the DG subalgebra of E of all H-invariant elements or $\mathfrak{h}$-invariant elements by E^H or $E^{\mathfrak{h}}$. The DG subalgebra of E of all H-basic elements or $\mathfrak{h}$-basic elements will be denoted by E_H or $E_{\mathfrak{h}}$. We define a decreasing filtration of E by

$$F^p(\mathfrak{g})E^n=F^pE^n=\{u\in E^n\,;\, i(x_1)\cdots i(x_r)u=0 \quad \text{for } x_i\in\mathfrak{g} \text{ with } r>n-p\}\,.$$

Then

$$F^pE=\sum_{n\geq 0} F^pE^n$$

are G-DG ideals of E with $(F^pE)(F^{p'}E)\subset F^{p+p'}E$. Thus the quotient algebra E/F^pE becomes a G-DGA in a natural way.

For example, let

$$W(\mathfrak{g})=\Lambda\mathfrak{g}^*\otimes S\mathfrak{g}^*$$

be the Weil algebra of $\mathfrak{g}$ with the graduation

$$W^p(\mathfrak{g})=\sum_{r+2s=p} W^{r,2s}(\mathfrak{g})\,, \qquad W^{r,2s}(\mathfrak{g})=\Lambda^r\mathfrak{g}^*\otimes S^s\mathfrak{g}^*\,.$$

Then

$$F^{2p-1}W(\mathfrak{g})=F^{2p}W(\mathfrak{g})=\sum_{s\geq p}\Lambda\mathfrak{g}^*\otimes S^s\mathfrak{g}^*\,.$$

The quotient G-DGA $W(\mathfrak{g})/F^{2(k+1)}W(\mathfrak{g})$ will be denoted by $W(\mathfrak{g})_k$. For a Lie subgroup H of G, the DGA $(W(\mathfrak{g})_k)_H$ will be denoted by $W(\mathfrak{g}, H)_k$. The algebra $W(\mathfrak{g})_{\mathfrak{g}}$ is identified with the algebra $I(\mathfrak{g})$ of $\mathfrak{g}$-invariant polynomials on $\mathfrak{g}$, and $W(\mathfrak{g})_G$ with the algebra $I(G)$ of G-invariant polynomials on $\mathfrak{g}$. The differential d is trivial on $W(\mathfrak{g})_{\mathfrak{g}}$.

Now we come back to a semi-simple flat homogeneous space G/U and recall some basic facts on G/U (cf. Nishikawa-Takeuchi [**11**]). There exists uniquely $h_0\in\mathfrak{g}_0$ such that

$$\mathfrak{g}_i=\{x\in\mathfrak{g}\,;\,[h_0, x]=ix\} \qquad (i=-1, 0, 1)\,.$$

Then G_0 is characterized by

$$G_0=\{a\in U\,;\,\mathrm{Ad}\,a\,h_0=h_0\}\,.$$

It follows that G_0 leaves each $\mathfrak{g}_i$ invariant, and hence each projection $\theta_i:\mathfrak{g}\to\mathfrak{g}_i$

is G_0-equivariant. Moreover, the projection $\theta: \mathfrak{g} \rightarrow \mathfrak{u}$ defined by the direct sum: $\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{u}$ is also G_0-equivariant. The group G_0 and the Lie algebra $\mathfrak{g}_0$ are identified with a reductive closed subgroup of $GL(q)$ and a reductive algebraic Lie subalgebra of $\mathfrak{gl}(q)$ respectively, where $q=\dim G/U$. Let K_0 be a maximal compact subgroup of G_0. Then there exists a Cartan involution τ of $\mathfrak{g}$ with $\tau h_0=-h_0$ such that if we set

$$\mathfrak{k}=\{x\in\mathfrak{g}\,;\,\tau x=x\},$$

then K_0 is given by

$$K_0=\{a\in G_0\,;\,\mathrm{Ad}\,a\mathfrak{k}=\mathfrak{k}\}.$$

Set

$$\mathfrak{p}=\{x\in\mathfrak{g}\,;\,\tau x=-x\},$$

$$\mathfrak{k}_0=\mathfrak{k}\cap\mathfrak{g}_0, \qquad \mathfrak{p}_0=\mathfrak{p}\cap\mathfrak{g}_0,$$

$$K=\{a\in G\,;\,\mathrm{Ad}\,a\mathfrak{k}=\mathfrak{k}\}.$$

Then one has $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ and $\mathrm{Lie}\,K=\mathfrak{k}$, $\mathrm{Lie}\,K_0=\mathfrak{k}_0$. The map $(a, x)\mapsto a\exp x$ $(a\in K_0,\ x\in\mathfrak{p}_0)$ defines a diffeomorphism: $K_0\times\mathfrak{p}_0\approx G_0$. Moreover the inclusion $K\rightarrow G$ induces a diffeomorphism: $K/K_0\approx G/U$. Making use of the Killing form B of $\mathfrak{g}$, we define a K-invariant inner product $\langle\,,\rangle$ on $\mathfrak{g}$ by

$$\langle x, y\rangle=-B(x, \tau y) \qquad \text{for} \quad x, y\in\mathfrak{g}.$$

In what follows, we shall identify $\mathfrak{g}_{-1}$ with $\boldsymbol{R}^q$ by an orthonormal basis for $\mathfrak{g}_{-1}$ with respect to $\langle\,,\rangle$, so that K_0 may be identified with a subgroup of the orthogonal group $O(q)$.

Now regard the Weil algebra $W(\mathfrak{g})$ of $\mathfrak{g}$ as a U-DGA and define a U-DGA $W(\mathfrak{g})_{(q)}$ by

$$W(\mathfrak{g})_{(q)}=W(\mathfrak{g})/F^{q+1}(\mathfrak{u})W(\mathfrak{g}),$$

and then define

$$W(\mathfrak{g}, K_0)_{(q)}=(W(\mathfrak{g})_{(q)})_{K_0}.$$

Note that if we identify $\mathfrak{g}_{-1}{}^*$ and $\mathfrak{u}^*$ with subspaces of $\mathfrak{g}^*$ by means of projections θ_{-1} and θ, then $F^{q+1}(\mathfrak{u})W(\mathfrak{g})$ is given by

$$F^{q+1}(\mathfrak{u})W(\mathfrak{g})=\sum_{r+2s\geq q+1}\Lambda\mathfrak{u}^*\otimes\Lambda^r\mathfrak{g}_{-1}{}^*\otimes S^s\mathfrak{g}^*. \tag{2.1}$$

Let I be the ideal of $W(\mathfrak{g})$ generated by

(a) $\alpha\tilde{\theta}_{-1}\in W^{0,2}(\mathfrak{g})=S^1\mathfrak{g}^*$ for $\alpha\in\mathfrak{g}_{-1}{}^*$, where $(\alpha\tilde{\theta}_{-1})(\xi)=\alpha\theta_{-1}(\xi)$ for $\xi\in\mathfrak{g}$;

(b) $\alpha[\theta_{-1}, \tilde{\theta}_0]\in W^{1,2}(\mathfrak{g})=\Lambda^1\mathfrak{g}^*\otimes S^1\mathfrak{g}^*$ for $\alpha\in\mathfrak{g}_{-1}{}^*$, where $(\alpha[\theta_{-1}, \tilde{\theta}_0])(x\otimes\xi)=\alpha[\theta_{-1}(x), \theta_0(\xi)]$ for $x, \xi\in\mathfrak{g}$;

(c) $\mathrm{Tr}\,[\theta_{-1}, \tilde{\theta}_1]\in W^{1,2}(\mathfrak{g})$, where $(\mathrm{Tr}\,[\theta_{-1}, \tilde{\theta}_1])(x\otimes\xi)=\mathrm{Tr}\,[\theta_{-1}(x), \theta_1(\xi)]$ for $x, \xi\in\mathfrak{g}$;

(d) $\mathrm{Tr}\,\tilde{\theta}_0\in W^{0,2}(\mathfrak{g})$, where $(\mathrm{Tr}\,\tilde{\theta}_0)(\xi)=\mathrm{Tr}\,\theta_0(\xi)$ for $\xi\in\mathfrak{g}$.

We shall prove later the following lemma.

LEMMA 2.1. *I is a G_0-DG ideal of $W(\mathfrak{g})$.*

Thus the ideal $I_{(q)}$ of $W(\mathfrak{g})$ generated by I and $F^{q+1}(\mathfrak{u})W(\mathfrak{g})$ is a G_0-DG ideal of $W(\mathfrak{g})$. We define a G_0-DGA $\widetilde{W}(\mathfrak{g})_{(q)}$ by

$$\widetilde{W}(\mathfrak{g})_{(q)}=W(\mathfrak{g})/I_{(q)}\,,$$

and then define

$$\widetilde{W}(\mathfrak{g},\, K_0)_{(q)}=(\widetilde{W}(\mathfrak{g})_{(q)})_{K_0}\,.$$

Note that the natural projection $\varpi: W(\mathfrak{g})_{(q)}\rightarrow\widetilde{W}(\mathfrak{g})_{(q)}$ is a G_0-DGA homomorphism and hence it induces a DGA homomorphism $\varpi: W(\mathfrak{g},\, K_0)_{(q)}\rightarrow\widetilde{W}(\mathfrak{g},\, K_0)_{(q)}$. The cohomology $H(\widetilde{W}(\mathfrak{g},\, K_0)_{(q)})$ of the DGA $\widetilde{W}(\mathfrak{g},\, K_0)_{(q)}$ is called the *characteristic algebra* for $\Gamma(Q)$-foliations.

PROOF OF LEMMA 2.1. Each element in (a), (b), (c), (d) is annihilated by $i(x)$ $(x\in\mathfrak{g}_0)$. For each element $a\in G_0$, one has $L(a)(\alpha\tilde{\theta}_{-1})=(L(a)\alpha)\tilde{\theta}_{-1}$, $L(a)(\alpha[\theta_{-1},\, \tilde{\theta}_0])=(L(a)\alpha)[\theta_{-1},\, \tilde{\theta}_0]$, $L(a)(\mathrm{Tr}\,[\theta_{-1},\, \tilde{\theta}_1])=\mathrm{Tr}\,[\theta_{-1},\, \tilde{\theta}_1]$ and $L(a)(\mathrm{Tr}\,\tilde{\theta}_0)=\mathrm{Tr}\,\tilde{\theta}_0$. Thus it remains to show that I is closed under the differential d of $W(\mathfrak{g})$. Recall that d is characterized by

$$d\alpha=d_{\Lambda}\alpha+\tilde{\alpha}\,,\qquad d\tilde{\alpha}=-\sum_i x_i^*\otimes(\mathrm{ad}\, x_i)^*\tilde{\alpha} \tag{2.2}$$

for each $\alpha\in\Lambda^1\mathfrak{g}^*$, where $\tilde{\alpha}$ is the element of $S^1\mathfrak{g}^*$ corresponding to α; d_{Λ} is the Chevalley-Eilenberg differential on $\Lambda\mathfrak{g}^*$; $\{x_i\}$ is a basis for $\mathfrak{g}$; $\{x_i^*\}$ is the dual basis for $\mathfrak{g}^*$; $(\mathrm{ad}\, x_i)^*$ is the transpose of $\mathrm{ad}\, x_i$.

(a) $d(\alpha\tilde{\theta}_{-1})\in W^{1,2}(\mathfrak{g})$ is computed by (2.2) to get

$$d(\alpha\tilde{\theta}_{-1})=-(\alpha[\theta_{-1},\, \tilde{\theta}_0]+\alpha[\theta_0,\, \tilde{\theta}_{-1}])\,, \tag{2.3}$$

where $\alpha[\theta_0,\, \tilde{\theta}_{-1}]\in W^{1,2}(\mathfrak{g})=\Lambda^1\mathfrak{g}^*\otimes S^1\mathfrak{g}^*$ is defined by

$$(\alpha[\theta_0,\, \tilde{\theta}_{-1}])(x\otimes\xi)=\alpha[\theta_0(x),\, \theta_{-1}(\xi)]\qquad\text{for}\quad x,\, \xi\in\mathfrak{g}\,.$$

Take basis $\{x_i\}$, $\{y_j\}$, $\{z_k\}$ for $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$, $\mathfrak{g}_1$ respectively and let $\{x_i^*\}$, $\{y_j^*\}$, $\{z_k^*\}$ be the dual basis for $\mathfrak{g}^*$. Define $\alpha_j\in\mathfrak{g}_{-1}^*$ by

$$\alpha_j(x)=\alpha[y_j,\, x]\qquad\text{for}\quad x\in\mathfrak{g}_{-1}\,.$$

Then one has

$$\alpha[\theta_0,\, \tilde{\theta}_{-1}]=\sum_j y_j^*(\alpha_j\tilde{\theta}_{-1})\,. \tag{2.4}$$

Thus $d(\alpha\tilde{\theta}_{-1})=-\alpha[\theta_{-1},\, \tilde{\theta}_0]-\sum_j y_j^*(\alpha_j\tilde{\theta}_{-1})\in I$.

