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A class of infinitesimal generators of one-dimensional Markov processes

II. Invariant measures

By H. LANGER and W. SCHENK

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It was shown in [4] that an operator of the form (1) below with boundary conditions of Feller-Wentzell type is the infinitesimal generator of a strongly continuous nonnegative contraction (s. c. n. c.) semigroup $(T_t)_{t\geqq 0}$ in $\boldsymbol{C}=C([0, 1])$*) or a subspace of $\boldsymbol{C}$. In this note we continue the study of these operators. The main result is that the semigroup $(T_t^*)_{t\geqq 0}$ or the corresponding Markov process have a unique invariant measure μ_0 with supp $\mu_0=[0, 1]$ if only the boundary conditions are "not too degenerated". This seems to be rather evident as the operator (1) contains a diffusion term $D_m D_x$. However the analytical proof of this fact we could give (Theorem 5) is not so short. Further it is shown that μ_0 is in $(0, 1)$ absolutely continuous with respect to the measure m.

In a following note we shall continue the study of this class of Markov processes along the lines of [6]. In particular, we shall investigate the limit behavior of the transition probabilities if $t\to\infty$ and derive Kolmogorov's equations for the densities of the transition probabilities (with respect to μ_0). As an important tool, the extension of the semigroup $(T_t)_{t\geqq 0}$ to $L^2(\mu_0)$ (with scalar product denoted by $[\cdot, \cdot]$) is considered. The explicit expressions of $[Af, f]$ and its real and imaginary parts, given at the end of this paper, will play an essential role in this investigation.

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1. Preliminaries.

Let m, b and the family of measures n_x, $x\in[0, 1]$, have the same properties as in [4], [5] that is m is a strongly increasing continuous function

*) In [4] only real spaces have been considered, here, however, $\boldsymbol{C}$ is supposed to be complex. It is easy to see ([5], p. 106), that the statements quoted above are true for the corresponding complex spaces.

on $[0, 1]$, b is a real continuous function on $[0, 1]$ and n_x, $x\in[0, 1]$, are nonnegative measures on $[0, 1]$ with the properties

(a) $n_x([0, 1])\leqq K<\infty \qquad (x\in[0, 1])$,

(b) $\xi\to x$ implies $n_\xi\to n_x$ *-weakly $(x, \xi\in[0, 1])$, that is

$$\int_0^1 f(y)n_\xi(dy)\to\int_0^1 f(y)n_x(dy) \quad \text{for all} \quad f\in \boldsymbol{C},$$

(c) $\sup_{x\in[0,1]}\int_{\substack{|x-y|\leqq\delta \\ y\in[0,1]}} n_x(dy)\to 0 \quad \text{if} \quad \delta\downarrow 0$.*)

The second order generalized differential operator $D_m D_x$ in $\boldsymbol{C}$ is defined in the usual way (see [4] and the references quoted there): Its domain $\mathfrak{D}(D_m D_x)$ is the set of all $f\in\boldsymbol{C}$ which admit a representation

$$f(x)=f_0+xf_0'+\int_0^x(x-s)\varphi(s)dm(s), \qquad x\in[0, 1],$$

with f_0, $f_0'\in C$**), $\varphi\in\boldsymbol{C}$, and for this function f we define

$$D_m D_x f:=\varphi.$$

With $D_x f$ denoting the first derivative of a continuously differentiable function f and $\varphi_x(y):=\int_x^y(y-s)dm(s)$, x, $y\in[0, 1]$, on $\mathfrak{D}(D_m D_x)$ we shall consider the following operator $\mathfrak{A}$:

$$(\mathfrak{A}f)(x):=(D_m D_x f)(x)+b(x)(D_x f)(x) \tag{1}$$
$$+\int_0^1(f(y)-f(x)-(y-x)(D_x f(x)))\frac{n_x(dy)}{\varphi_x(y)}, \quad x\in[0, 1], \quad f\in\mathfrak{D}(D_m D_x).$$

The integral on the right hand side of (1) is possibly an improper integral with respect to the singularity at $y=x$, but it is easy to see that it exists for all $f\in\mathfrak{D}(D_m D_x)$. In the following, by $\tilde{n}_x$ we denote the measure

$$\tilde{n}_x(dy):=\frac{n_x(dy)}{\varphi_x(y)} \quad \text{on } [0, 1]\setminus\{x\}.$$

