J. Math. Soc. Japan
Vol. 32, No. 1, 1980

Equivariant homotopy equivalence of group representations

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(Received May 15, 1978)

1. Introduction.

In [6], we defined $J_G(X)$ for a compact group G and for a compact G-space X. When X is a point, we denote it by $J_G(*)$. Similar groups $JO(G)$ were defined and studied by Atiyah and Tall [2], Snaith [13], and Lee and Wasserman [9]. Our definition is more rigid than those of $JO(G)$ and is given from the geometrical point of view as follows.

Two orthogonal representation spaces V, W of a compact topological group G are said to be J-equivalent if there exist an orthogonal representation space U and a G-homotopy equivalence $f: S(V\oplus U)\rightarrow S(W\oplus U)$ where $S(V\oplus U)$ and $S(W\oplus U)$ denote the unit spheres in $V\oplus U$ and $W\oplus U$ respectively. Then the group $J_G(*)$ is defined as the quotient of the orthogonal representation ring $RO(G)$ by the subgroup

$$T_G(*)=\{V-W \mid V \text{ is } J\text{-equivalent to } W\}.$$

The natural epimorphism $RO(G)\rightarrow J_G(*)$ is denoted by J_G.

The purpose of the present paper is to determine the group structure of $J_G(*)$ for G an arbitrary compact abelian topological group as promised in [7] (see Theorem 4.1).

Essential part of the computation is to determine the group structure of $J_{Z_n}(*)$ for every cyclic group Z_n. Let V, W be Z_n-representation spaces such that Z_n acts freely on $S(V)$ and $S(W)$. Then we shall obtain the following unexpected theorem (see Theorem 2.6) which is crucial for determining the group structure of $J_{Z_n}(*)$; *$S(V)$ and $S(W)$ are Z_n-homotopy equivalent if and only if V is J-equivalent to W.*

More generally we have as a corollary to the main theorem that *for a compact abelian topological group G, two G-representations V and W are J-equivalent if and only if $S(V)$ and $S(W)$ are G-homotopy equivalent* (Corollary 4.2).

Research supported in part by Sonderforschungsbereich "Theoretische Mathematik" grant Universität Bonn and Fūjukai grant Tokyo.

This result together with [6] yields the following. Let G be a compact abelian Lie group and M_1, M_2 be closed smooth G-manifolds. Suppose that we are given a G-homotopy equivalence $f: M_1 \to M_2$. Denote by F_1^μ each component of the fixed point set of M_1. Set $F_2^\mu = f(F_1^\mu)$. Then the union $\bigcup_\mu F_2^\mu$ is the fixed point set of M_2 and each F_2^μ is a component of $\bigcup_\mu F_2^\mu$. Denote by V_i^μ the normal representation of F_i^μ in M_i $(i=1, 2)$. Then we have that *$S(V_1^\mu)$ is G-homotopy equivalent to $S(V_2^\mu)$* (Theorem 5.1).

Remark that we showed in [6] that $S(V_1^\mu)$ and $S(V_2^\mu)$ are *stably* G-homotopy equivalent for G an arbitrary compact *Lie* group.

Last of all, we shall express $J_G(*)$ in terms of the equivariant Adams operations for G an abelian p-group.

In a forthcoming paper, we shall study $J_G(*)$ for G an arbitrary p-group.

The author is very grateful to Professor T. tom Dieck for stimulating conversations.

2. Stable and unstable Z_n-homotopy equivalences.

We begin by fixing some notations. Let G be a compact topological group and X be a G-space. Then for a subgroup H of G, we set

$$X^H = \{x \in X \mid hx = x \quad \text{for all} \quad h \in H\} .$$

For a G-map $f: X_1 \to X_2$, we denote by f^H the restriction $f|X_1^H : X_1^H \to X_2^H$. We promise that a representation indicates its representation space as well. R^2 will denote the trivial two dimensional representation. Given an orthogonal representation V of G, we denote by $S(V)$ the unit sphere in V. Denote by Z_n the cyclic group Z/nZ of order n where n is a positive integer. When n is divisible by m, we often regard Z_m as the canonical subgroup of Z_n.

Our computation of $J_G(*)$ is based on the following lemma which is a generalization of Corollary 2.4 in [9].

LEMMA 2.1. *Let V be an orthogonal Z_n-representation space which involves R^2 as a direct summand. Let $f: S(V) \to S(V)$ be a Z_n-map such that the degree of the restricted map $f^H: S(V)^H \to S(V)^H$ is ± 1 for every non-zero subgroup H of Z_n.*

Then the degree of f is ± 1 mod n.