(b) Differentiate (2.3) to get

$$\begin{aligned} d(\alpha[\theta_{-1},\, \tilde{\theta}_0])&=-d(\alpha[\theta_0,\, \tilde{\theta}_{-1}])=-d\sum_j y_j^*(\alpha_j\tilde{\theta}_{-1})\qquad\text{by (2.4)}\\ &=-\sum_j(dy_j^*)(\alpha_j\tilde{\theta}_{-1})+\sum_j y_j^* d(\alpha_j\tilde{\theta}_{-1})\in I\,. \end{aligned}$$

(d) $d(\mathrm{Tr}\,\tilde{\theta}_0)\in W^{1,2}(\mathfrak{g})=\Lambda^1\mathfrak{g}^*\otimes S^1\mathfrak{g}^*$ is computed by (2.2) to get

$$\begin{aligned} d(\mathrm{Tr}\,\tilde{\theta}_0)(x\otimes\xi) & \\ &=-\mathrm{Tr}([\theta_{-1}(x),\ \theta_1(\xi)]+[\theta_1(x),\ \theta_{-1}(\xi)]+[\theta_0(x),\ \theta_0(\xi)]) \\ &=-\mathrm{Tr}([\theta_{-1}(x),\ \theta_1(\xi)]+[\theta_1(x),\ \theta_{-1}(\xi)]),\end{aligned}$$

and hence

$$d(\mathrm{Tr}\,\tilde{\theta}_0)=-(\mathrm{Tr}\,[\theta_{-1},\ \tilde{\theta}_1]+\mathrm{Tr}\,[\theta_1,\ \tilde{\theta}_{-1}]),$$

where $\mathrm{Tr}\,[\theta_1,\ \tilde{\theta}_{-1}]\in W^{1,2}(\mathfrak{g})$ is defined by

$$(\mathrm{Tr}\,[\theta_1,\ \tilde{\theta}_{-1}])(x\otimes\xi)=\mathrm{Tr}\,[\theta_1(x),\ \theta_{-1}(\xi)] \qquad \text{for} \quad x,\ \xi\in\mathfrak{g}.$$

Define $t_k\in\mathfrak{g}_{-1}{}^*$ by

$$t_k(x)=\mathrm{Tr}\,[z_k,\ x] \qquad \text{for} \quad x\in\mathfrak{g}_{-1}.$$

Then one has

$$\mathrm{Tr}\,[\theta_1,\ \tilde{\theta}_{-1}]=\sum_k z_k{}^*(t_k\tilde{\theta}_{-1}).$$

Thus $d(\mathrm{Tr}\,\tilde{\theta}_0)=-\mathrm{Tr}\,[\theta_{-1},\ \tilde{\theta}_1]-\sum_k z_k{}^*(t_k\tilde{\theta}_{-1})\in I.$

(c) In the same way as (b), we get

$$d(\mathrm{Tr}\,[\theta_{-1},\ \tilde{\theta}_1])=-\sum_k(dz_k{}^*)(t_k\tilde{\theta}_{-1})+\sum_k z_k{}^*d(t_k\tilde{\theta}_{-1})\in I. \qquad \text{q. e. d.}$$

Now assume $H^{2,1}(\mathfrak{g})=\{0\}$ and let $\mathcal{F}$ be a $\Gamma(Q)$-foliation on M. Let $k(\omega)$: $W(\mathfrak{g})\to A(Q(\mathcal{F})^G)$ be the Weil homomorphism for the G-connection form ω in Lemma 1.4. Here $A(*)$ means the de Rham complex of $*$. Recall that $k(\omega)$ is the G-DGA homomorphism characterized by

$$k(\omega)(\alpha)=\alpha\circ\omega, \qquad k(\omega)(\tilde{\alpha})=\alpha\circ\Omega \qquad \text{for} \quad \alpha\in\Lambda^1\mathfrak{g}^*.$$

Let $j: Q(\mathcal{F})\to Q(\mathcal{F})^G$ be the inclusion and $j^*: A(Q(\mathcal{F})^G)\to A(Q(\mathcal{F}))$ the restriction. Define a U-DGA homomorphism $l(\omega): W(\mathfrak{g})\to A(Q(\mathcal{F}))$ by the composition

$$l(\omega): W(\mathfrak{g})\xrightarrow{k(\omega)} A(Q(\mathcal{F})^G)\xrightarrow{j^*} A(Q(\mathcal{F})).$$

Since Lemma 1.6 and (2.1) imply $l(\omega)F^{q+1}(\mathfrak{u})W(\mathfrak{g})=\{0\}$, $l(\omega)$ induces a U-DGA homomorphism $l(\omega): W(\mathfrak{g})_{(q)}\to A(Q(\mathcal{F}))$. Moreover, since $k(\omega)$ maps $\alpha\tilde{\theta}_{-1}$, $\alpha[\theta_{-1},\tilde{\theta}_0]$, $\mathrm{Tr}\,[\theta_{-1},\ \tilde{\theta}_1]$, $\mathrm{Tr}\,\tilde{\theta}_0$ into $\alpha\Omega_{-1}$, $\alpha[\omega_{-1},\ \Omega_0]$, $\mathrm{Tr}\,[\omega_{-1},\ \Omega_1]$, $\mathrm{Tr}\,\Omega_0$ respectively, it follows from (1.12)-(1.15) that $l(\omega)I=\{0\}$. Thus there exists a G_0-DGA homomorphism $\tilde{l}(\omega): \widetilde{W}(\mathfrak{g})_{(q)}\to A(Q(\mathcal{F}))$ such that the diagram

$$\begin{array}{ccc} W(\mathfrak{g})_{(q)} & \xrightarrow{\ l(\omega)\ } & A(Q(\mathcal{F})) \\ \varpi\downarrow & \nearrow_{\tilde{l}(\omega)} & \\ \widetilde{W}(\mathfrak{g})_{(q)} & & \end{array}$$

is commutative. This induces a commutative diagram

$$\begin{array}{ccc} W(\mathfrak{g}, K_0)_{(q)} & \xrightarrow{\ l(\omega)\ } & A(Q(\mathcal{F})/K_0) \\ \Big\downarrow \varpi & \nearrow \tilde{l}(\omega) & \\ \tilde{W}(\mathfrak{g}, K_0)_{(q)} & & \end{array}$$

of DGA homomorphisms. Since $U \approx G_0 \times \mathfrak{g}_1 \approx K_0 \times \mathfrak{p}_0 \times \mathfrak{g}_1$, the bundle $U/K_0 \to Q(\mathcal{F})/K_0 \to M$ has a smooth section $\sigma : M \to Q(\mathcal{F})/K_0$ (unique up to homotopy). Let $\sigma^* : A(Q(\mathcal{F})/K_0) \to A(M)$ be the pull back by σ and define a DGA homomorphism $\Delta(\mathcal{F}) : \tilde{W}(\mathfrak{g}, K_0)_{(q)} \to A(M)$ by the composition

$$\Delta(\mathcal{F}) : \tilde{W}(\mathfrak{g}, K_0)_{(q)} \xrightarrow{\tilde{l}(\omega)} A(Q(\mathcal{F})/K_0) \xrightarrow{\sigma^*} A(M).$$

The induced GA homomorphim

$$\Delta_*(\mathcal{F}) : H(\tilde{W}(\mathfrak{g}, K_0)_{(q)}) \longrightarrow H(M)$$

is called the *characteristic homomorphism* for the $\Gamma(Q)$-foliation $\mathcal{F}$.

On the other hand, making use of the torsion free $GL(q)$-connection form $\bar{\omega}_0$ on $P^1(\bar{\mathcal{F}})$, we can define the characteristic homomorphism for the smooth foliation $\bar{\mathcal{F}}$ in the following way. Let $k(\bar{\omega}_0) : W(\mathfrak{gl}(q)) \to A(P^1(\bar{\mathcal{F}}))$ be the Weil homomorphism for $\bar{\omega}_0$. Since Lemma 1.5 implies $k(\bar{\omega}_0)F^{2(q+1)}W(\mathfrak{gl}(q)) = \{0\}$, $k(\bar{\omega}_0)$ induces a $GL(q)$-DGA homomorphism $k(\bar{\omega}_0) : W(\mathfrak{gl}(q))_q \to A(P^1(\bar{\mathcal{F}}))$. The bundle $G^2(q)/O(q) \to P^2(\bar{\mathcal{F}})/O(q) \to M$ has also a smooth section $\bar{\sigma} : M \to P^2(\bar{\mathcal{F}})/O(q)$ (unique up to homotopy), since $G^2(q)/O(q)$ is contractible. We define a DGA homomorphism $\bar{\Delta}(\bar{\mathcal{F}}) : W(\mathfrak{gl}(q), O(q))_q \to A(M)$ by the composition

$$\bar{\Delta}(\bar{\mathcal{F}}) : W(\mathfrak{gl}(q), O(q))_q \xrightarrow{k(\bar{\omega}_0)} A(P^1(\bar{\mathcal{F}})/O(q)) \xrightarrow{(\pi_2^1)^*} A(P^2(\bar{\mathcal{F}})/O(q)) \xrightarrow{\bar{\sigma}^*} A(M).$$

Then the induced GA homomorphism

$$\bar{\Delta}_*(\bar{\mathcal{F}}) : H(W(\mathfrak{gl}(q), O(q))_q) \longrightarrow H(M)$$

coincides with the Bott-Haefliger's characteristic homomorphism [5] for the smooth foliation $\bar{\mathcal{F}}$.

§ 3. Connecting homomorphisms.

In this section we shall define a connecting homomorphism

$$\tilde{\Phi}_* : H(W(\mathfrak{gl}(q), O(q))_q) \longrightarrow H(\tilde{W}(\mathfrak{g}, K_0)_{(q)}) \qquad \text{with} \quad \bar{\Delta}_*(\bar{\mathcal{F}}) = \Delta_*(\mathcal{F}) \circ \tilde{\Phi}_*$$

and study $\tilde{\Phi}_*$ to prove the vanishing theorem.

We shall first recall the notion of the Weil homomorphism $k(\theta)$ and the difference map $\lambda(\theta)$ for a splitting θ (cf. Kamber-Tondeur [6]). Let G be a Lie group and $K\subset H\subset G$ Lie subgroups of G. Let $i: H\to G$ denote the inclusion homomorphism. Let $\mathfrak{g}=\text{Lie } G$, $\mathfrak{h}=\text{Lie } H$ and $\mathfrak{k}=\text{Lie } K$. In what follows, for a homomorphism φ of Lie groups, its differential will be denoted by the same letter φ. Let $\theta: \mathfrak{g}\to\mathfrak{h}$ be a K-equivariant splitting of the exact sequence

$$0 \longrightarrow \mathfrak{h} \stackrel{i}{\longrightarrow} \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0 .$$

Then there exists a unique K-DGA homomorphism $k(\theta): W(\mathfrak{h})\to W(\mathfrak{g})$ such that $k(\theta)=\theta^*$ (transpose of θ) on $\Lambda^1\mathfrak{h}^*$. It is called the *Weil homomorphism* for θ. We define next a linear map $\lambda(\theta): W(\mathfrak{g})\to W(\mathfrak{g})$ of degree -1 by the composition

$$\lambda(\theta): W(\mathfrak{g}) \stackrel{\lambda}{\longrightarrow} W(\mathfrak{g})\otimes W(\mathfrak{g}) \xrightarrow{id\otimes k(\theta)\circ W(i)} W(\mathfrak{g})\otimes W(\mathfrak{g}) \stackrel{\mu}{\longrightarrow} W(\mathfrak{g}),$$

where λ is the universal homotopy operator of Kamber-Tondeur; $W(i): W(\mathfrak{g})\to W(\mathfrak{h})$ is the DGA homomorphism characterized by that $W(i)=i^*$ on $\Lambda^1\mathfrak{g}^*$; μ is the algebra multiplication. It is called the *difference map* for θ and satisfies

(3.1) $$\lambda(\theta)\circ i(x)=-i(x)\circ\lambda(\theta) \quad \text{for each} \quad x\in\mathfrak{k},$$

(3.2) $$\lambda(\theta)\circ L(a)=L(a)\circ\lambda(\theta) \quad \text{for each} \quad a\in K,$$

(3.3) $$\lambda(\theta)\circ d+d\circ\lambda(\theta)=k(\theta)\circ W(i)-id .$$

These maps have the following properties for filtrations.

LEMMA 3.1. (Kamber-Tondeur [7])

1) $k(\theta)F^{2p}(\mathfrak{h})W(\mathfrak{h})\subset F^{2p-\min(p,q)}(\mathfrak{h})W(\mathfrak{g})$;

2) $\lambda(\theta)I^{2p}(\mathfrak{g})\subset F^{2p-\min(p,q)}(\mathfrak{h})W(\mathfrak{g})$,

where $q=\dim\mathfrak{g}/\mathfrak{h}$.