If $f\in\mathfrak{D}(D_m D_x)$ we define

$$\Phi_0(f):=\kappa_0 f(0)+\int_0^1\frac{f(0)-f(x)}{x}dq_0(x)+\sigma_0(\mathfrak{A}f)(0),$$

$$\Phi_1(f):=\kappa_1 f(1)+\int_0^1\frac{f(1)-f(x)}{1-x}dq_1(x)+\sigma_1(\mathfrak{A}f)(1),$$

where the constants κ_0, κ_1, σ_0, σ_1 are nonnegative, q_0 and q_1 are nonnegative measures on $[0, 1]$ and $\kappa_i+\sigma_i+\int_0^1 dq_i>0$, $i=0, 1$. If q_i has concentrated mass

*) The conditions (a-c) are equivalent to (b) and (a′) $n_x(\{x\})=0$ for all $x\in[0,1]$.

**) C denotes the set of complex numbers, := is used to define new symbols.

at the point i, $i=0, 1$, it is understood that

$$\left.\frac{f(0)-f(x)}{x}\right|_{x=0}=-(D_x f)(0), \qquad \left.\frac{f(1)-f(x)}{1-x}\right|_{x=1}=(D_x f)(1).$$

We always suppose that the equations

$$\Phi_0(f)=0, \qquad \Phi_1(f)=0 \tag{2}$$

are not equivalent to $f(0)=f(1)$. The restriction A of $\mathfrak{A}$ by the boundary conditions (2), that is $\mathfrak{D}(A):=\{f\in\mathfrak{D}(D_m D_x): \Phi_0(f)=\Phi_1(f)=0\}$ and $Af:=\mathfrak{A}f$ for $f\in\mathfrak{D}(A)$, is the infinitesimal generator of a s.c.n.c. semigroup in $\boldsymbol{C}$ or the subspace of $\boldsymbol{C}$ determined by the boundary conditions (2), see [**4**], [**5**]. For simplicity we shall always suppose in the following, that the functionals Φ_i, $i=0, 1$, are not continuous on $\boldsymbol{C}$ that is

$$\int_0^1 |i-x|^{-1} dq_i(x)=\infty \quad \text{or} \quad \sigma_i>0, \quad i=0, 1. \tag{3}$$

In this case the domain $\mathfrak{D}(A)$ of A is dense in $\boldsymbol{C}$.

LEMMA 1. *The spectrum* $\sigma(A)$*) *is discrete in the finite complex plane.*

PROOF. Suppose first that the functionals Φ_i are

$$\Phi_i(f):=(-1)^{i+1}(D_x f)(i), \qquad i=0, 1,$$

and denote by A_1 the corresponding restriction of $\mathfrak{A}$ by the boundary conditions (2). Then with the operators $A_0: \mathfrak{D}(A_0)=\mathfrak{D}(A_1)$,

$$A_0 f:=D_m D_x f, \qquad f\in\mathfrak{D}(A_0),$$

and $B: \mathfrak{D}(B)=\mathfrak{D}(A_1)$,

$$(Bf)(x):=b(x)(D_x f)(x)$$
$$+\int_0^1 (f(y)-f(x)-(y-x)(D_x f)(x))\tilde{n}_x(dy), \quad x\in[0, 1], \quad f\in\mathfrak{D}(B),$$

we have for the resolvents $R_\lambda^{(0)}:=(\lambda I-A_0)^{-1}$, $R_\lambda^{(1)}:=(\lambda I-A_1)^{-1}$:

$$R_\lambda^{(1)}=R_\lambda^{(0)}(I-BR_\lambda^{(0)})^{-1}, \qquad \lambda\in\rho(A_1)\cap\rho(A_0),$$

and $BR_\lambda^{(0)}$ is compact in $\boldsymbol{C}$ ([**4**]). Evidently $BR_\lambda^{(0)}$ is a holomorphic function of λ in $\rho(A_0)$ and the positive half axis belongs to $\rho(A_1)\cap\rho(A_0)$, hence $1\notin\sigma_p(BR_\lambda^{(0)})$ if $\lambda>0$. By a theorem of I.C. Gohberg ([**2**]), $\sigma(A_1)$ is discrete.

*) The spectrum $\sigma(A)$, resolvent set $\rho(A)$ and point spectrum $\sigma_p(A)$ of a linear operator A are defined as in [**1**].

Let now A be the operator (1) with general boundary conditions. For fixed $\lambda_0>0$ the difference $(\lambda_0 I-A)^{-1}-(\lambda_0 I-A_1)^{-1}$ is two-dimensional (see [4], p. 248). On the other hand $\sigma(R_{\lambda_0}^{(1)})$ $(\sigma(R_{\lambda_0}))$ is discrete in $C\setminus\{0\}$ if and only if $\sigma(A_1)$ $(\sigma(A)$ resp.) is discrete in C. Therefore the statement follows from the first part of the proof.