PROOF. The present proof is also a generalization of those of Corollaries 2.3 and 2.4 in [9].

Let x_0 (resp. y_0) be the point $(1, 0)$ (resp. $(-1, 0)$) of $R^2 \subset V$. Then the equivariant homotopy classes of Z_n-maps from $S(V)$ to $S(V)$ preserving x_0 form a group.

We divide the proof of Lemma 2.1 into three cases.

Case 1. $n = 2^k$. Strictly speaking, we show the following more general relations in this case.

ASSERTION 2.2.

$$\deg f \equiv \deg f^{Z_2} \qquad \mathrm{mod}\ n\,.$$

If $k \geqq 3$, $\deg f^{Z_2} = \deg f^{Z_{2^i}}$ *for* $1 \leqq i \leqq k-1$.

PROOF OF ASSERTION 2.2. Let $V = V^{Z_2} \oplus V_1$ be the orthogonal decomposition. Then we set

$$'f = f^{Z_2} * id : S(V) = S(V)^{Z_2} * S(V_1) \longrightarrow S(V) = S(V)^{Z_2} * S(V_1)$$

where $*$ denotes the join. Since $f^{Z_2} = {'f}^{Z_2}$, $(f - {'f})^{Z_2} : S(V)^{Z_2} \to S(V)^{Z_2}$ is Z_n-homotopic to the constant map $h_1(S(V)^{Z_2}) = x_0$. It follows from the equivariant homotopy extension theorem that $f - {'f}$ is Z_n-homotopic to a map $h_2 : S(V) \to S(V)$ with $h_2(S(V)^{Z_2}) = x_0$. In view of Proposition 2.2 in [**9**], h_2 is Z_n-homotopic to a map h_3 transverse regular to y_0 in $S(V)$ and close to h_2. Since Z_n acts freely on $h_3^{-1}(y_0)$, the number of points $x \in h_3^{-1}(y_0)$ is divisible by n. Let a^+ (resp. a^-) be the number of points $x \in h_3^{-1}(y_0)$ for which the differential $dh_{3x} : T_x(S(V)) \to T_{y_0}(S(V))$ is orientation preserving (resp. reversing) with respect to a fixed orientation on $S(V)$ where $T_x(S(V))$ and $T_{y_0}(S(V))$ denote the tangent spaces. For every $g \in Z_n$, we note that

$$dh_{3gx} = dg_{h_3(x)} \circ dh_{3x} \circ d(g^{-1})_{gx}\,.$$

Note that dg^{-1} is orientation preserving if and only if dg is orientation preserving. Hence dh_{3gx} is orientation preserving if and only if dh_{3x} is orientation preserving. Thus a^+ and a^- are both divisible by n; hence $\deg h_3 = a^+ - a^- \equiv 0 \ \mathrm{mod}\ n$. Recall that

$$\deg h_3 = \deg f - \deg {'f} = \deg f - \deg f^{Z_2}\,,$$

which proves the first formula.

Assume next that k is greater than two. Regarding $S(V)^{Z_2}$ as a Z_n/Z_2 $(\cong Z_{n/2})$-manifold, the first formula in Assertion 2.2 reads

$$\begin{aligned} \deg f^{Z_2} &\equiv \deg (f^{Z_2})^{(Z_4/Z_2)} \qquad \mathrm{mod}\ n/2 \\ &= \deg f^{Z_4}\,. \end{aligned}$$

Recall that $\deg f^{Z_2} = \pm 1$, $\deg f^{Z_4} = \pm 1$. Therefore we can conclude that

$$\deg f^{Z_2} = \deg f^{Z_4}\,.$$

Continuing this argument, we get the second formula in Assertion 2.2.

REMARK 2.3. $\deg f^{Z_n}$ is not equal to $\deg f^{Z_2}$ in general.

Case 2. $n = p_1^{r(1)} \cdots p_t^{r(t)}$, p_i : odd prime.

In this case, we show the following

Assertion 2.4.

$$\deg f \equiv \deg f^{Z_n} \qquad \text{mod } n\,.$$

Proof of Assertion 2.4. We prove Assertion 2.4 by induction on t. Let t be one, that is, n is of the form p^r. Then we can employ the argument in Case 1 and get

$$\deg f \equiv \deg f^{Z_p} \qquad \text{mod } n$$

and

$$\deg f^{Z_p} = \deg f^{Z_{p^i}} \qquad \text{for} \quad 1 \leqq i \leqq r-1\,.$$

Moreover, since $p \geqq 3$, we obtain even the following formula

$$\deg f^{Z_p} = \deg f^{Z_n}\,.$$

Namely Assertion 2.4 holds for $t=1$.