Next we shall recall the definition of Vey basis for $H(W(\mathfrak{gl}(q), O(q))_q)$ (cf. Godbillon [4]). We define $\bar{c}_j\in I^{2j}(GL(q))$ $(1\leqq j\leqq q)$ by

$$\det\left(1_q+\frac{1}{2\pi}x\right)=1+\bar{c}_1(x)+\bar{c}_2(x)+\cdots+\bar{c}_q(x) \qquad \text{for} \quad x\in\mathfrak{gl}(q).$$

If we denote by $\mathfrak{s}(q)$ the space of symmetric matrices in $\mathfrak{gl}(q)$, we have the direct sum decomposition: $\mathfrak{gl}(q)=\mathfrak{o}(q)+\mathfrak{s}(q)$, where $\mathfrak{o}(q)=\text{Lie } O(q)$. Denote by $\bar{\theta}: \mathfrak{gl}(q)\to\mathfrak{o}(q)$ the projection with respect to the above decomposition, and let $\lambda(\bar{\theta}): W(\mathfrak{gl}(q))\to W(\mathfrak{gl}(q))$ be the difference map for $\bar{\theta}$. We set

$$\bar{u}_i=-\lambda(\bar{\theta})\bar{c}_i \qquad \text{for odd } i \text{ with} \quad 1\leqq i\leqq q .$$

Then $\bar{u}_i\in W(\mathfrak{gl}(q))_{O(q)}$ by (3.1), (3.2) and $d\bar{u}_i=\bar{c}_i$ by (3.3). Let $\boldsymbol{R}[\bar{c}_1, \cdots, \bar{c}_q]_q$ denote the quotient algebra of the polynomial algebra $\boldsymbol{R}[\bar{c}_1, \cdots, \bar{c}_q]$ modulo the ideal of elements of degree$\geqq 2(q+1)$. Then the DGA

$$WO_q = \Lambda(\bar{u}_1, \bar{u}_3, \cdots)\otimes \boldsymbol{R}[\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_q]_q ,$$

with the differential $d(\bar{u}_i)=\bar{c}_i$, $d(\bar{c}_j)=0$, may be identified in a natural manner with a DG subalgebra of $W(\mathfrak{gl}(q), O(q))_q$ such that $H(WO_q)\cong H(W(\mathfrak{gl}(q), O(q))_q)$. A basis for $H(WO_q)$ is given as follows. Let $\mathcal{I}$ be the set of odd integers i with $1\leqq i\leqq q$. For a subset $I=(i_1, \cdots, i_k)$ of $\mathcal{I}$ and a series $J=(j_1, \cdots, j_l)$ of integers with $1\leqq j_1\leqq \cdots \leqq j_l\leqq q$; $|J|=j_1+\cdots+j_l\leqq q$, we set

$$\bar{u}_I\bar{c}_J=\bar{u}_{i_1}\cdots \bar{u}_{i_k}\bar{c}_{j_1}\cdots \bar{c}_{j_l}.$$

Let i^0 be the smallest element in I if $I\neq\emptyset$, $i^0=\infty$ if $I=\emptyset$, and let j^0 be the smallest element in $J\cap\mathcal{I}$ if $J\cap\mathcal{I}\neq\emptyset$, $j^0=\infty$ if $J\cap\mathcal{I}=\emptyset$. Then the set

$$\{[\bar{u}_I\bar{c}_J]\,;\, i^0+|J|\geqq q+1,\ i^0\leqq j^0\}$$

is a basis for $H(WO_q)$ and called the *Vey basis*. In particular, the class $[\bar{u}_I\bar{c}_J]$ with $i^0+|J|\geqq q+2$ is called a *rigid class*.

Now we come back to a semi-simple flat homogeneous space G/U of $\dim G/U=q$ with $H^{2,1}(\mathfrak{g})=\{0\}$ and a $\Gamma(Q)$-foliation $\mathcal{F}$ on M. Let $i: U\to G$ be the inclusion. Consider the G_0-equivariant splitting $\theta:\mathfrak{g}\to\mathfrak{u}$ of the exact sequence

$$0\longrightarrow \mathfrak{u}\overset{i}{\longrightarrow}\mathfrak{g}\longrightarrow \mathfrak{g}/\mathfrak{u}\longrightarrow 0 .$$

Let $k(\theta): W(\mathfrak{u})\to W(\mathfrak{g})$ be the Weil homomorphism for θ, which is a G_0-DGA homomorphism. Let $W(\rho): W(\mathfrak{gl}(q))\to W(\mathfrak{u})$ be the DGA homomorphism characterized by that $W(\rho)=\rho^*$ on $\Lambda^1\mathfrak{gl}(q)^*$. We define a DGA homomorphism $\Phi: W(\mathfrak{gl}(q))\to W(\mathfrak{g})$ by the composition

$$\Phi: W(\mathfrak{gl}(q))\overset{W(\rho)}{\longrightarrow} W(\mathfrak{u})\overset{k(\theta)}{\longrightarrow} W(\mathfrak{g}) .$$

Since $W(\rho)F^{2(q+1)}W(\mathfrak{gl}(q))\subset F^{2(q+1)}W(\mathfrak{u})$ and $k(\theta)F^{2(q+1)}W(\mathfrak{u})\subset F^{q+1}(\mathfrak{u})W(\mathfrak{g})$ by Lemma 3.1, 1), Φ induces a DGA homomorphism $\Phi: W(\mathfrak{gl}(q))_q\to W(\mathfrak{g})_{(q)}$. This induces a DGA homomorphism $\Phi: W(\mathfrak{gl}(q), O(q))_q\to W(\mathfrak{g}, K_0)_{(q)}$, since $\rho(K_0)\subset O(q)$. We define further a DGA homomorphism $\tilde{\Phi}: W(\mathfrak{gl}(q), O(q))_q\to \widetilde{W}(\mathfrak{g}, K_0)_{(q)}$ by the composition

$$\tilde{\Phi}: W(\mathfrak{gl}(q), O(q))_q\overset{\Phi}{\longrightarrow} W(\mathfrak{g}, K_0)_{(q)}\overset{\varpi}{\longrightarrow}\widetilde{W}(\mathfrak{g}, K_0)_{(q)} .$$

The induced GA homomorphisms are denoted by

$$\tilde{\Phi}_*: H(W(\mathfrak{gl}(q), O(q))_q)\overset{\Phi_*}{\longrightarrow} H(W(\mathfrak{g}, K_0)_{(q)})\overset{\varpi_*}{\longrightarrow} H(\widetilde{W}(\mathfrak{g}, K_0)_{(q)}) .$$

THEOREM 3.1. *Suppose $H^{2,1}(\mathfrak{g})=\{0\}$. Then the diagram*

$$\begin{array}{ccc} H(W(\mathfrak{gl}(q), O(q))_q) & \xrightarrow{\bar{\Delta}_*(\bar{\mathcal{F}})} & H(M) \\ \Big\downarrow \tilde{\Phi}_* & \nearrow \Delta_*(\mathcal{F}) & \\ H(\widetilde{W}(\mathfrak{g}, K_0)_{(q)}) & & \end{array}$$

is commutative for each $\Gamma(Q)$-foliation $\mathcal{F}$ on M.

PROOF. Let $\bar{j}: Q(\mathcal{F}) \to P^2(\bar{\mathcal{F}})$ be the inclusion map. The map $Q(\mathcal{F})/K_0 \to P^2(\bar{\mathcal{F}})/O(q)$ induced by $\bar{j}$ will be denoted also by $\bar{j}$. We shall first show that the diagram

$$\begin{array}{ccccc} W(\mathfrak{gl}(q), O(q))_q & \xrightarrow{k(\bar{\omega}_0)} & A(P^1(\bar{\mathcal{F}})/O(q)) & \xrightarrow{(\pi_2^1)^*} & A(P^2(\bar{\mathcal{F}})/O(q)) \\ \Big\downarrow \Phi & & & & \Big\downarrow \bar{j}^* \\ W(\mathfrak{g}, K_0)_{(q)} & & \xrightarrow[l(\omega)]{\hspace{6em}} & & A(Q(\mathcal{F})/K_0) \end{array}$$

is commutative in cohomology. We define two DGA homomorphisms k_0, k_1: $W(\mathfrak{gl}(q)) \to A(Q(\mathcal{F}))$ by

$$k_0 = \bar{j}^* \circ (\pi_2^1)^* \circ k(\bar{\omega}_0), \qquad k_1 = l(\omega) \circ \Phi,$$

and then define a linear map $\lambda(k_0, k_1): W(\mathfrak{gl}(q)) \to A(Q(\mathcal{F}))$ of degree -1 by the composition

$$\lambda(k_0, k_1): W(\mathfrak{gl}(q)) \xrightarrow{\lambda} W(\mathfrak{gl}(q)) \otimes W(\mathfrak{gl}(q)) \xrightarrow{k_0 \otimes k_1}$$
$$A(Q(\mathcal{F})) \otimes A(Q(\mathcal{F})) \xrightarrow{\mu} A(Q(\mathcal{F})),$$

where λ is the universal homotopy operator of Kamber-Tondeur and μ is the algebra multiplication. It satisfies

(3.4) $\lambda(k_0, k_1) \circ i(\rho_0(x)) = -i(x) \circ \lambda(k_0, k_1)$ for each $x \in \mathfrak{k}_0$,

(3.5) $\lambda(k_0, k_1) \circ L(\rho_0(a)) = L(a) \circ \lambda(k_0, k_1)$ for each $a \in K_0$,

(3.6) $\lambda(k_0, k_1) \circ d + d \circ \lambda(k_0, k_1) = k_0 - k_1$.

We shall prove

(3.7) $\lambda(k_0, k_1) F^{2(q+1)} W(\mathfrak{gl}(q)) = \{0\}$.

Then this together with (3.4), (3.5) implies that $\lambda(k_0, k_1)$ induces a linear map $\lambda(k_0, k_1): W(\mathfrak{gl}(q), O(q))_q \to A(Q(\mathcal{F})/K_0)$ satisfying the same equality as (3.6) on

$W(\mathfrak{gl}(q), O(q))_q$. Thus we get the cohomology commutativity of the above diagram.

Recall (cf. Kamber-Tondeur [6]) that each element of $\lambda F^{2(q+1)}W(\mathfrak{gl}(q))$ is the sum of elements of the form

$$w(\alpha_1 \otimes 1 - 1 \otimes \alpha_1) \cdots (\alpha_r \otimes 1 - 1 \otimes \alpha_r)(\tilde{\beta}_1 \cdots \tilde{\beta}_s \otimes \tilde{\gamma}_1 \cdots \tilde{\gamma}_t),$$

with $w \in W(\mathfrak{gl}(q)) \otimes W(\mathfrak{gl}(q))$, $\alpha_i, \beta_i, \gamma_i \in \mathfrak{gl}(q)^*$, $r+s+t \geqq q+1$. Since the sub-bundle $F^{(2)}$ of $T(Q(\mathcal{F}))$ defined in Lemma 1.6 has the codimension q, it suffices for the proof of (3.7) to show that for each $\alpha \in \mathfrak{gl}(q)^*$ and $v \in Q(\mathcal{F})$ one has

$$\mu((k_0 \otimes k_1)(\alpha \otimes 1 - 1 \otimes \alpha))(X) = 0 \quad \text{for} \quad X \in F_v^{(2)},$$
$$k_0(\tilde{\alpha})(X, Y) = 0 \qquad \text{for} \quad X, Y \in F_v^{(2)},$$
$$k_1(\tilde{\alpha})(X, Y) = 0 \qquad \text{for} \quad X, Y \in F_v^{(2)}.$$

For $X \in F_v^{(2)}$ one has

$$k_0(\alpha)(X) = (k(\bar{\omega}_0)\alpha)(\pi^1_{2*}X) = \alpha(\bar{\omega}_0(\pi^1_{2*}X))$$
$$= \alpha(\rho_0(s^*\omega_0))(\pi^1_{2*}X) = \alpha(\rho_0\omega_0(s_*\pi^1_{2*}X)).$$

Let $v = s(u)\exp A$ with $u = \pi^1_2(v)$, $A \in \mathfrak{g}_1$. Then, since

$$\pi^1_{2*}((R_{\exp A})^{-1}_* X - s_*\pi^1_{2*}X) = \pi^1_{2*}X - \pi^1_{2*}X = 0,$$

we find $B \in \mathfrak{g}_1$ such that $(R_{\exp A})^{-1}_* X - s_*\pi^1_{2*}X = B^*_{s(u)}$, where B^* is the fundamental vector field on $Q(\mathcal{F})$ generated by B. Evaluating ω at the both sides, one has $\mathrm{Ad}(\exp A)\omega(X) - \omega(s_*\pi^1_{2*}X) = B$. Taking $\mathfrak{g}_0$-components we get $[A, \omega_{-1}(X)] + \omega_0(X) - \omega_0(s_*\pi^1_{2*}X) = 0$. Now Lemma 1.6 implies $\omega_0(s_*\pi^1_{2*}X) = \omega_0(X)$, and hence $k_0(\alpha)(X) = \alpha(\rho_0\omega_0(X))$. On the other hand, one has

$$k_1(\alpha)(X) = (k(\omega)k(\theta)W(\rho)\alpha)(X) = (k(\omega)\theta^*\rho^*\alpha)(X)$$
$$= (\theta^*\rho^*\alpha)(\omega(X)) = \alpha(\rho\theta\omega(X)) = \alpha(\rho_0\omega_0(X)).$$