In the following the s. c. n. c. semigroup in $\boldsymbol{C}$ generated by the operator A will be denoted by $(T_t)_{t\geq 0}$, its adjoint semigroup in $\boldsymbol{C}^*$ by $(T_t^*)_{t\geq 0}$. The corresponding transition function is $P(t; x, \Gamma)$ $(t>0,\ x\in[0, 1],\ \Gamma\in\mathfrak{B}_{[0,1]})$. A nonnegative measure $\mu\in\boldsymbol{C}^*$, $\mu\neq 0$, is said to be *invariant* (*subinvariant*) under $(T_t^*)_{t\geq 0}$ if $T_t^*\mu=\mu$ $(T_t^*\mu\leqq\mu$ resp.) for all $t\geqq 0$.

The following lemma is well-known for arbitrary strongly continuous semigroups $(T_t)_{t\geq 0}$ in a Banach space. It is reproduced here only for the sake of completeness.*)

LEMMA 2. *The following statements are equivalent:*

1) $\mu_0\in\boldsymbol{C}^*$ *is an invariant measure of the semigroup* $(T_t^*)_{t\geq 0}$;

2) *for some* $\lambda\in\rho(A)$ *we have* $\lambda R_\lambda^*\mu_0=\mu_0$;

3) *for all* $\lambda\in\rho(A)$ *we have* $\lambda R_\lambda^*\mu_0=\mu_0$;

4) μ_0 *is orthogonal to the range* $\mathfrak{R}(A)$.**)

PROOF. Evidently $\lambda R_\lambda^*\mu_0=\mu_0$ is equivalent to $\mu_0(\lambda R_\lambda f-f)=0$ for all $f\in\boldsymbol{C}$. If $\lambda', \lambda\in\rho(A)$, we get therefore

$$0=\mu_0(\lambda R_\lambda R_{\lambda'} f-R_{\lambda'} f)=\mu_0(\lambda(\lambda-\lambda')^{-1}(R_{\lambda'}-R_\lambda)f-R_{\lambda'} f)$$
$$=(\lambda-\lambda')^{-1}\mu_0(-f+\lambda' R_{\lambda'} f),$$

hence the eigenspace of λR_λ^* to the eigenvalue one is independent of λ. It is obvious from the definition of R_λ that $T_t^*\mu_0=\mu_0$ for all $t\geqq 0$ implies $\lambda R_\lambda^*\mu_0=\mu_0$. On the other hand, the relation

$$T_t f-f=\int_0^t T_s A f ds \qquad (f\in\mathfrak{D}(A))$$

implies $T_t f-f\in\overline{\mathfrak{R}(A)}$ for arbitrary $f\in\boldsymbol{C}$. Suppose now

$$0=\mu_0(\lambda R_\lambda f-f)=\mu_0(A(\lambda I-A)^{-1}f)$$

for all $f\in\boldsymbol{C}$, that is $\mu_0(g)=0$ for all $g\in\mathfrak{R}(A)$. Then $\mu_0(T_t f-f)=0$ for $f\in\boldsymbol{C}$, $t\geqq 0$, and the statement follows.

2. Invariant measures.

In this section we suppose $\kappa_0=\kappa_1=0$. Then the transition function (or the corresponding Markov process) is conservative, that is we have

*) We thank our colleague Dr. R. Kühne for pointing out the properties 2), 3) to us.
**) $\mathfrak{R}(A) := \{Af : f\in\mathfrak{D}(A)\}$.

$$P(t\,;x,[0,1])=1\,,\qquad t>0,\ x\in[0,1]\,.$$

LEMMA 3. *If the nonnegative measure* μ_0 $(\neq 0)$ *is invariant under* $(T_t^*)_{t\geq 0}$ *and* $\operatorname{supp}\mu_0\not\subset\{0,1\}$ *then* $\operatorname{supp}\mu_0=[0,1]$.