When $t \geqq 2$, we suppose that Assertion 2.4 is true for $s<t$, $n=p_1^{r(1)} \cdots p_s^{r(s)}$. Regarding

$$f^{Z_{p_1^{r(1)}}} : S(V)^{Z_{p_1^{r(1)}}} \longrightarrow S(V)^{Z_{p_1^{r(1)}}}$$

as $Z_n/Z_{p_1^{r(1)}}(\cong Z_{p_2^{r(2)} \cdots p_t^{r(t)}})$-map, we have

$$\begin{aligned} \deg f^{Z_{p_1^{r(1)}}} &\equiv \deg \left(f^{Z_{p_1^{r(1)}}}\right)^{(Z_n/Z_{p_1^{r(1)}})} \qquad \text{mod } p_2^{r(2)} \cdots p_t^{r(t)} \\ &= \deg f^{Z_n} \end{aligned}$$

by the inductive assumption. Recall that

$$\deg f^{Z_{p_1^{r(1)}}} = \pm 1 \qquad \text{and} \quad \deg f^{Z_n} = \pm 1\,.$$

Hence $\deg f^{Z_{p_1^{r(1)}}} = \deg f^{Z_n}$. It follows that

$$\deg f \equiv \deg f^{Z_n} \qquad \text{mod } p_1^{r(1)}\,.$$

Since this argument is valid for any $p_i^{r(i)}$, we can conclude that

$$\deg f \equiv \deg f^{Z_n} \qquad \text{mod } n\,.$$

This completes the inductive proof of Assertion 2.4.

Case 3. $n=2^k \cdot p_1^{r(1)} \cdots p_t^{r(t)}$, $k \geqq 1$, $t \geqq 1$.

Assertion 2.5.

$$\deg f \equiv \deg f^{Z_{n/2}} \qquad \text{mod } n\,.$$

Proof of Assertion 2.5. It follows from Assertions 2.2 and 2.4 that

$$\deg f \equiv \begin{cases} 1 & \text{mod } 2 \quad \text{if} \quad k=1\,, \\ \deg f^{Z_{2^{k-1}}} & \text{mod } 2^k \quad \text{if} \quad k \geqq 2 \end{cases}$$

and

$$\deg f \equiv \deg f^{Z_{p_1^{r(1)} \cdots p_t^{r(t)}}} \qquad \bmod p_1^{r(1)} \cdots p_t^{r(t)} .$$

Thus, if $k=1$,

$$\deg f \equiv \deg f^{Z_{n/2}} = \pm 1 \qquad \bmod n .$$

We next suppose that $k \geqq 2$. Regarding $S(V)^{Z_{p_1^{r(1)} \cdots p_t^{r(t)}}}$ as a $Z_n / Z_{p_1^{r(1)} \cdots p_t^{r(t)}}$ $(\cong Z_{2^k})$-manifold, we can prove by the same argument used often above that

$$\deg f^{Z_{p_1^{r(1)} \cdots p_t^{r(t)}}} = \deg f^{Z_{n/2}} .$$

Similarly one verifies that

$$\deg f^{Z_{2^{k-1}}} = \deg f^{Z_{n/2}} .$$

Putting all this together, we get Assertion 2.5.

This makes the proof of Lemma 1 complete.

When $2q \not\equiv 0 \bmod n$, we define $\rho(q): Z_n \to SO(2)$ by

$$\rho(q)(j) = \begin{pmatrix} \cos \frac{2qj\pi}{n} & \sin \frac{2qj\pi}{n} \\ -\sin \frac{2qj\pi}{n} & \cos \frac{2qj\pi}{n} \end{pmatrix} \qquad \text{for} \quad j \in Z_n .$$

When $2q \equiv 0 \bmod n$, we define $\rho(q): Z_n \to O(1)$ by $\rho(q)(j) = (-1)^{2qj/n}$ for $j \in Z_n$. Hereafter we denote by m the integer part of $n/2$. Let $c(q)$, $d(q)$ be non negative integers with $c(q) = d(q) = 0$ for $(q, n) \neq 1$ where $q = 1, \cdots, m$.