Thus one has

$$\mu((k_0 \otimes k_1)(\alpha \otimes 1 - 1 \otimes \alpha))(X) = k_0(\alpha)(X) - k_1(\alpha)(X) = 0.$$

Let $X, Y \in F_v^{(2)}$. Since $\pi^1_{2*}X, \pi^1_{2*}Y \in F_u^{(1)}$, one has

$$k_0(\tilde{\alpha})(X, Y) = \alpha(\bar{\Omega}_0(\pi^1_{2*}X, \pi^1_{2*}Y)) = 0$$

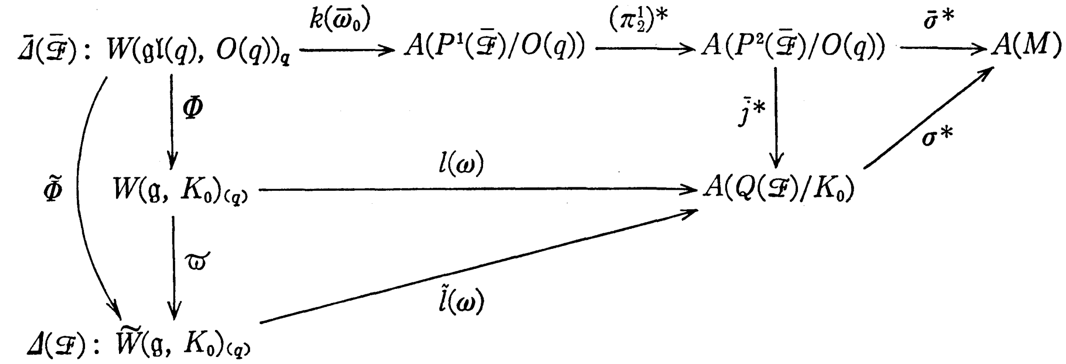
by Lemma 1.5. Furthermore by Lemma 1.6 one has

$$k_1(\tilde{\alpha})(X, Y) = \alpha(\rho\theta\Omega(X, Y)) = 0.$$

Thus we have done.

Now take a smooth section $\sigma: M \to Q(\mathcal{F})/K_0$ and define a smooth section $\bar{\sigma}: M \to P^2(\mathcal{F})/O(q)$ by $\bar{\sigma} = \bar{j} \circ \sigma$. Then we may use these sections to define $\Delta(\mathcal{F})$

and $\bar{\Delta}(\mathcal{F})$, and hence the following diagram is commutative in cohomology



This implies the theorem. q. e. d.

We use the notation in Introduction for invariant polynomials. Moreover, for a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(q)$, let $\mathrm{Pont}(\mathfrak{h})$ denote the subalgebra of $I(\mathfrak{h})$ generated by polynomials $\mathrm{Tr}\, x^{2k}$ with $1 \leqq k \leqq [q/2]$. Then we have the following

THEOREM 3.2. 1) *Suppose* $\mathrm{Char}(\mathfrak{g}_0') \subset I_G(\mathfrak{g}_0')$. *Then* $\Phi_*[\bar{u}_I \bar{c}_J] = 0$ *if*

(1) $I \neq \emptyset,\ i^0 + |J| \geqq q+2$;

(2) $2|J| \geqq q+1$.

2) *Suppose* $\mathrm{Pont}(\mathfrak{g}_0') \subset I_G(\mathfrak{g}_0')$. *Then* $\Phi_*[\bar{c}_J] = 0$ *if* $2|J| \geqq q+1$.

The first statement 1) together with Theorem 3.1 implies the vanishing theorem in Introduction. For the proof of this theorem we need the following two lemmas.

LEMMA 3.2. *We define* $c_1 \in I^2(G_0)$ *by*

$$c_1(x) = \frac{1}{2\pi} \mathrm{Tr}\, \rho_0(x) \qquad \textit{for} \quad x \in \mathfrak{g}_0 .$$

Then $\mathrm{Char}(\mathfrak{g}_0') \subset I_G(\mathfrak{g}_0')$ *if and only if* $\mathrm{Char}(\mathfrak{g}_0) \subset I_G(\mathfrak{g}_0) \bmod c_1 I(G_0)$.

PROOF. Let $i_0 : \mathfrak{g}_0 \to \mathfrak{g}$, $i_0' : \mathfrak{g}_0' \to \mathfrak{g}$ be the inclusions, and let $\rho_0' = \rho_0 | \mathfrak{g}_0' : \mathfrak{g}_0' \to \mathfrak{gl}(q)$. Suppose $\mathrm{Char}(\mathfrak{g}_0) \subset I_G(\mathfrak{g}_0) \bmod c_1 I(G_0)$. Then, for each $\bar{c} \in I(GL(q))$ there exists $f \in I(G)$ such that $\rho_0^* \bar{c} \equiv i_0^* f \bmod c_1 I(G_0)$. Restricting the both sides on $\mathfrak{g}_0'$, we get $\rho_0'^* \bar{c} = i_0'^* f$. This shows $\mathrm{Char}(\mathfrak{g}_0') \subset I_G(\mathfrak{g}_0')$.

Assume conversely $\mathrm{Char}(\mathfrak{g}_0') \subset I_G(\mathfrak{g}_0')$. The decomposition: $\mathfrak{g}_0 = \boldsymbol{R} h_0 \oplus \mathfrak{g}_0'$ implies $I(\mathfrak{g}_0) = \boldsymbol{R}[c_1] \otimes I(\mathfrak{g}_0')$. Thus for each $\bar{c} \in I(GL(q))$ there exists $f \in \mathrm{Char}(\mathfrak{g}_0')$ and $g \in I(G_0)$ such that $\rho_0^* \bar{c} = 1 \otimes f + c_1 g$, and hence $\rho_0^* \bar{c} \equiv 1 \otimes f \bmod c_1 I(G_0)$. From the assumption, there exists $\tilde{f} \in I(G)$ such that $i_0'^* \tilde{f} = f$. Now $i_0^* \tilde{f} - 1 \otimes f$ vanishes on $\mathfrak{g}_0'$, and hence $i_0^* \tilde{f} \equiv 1 \otimes f \bmod c_1 I(G_0)$. Thus $\rho_0^* \bar{c} \equiv i_0^* \tilde{f} \bmod c_1 I(G_0)$. This shows $\mathrm{Char}(\mathfrak{g}_0) \subset I_G(\mathfrak{g}_0) \bmod c_1 I(G_0)$. q. e. d.

LEMMA 3.3. *The restriction homomorphism*: $I(U) \to I(G_0)$ *is an isomorphism.*

PROOF. The decomposition: $\mathfrak{u}=\mathfrak{g}_0+\mathfrak{g}_1$ induces the identification: $S\mathfrak{u}^*=S\mathfrak{g}_0^*\otimes S\mathfrak{g}_1^*$, and hence

$$S^p\mathfrak{u}^*=\sum_{r+s=p} S^r\mathfrak{g}_0^*\otimes S^s\mathfrak{g}_1^*.$$

Since the operator $L(h_0)$ is the scalar $-s$ on $S^r\mathfrak{g}_0^*\otimes S^s\mathfrak{g}_1^*$, we have $I(U)\subset I(G_0)\otimes 1$. Conversely, since $L(\mathfrak{g}_1)\mathfrak{g}_0^*=\{0\}$ we have $I(G_0)\otimes 1\subset I(U)$. Thus we get $I(U)=I(G_0)\otimes 1$, which implies the lemma. q. e. d.

PROOF OF THEOREM 3.2. 1) We have $\rho_0^*\bar{c}_1=c_1$ from definitions. It follows from Lemmas 3.2, 3.3 that $\rho^*I(GL(q))\subset i^*I(G) \bmod (\rho^*\bar{c}_1)I(U)$. Hence there exist $f_J\in I^{2|J|}(G)$ and $g_J\in I^{2(|J|-1)}(U)$ such that $\rho^*\bar{c}_J=i^*f_J+(\rho^*\bar{c}_1)g_J$. It follows

$$\begin{aligned}\Phi(\bar{c}_J)&=k(\theta)\rho^*\bar{c}_J=k(\theta)i^*f_J+(k(\theta)\rho^*\bar{c}_1)(k(\theta)g_J)\\&=k(\theta)W(i)f_J+\Phi(\bar{c}_1)k(\theta)g_J=f_J+dw_J+\Phi(\bar{c}_1)k(\theta)g_J,\end{aligned}$$

where $w_J=\lambda(\theta)f_J$, by (3.3). Note that $w_J\in W(\mathfrak{g})_{K_0}$ by (3.1), (3.2). It follows

$$\Phi(\bar{u}_I\bar{c}_J)=\Phi(\bar{u}_I)f_J+\Phi(\bar{u}_I)dw_J+\Phi(\bar{u}_I\bar{c}_1)k(\theta)g_J.$$

Here

(3.8) $$f_J\in F^{2|J|}(\mathfrak{u})W(\mathfrak{g}),$$

and Lemma 3.1 implies

(3.9) $$w_J\in F^{|J|}(\mathfrak{u})W(\mathfrak{g}),$$

(3.10) $$k(\theta)g_J\in F^{|J|-1}(\mathfrak{u})W(\mathfrak{g}),$$

(3.11) $$\Phi(\bar{c}_j)\in F^j(\mathfrak{u})W(\mathfrak{g}) \quad \text{for} \quad 1\leqq j\leqq q.$$

Now (3.8) and the condition (2) implies $\Phi(\bar{u}_I)f_J\in F^{q+1}(\mathfrak{u})W(\mathfrak{g})$, and hence $\Phi(\bar{u}_I)f_J=0$ in $W(\mathfrak{g})_{(q)}$. If we write $I_p=(i_1,\cdots,\hat{i}_p,\cdots,i_k)$ for $1\leqq p\leqq k$, we have

$$d(\Phi(\bar{u}_I)w_J)=\sum_p \pm\Phi(\bar{u}_{I_p})\Phi(\bar{c}_{i_p})w_J\pm\Phi(\bar{u}_I)dw_J.$$

Here $\Phi(\bar{u}_{I_p})\Phi(\bar{c}_{i_p})w_J\in F^{i_p+|J|}(\mathfrak{u})W(\mathfrak{g})$ by (3.11), (3.9), with $i_p+|J|\geqq i^0+|J|\geqq q+1$, and hence $\Phi(\bar{u}_{I_p})\Phi(\bar{c}_{i_p})w_J=0$ in $W(\mathfrak{g})_{(q)}$. But $\Phi(\bar{u}_I)w_J\in W(\mathfrak{g})_{K_0}$ since $\bar{u}_I\in W(\mathfrak{gl}(q))_{O(q)}$. Thus $\Phi(\bar{u}_I)dw_J$ is exact in $W(\mathfrak{g}, K_0)_{(q)}$. Furthermore, one has

$$d(\Phi(\bar{u}_1\bar{u}_I)k(\theta)g_J)=\Phi(\bar{u}_I\bar{c}_1)k(\theta)g_J+\sum_p \pm\Phi(\bar{u}_1\bar{u}_{I_p})\Phi(\bar{c}_{i_p})k(\theta)g_J.$$

Here $\Phi(\bar{u}_1\bar{u}_{I_p})\Phi(\bar{c}_{i_p})k(\theta)g_J\in F^{i_p+|J|-1}(\mathfrak{u})W(\mathfrak{g})$ by (3.11), (3.10), and $i_p+|J|-1\geqq i^0+|J|-1\geqq q+1$ by the condition (1), and hence the second terms$=0$ in $W(\mathfrak{g})_{(q)}$. Since $\Phi(\bar{u}_1\bar{u}_I)k(\theta)g_J\in W(\mathfrak{g})_{K_0}$, $\Phi(\bar{u}_I\bar{c}_1)k(\theta)g_J$ is also exact in $W(\mathfrak{g}, K_0)_{(q)}$. These prove that $\Phi(\bar{u}_I\bar{c}_J)$ is exact in $W(\mathfrak{g}, K_0)_{(q)}$.

2) Follows from the same arguments as 1). q. e. d.

REMARKS. 1) We may apply our method to $\Gamma(Q)$-foliations with trivial normal bundles to show the vanishing of all rigid classes.

2) Theorem 3.1 and the second statement of Theorem 3.2 imply a vanishing theorem for Pontrjagin classes: *If* $H^{2,1}(\mathfrak{g})=\{0\}$ *and* $\mathrm{Pont}(\mathfrak{g}_0')\subset I_G(\mathfrak{g}_0')$, *then* $[\bar{c}_J](\mathcal{F})=0$ *for each* J *with* $2|J|\geqq q+1$. This has been also proved by Nishikawa-Takeuchi [11] under a less restrictive condition: $\mathrm{Pont}(\mathfrak{k}_0)\subset I_G(\mathfrak{k}_0)$.

§ 4. Characteristic algebras.