PROOF. If $S_0:=\operatorname{supp}\mu_0$, we have

$$\int_{S_0}P(t\,;x,\Gamma)\mu_0(dx)=\mu_0(\Gamma)\quad\text{for all}\quad \Gamma\in\mathfrak{B}_{[0,1]}\,,\qquad t\geqq 0\,,$$

hence $P(t\,;x,\Gamma)=0$ for μ_0-almost all $x\in S_0$, if $\Gamma\cap S_0=\emptyset$. Consider a continuous function f on $[0,1]$ vanishing on S_0. Then we have with $\Delta_0:=[0,1]\backslash S_0$

$$\int_{\Delta_0}P(T\,;x,dy)f(y)=0$$

for μ_0-almost all $x\in S_0$. But the integral on the left hand side is a continuous function of x, hence it vanishes identically on S_0. This implies $P(t\,;x,\Delta_0)=0$, or $P(t\,;x,S_0)=1$ for all $x\in S_0$, $t\geqq 0$. Therefore for the corresponding canonical Feller process X with P_x-probability one the paths starting in a point $x\in S_0$ always remain in S_0. Hence if $\Delta\subset\Delta_0$, $\Delta\in\mathfrak{B}_{[0,1]}$, $x\in S_0$:

$$\tilde{n}_x(\Delta)=\lim_{U\downarrow x}\frac{P_x(X_\tau\in\Delta)}{E_x\tau}=0\,,$$

where $\tau:=\tau_U$ denotes the first exit time of the neighbourhood U of x.

Assume now $S_0\neq[0,1]$, $S_0\not\subset\{0,1\}$ and consider a boundary point x_0 of Δ_0, $x_0\in(0,1)$. Suppose e.g. that for some $\delta>0$ the interval $(x_0-\delta,x_0)$ belongs to Δ_0 and $x_0+\delta<1$. Then it follows easily that there exists a nonnegative function $f_0\in\mathfrak{D}(D_mD_x)$ with the properties

$$f_0(x)=0\quad\text{if}\quad |x-x_0|\geqq\delta\,,\qquad (D_mD_xf_0)(x)\geqq 0\quad\text{if}\quad x\geqq x_0\,,$$

$$(D_mD_xf_0)(x_0)>|b(x_0)|\,|(D_xf_0)(x_0)|$$
$$+\left|\int_0^{x_0-\delta}\{-f_0(x_0)-(y-x_0)(D_xf_0)(x_0)\}\,\tilde{n}_{x_0}(dy)\right|\,.$$

Hence

$$(\mathfrak{A}f_0)(x_0)\geqq(D_mD_xf_0)(x_0)-|b(x_0)|\,|(D_xf_0)(x_0)|=:\gamma_0>0\,.$$

Moreover, by the discontinuity of the functionals Φ_0, Φ_1 we can choose real functions g_0, $g_1\in\mathfrak{D}(D_mD_x)$ vanishing on $(x_0-\delta,1)$ and $[0,x_0+\delta)$ resp. and with the properties

$$\Phi_0(g_0)=-\Phi_0(f_0)\,,\quad \Phi_1(g_0)=0\,,$$

$$\Phi_0(g_1)=0\,,\quad \Phi_1(g_1)=-\Phi_1(f_0)\,,$$

$$\|g_i\|\leqq f_0(x_0)/2\,,\quad |(Bg_i)(x_0)|\leqq\gamma_0/4\,,\quad i=0,1\,.$$

Then for $f:=f_0+g_0+g_1\in\mathfrak{D}(A)$, we have

$$f(x_0)\geqq f(x) \quad \text{for} \quad x\in S_0\,, \tag{4}$$

$$(Af)(x_0)\geqq\gamma_0-|(Bg_0)(x_0)|-|(Bg_1)(x_0)|\geqq\gamma_0/2>0\,.$$

On the other hand we have from (4)

$$(Af)(x_0)=\lim_{t\downarrow 0} t^{-1}\Bigl(\int_{S_0} P(t\,;x_0,dy)f(y)-f(x_0)\Bigr)\leqq 0\,.$$

LEMMA 4. *Suppose the functionals Φ_i in the boundary conditions* (2) *satisfy the following hypotheses:*

1) $\kappa_0=\kappa_1=0$;
2) $\Phi_i(f)\neq\sigma_i(\mathfrak{A}f)(i),\quad i=0,1$;
3) *for at least one index $i=0$ or 1 the functional Φ_i is not of the form*

$$\Phi_i(f)=\sigma_i(\mathfrak{A}f)(i)+(f(i)-f(j))\delta_i\,,\quad i\neq j\,,\quad j=0,1\,,\quad \sigma_i+\delta_i>0\,.$$

Then $f=\mathbf{1}$ is (*up to scalar multiples*) *the unique solution of the equation $Af=0$.*