Then we set

$$V = \sum_{q=1}^{m} c(q)\rho(q) \qquad \text{and} \quad W = \sum_{q=1}^{m} d(q)\rho(q) .$$

THEOREM 2.6. *The following three conditions are equivalent;*

(i) *$S(V)$ and $S(W)$ are Z_n-homotopy equivalent,*

(ii)
$$\sum_{q=1}^{m} c(q) = \sum_{q=1}^{m} d(q) ,$$

$$\prod_{q=1}^{m} q^{c(q)} \equiv \pm \prod_{q=1}^{m} q^{d(q)} \qquad \bmod n ,$$

(iii) *there exists a Z_n-representation U such that $S(V \oplus U)$ and $S(W \oplus U)$ are Z_n-homotopy equivalent.*

PROOF. It is well-known that the conditions (i) and (ii) are equivalent (see for examples [4], [8]). Clearly (i) implies (iii). Therefore it suffices to show that (iii) implies (ii). Let $f: S(V \oplus U) \to S(W \oplus U)$ be the Z_n-homotopy equivalence. Then the map

$$f*id: S(V\oplus U\oplus R^2)=S(V\oplus U)*S(R^2) \longrightarrow S(W\oplus U\oplus R^2)$$

is also a Z_n-homotopy equivalence. Therefore we may assume without loss of generality that U includes R^2 as a direct summand. When we write U as $\sum_{q=0}^{m} u(q)\rho(q)$, we put

$$a(q)=\begin{cases} u(q) & \text{for } (q, n)=1 \\ 0 & \text{for } (q, n)\neq 1 \end{cases}$$

and

$$b(q)=\begin{cases} u(q) & \text{for } (q, n)\neq 1 \\ 0 & \text{for } (q, n)=1 . \end{cases}$$

Then we have a direct sum decomposition

$$U=U_a\oplus U_b$$

where

$$U_a=\sum_{q=1}^{m} a(q)\rho(q) \quad \text{and} \quad U_b=\sum_{q=0}^{m} b(q)\rho(q).$$

We set

$$X=\{x\in S(U_b) \mid (Z_n)_x\neq\{0\}\} .$$

If $X=S(U_b)$, then $f(S(U_b))=S(U_b)$ and the restriction $f|S(U_b): S(U_b)\to S(U_b)$ is a Z_n-homotopy equivalence. If $X\neq S(U_b)$, $f(S(U_b))$ is not included in $S(U_b)$ in general and we have to contrive as follows. Obviously

$$f(X)\subset X\subset S(U_b)\subset S(W\oplus U) .$$

Hence $f(X)\cap S(W\oplus U_a)=\emptyset$. It follows that $f(B)\cap S(W\oplus U_a)=\emptyset$ for some open neighborhood B of X in $S(U_b)$. Since Z_n acts freely on $S(U_b)-X$, the restricted map

$$f\mid S(U_b): S(U_b) \longrightarrow S(W\oplus U)$$

is Z_n-homotopic to a map f_1 transverse regular to $S(W\oplus U_a)$ in $S(W\oplus U)$ and close to $f|S(U_b)$ [**9**]. Since the codimension of $S(W\oplus U_a)$ in $S(W\oplus U)$ is equal to $\dim S(U_b)+1$, we have that

$$f_1(S(U_b))\cap S(W\oplus U_a)=\emptyset .$$

It follows that f_1 is Z_n-homotopic to a map f_2 whose image $f_2(S(U_b))$ is contained in $S(U_b)$. By the equivariant homotopy extension theorem, these homotopies can be extended to $S(V\oplus U)$. Thus we have shown that f is Z_n-homotopic to a map $f_3: S(V\oplus U)\to S(W\oplus U)$ satisfying $f_3(S(U_b))\subset S(U_b)$. Note that f_3 is also a Z_n-homotopy equivalence. Since

$$S(V \oplus U)^H = S(W \oplus U)^H = S(U_b)^H$$

for every non zero subgroup H of Z_n, the map

$$f_3^H : S(U_b)^H \longrightarrow S(U_b)^H$$

is a homotopy equivalence. In particular $\deg f_3^H = \pm 1$.

We are now in a position to apply Lemma 2.1 and have that the degree of the restricted map

$$f_3 | S(U_b) : S(U_b) \longrightarrow S(U_b)$$

is $\pm 1 \bmod n$.