In this section we shall study the structure of the characteristic algebra $H(\widetilde{W}(\mathfrak{g}, K_0)_{(q)})$.

We shall first recall some facts on the cohomology of a G-DGA (cf. Kamber-Tondeur [6]). Let G be a Lie group such that $\mathfrak{g}=\mathrm{Lie}\, G$ is reductive. We assume the following conditions:

$$I(G)=I(\mathfrak{g})\,, \tag{4.1}$$

$$(\Lambda\mathfrak{g}^*)^G=(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}\,. \tag{4.2}$$

Let K be a Lie subgroup of G with finite components such that $\mathfrak{k}=\mathrm{Lie}\, K$ is reductive in $\mathfrak{g}$, and let $i: K\rightarrow G$ be the inclusion. Then there exists a K-equivariant splitting $\theta: \mathfrak{g}\rightarrow\mathfrak{k}$ of the exact sequence

$$0\longrightarrow\mathfrak{k}\overset{i}{\longrightarrow}\mathfrak{g}\longrightarrow\mathfrak{g}/\mathfrak{k}\longrightarrow 0\,.$$

Choose a such splitting θ and fix it once for all. Let $P(\mathfrak{g})\subset(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$ be the graded subspace of all primitive elements, and $P(\mathfrak{g}, \mathfrak{k})\subset P(\mathfrak{g})$ the Samelson subspace for the reductive pair $(\mathfrak{g}, \mathfrak{k})$. Set

$$I^+(\mathfrak{g})=\sum_{p>0} I^{2p}(\mathfrak{g})\,, \qquad I^+(G)=I(G)\cap I^+(\mathfrak{g})\,,$$

$$I(\mathfrak{g}, \mathfrak{k})=\{f\in I(\mathfrak{g})\,;\, i^*f=0\}\,.$$

Note that we have $I^+(G)=I^+(\mathfrak{g})$ and $I(K)$ is an $I^+(G)$-module by the restriction i^*. We assume that the reductive pair $(\mathfrak{g}, \mathfrak{k})$ is a *special Cartan pair* in the sense that there exists a transgression $\tau: P(\mathfrak{g})\rightarrow I^+(\mathfrak{g})$ such that $\tau P(\mathfrak{g}, \mathfrak{k})$ generates the ideal $I(\mathfrak{g}, \mathfrak{k})$. For example, a symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is always a special Cartan pair. A transgression τ such that $\tau P(\mathfrak{g}, \mathfrak{k})\subset I(\mathfrak{g}, \mathfrak{k})$ is said to be *adapted* to $(\mathfrak{g}, \mathfrak{k})$. Choose an adapted transgression τ and fix it once for all.

Let

$$E=\sum_{p\geqq 0} E^p\,, \qquad \text{with the differential } d$$

be a G-DGA such that $\dim E^p<\infty$ for each $p\geqq 0$ and

$$E^G=E^{\mathfrak{g}}\,. \tag{4.3}$$

Assume a G-DGA homomorphism $k: W(\mathfrak{g})\rightarrow E$ is given. Note that then k induces a DGA homomorphism $k: I(G)\rightarrow E_G$. We set

$$\hat{A}(E)=\Lambda P(\mathfrak{g}, \mathfrak{k})\otimes E_G,$$

and define a differential $d_{\hat{A}}$ on $\hat{A}(E)$ by

$$d_{\hat{A}}(y\otimes 1)=1\otimes k\tau y \qquad \text{for} \quad y\in P(\mathfrak{g}, \mathfrak{k}),$$
$$d_{\hat{A}}(1\otimes u)=1\otimes du \qquad \text{for} \quad u\in E_G,$$

to get a DGA $\hat{A}(E)$. Then we can define a DGA homomorphism $\alpha:\hat{A}(E)\rightarrow E_K$ by

$$\alpha(y\otimes 1)=-k\lambda(\theta)\tau y \qquad \text{for} \quad y\in P(\mathfrak{g}, \mathfrak{k}),$$
$$\alpha(1\otimes u)=u \qquad \text{for} \quad u\in E_G\subset E_K.$$

Let $\alpha_*: H(\hat{A}(E))\rightarrow H(E_K)$ denote the induced GA homomorphism. On the other hand, we can define a DGA homomorphism $\beta: I(K)\rightarrow E_K$ by

$$\beta(f)=k(k(\theta)f) \qquad \text{for} \quad f\in I(K).$$

Let $\beta_*: I(K)\rightarrow H(E_K)$ denote the induced GA homomorphism. Choose next a graded linear splitting $\nu: I(K)/I^+(G)I(K)\rightarrow I(K)$ of the exact sequence

$$0\longrightarrow I^+(G)I(K)\longrightarrow I(K)\longrightarrow I(K)/I^+(G)I(K)\longrightarrow 0.$$

Finally let $\mu: H(E_K)\otimes H(E_K)\rightarrow H(E_K)$ be the algebra multiplication. Then we have the following

THEOREM 4.1. *$\mu\circ(\alpha_*\otimes\beta_*\circ\nu): H(\hat{A}(E))\otimes(I(K)/I^+(G)I(K))\rightarrow H(E_K)$ is a graded linear isomorphism.*

COROLLARY. *If further $i^*: I(G)\rightarrow I(K)$ is surjective, then $\alpha_*: H(\hat{A}(E))\rightarrow H(E_K)$ is a GA isomorphism.*

We shall apply Theorem 4.1 to $G=G_0$, $K=K_0$ and $E=\widetilde{W}(\mathfrak{g})_{\langle q\rangle}$. Assume the condition

$$\text{(Z)} \qquad (G_0{}^Z)^0\subset G_0\subset G_0{}^Z,$$

where $G_0{}^Z\subset GL(q)$ denotes the Zariski-connected real algebraic group with $\operatorname{Lie} G_0{}^Z=\mathfrak{g}_0$ and $(G_0{}^Z)^0$ the identity-component of $G_0{}^Z$. Then G_0 satisfies the conditions (4.1), (4.2), (4.3). Let $i': K_0\rightarrow G_0$ be the inclusion and $\theta': \mathfrak{g}_0\rightarrow\mathfrak{k}_0$ the K_0-equivariant projection with respect to the decomposition: $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$. The pair $(\mathfrak{g}_0, \mathfrak{k}_0)$ is a symmetric pair, and hence it is a special Cartan pair. Choose an adapted transgression $\tau: P(\mathfrak{g}_0)\rightarrow I^+(\mathfrak{g}_0)$. Composing the natural projection: $W(\mathfrak{g})\rightarrow\widetilde{W}(\mathfrak{g})_{\langle q\rangle}$ to the G_0-DGA homomorphism $k(\theta_0): W(\mathfrak{g}_0)\rightarrow W(\mathfrak{g})$, we get a G_0-DGA homomorphism $\tilde{k}(\theta_0): W(\mathfrak{g}_0)\rightarrow\widetilde{W}(\mathfrak{g})_{\langle q\rangle}$. We set

$$\hat{A}=\Lambda P(\mathfrak{g}_0, \mathfrak{k}_0)\otimes\widetilde{W}(\mathfrak{g}, G_0)_{\langle q\rangle}, \qquad \text{where} \quad \widetilde{W}(\mathfrak{g}, G_0)_{\langle q\rangle}=(\widetilde{W}(\mathfrak{g})_{\langle q\rangle})_{G_0},$$

with the differential $d_{\hat{A}}$ defined by

$$d_{\hat{A}}(y\otimes 1)=1\otimes\tilde{k}(\theta_0)\tau y \qquad \text{for} \quad y\in P(\mathfrak{g}_0, \mathfrak{k}_0),$$

$$d_{\hat{A}}(1\otimes w)=1\otimes dw \qquad \text{for} \quad w\in\widetilde{W}(\mathfrak{g},\ G_0)_{(q)}\,.$$

Let $\alpha:\hat{A}\to\widetilde{W}(\mathfrak{g},\ K_0)_{(q)}$ be the DGA homomorphism defined by

$$\alpha(y\otimes 1)=-\tilde{k}(\theta_0)\lambda(\theta')\tau y \qquad \text{for} \quad y\in P(\mathfrak{g}_0,\ \mathfrak{k}_0)\,,$$

$$\alpha(1\otimes w)=w \qquad \text{for} \quad w\in\widetilde{W}(\mathfrak{g},\ G_0)_{(q)}\,,$$

and let $\beta: I(K_0)\to\widetilde{W}(\mathfrak{g},\ K_0)_{(q)}$ be the DGA homomorphism defined by

$$\beta(f)=\tilde{k}(\theta_0)k(\theta')f \qquad \text{for} \quad f\in I(K_0)\,.$$

Choose a graded linear splitting $\nu: I(K_0)/I^+(G_0)I(K_0)\to I(K_0)$ of the exact sequence

$$0\longrightarrow I^+(G_0)I(K_0)\longrightarrow I(K_0)\longrightarrow I(K_0)/I^+(G_0)I(K_0)\longrightarrow 0\,.$$

Then we have the following

THEOREM 4.2. *Assume G_0 satisfies the condition* (Z). *Then* $\mu\circ(\alpha_*\otimes\beta_*\circ\nu)$: $H(\Lambda P(\mathfrak{g}_0,\ \mathfrak{k}_0)\otimes\widetilde{W}(\mathfrak{g},\ G_0)_{(q)})\otimes(I(K_0)/I^+(G_0)I(K_0))\to H(\widetilde{W}(\mathfrak{g},\ K_0)_{(q)})$ *is a graded linear isomorphism. If further* $i'^*: I(G_0)\to I(K_0)$ *is surjective, then* $\alpha_*: H(\Lambda P(\mathfrak{g}_0,\ \mathfrak{k}_0)\otimes\widetilde{W}(\mathfrak{g},\ G_0)_{(q)})\to H(\widetilde{W}(\mathfrak{g},\ K_0)_{(q)})$ *is a GA isomorphism.*

REMARK. The characteristic algebra $H(\widetilde{W}(\mathfrak{g})_{(q)})$ for $\Gamma(Q)$-foliations with trivial normal bundles is isomorphic with the cohomology algebra $H(A)$ of the DGA A defined by

$$A=\Lambda P(\mathfrak{g}_0)\otimes\widetilde{W}(\mathfrak{g},\ \mathfrak{g}_0)_{(q)}\,, \qquad \text{where} \quad \widetilde{W}(\mathfrak{g},\ \mathfrak{g}_0)_{(q)}=(\widetilde{W}(\mathfrak{g})_{(q)})_{\mathfrak{g}_0}\,,$$

with the differential d_A defined by

$$d_A(y\otimes 1)=1\otimes\tilde{k}(\theta_0)\tau y \qquad \text{for} \quad y\in P(\mathfrak{g}_0)\,,$$

$$d_A(1\otimes w)=1\otimes dw \qquad \text{for} \quad w\in\widetilde{W}(\mathfrak{g},\ \mathfrak{g}_0)_{(q)}$$

(cf. Kamber-Tondeur [**6**]).

EXAMPLE 1. Let $G=GL(q+1)/\boldsymbol{R}^*1_{q+1}$ and

$$U=\left\{\left(\begin{array}{c|c} a & b \\ \hline 0 & d \end{array}\right)\in GL(q+1)\right\}\Big/\boldsymbol{R}^*1_{q+1}\,.$$

Then $\mathfrak{g}=\mathfrak{sl}(q+1)$,

$$\mathfrak{g}_0=\left\{\begin{pmatrix} \alpha & 0 \\ 0 & B \end{pmatrix};\ B\in\mathfrak{gl}(q),\ \alpha=-\mathrm{Tr}\ B\right\},$$

$$\mathfrak{g}_{-1}=\left\{\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix};\ x\in\boldsymbol{R}^q\right\}, \qquad \mathfrak{g}_1=\left\{\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix};\ {}^t\xi\in\boldsymbol{R}^q\right\},$$

$$\mathfrak{g}_0'=\left\{\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix};\ B\in\mathfrak{sl}(q)\right\},$$

$$G_0=\left\{\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix};\, b\in GL(q),\ a\in \boldsymbol{R}^*\right\}\Big/\boldsymbol{R}^*1_{q+1}\,.$$

Let

$$K_0=\left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & b\end{pmatrix};\, b\in O(q)\right\}\Big/\{\pm 1_{q+1}\}\,.$$

The homomorphism $\rho_0: G_0\to GL(q)$ is given by

$$\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} \bmod \boldsymbol{R}^*1_{q+1} \longmapsto a^{-1}b\,,$$

and hence $\rho_0: G_0 \xrightarrow{\cong} GL(q)$, $\rho_0: K_0 \xrightarrow{\cong} O(q)$. It follows that $(G_0{}^Z)^0=G_0=G_0{}^Z$, $i'^*: I(G_0)\to I(K_0)$ is surjective and $\mathrm{Char}(\mathfrak{g}_0')\subset I_G(\mathfrak{g}_0')$. Moreover $H^{2,1}(\mathfrak{g})=\{0\}$ for $q\geqq 2$. In this case, $\Gamma(Q)$ is the pseudogroup of local projective transformations of a torsion free connection on a smooth manifold B of $\dim B=q$.