PROOF. Condition 1) evidently implies $A\mathbf{1}=0$. By the spectral mapping theorem, if $Af_0=0$ we have $\lambda R_\lambda f_0=f_0$, hence

$$\lambda R_\lambda|f_0|\geqq|f_0| \quad \text{if} \quad \lambda>0\,. \tag{5}$$

Moreover, by a theorem of Mazur [7] there exists a $\mu_0\in\boldsymbol{C}^*$, $\mu_0\neq0$, such that $\lambda R_\lambda^*\mu_0=\mu_0$, and it follows $\lambda R_\lambda^*|\mu_0|\geqq|\mu_0|$. Now $\lambda R_\lambda^*|\mu_0|(\Gamma)>|\mu_0|(\Gamma)$ for some Borel set Γ would imply $\lambda\|R_\lambda^*|\mu_0|\|>\||\mu_0|\|$*), which is impossible because of $\|\lambda R_\lambda^*\|\leqq1$. Therefore $\lambda R_\lambda^*|\mu_0|=|\mu_0|$.

Assume $S_0:=\operatorname{supp}|\mu_0|\subset\{0,1\}$. Then if a path of the Markov process with initial distribution $|\mu_0|/\|\mu_0\|$ starts in $x\in S_0$, is always remains there with P_x-probability one. Hence if S_0 consists of one point i only ($i=0$ or 1), the boundary condition $(\mathfrak{A}f)(i)=0$ must hold, a contradiction to 2). If $S_0=\{0,1\}$, both boundary conditions must be of the form

$$\sigma_i(\mathfrak{A}f)(i)+(f(i)-f(j))\delta_i=0\,,\quad i,j=0,1\,,\quad i\neq j\,,$$

where σ_i, $\delta_i\geqq0$, $\sigma_i+\delta_i>0$, $i=0,1$, which is a contradiction to 3).

From Lemma 3, $\operatorname{supp}|\mu_0|=[0,1]$. Integrating the inequality in (5) with respect to $|\mu_0|$ we get $\lambda R_\lambda|f_0|=|f_0|$, hence

$$|f_0|\in\mathfrak{D}(A)\,. \tag{6}$$

Assume now $(D_mf_0)(x_0)\neq0$ for some $x_0\in(0,1)$. Then (6) applied to the function $f_0-f_0(x_0)\mathbf{1}$ instead of f_0 gives the existence of the derivative of

*) Here $\|\mu\|$ denotes the norm of $\mu\in\boldsymbol{C}^*$, that is the total variation of μ on [0,1]. For the norm of a bounded linear operator in $\boldsymbol{C}^*$ we use the same symbol.

$|f_0-f_0(x_0)\mathbf{1}|$ at x_0, which is impossible. Hence $D_x f_0=0$, that is $f_0=$const.

THEOREM 5. *For the semigroup* $(T_t^*)_{t\geq 0}$ *there exists a unique (up to positive multiples) invariant nonnegative measure* μ_0 *and this measure has the property* $\operatorname{supp}\mu_0=[0, 1]$ *if and only if the conditions* 1)-3) *of Lemma* 4 *are satisfied.*

PROOF. If the conditions 1)-3) are satisfied, the eigenspace of λR_λ $(\lambda>0)$ to the eigenvalue one is one-dimensional. By Lemma 1, the same is true for the eigenspace of λR_λ^*. As for the nonnegative contraction λR_λ^* the equation $\lambda R_\lambda^*\mu=\mu$ implies $\lambda R_\lambda^*|\mu|=|\mu|$, the existence and uniqueness of the invariant measure μ_0 follows. The relation $\operatorname{supp}\mu_0=[0, 1]$ was shown in the proof of Lemma 3.

Suppose now that there exists a unique invariant measure μ_0 which has, moreover, the property $\operatorname{supp}\mu_0=[0, 1]$. Then, if e.g. $\Phi_0(f)=(\mathfrak{A}f)(0)$ $(f\in\mathfrak{D}(\mathfrak{A}))$, the point measure at zero is invariant, which is impossible. If both functionals Φ_i, $i=0, 1$, are of the form

$$\Phi_i(f)=\sigma_i(\mathfrak{A}f)(i)+(f(i)-f(j))\delta_i\,,\quad i\neq j\,,\quad \sigma_i+\delta_i>0\,,\quad i,\ j=0,\ 1\,,$$

there exists an invariant measure concentrated on the boundary, which is also impossible. The proof of the theorem will be completed if it is shown, that in case $\kappa_0+\kappa_1>0$ the support of a nontrivial invariant measure is contained in $\{0, 1\}$. This is a consequence of Corollary 7 in the following section.