Let q and q' be positive integers with $(q, n) = (q', n) = 1$. Then it is easy to see that there exists a Z_n-map

$$j_1 : S(\rho(q)) \longrightarrow S(\rho(q'))$$

with $\deg j_1 \equiv q'/q \bmod n$. Hence by joining these maps, we get a Z_n-map $j_2 : S(V) \to S(W)$ with

$$\deg j_2 \equiv \prod_{q=1}^{m} q^{(d(q) - c(q))} \qquad \bmod n \, .$$

Consider the Z_n-map

$$j_3 = j_2 * id * (f_3 | S(U_b)) :$$
$$S(V \oplus U) = S(V) * S(U_a) * S(U_b) \longrightarrow S(W \oplus U) = S(W) * S(U_a) * S(U_b) \, .$$

Since $f_3 | S(U_b) = j_3 | S(U_b)$, $f_3 - j_3$ is Z_n-homotopic to a map h with $h(S(U_b)) = x_0$ $(\in R^2 \subset U_b)$. We fix orientations on $S(V)$, $S(W)$ and $S(U)$ respectively. Accordingly $S(V \oplus U)$ and $S(W \oplus U)$ are oriented. Then it follows by the argument used often before that $\deg h \equiv 0 \bmod n$. Note that $\deg f_3 = \pm 1$, because f_3 is a homotopy equivalence.

Putting all this together, we obtain

$$\prod_{q=1}^{m} q^{c(q)} \equiv \pm \prod_{q=1}^{m} q^{d(q)} \qquad \bmod n \, .$$

On the other hand, the equation

$$\sum_{q=1}^{m} c(q) = \sum_{q=1}^{m} d(q)$$

holds trivially from (iii).

This makes the proof of Theorem 2.6 complete.

3. The groups $J'_{Z_n}(*)$ and $J''_{Z_n}(*)$.

Let $FRO(Z_n)$ be the subgroup of the orthogonal representation ring $RO(Z_n)$ generated by

$$\{\rho(q) \mid 1 \leqq q \leqq m, (q, n)=1\}.$$

Then we set

$$J'_{Z_n}(*)=J_{Z_n}(FRO(Z_n)).$$

The restricted homomorphism $J_{Z_n}|FRO(Z_n)$ is denoted by J'_{Z_n}.

On the other hand we define another group $J''_{Z_n}(*)$ for $n>1$ as follows. Let $n=2^k \cdot p_1^{r(1)} \cdots p_t^{r(t)}$ be the prime decomposition of n.

Case 1. $k \geqq 2$. We set

$$J''_{Z_n}(*)=Z \oplus Z_{2^{k-2}} \oplus \bigoplus_{i=1}^{t} Z_{(p_i^{r(i)}-p_i^{r(i)-1})}.$$

Case 2. $k=0$ or 1. We set

$$J''_{Z_n}(*)=Z \oplus \left\{ \bigoplus_{i=1}^{t} Z_{(p_i^{r(i)}-p_i^{r(i)-1})} \right\} \Big/ Z_2$$

where the inclusion of Z_2 into $\bigoplus_{i=1}^{t} Z_{(p_i^{r(i)}-p_i^{r(i)-1})}$ is given by $1 \mapsto \bigoplus_{i=1}^{t} (p_i^{r(i)}-p_i^{r(i)-1})/2$.

Then we define a homomorphism

$$J''_{Z_n}: FRO(Z_n) \longrightarrow J''_{Z_n}(*)$$

as follows. As is well-known, there exist integers $\alpha(i)$ with $1 \leqq \alpha(i) < n$ for $i=-1, 0, 1, \cdots, t$ such that

(a) $\alpha(-1) \equiv -1 \bmod 2^k$, $\alpha(-1) \equiv 1 \bmod p_j^{r(j)}$ for every $j \geqq 1$,

(b) $\alpha(0) \equiv 5 \bmod 2^k$, $\alpha(0) \equiv 1 \bmod p_j^{r(j)}$ for every $j \geqq 1$,

(c) for $i \geqq 1$, $\alpha(i)$ is a primitive root mod $p_i^{r(i)}$, $\alpha(i) \equiv 1 \bmod 2^k$, $\alpha(i) \equiv 1 \bmod p_j^{r(j)}$ for every $j \geqq 1$ with $j \neq i$.