EXAMPLE 2. Let

$$O(S)=\{a\in GL(q+2)\,;\,{}^taSa=S\}\,,$$

where

$$S=\begin{pmatrix} & & 1 \\ & S_{r,s} & \\ 1 & & \end{pmatrix},\qquad S_{r,s}=\begin{pmatrix} 1_r & \\ & -1_s\end{pmatrix},\qquad r+s=q,\ r\geqq s\geqq 0\,,$$

and let

$$CO(r,s)=\{a\in GL(q)\,;\,\exists\alpha>0 \text{ with } {}^taS_{r,s}a=\alpha S_{r,s}\}\,.$$

Set

$$G=O(S)/\{\pm 1_{q+2}\}\,,$$

$$U=\left\{\left(\begin{array}{c|c} a & b \\ \hline 0 & d\end{array}\right)\in O(S)\right\}\Big/\{\pm 1_{q+2}\}\,.$$

Then

$$\mathfrak{g}=\{X\in\mathfrak{gl}(q+2)\,;\,{}^tXS+XS=0\}\,,$$

$$\mathfrak{g}_0=\left\{\begin{pmatrix}\alpha & & \\ & B & \\ & & -\alpha\end{pmatrix};\,\alpha\in\boldsymbol{R},\ {}^tBS_{r,s}+S_{r,s}B=0\right\},$$

$$\mathfrak{g}_{-1}=\left\{\left(\begin{array}{c|cc|c} & & & \\ \hline x' & & & \\ x'' & & & \\ \hline & {}^tx' & -{}^tx'' & \end{array}\right);\ \begin{array}{l} x'\in\boldsymbol{R}^r \\ x''\in\boldsymbol{R}^s\end{array}\right\},\qquad \mathfrak{g}_1=\left\{\left(\begin{array}{c|cc|c} & \xi' & \xi'' & \\ \hline & & & {}^t\xi' \\ & & & -{}^t\xi'' \\ \hline & & & \end{array}\right);\ \begin{array}{l} {}^t\xi'\in\boldsymbol{R}^r \\ {}^t\xi''\in\boldsymbol{R}^s\end{array}\right\},$$

$$\mathfrak{g}_0'=\left\{\begin{pmatrix}0 & & \\ & B & \\ & & 0\end{pmatrix};\,{}^tBS_{r,s}+S_{r,s}B=0\right\},$$

$$G_0=\left\{\begin{pmatrix}a & & \\ & b & \\ & & a^{-1}\end{pmatrix};\,a\in\boldsymbol{R}^*,\ {}^tbS_{r,s}b=S_{r,s}\right\}\Big/\{\pm 1_{q+2}\}\,.$$

Let

$$K_0=\left\{\begin{pmatrix} \pm 1 & & \\ & b & \\ & & \pm 1 \end{pmatrix}; b\in O(r)\times O(s)\right\}\Big/\{\pm 1_{q+2}\}.$$

The homomorphism $\rho_0: G_0\to GL(q)$ is given by

$$\begin{pmatrix} a & & \\ & b & \\ & & a^{-1} \end{pmatrix} \bmod \{1_{q+2}\} \longmapsto a^{-1}b,$$

and hence $\rho_0: G_0 \xrightarrow{\cong} CO(r, s)$, $\rho_0: K_0 \xrightarrow{\cong} O(r)\times O(s)$. We have

$$G_0{}^Z=\{a\in GL(q); \exists\, \alpha\in \boldsymbol{R}^* \text{ with } {}^t a S_{r,s} a=\alpha S_{r,s}\},$$

and hence $(G_0{}^Z)^0\subset G_0\subset G_0{}^Z$. It is seen that $i'^*: I(G_0)\to I(K_0)$ is surjective for $s\leqq 1$ and $\mathrm{Char}(\mathfrak{g}_0')\subset I_G(\mathfrak{g}_0')$. Moreover $H^{2,1}(\mathfrak{g})=\{0\}$ for $q\geqq 3$. In this case, $\Gamma(Q)$ is the pseudogroup of local conformal transformations of a pseudo-Riemannian metric of signature (r, s) on a smooth manifold B of $\dim B=q$.

Thus, in projective and conformal cases, $H(\widetilde{W}(\mathfrak{g}, K_0)_{(q)})$ is isomorphic with $H(\Lambda P(\mathfrak{g}_0, \mathfrak{k}_0)\otimes \widetilde{W}(\mathfrak{g}, G_0)_{(q)})$, which may be determined by means of the theory of Weyl on tensor invariants (cf. Morita [10]).

§ 5. Locally homogeneous foliations.

In this section we shall give a geometric construction of locally homogeneous foliations by means of the notion of asympote geodesics.

Let (S, g) be a simply connected complete Riemannian manifold of $\dim S\geqq 2$ with non-positive sectional curvatures. We denote by d the Riemannian distance of (S, g). For a vector x in the unit tangent bundle $T^1(S, g)=T^1S$, let $\gamma_x: \boldsymbol{R}\to S$ denote the geodesic of (S, g) with $\gamma_x'(0)=x$. We denote by ϕ_t the geodesic flow on T^1S:

$$\phi_t(x)=\gamma_x'(t) \qquad \text{for} \quad t\in \boldsymbol{R},\ x\in T^1S.$$

Let $I(S, g)$ be the group of isometries of (S, g). The natural action of $I(S, g)$ on the tangent bundle TS of S will be denoted by

$$\tau_a(x)=a\cdot x \qquad \text{for} \quad a\in I(S, g),\ x\in TS.$$

The group $I(S, g)$ leaves T^1S invariant and commutes with the geodesic flow ϕ_t on T^1S, i.e., one has

$$\phi_t\circ\tau_a=\tau_a\circ\phi_t \qquad \text{for} \quad a\in I(S, g),\ t\in \boldsymbol{R}. \tag{5.1}$$

Let $x, y\in T^1S$. For $t\in \boldsymbol{R}$, let $\gamma_y(s(t))$ be the foot of the unique perpendicular from $\gamma_x(t)$ to the geodesic γ_y. Then the following facts are known (Cartan [3]):

1) There exists $\lim_{t\to+\infty} d(\gamma_x(t), \gamma_y(s(t)))$ in $\boldsymbol{R}^+\cup\{\infty\}$, where $\boldsymbol{R}^+$ denotes the set of non-negative real numbers;

2) If $\lim_{t\to+\infty} d(\gamma_x(t), \gamma_y(s(t)))\in\boldsymbol{R}^+$, then the function $s(t)$ is a monotone function of t and tends to $\pm\infty$ as $t\to+\infty$.

We define a function $\rho: T^1S\times T^1S\to\boldsymbol{R}^+\cup\{\infty\}$ as follows. Let $x, y\in T^1S$. If $\lim_{t\to+\infty} d(\gamma_x(t), \gamma_y(s(t)))\in\boldsymbol{R}^+$ and $s(t)$ is a monotone increasing function of t, then we set $\rho(x, y)=\lim_{t\to+\infty} d(\gamma_x(t), \gamma_y(s(t)))$. Otherwise, we set $\rho(x, y)=\infty$. It is clear from the definition that

$$\rho(\phi_t x, y)=\rho(x, \phi_t y)=\rho(x, y) \quad \text{for} \quad t\in\boldsymbol{R}, \tag{5.2}$$

$$\rho(a\cdot x, a\cdot y)=\rho(x, y) \quad \text{for} \quad a\in I(S, g). \tag{5.3}$$

Furthermore, it is known (Karpelevič [8]) that the function $\rho: T^1S\times T^1S\to \boldsymbol{R}^+\cup\{\infty\}$ satisfies the axioms of the "pseudo-distance", i.e., one has

(1) $\rho(x, y)=\rho(y, x)$;

(2) $\rho(x, y)+\rho(y, z)\geqq\rho(x, z)$;

(3) $\rho(x, x)=0$.

Let $x, y\in T^1S$. If $\rho(x, y)\in\boldsymbol{R}^+$, then the geodesic γ_x, γ_y are said to be *asymptote* and the vectors x, y are said to be *a-equivalent*, which will be denoted by $x\sim y$. It follows from the above (1), (2), (3) that $x\sim y$ is an equivalence relation on T^1S. An a-equivalence class $\mathcal{X}\subset T^1S$ is called an *a-component*. We denote by $\mathcal{A}(S)$ the set of all a-components of T^1S. It follows from (5.2), (5.3) that $\phi_t\mathcal{X}=\mathcal{X}$ for $t\in\boldsymbol{R}$, $\mathcal{X}\in\mathcal{A}(S)$ and that $a\cdot\mathcal{X}\in\mathcal{A}(S)$ for $a\in I(S, g)$, $\mathcal{X}\in\mathcal{A}(S)$, and hence $I(S, g)$ acts on $\mathcal{A}(S)$ in a natural manner.

In what follows, we assume that (S, g) is a Riemmanian symmetric space of non-compact type. Let G be the identity-component $I^0(S, g)$ of $I(S, g)$ and $\mathfrak{g}=\mathrm{Lie}\, G$. In general, for $H\in\mathfrak{g}$, $\lambda\in\boldsymbol{R}$ we set

$$\mathfrak{g}(\mathrm{ad}\, H, \lambda)=\{X\in\mathfrak{g}\,; [H, X]=\lambda X\}.$$

Let $x\in T^1S$. We denote by $h_x(t)\in G$ for $t\in\boldsymbol{R}$ the transvection based on the geodesic $\gamma_x|[0, t]$, and then by $H_x\in\mathfrak{g}$ the element corresponding to the one parameter subgroup $\{h_x(t)\,; t\in\boldsymbol{R}\}$ of G, so that $h_x(t)=\exp tH_x$. Set

$$K_x=\{a\in G\,; a\cdot x=x\},$$

$$\mathfrak{u}_x=\sum_{\lambda\geqq 0}\mathfrak{g}(\mathrm{ad}\, H_x, \lambda),$$

$$U_x=\{a\in G\,; \mathrm{Ad}\, a\mathfrak{u}_x=\mathfrak{u}_x\}.$$

Then we have $\mathrm{Lie}\, U_x=\mathfrak{u}_x$. With these notations, we have the following

LEMMA 5.1. (Karpelevič [8]) *Let* $x, y\in T^1S$.

1) *If $x\sim y$, then there exists $a\in G$ such that $a\cdot x=y$.*

2) *For $a\in G$, one has*

$$a\cdot x \sim x \Leftrightarrow a\in U_x .$$

Now we take a point $o\in S$ and a vector $x_0\in T_o^1S$, and fix them once for all. We set

$$U=U_{x_0}, \qquad K_0=K_{x_0}, \qquad H_0=H_{x_0},$$
$$K=\{a\in G\,;\,ao=o\}.$$

Then K is a maximal compact subgroup of G and one has an identification: $S=G/K$. Let $\mathfrak{u}=\mathrm{Lie}\,U$, $\mathfrak{k}_0=\mathrm{Lie}\,K_0$, $\mathfrak{k}=\mathrm{Lie}\,K$ and define

$$\mathfrak{p}=\{X\in\mathfrak{g}\,;\,B(X,\mathfrak{k})=\{0\}\},$$

where B denotes the Killing form of $\mathfrak{g}$. Then $H_0\in\mathfrak{p}$ and one has the Cartan decomposition: $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, and hence $\mathfrak{p}$ is identified with T_oS. Let $\mathcal{X}_0\in\mathcal{A}(S)$ be the a-component containing x_0 and set

$$B=\{a\cdot\mathcal{X}_0\,;\,a\in G\}\subset\mathcal{A}(S),$$
$$M=\bigcup_{\mathcal{X}\in B}\mathcal{X}\subset T^1S.$$

We denote by $\psi: M\to B$ the natural projection, which satisfies

$$\psi(a\cdot x)=a\cdot\psi(x) \qquad \text{for} \quad a\in G,\ x\in M.$$

The following theorem describes the structure of these sets M, B, $\mathcal{X}_0$.

THEOREM 5.1. 1) *The subset $M\subset T^1S$ is a closed connected submanifold of T^1S, which is diffeomorphic with G/K_0. It is invariant under the action of the geodesic flow ϕ_t.*

2) *The a-component $\mathcal{X}_0$ is a connected closed submanifold of T^1S, which is diffeomorphic with U/K_0 and S.*

3) *The set B has the structure of a compact connected smooth manifold which is diffeomorphic with G/U and K/K_0, such that the fibering $\mathcal{X}_0\to M\xrightarrow{\psi}B$ is diffeomorphic with the smooth fibering $U/K_0\to G/K_0\to G/U$.*

PROOF. Lemma 5.1, 1) implies that G acts transitively on M. This implies 1) except the closedness of M. Lemma 5.1 implies also that $\mathcal{X}_0$ is a submanifold of T^1S which is diffeomorphic with U/K_0. Moreover G acts transitively on B and one has $a\cdot\mathcal{X}_0=\mathcal{X}_0\Leftrightarrow a\in U$, by Lemma 5.1, 2). Thus B is identified with G/U as the sets. We define the structure of a smooth manifold on B from that of G/U.