3. Subinvariant measures.

LEMMA 6. *If for* $i=0$ *or* 1 *we have* $\kappa_i>0$, *each invariant measure* μ_0 *of* $(T_t^*)_{t\geq 0}$ *has the property* $i\notin\operatorname{supp}\mu_0$.

PROOF. If $\kappa_0+\kappa_1>0$, we consider the boundary conditions given by the functionals

$$\hat{\Phi}_i(f):=\Phi_i(f)-\kappa_i f(i)\,,\quad f\in\mathfrak{D}(\mathfrak{A})\,,\quad i=0,\ 1\,. \tag{7}$$

The hypothesis that $\mathfrak{D}(A)$ is dense in $\boldsymbol{C}$ implies that the operator given by (1) and the boundary conditions $\hat{\Phi}_i(f)=0$, $i=0, 1$, is the infinitesimal generator (denoted by $\hat{A}$) of a s.c.n.c. semigroup in $\boldsymbol{C}$.

Fix $\lambda>0$ and consider the (nonnegative) solutions f_0, f_1 of the equation $\mathfrak{A}f-\lambda f=0$, satisfying the conditions $f_0(0)=1$, $f_0(1)=0$, $f_1(0)=0$, $f_1(1)=1$. In [**4**], p. 247, it was shown that

$$F(x):=1-f_0(x)-f_1(x)>0\qquad(0<x<1)\,. \tag{8}$$

Moreover

$$(D_xF)(0)>0\,,\qquad (D_xF)(1)<0\,. \tag{9}$$

To show e.g. the first relation, assume $(D_xF)(0)=0$. Together with $F(0)=0$ this implies

$$(D_m D_x F)(0) = -\lambda - \int_0^1 F(y)\tilde{n}_0(dy) \leqq -\lambda < 0\,,$$

a contradiction to (8).

From the inequalities (8) and (9) we get

$$\hat{\Phi}_0(f_0) + \hat{\Phi}_0(f_1) = \int_0^1 x^{-1}F(x)dq_0(x) + \sigma_0\lambda > 0$$

and a corresponding relation for $\hat{\Phi}_1$, therefore

$$\hat{\Phi}_0(f_0) > -\hat{\Phi}_0(f_1) \geqq 0\,, \qquad \hat{\Phi}_1(f_1) > -\hat{\Phi}_1(f_0) \geqq 0\,. \tag{10}$$

The resolvents R_λ and $\hat{R}_\lambda$ of A and $\hat{A}$ resp. are connected by the relation

$$R_\lambda f = \hat{R}_\lambda f - c_0(f)f_0 - c_1(f)f_1\,, \qquad f \in \boldsymbol{C}\,, \tag{11}$$

with

$$c_0(f) := \frac{1}{\Delta}\begin{vmatrix} \kappa_0(\hat{R}_\lambda f)(0) & \hat{\Phi}_0(f_1) \\ \kappa_1(\hat{R}_\lambda f)(1) & \hat{\Phi}_1(f) + \kappa_1 \end{vmatrix}\,,$$

$$c_1(f) := \frac{1}{\Delta}\begin{vmatrix} \hat{\Phi}_0(f_0) + \kappa_0 & \kappa_0(\hat{R}_\lambda f)(0) \\ \hat{\Phi}_1(f_0) & \kappa_1(\hat{R}_\lambda f)(1) \end{vmatrix}\,,$$

$$\Delta := \kappa_0\kappa_1 + \kappa_1\hat{\Phi}_0(f_0) + \kappa_0\hat{\Phi}_1(f_1) + \hat{\Phi}_0(f_0)\hat{\Phi}_1(f_1) - \hat{\Phi}_0(f_1)\hat{\Phi}_1(f_0) > 0\,.$$

If $f \geqq 0$, we find from (10) and $\hat{R}_\lambda f \geqq 0$ that $c_0(f) \geqq 0$, $c_1(f) \geqq 0$, hence

$$R_\lambda f \leqq \hat{R}_\lambda f\,. \tag{12}$$

Suppose now e.g. $\kappa_0 > 0$. Then $c_0(\mathbf{1}) \geqq (\lambda\Delta)^{-1}\kappa_0\hat{\Phi}_1(f_1) > 0$ and

$$(R_\lambda \mathbf{1})(0) = (\hat{R}_\lambda \mathbf{1})(0) - c_0(\mathbf{1}) < (\hat{R}_\lambda \mathbf{1})(0) = \lambda^{-1}\,. \tag{13}$$

Assume $0 \in \operatorname{supp} \mu_0$ for the invariant measure μ_0 of $(T_t^*)_{t\geqq 0}$. Then (13) implies

$$\lambda^{-1}\int_0^1 d\mu_0 = \int_0^1 (R_\lambda \mathbf{1})d\mu_0 < \lambda^{-1}\int_0^1 d\mu_0\,,$$

which is impossible.