Then for every integer q with $(q, n)=1$, there exist unique $\mu(q, -1) \in Z_2$, $\mu(q, 0) \in Z_{2^{k-2}}$ and $\mu(q, i) \in Z_{(p_i^{r(i)}-p_i^{r(i)-1})}$ for $1 \leqq i \leqq t$ such that

$$q \equiv \prod_{i=-1}^{t} \alpha(i)^{\mu(q, i)} \quad \bmod n.$$

When $k=0$ or 1, we consider $\alpha(i)$ and $\mu(q, i)$ only for $1 \leqq i \leqq t$. Let

$$j_i: Z_2 \longrightarrow Z_{(p_i^{r(i)}-p_i^{r(i)-1})}$$

be the natural inclusion map given by $1 \mapsto (p_i^{r(i)}-p_i^{r(i)-1})/2$. Let $\sum_{q=1}^{m} a(q)\rho(q)$ be an arbitrary element of $FRO(Z_n)$, that is, $a(q) \in Z$ and $a(q)=0$ for $(q, n) \neq 1$. In the following, $[x]$ indicates the equivalence class represented by x in respective context.

Case 1. $k \geqq 2$. We set

$$J''_{Z_n}\Big(\sum_{q=1}^{m} a(q)\rho(q)\Big)=\sum_{q=1}^{m} a(q)\oplus \sum_{q=1}^{m} a(q)\mu(q, 0)$$

$$\oplus \bigoplus_{i=1}^{t}\Big\{j_i\Big(\sum_{q=1}^{m} a(q)\mu(q, -1)\Big)+\sum_{q=1}^{m} a(q)\mu(q, i)\Big\}.$$

Case 2. $k=0$ or 1. We set

$$J''_{Z_n}\Big(\sum_{q=1}^{m} a(q)\rho(q)\Big)=\sum_{q=1}^{m} a(q)\oplus\Big[\bigoplus_{i=1}^{t}\sum_{q=1}^{m} a(q)\mu(q, i)\Big].$$

Then we have

THEOREM 3.1. *J''_{Z_n} is a well-defined epimorphism and* $\mathrm{Ker}\, J'_{Z_n}=\mathrm{Ker}\, J''_{Z_n}$. *Hence we have that*

$$J'_{Z_n}(*)\cong J''_{Z_n}(*).$$

PROOF. Obviously J''_{Z_n} is a well-defined homomorphism. First we show that J''_{Z_n} is an epimorphism. Define $\beta(i)$ by

$$\beta(i)=\begin{cases} \alpha(i) & \text{if } 1\leqq \alpha(i)\leqq m \\ n-\alpha(i) & \text{if } m<\alpha(i)<n. \end{cases}$$

Case 1. $k\geqq 2$. Let x, x_i $(i=0, 1, \cdots, t)$ be arbitrary integers. Then one verifies that

$$J''_{Z_n}\Big(\Big(x-\sum_{i=0}^{t} x_i\Big)\rho(1)+\sum_{i=0}^{t} x_i\rho(\beta(i))\Big)$$

$$=x\oplus[x_0]\oplus\bigoplus_{i=1}^{t}[x_i]\in J''_{Z_n}(*)=Z\oplus Z_{2^{k-2}}\oplus\bigoplus_{i=1}^{t} Z_{(p_i^{r(i)}-p_i^{r(i)-1})}.$$

Namely J''_{Z_n} is surjective in this case.

Case 2. $k=0$ or 1. Let x, x_i $(i=1, \cdots, t)$ be arbitrary integers. Then one verifies similarly that

$$J''_{Z_n}\Big(\Big(x-\sum_{i=1}^{t} x_i\Big)\rho(1)+\sum_{i=1}^{t} x_i\rho(\beta(i))\Big)$$

$$=x\oplus\Big[\bigoplus_{i=1}^{t}[x_i]\Big]\in J''_{Z_n}(*)=Z\oplus\Big\{\bigoplus_{i=1}^{t} Z_{(p_i^{r(i)}-p_i^{r(i)-1})}\Big\}\Big/Z_2.$$

Namely J''_{Z_n} is surjective in this case too.

Next we shall show that $\mathrm{Ker}\, J'_{Z_n}=\mathrm{Ker}\, J''_{Z_n}$. Given an integer $a(q)$, we set

$$a(q)'=\begin{cases} a(q) & \text{for } a(q)>0 \\ 0 & \text{for } a(q)\leqq 0 \end{cases}$$

and

$$a(q)''=\begin{cases} -a(q) & \text{for } a(q)<0 \\ 0 & \text{for } a(q)\geqq 0 . \end{cases}$$

Case 1. $k\geqq 2$. Let $\sum_{q=1}^{m} a(q)\rho(q)$ be an arbitrary element of $\operatorname{Ker} J''_{Z_n}$, that is,

$$\sum_{q=1}^{m} a(q)'=\sum_{q=1}^{m} a(q)'',\ \sum_{q=1}^{m} a(q)'\mu(q,0)=\sum_{q=1}^{m} a(q)''\mu(q,0),$$

$$(3.2)\qquad j_i\Big(\sum_{q=1}^{m} a(q)'\mu(q,-1)\Big)+\sum_{q=1}^{m} a(q)'\mu(q,i) =j_i\Big(\sum_{q=1}^{m} a(q)''\mu(q,-1)\Big)+\sum_{q=1}^{m} a(q)''\mu(q,i)$$

for every i with $1\leqq i\leqq t$.