Take a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ with $H_0\in\mathfrak{a}$, and choose a lexicographic order $>$ on the dual space of $\mathfrak{a}$ such that $\gamma(H_0)\geqq 0$ for each positive root γ. Denoting by $\mathfrak{g}_\gamma$ the root space for a root γ, we define

$$\mathfrak{n}=\sum_{\gamma>0}\mathfrak{g}_\gamma, \qquad \mathfrak{n}_0=\sum_{\gamma>0,\,\gamma(H_0)=0}\mathfrak{g}_\gamma,$$

$$\mathfrak{g}_+=\sum_{\gamma(H_0)>0}\mathfrak{g}_\gamma .$$

The connected Lie subgroups of G generated by $\mathfrak{n}$, $\mathfrak{n}_0$, $\mathfrak{g}_+$, $\mathfrak{a}$ are denoted by N, N_0, G_+, A respectively. Set

$$G_0=\{a\in G\,;\,\mathrm{Ad}\,aH_0=H_0\}\,.$$

Then we have Iwasawa decompositions: $G=KAN$ and $G_0=K_0AN_0$. Moreover it is known (Takeuchi [13]) that $U=G_0G_+$ (semi-direct) and $U\cap K=K_0$. Thus one has $U=G_0G_+=K_0AN_0G_+=K_0AN$ and $G=KU$. It follows that $U/K_0\approx AN\approx G/K=S$ and $K/K_0\approx G/U$.

It remains to show the closedness of M and $\mathcal{X}_0$ in T^1S. Since the map $AN\ni s\mapsto so\in S$ is a diffeomorphism, the map $\varphi: AN\times T_o^1S\to T^1S$ defined by

$$\varphi(s,\,x)=s\cdot x \qquad \text{for} \quad s\in AN,\ x\in T_o^1S$$

is a diffeomorphism. Then one has $M=\varphi(AN\times K\cdot x_0)$ and $\mathcal{X}_0=\varphi(AN\times\{x_0\})$, which implies the required closedness. q. e. d.

Since the choice of x_0 is arbitrary, we have the following

COROLLARY. *Each $\mathfrak{a}$-component in T^1S is a closed submanifold of T^1S which is diffeomorphic with S.*

Now the fibering $\mathcal{X}_0\to M\overset{\psi}{\to}B$ defines a G-invariant smooth foliation $\mathcal{F}$ on M such that each leaf of $\mathcal{F}$ is an $\mathfrak{a}$-component in T^1S. Take a uniform discrete subgroup D of G acting on G/K_0 properly discontinuously and without fixed points (cf. Borel [2]). Then $\mathcal{F}$ induces in a natural way a foliation on the quotient $D\backslash M$, which will be called a *locally homogeneous foliation* associated to the symmetric space S. We denote by $\Gamma(G)$ the pseudogroup of all local diffeomorphisms of B which is extendable to the action of an element of G. Then it is seen that the above locally homogeneous foliation is a $\Gamma(G)$-foliation on $D\backslash M$.

§ 6. Canonical Riemannian metrics on tangent bundles.

In this section we shall compute the Riemannian metric on M induced by the canonical Riemannian metric on T^1S.

We shall first recall the definition of the canonical Riemannian metric g^T on the tangent bundle TS of a general Riemannian manifold (S, g). Let π: $TS\to S$ be the natural projection, and let $x\in TS$ with $\pi(x)=p\in S$. We denote by V_x the kernel of $(\pi_*)_x: T_x(TS)\to T_pS$ and by $\iota_x: V_x\to T_x(TS)$ the inclusion. Note that $V_x=T_x(T_pS)\cong T_pS$. The connection map $\kappa_x: T_x(TS)\to V_x$ is a splitting of the exact sequence

$$0\longrightarrow V_x\overset{\iota_x}{\longrightarrow}T_x(TS)\overset{(\pi_*)_x}{\longrightarrow}T_pS\longrightarrow 0\,.$$

Thus the map $\xi \mapsto \kappa_x \xi + (\pi_*)_x \xi$ ($\xi \in T_x(TS)$) defines a linear isomorphism φ_x: $T_x(TS) \to V_x \oplus T_p S \cong T_p S \oplus T_p S$. The *canonical Riemannian metric* g^T on $T_x(TS)$ is defined to be the inner product which corresponds by φ_x to the orthogonal direct sum $g \oplus g$ on $T_p S \oplus T_p S$.

The connection map κ_x in the above definition is defined as follows. Let ∇ be the Riemannian connection for g. For $\xi \in T_x(TS)$ choose a smooth curve $x(t)$ in TS such that $x(0)=x$ and $x'(0)=\xi$. Then $p(t)=\pi(x(t))$ is a smooth curve in S such that $p(0)=p$ and $p'(0)=\pi_*\xi$. Since $x(t)$ is a smooth vector field along $p(t)$, we can differentiate it by $\pi_*\xi$. Now $\kappa_x(\xi)$ is defined by

$$\kappa_x(\xi)=\nabla_{\pi_*\xi} x(t) \in T_x(T_p S)=V_x .$$

It is seen that this does not depend on the choice of a curve $x(t)$ and that $\kappa_x \circ \iota_x = id$.

The canonical Riemannian metric g^T on TS is invariant under $I(S, g)$, so that the induced Riemannian metric g^T on T^1S is also invariant under $I(S, g)$.

Now we come back to our Riemannian symmetric space (S, g) of non-compact type and compute the G-invariant Riemannian metric g^T on M induced by the canonical Riemannian metric on T^1S. Let

$$(S, g)=(S_1, g_1)\times \cdots \times (S_s, g_s)$$

be the de Rham decomposition of (S, g), and let

$$G=G_1\times \cdots \times G_s ,$$

$$\mathfrak{g}=\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s ,$$

$$\mathfrak{p}=\mathfrak{p}_1+ \cdots +\mathfrak{p}_s$$

be the corresponding decompositions. Then, for each k $(1\leqq k\leqq s)$ there exists $c_k>0$ such that $g_o|\mathfrak{p}_k\times\mathfrak{p}_k=c_k B_k|\mathfrak{p}_k\times\mathfrak{p}_k$, where B_k denotes the Killing form of $\mathfrak{g}_k$. Thus the symmetric bilinear form $c_1B_1\oplus \cdots \oplus c_sB_s$ on $\mathfrak{g}$ coincides with g_o on $\mathfrak{p}$. This form will be denoted by $(\ ,\ )$. It is G-invariant and non-degenerate. Let τ be the Cartan involution of $\mathfrak{g}$ associated to the Cartan decomposition: $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. We define an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by

$$\langle X, Y\rangle=-(X, \tau Y) \qquad \text{for} \quad X, Y\in\mathfrak{g} .$$

It satisfies $\langle \mathrm{Ad}\, kX, \mathrm{Ad}\, kY\rangle=\langle X, Y\rangle$ for $k\in K$ and $\langle \tau X, \tau Y\rangle=\langle X, Y\rangle$. We set $\mathfrak{g}_\lambda=\mathfrak{g}(\mathrm{ad}\, H_0, \lambda)$ for $\lambda\in\boldsymbol{R}$ and define

$$\mathfrak{g}_*=\sum_{\lambda\neq 0}\mathfrak{g}_\lambda ,$$

$$\mathfrak{k}_*=\mathfrak{k}\cap\mathfrak{g}_* , \qquad \mathfrak{p}_*=\mathfrak{p}\cap\mathfrak{g}_* ,$$

$$\mathfrak{g}_+=\sum_{\lambda>0}\mathfrak{g}_\lambda , \qquad \mathfrak{g}_-=\sum_{\lambda<0}\mathfrak{g}_\lambda ,$$

$$\mathfrak{p}_0=\mathfrak{p}\cap\mathfrak{g}_0, \qquad \mathfrak{m}=\mathfrak{p}_0+\mathfrak{g}_*.$$

Then we have the following orthogonal decompositions with respect to $\langle , \rangle$.

$$\begin{aligned}\mathfrak{g}&=\mathfrak{g}_0+\sum_{\lambda\neq 0}\mathfrak{g}_\lambda=\mathfrak{g}_0+\mathfrak{g}_++\mathfrak{g}_-\\ &=\mathfrak{k}+\mathfrak{p}=\mathfrak{k}_0+\mathfrak{m},\\ \mathfrak{g}_0&=\mathfrak{k}_0+\mathfrak{p}_0,\\ \mathfrak{g}_*&=\mathfrak{g}_++\mathfrak{g}_-=\mathfrak{k}_*+\mathfrak{p}_*,\\ \mathfrak{k}&=\mathfrak{k}_0+\mathfrak{k}_*, \qquad \mathfrak{p}=\mathfrak{p}_0+\mathfrak{p}_*,\\ \mathfrak{m}&=\mathfrak{p}_0+\mathfrak{g}_++\mathfrak{g}_-.\end{aligned}$$

Note that K_0 leaves $\mathfrak{p}_0$, $\mathfrak{g}_\lambda$ $(\lambda\neq 0)$ invariant, and hence it leaves $\mathfrak{g}_+$, $\mathfrak{g}_-$, $\mathfrak{m}$ invariant. We shall identify $\mathfrak{m}$ with $T_{x_0}M$. The $\mathfrak{k}$-component and $\mathfrak{p}$-component of $X\in\mathfrak{g}$ with respect to the decomposition: $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ will be denoted by $X_\mathfrak{k}$ and $X_\mathfrak{p}$ respectively.

LEMMA 6.1. *For $X\in\mathfrak{g}$ we define a smooth curve $p(t)$ in S by $p(t)=(\exp tX)o$. For $Y\in\mathfrak{p}=T_oS$, let $Y(t)$ denote the smooth vector field along $p(t)$ defined by $Y(t)=(\exp tX)\cdot Y$. Then*

$$\nabla_{X_\mathfrak{p}}Y(t)=[X_\mathfrak{k}, Y].$$

PROOF. Let Y^* denote the vector field on S generated by Y. Then

$$Y^*{}_{p(t)}=\frac{d}{ds}(\exp sY\exp tXo)|_{s=0},$$

where $\exp sY\exp tX=\exp tX(\exp tX)^{-1}\exp sY\exp tX=\exp tX\exp s[\mathrm{Ad}(\exp tX)^{-1}Y]$. Thus

$$\begin{aligned}Y^*{}_{p(t)}&=(\exp tX)\cdot[\mathrm{Ad}(\exp tX)^{-1}Y]_\mathfrak{p}\\ &=(\exp tX)\cdot(Y-t[X, Y]+(t^2/2)[X, [X, Y]]-\cdots)_\mathfrak{p}\\ &=Y(t)-t(\exp tX)\cdot[X_\mathfrak{k}, Y]+t^2(\exp tX)\cdot Z(t),\end{aligned}$$

where $Z(t)$ is a smooth curve in $\mathfrak{p}$. It follows

$$\nabla_{X_\mathfrak{p}}Y^*{}_{p(t)}=\nabla_{X_\mathfrak{p}}Y(t)-[X_\mathfrak{k}, Y],$$

where $\nabla_{X_\mathfrak{p}}Y^*{}_{p(t)}=0$ since (S, g) is symmetric. This proves the lemma.

q. e. d.

LEMMA 6.2. *We identify the injective map $\varphi_{x_0}: T_{x_0}M\to T_oS\oplus T_oS$ with the map $\varphi_{x_0}: \mathfrak{m}\to\mathfrak{p}\oplus\mathfrak{p}$. Then*

$$\varphi_{x_0}(X)=[X_\mathfrak{k}, H_0]\oplus X_\mathfrak{p} \qquad \textit{for} \quad X\in\mathfrak{m}.$$

PROOF. Since $(\pi_*)_{x_0}X=X_\mathfrak{p}$, it suffices to show that $\kappa_{x_0}(X)=[X_\mathfrak{k}, H_0]$. We define a smooth curve $x(t)$ in T^1S with $x(0)=x_0$, $x'(0)=X\in T_{x_0}M$ by $x(t)=$

$(\exp tX)\cdot H_0$. Then Lemma 6.1 implies

$$\kappa_{x_0}(X)=\nabla_{X_{\mathfrak{p}}}x(t)=[X_{\mathfrak{k}}, H_0]. \qquad \text{q.e.d.}$$

Note that ad H_0 induces linear isomorphisms $\mathfrak{k}_*\to\mathfrak{p}_*$ and $\mathfrak{p}_*\to\mathfrak{k}_*$, and hence $(\operatorname{ad} H_0)^2$ induces a linear automorphism of $\mathfrak{k}_*$, which is symmetric positive definite with respect to $\langle\,,\rangle$.