COROLLARY 7. *If κ_0, $\kappa_1 > 0$, the semigroup $(T_t^*)_{t\geqq 0}$ does not have an invariant measure. If* e.g. *$\kappa_1 > 0$, there exists an invariant measure μ_0 of $(T_t^*)_{t\geqq 0}$ if and only if $\Phi_0(f) = \sigma_0(\mathfrak{A}f)(0)$; in this case μ_0 is the point measure concentrated at* 0.

Indeed, (12) implies

$$R_\lambda^* \leqq \hat{R}_\lambda^*\,. \tag{14}$$

Suppose now μ_0 is an invariant measure of $(T_t^*)_{t\geq 0}$. Then $\mu_0=\lambda R_\lambda^*\mu_0\leqq\lambda\hat{R}_\lambda^*\mu_0$, which implies $\mu_0=\lambda\hat{R}_\lambda^*\mu_0$. By Lemma 3 we have $\operatorname{supp}\mu_0=[0, 1]$ or $\operatorname{supp}\mu_0\subset\{0, 1\}$. If $\kappa_0+\kappa_1>0$, the first case is excluded by Lemma 6. Now the first statement of the corollary follows immediately. If, in particular, $\kappa_1>0$, $\kappa_0=0$ and μ_0 is an invariant measure of $(T_t^*)_{t\geq 0}$, it must be a point measure at 0. Hence 0 is absorbing and $(Af)(0)=0$.

The inequality (12) and Theorem 5 have the following consequence.

THEOREM 8. *Suppose the functionals $\hat{\Phi}_i$ in* (7) *satisfy the following conditions:*

1) $\hat{\Phi}_i(f)\neq\sigma_i(\mathfrak{A}f)(i)$, $i=0, 1$;

2) *for at least one index $i=0$ or 1 the functional $\hat{\Phi}_i$ is not of the form*

$$\hat{\Phi}_i(f)=\sigma_i(\mathfrak{A}f)(i)+(f(i)-f(j))\delta_i\,,\quad j\neq i\,,\quad \sigma_i+\delta_i>0\,.$$

Then there exists a subinvariant measure μ_0 of $(T_t^*)_{t\geq 0}$ with the property

$$\operatorname{supp}\mu_0=[0, 1]\,.$$

Indeed, by Theorem 5, the semigroup $(\hat{T}_t^*)_{t\geq 0}$ corresponding to the operator $\hat{A}$ has an invariant measure μ_0 with $\operatorname{supp}\mu_0=[0, 1]$ and from (14) we get

$$\lambda R_\lambda^*\mu_0\leqq\lambda\hat{R}_\lambda^*\mu_0=\mu_0\,.$$

Now if $f\in\boldsymbol{C}$, $f\geqq 0$, it follows for $t>0$ (see [3]):

$$\begin{aligned}(T_t^*\mu_0)(f)&=\mu_0(T_tf)=\lim_{k\uparrow\infty}k\cdot t^{-1}\mu_0(R_{k/t}^kf)\\&=\lim_{k\uparrow\infty}k\cdot t^{-1}(R_{k/t}^{*k}\mu_0)(f)\leqq\mu_0(f)\,.\end{aligned}$$

4. Absolute continuity of the invariant measure.

In this section we suppose that the conditions of Lemma 4 are satisfied. Then our general hypothesis (3) about the boundary condition implies $Q_i:=\int_0^1 dq_i>0$, and we can assume $Q_i=1$ $(i=0, 1)$. Let m_0 and M denote the following measures on $[0, 1]$:

$$dm_0(x):=\sigma_0 d\delta_0(x)+\sigma_1 d\delta_1(x)+dm(x)\,,$$

$$dM(x):=\sigma_0 d\delta_0(x)+\sigma_1 d\delta_1(x)+\rho(x)dm(x)\,,$$

$$\rho(x):=1-\int_x^1(y-x)y^{-1}dq_0(y)-\int_0^x(x-y)(1-y)^{-1}dq_1(y)\,,$$

where δ_i is the unit measure concentrated at i, $i=0, 1$. The measure M was introduced in [6]. It is the invariant measure of the adjoint of the semigroup generated by D_mD_x with boundary conditions (2) in $\boldsymbol{C}$.