It is easy to see that the condition (3.2) is equivalent to the following condition (3.3);

$$(3.3)\qquad \begin{aligned} &\sum_{q=1}^{m} a(q)'=\sum_{q=1}^{m} a(q)'', \\ &\prod_{q=1}^{m} q^{a(q)'}\equiv\pm\prod_{q=1}^{m} q^{a(q)''} \mod n . \end{aligned}$$

Therefore Theorem 3.1 follows from Theorem 2.6 in this case.

Case 2. $k=0$ or 1. Since the proof is analogous to that of Case 1, we omit it.

This makes the proof of Theorem 3.1 complete.

4. The group $J_G(*)$.

Let G be a compact abelian topological group and F_0 (resp. F_1) be the family of all closed proper subgroups H of G such that G/H is isomorphic to the circle group S^1 (resp. a finite cyclic group). As is well-known, there is a canonical isomorphism

$$RO(G)\cong Z\oplus Z(F_0)\oplus\bigoplus_{H\in F_1} FRO(G/H)$$

of groups where $Z(F_0)$ denotes the free abelian group generated by F_0. Then it is not difficult to see

$$T_G(*)=\{0\}\oplus\{0\}\oplus\bigoplus_{H\in F_1}\operatorname{Ker} J'_{G/H}$$

under the above correspondence. It follows that

$$J_G(*)\cong Z\oplus Z(F_0)\oplus\bigoplus_{H\in F_1} J'_{G/H}(*)\,.$$

Therefore we obtain the following main theorem from Theorem 3.1.

THEOREM 4.1. *We have the following isomorphism,*

$$J_G(*)\cong Z\oplus Z(F_0)\oplus\bigoplus_{H\in F_1} J''_{G/H}(*)$$

where $J''_{G/H}(*)$ is the group given in § 3.

COROLLARY 4.2. *Let V, W be orthogonal G-representations. Then $S(V)$ is G-homotopy equivalent to $S(W)$ if and only if V is J-equivalent to W.*

5. Normal representations of fixed point sets of G-homotopy equivalent manifolds.

Let G be a compact Lie group and M_1, M_2 be closed smooth G-manifolds. Suppose that we are given a G-homotopy equivalence $f: M_1\to M_2$. Denote by F_1^μ each component of the fixed point set of M_1. Set $F_2^\mu=f(F_1^\mu)$. Then F_2^μ is a component of the fixed point set of M_2 and the union $\bigcup_\mu F_2^\mu$ is exactly the fixed point set of M_2. Denote by V_i^μ the normal representation of F_i^μ in M_i $(i=1, 2)$.

Then we showed in [6] that *V_1^μ is J-equivalent to V_2^μ.* Therefore this together with Corollary 4.2 brings about;

THEOREM 5.1. *If G is a compact abelian Lie group, $S(V_1^\mu)$ and $S(V_2^\mu)$ are themselves G-homotopy equivalent.*

6. Equivariant Adams conjecture.

Let p be a positive prime integer and r be a positive integer. When p is odd, we denote by α a primitive root mod p^r. When $p=2$, we set $\alpha=5$. Let G be an abelian p-group of order p^r and $\Psi^s: RO(G)\to RO(G)$ be the equivariant s-th Adams operation [2], [9], [13].

DEFINITION 6.1. Denote by $WO(G)$ the subgroup

$$\{(1-\Psi^\alpha)^2(x)\mid x\in RO(G)\}$$

of $RO(G)$.

REMARK 6.2. Although the definition of $WO(G)$ seems to be different from that in [7], they are equivalent.