THEOREM 6.1. *Let $X, Y\in\mathfrak{m}=T_{x_0}M$. Then $X_{\mathfrak{k}}, Y_{\mathfrak{k}}\in\mathfrak{k}_*$ and $g^T(X, Y)$ is given by*

$$g^T(X, Y)=\langle(\operatorname{ad} H_0)^2X_{\mathfrak{k}}, Y_{\mathfrak{k}}\rangle+\langle X_{\mathfrak{p}}, Y_{\mathfrak{p}}\rangle.$$

If further $X\in\mathfrak{g}_\lambda$, then

$$g^T(X, Y)=\lambda^2\langle X_{\mathfrak{k}}, Y_{\mathfrak{k}}\rangle+\langle X_{\mathfrak{p}}, Y_{\mathfrak{p}}\rangle.$$

PROOF. That $X_{\mathfrak{k}}, Y_{\mathfrak{k}}\in\mathfrak{k}_*$ follows from the decomposition: $\mathfrak{m}=\mathfrak{k}_*+(\mathfrak{p}_0+\mathfrak{p}_*)$. By Lemma 6.2, one has

$$\begin{aligned} g^T(X, Y)&=([X_{\mathfrak{k}}, H_0], [Y_{\mathfrak{k}}, H_0])+(X_{\mathfrak{p}}, Y_{\mathfrak{p}})\\ &=([H_0, X_{\mathfrak{k}}], [H_0, Y_{\mathfrak{k}}])+(X_{\mathfrak{p}}, Y_{\mathfrak{p}})\\ &=-((\operatorname{ad} H_0)^2X_{\mathfrak{k}}, Y_{\mathfrak{k}})+(X_{\mathfrak{p}}, Y_{\mathfrak{p}})\\ &=\langle(\operatorname{ad} H_0)^2X_{\mathfrak{k}}, Y_{\mathfrak{k}}\rangle+\langle X_{\mathfrak{p}}, Y_{\mathfrak{p}}\rangle. \end{aligned}$$

If $X\in\mathfrak{g}_\lambda$, then $[H_0, X_{\mathfrak{p}}]=\lambda X_{\mathfrak{k}}$, $[H_0, X_{\mathfrak{k}}]=\lambda X_{\mathfrak{p}}$ and hence $(\operatorname{ad} H_0)^2X_{\mathfrak{k}}=\lambda^2X_{\mathfrak{k}}$. This implies the second statement. q.e.d.

COROLLARY. 1) $g^T(\mathfrak{p}_0, \mathfrak{g}_*)=\{0\}$.

2) $g^T(X, Y)=\langle X, Y\rangle$ *for* $X, Y\in\mathfrak{p}_0$.

3) *If* $X\in\mathfrak{g}_\lambda$ $(\lambda\neq 0)$, $Y\in\mathfrak{g}_\mu$ $(\mu\neq 0)$, *then*

$$g^T(X, Y)=\begin{cases}\dfrac{1}{2}(\lambda^2+1)\langle X, Y\rangle & \lambda=\mu\\ \dfrac{1}{2}(\lambda^2-1)\langle X, \tau Y\rangle & \lambda+\mu=0\\ 0 & \textit{otherwise}.\end{cases}$$

PROOF. 1) Let $X\in\mathfrak{p}_0$, $Y\in\mathfrak{g}_*$. Then $X_{\mathfrak{p}}=X\in\mathfrak{p}_0$ and $Y_{\mathfrak{p}}\in\mathfrak{p}_*$. Now $\langle\mathfrak{p}_0, \mathfrak{p}_*\rangle=\{0\}$ implies $g^T(X, Y)=0$.

2) Follows from $X_{\mathfrak{p}}=X$, $Y_{\mathfrak{p}}=Y$.

3) We have

$$X_{\mathfrak{k}}=(1/2)(X+\tau X), \qquad Y_{\mathfrak{k}}=(1/2)(Y+\tau Y),$$
$$X_{\mathfrak{p}}=(1/2)(X-\tau X), \qquad Y_{\mathfrak{p}}=(1/2)(Y-\tau Y).$$

It follows from $\tau Y\in\mathfrak{g}_{-\mu}$ that

$$\begin{aligned}\langle X_{\mathfrak{k}}, Y_{\mathfrak{k}}\rangle&=(1/4)\{\langle X, Y\rangle+\langle X, \tau Y\rangle+\langle\tau X, Y\rangle+\langle\tau X, \tau Y\rangle\}\\ &=(1/2)\{\langle X, Y\rangle+\langle X, \tau Y\rangle\}\end{aligned}$$

$$=\begin{cases}(1/2)\langle X, Y\rangle & \lambda=\mu\\ (1/2)\langle X, \tau Y\rangle & \lambda+\mu=0\\ 0 & \text{otherwise,}\end{cases}$$

$$\langle X_{\mathfrak{p}}, Y_{\mathfrak{p}}\rangle=(1/4)\{\langle X, Y\rangle-\langle X, \tau Y\rangle-\langle \tau X, Y\rangle+\langle \tau X, \tau Y\rangle\}$$
$$=(1/2)\{\langle X, Y\rangle-\langle X, \tau Y\rangle\}$$
$$=\begin{cases}(1/2)\langle X, Y\rangle & \lambda=\mu\\ -(1/2)\langle X, \tau Y\rangle & \lambda+\mu=0\\ 0 & \text{otherwise.}\end{cases}$$

These together with the above theorem imply the third assertion. q. e. d.

§7. Locally homogeneous foliations and Anosov flows.

In this section we shall study the flow ϕ_t on M induced by the geodesic flow on T^1S.

Recall that K_0 leaves $\mathfrak{p}_0$ and $\mathfrak{g}_\lambda$ ($\lambda\neq 0$) invariant. This implies that for each eigenvalue λ of $\mathrm{ad}\, H_0$, there exists a unique G-invariant subbundle F_λ of TM such that

$$(F_\lambda)_{x_0}=\begin{cases}\mathfrak{p}_0 & \lambda=0\\ \mathfrak{g}_\lambda & \lambda\neq 0.\end{cases}$$

Define

$$F_+=\sum_{\lambda>0}\oplus F_\lambda, \qquad F_-=\sum_{\lambda<0}\oplus F_\lambda,$$

which are G-invariant subbundles of TM such that $(F_\pm)_{x_0}=\mathfrak{g}_\pm$. Note that F_0, $F_\pm$ and $F_0\oplus F_+$ are integrable subbundles of TM, since $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$, $\mathfrak{g}_\pm$ and $\mathfrak{u}=\mathfrak{k}_0+\mathfrak{p}_0+\mathfrak{g}_+$ are Lie subalgebras of $\mathfrak{g}$.

LEMMA 7.1. *$g^T((F_\lambda)_x, (F_\mu)_x)=\{0\}$ at each point $x\in M$ if $\lambda\neq\pm\mu$.*

PROOF. Follows from Corollary of Theorem 6.1 and the G-invariance of g^T. q. e. d.

LEMMA 7.2. 1) *Each F_λ is invariant under ϕ_{t*}.*

2) *Let $\xi\in(F_\lambda)_x$, $\eta\in(F_\mu)_x$ ($x\in M$). Then*

$$g^T(\phi_{t*}\xi, \phi_{t*}\eta)=e^{-(\lambda+\mu)t}g^T(\xi, \eta) \qquad \text{for} \quad t\in\mathbf{R}.$$

In particular, we have

$$g^T(\phi_{t*}\xi, \phi_{t*}\eta)=e^{-2\lambda t}g^T(\xi, \eta) \qquad \text{if} \quad \lambda=\mu,$$
$$g^T(\phi_{t*}\xi, \phi_{t*}\eta)=0 \qquad \text{if} \quad \lambda\mu>0,\ \lambda\neq\mu.$$

PROOF. We shall show first that for $\xi\in(F_\lambda)_x$ with $x=a\cdot x_0\in M$, $a\in G$, one has

$$\phi_{t*}\xi=e^{-\lambda t}(\tau_{a\exp tH_0a^{-1}})_*\xi\,.$$

Set $X=(\tau_a)_*^{-1}\xi\in\mathfrak{g}_\lambda\cap\mathfrak{m}$. Then

$$\phi_{t*}\xi=\phi_{t*}(\tau_a)_*X=\frac{d}{ds}\phi_t((a\exp sX)\cdot x_0)|_{s=0}\,,$$

where by (5.1) one has

$$\begin{aligned}\phi_t((a\exp sX)\cdot x_0)&=(a\exp sX)\cdot\phi_t(x_0)=(a\exp sX\exp tH_0)\cdot x_0\\&=(a\exp tH_0(\exp tH_0)^{-1}\exp sX\exp tH_0)\cdot x_0\\&=(a\exp tH_0\exp s[\mathrm{Ad}(\exp tH_0)^{-1}X])\cdot x_0\\&=(a\exp tH_0\exp se^{-\lambda t}X)\cdot x_0\,.\end{aligned}$$

It follows

$$\begin{aligned}\phi_{t*}\xi&=(\tau_{a\exp tH_0})_*e^{-\lambda t}X=e^{-\lambda t}(\tau_{a\exp tH_0})_*(\tau_a)_*^{-1}\xi\\&=e^{-\lambda t}(\tau_{a\exp tH_0a^{-1}})_*\xi\,.\end{aligned}$$

Now 1) follows from the G-invariance of F_λ. The assertion 2) follows from the G-invariance of g^T and Lemma 7.1. q. e. d.

THEOREM 7.1. *The flow ϕ_t on (M, g^T) induced by the geodesic flow on T^1S is an Anosov flow, whose invariant bundle, contracting bundle, expanding bundle are given by F_0, F_+, F_- respectively. Moreover, the Whitney sum $F_0\oplus F_+$ coincides with the tangent bundle $\tau(\mathcal{F})$ for the foliation $\mathcal{F}$ on M defined by the natural fibering $\psi: M\to B$.*

PROOF. Let C_1 be the smallest positive eigenvalue of $\mathrm{ad}\,H_0$ and let $C_2=1$. Then the flow ϕ_t together with subbundles F_0, F_+, F_- satisfies the conditions for an Anosov flow stated in Introduction, in virtue of Lemma 7.2. The second statement follows from Theorem 5.1, 3). q. e. d.

COROLLARY. *The flow ϕ_t on M, the Riemannian metric g^T on M, the subbundles F_0, F_+, F_- of TM and the smooth foliation $\mathcal{F}$ on M, which are all G-invariant, induce a flow on $D\backslash M$, a Riemannian metric on $D\backslash M$, subbundles of $T(D\backslash M)$ and a locally homogeneous foliation on $D\backslash M$. Then the same results as in Theorem 7.1 hold for them.*

We shall state here on the relationship between our locally homogeneous foliations and semi-simple flat homogeneous spaces. Let

$$\begin{aligned}&x_0=x_1+\cdots+x_s\,, &&x_k\in TS_k\,,\\&H_0=H_1+\cdots+H_s\,, &&H_k\in\mathfrak{p}_k\end{aligned}$$

be the decompositions corresponding to the de Rham decomposition: $(S, g)=(S_1, g_1)\times\cdots\times(S_s, g_s)$. We assume

(*) $H_k\neq 0$ and $\mathrm{ad}\,H_k$ has only one positive eigenvalue λ_k for each k.

We set

$$x_0' = \lambda_1^{-1}x_1 + \cdots + \lambda_s^{-1}x_s \in T_oS\,,$$

so that the corresponding $H_0' \in \mathfrak{p}$ is given by

$$H_0' = \lambda_1^{-1}H_1 + \cdots + \lambda_s^{-1}H_s\,.$$

We define a new Riemannian metric g' on S by

$$g' = \lambda_1^2 g_1 \oplus \cdots \oplus \lambda_s^2 g_s\,.$$

Then (S, g') is also a Riemannian symmetric space of noncompact type with $x_0' \in T^1(S, g')$. Moreover one has

$$I^0(S, g') = G\,, \qquad U_{x_0'} = U\,, \qquad K_{x_0'} = K_0\,.$$

Write g, x_0, H_0 for these g', x_0', H_0' and make our construction starting from these new metric g and unit vector x_0. Then eigenvalues of $\mathrm{ad}\, H_0$ are 0, 1, -1, and G/U is a semi-simple flat homogeneous space. The decomposition

$$TM = F_0 \oplus F_+ \oplus F_-$$

is the orthogonal Whitney sum with respect to the metric g^T, in virtue of Corollary of Theorem 6.1. Furthermore one has $\Gamma(G) \subset \Gamma(Q_G)$ in general, and $\Gamma(G) = \Gamma(Q_G)$ if $H^{2,1}(\mathfrak{g}) = \{0\}$ (cf. Ochiai [**12**]). Thus our locally homogeneous foliation on $D \backslash M$ is a $\Gamma(Q_G)$-foliation.

For example, the hyperbolic space H^{q+1} of dimension $q+1$ satisfies the condition (*), and one has $M = T^1H^{q+1}$ in this case. Our locally homogeneous foliation associated to H^{q+1} is nothing but the locally homogeneous conformal foliation of Yamato [**15**]. In particular, in the case $q=1$, it is the foliation given by Roussarie. Our flow ϕ_t on $D \backslash T^1H^{q+1} = T^1(D \backslash H^{q+1})$ coincides with the original Anosov flow constructed by Anosov [**1**]. It should be noted that in this case each maximal integral submanifold of the integrable subbundle F_+ of $T(T^1H^{q+1})$ is diffeomorphic with a horocycle of the hyperbolic space H^{q+1}.

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