As a special case of Theorem 4.1, we have

THEOREM 6.3.

$$J_G(*)\cong RO(G)/WO(G)\,.$$

PROOF. Consider first the following commutative diagram

$$
\begin{array}{ccc}
RO(G) \cong Z \oplus \bigoplus_{H \in F_1} FRO(G/H) & & \\
\Big\downarrow J_G & \Big\downarrow id \oplus \bigoplus_{H \in F_1} J'_{G/H} & \\
J_G(*) \cong Z \oplus \bigoplus_{H \in F_1} J'_{G/H}(*) . & &
\end{array}
$$

Then it suffices to show that

$$WO(G) \cong \{0\} \oplus \bigoplus_{H \in F_1} WO(G/H) \tag{6.4}$$

and

$$\operatorname{Ker} J'_{G/H} = WO(G/H) \qquad \text{for} \quad H \in F_1 . \tag{6.5}$$

Since $(1-\Psi^{\alpha})^2(FRO(G/H)) \subset FRO(G/H)$ under the above correspondence, (6.4) holds.

The proof of (6.5) is divided into two cases.

Case 1. p: odd prime. Let p^s be the order of the group G/H and $\rho(q)$ be the representation of G/H defined in § 2. Then an element x of $FRO(G/H)$ can be written as

$$x = \sum_{i=0}^{\frac{p^s - p^{s-1}}{2}} c(i) \rho(\alpha^i)$$

where $c(i)$ are integers. In virtue of Theorem 2.6, x belongs to $\operatorname{Ker} J'_{G/H}$ if and only if

$$
\begin{aligned}
&\textstyle\sum c(i) = 0 \\
&\textstyle\prod \alpha^{ic(i)} \equiv \pm 1 \qquad \bmod p^s .
\end{aligned} \tag{6.7}
$$

The condition (6.7) is equivalent to the following condition (6.8);

$$
\begin{aligned}
&\textstyle\sum c(i) = 0 \\
&\textstyle\sum ic(i) \equiv 0 \qquad \bmod \frac{p^s - p^{s-1}}{2} .
\end{aligned} \tag{6.8}
$$

We now consider a polynomial

$$f(X) = \sum c(i) X^i$$

where X is an indeterminate. If (6.8) holds, $f(1)=0$ and $f'(1) = v(p^s - p^{s-1})/2$ for some integer v. Then we define another polynomial $F(X)$ by

$$F(X) = f(X) + v - vX^{(p^s - p^{s-1})/2} .$$

Since $F(1)=0$, $F'(1)=0$, there exists a polynomial $f_1(X)$ with integer coefficients satisfying

$$F(X)=(1-X)^2 f_1(X).$$

Note that

$$\Psi^{\alpha(p^s-p^{s-1})/2} | FRO(G/H)=\text{identity}.$$

Hence we have

$$\begin{aligned} x &= f(\Psi^\alpha)\rho(1) \\ &= F(\Psi^\alpha)\rho(1) \\ &= (1-\Psi^\alpha)^2 f_1(\Psi^\alpha)\rho(1). \end{aligned}$$

Namely

$$\operatorname{Ker} J'_{G/H} \subset WO(G/H).$$

Conversely every element x of $WO(G/H)$ can be written as

$$\begin{aligned} x &= (1-\Psi^\alpha)^2(\sum a(i)\rho(\alpha^i)) \\ &= \sum a(i)\{\rho(\alpha^i)-2\rho(\alpha^{i+1})+\rho(\alpha^{i+2})\}. \end{aligned}$$

In view of Theorem 2.6,

$$\rho(\alpha^i)-2\rho(\alpha^{i+1})+\rho(\alpha^{i+2})$$

belongs to $\operatorname{Ker} J'_{G/H}$, and hence

$$WO(G/H) \subset \operatorname{Ker} J'_{G/H}.$$

Case 2. $p=2$. Since the proof is quite similar to that of Case 1, we omit it.

This makes the proof of Theorem 6.3 complete.

REMARK 6.4. Theorem 4.1 and Theorem 6.3 show that $J_G(*)$ involves many torsion groups in general and the equivariant Adams conjecture does not hold in general in the form similar to the non equivariant case [**1**], [**3**], [**10**]. These phenomena contrast with those of $JO(G)$ [**2**], [**9**], [**13**].

7. A concluding remark.

Let G be a compact connected Lie group and T^n be a maximal torus of G. Then the natural homomorphism $RO(G)\to RO(T^n)$ is injective. On the other hand, $J_{T^n}: RO(T^n)\to J_{T^n}(*)$ is an isomorphism by Theorem 4.1. Thus we have

REMARK 7.1. For a compact connected Lie group G, $J_G(*)$ is isomorphic to $RO(G)$. This phenomenon occurs even for $JO(G)$ [**9**].

Added in proof. Professor T. tom Dieck kindly informed me that he obtained Theorem 6.3 for G an arbitrary p-group [**5**].

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