By Γ we denote the kernel

$$\Gamma(x, s) := \begin{cases} \int_{y=0}^{s} (s-y)\tilde{n}_x(dy), & 0 \leqq s < x \leqq 1, \\ \int_{y=s}^{1} (y-s)\tilde{n}_x(dy), & 1 \geqq s > x \geqq 0. \end{cases}$$

Evidently, $\Gamma(x, 0)=\Gamma(x, 1)=0$ $(0<x<1)$, and it is easy to see that for x fixed $\Gamma(x, \cdot)$ is m-summable, and $\int_0^1 \Gamma(x, s)\varphi(s)dm(s)$ is a continuous function of x if $\varphi \in \boldsymbol{C}$.

THEOREM 9. *Suppose the conditions* 1)-3) *of Lemma* 4 *are satisfied and* $Q_i=1$ $(i=0, 1)$. *Then the invariant measure* μ_0 *of* $(T_t^*)_{t\geqq 0}$ *of Theorem* 5 *is absolutely continuous with respect to* m_0 *and its density* $g_0 := d\mu_0/dm_0$ *belongs to* $L^\infty(m_0)$.

PROOF. If $f \in \mathfrak{D}(A)$, $f(x)=f_0+f_0'+\int_0^x (x-s)\varphi(s)dm(s)$, we have

$$(Af)(x)=\varphi(x)+b(x)(f_0'+\int_0^x \varphi(s)dm(s))$$
$$+\int_0^1\int_x^y (y-s)\varphi(s)dm(s)\tilde{n}_x(dy).$$

Integration by parts shows that the relation $\int_0^1 Afd\mu_0=0$ $(f \in \mathfrak{D}(A))$ is equivalent to

$$\int_0^1 \varphi(s)\Big[\int_0^1 \Gamma(x, s)d\mu_0(x)dm(s)+d\mu_0(s)+\int_s^1 bd\mu_0 dm(s)\Big]$$
$$+f_0'\int_0^1 b(x)d\mu_0(x)=0. \qquad (15)$$

The boundary conditions are equivalent to the following relations:

$$\sigma_0\Big(\varphi(0)+b(0)f_0'+\int_0^1\int_0^y (y-s)\varphi(s)dm(s)\tilde{n}_0(dy)\Big)$$
$$-\int_0^1 x^{-1}\int_0^x (x-s)\varphi(s)dm(s)dq_0(x)-f_0'=0,$$

$$\sigma_1\Big(\varphi(1)+b(1)\Big(f_0'+\int_0^1 \varphi dm\Big)+\int_0^1\int_1^y (y-s)\varphi(s)dm(s)\tilde{n}_1(dy)\Big)$$
$$+\int_0^1\int_x^1 (x-s)(1-x)^{-1}\varphi(s)dm(s)dq_1(x)+f_0'+\int_0^1 \varphi(s)dm(s)=0, \qquad (16)$$

which can be written as

$$f_0'(\sigma_0 b(0)-1)+\int_0^1 \varphi(s)d\nu_0(s)=0\,,$$

$$f_0'(\sigma_1 b(1)+1)+\int_0^1 \varphi(s)d\nu_1(s)=0\,.$$

Here ν_0, ν_1 are measures on [0, 1] which can easily be calculated from (16). They are absolutely continuous with respect to m_0.

Suppose first $\sigma_0 b(0)-1\neq 0$. Then

$$f_0'=-(\sigma_0 b(0)-1)^{-1}\int_0^1 \varphi(s)d\nu_0(s)\,,$$

and (15) gives

$$\begin{aligned}\int_0^1 \varphi(s)\Big[\int_0^1 \Gamma(x,\ s)d\mu_0(x)dm(s)+d\mu_0(s)+\int_s^1 bd\mu_0 dm(s)\Big]\\ -(\sigma_0 b(0)-1)^{-1}\int_0^1 \varphi(s)d\nu_0(s)\int_0^1 bd\mu_0=0\end{aligned} \tag{17}$$

for all functions $\varphi\in \boldsymbol{C}$ with the property

$$(\sigma_1 b(1)+1)\int_0^1 \varphi d\nu_0-(\sigma_0 b(0)-1)\int_0^1 \varphi d\nu_1=0\,.$$

Hence, with a suitable choice of μ_0, we have

$$\begin{aligned}\int_0^1 \Gamma(x,\ s)d\mu_0(x)dm(s)+d\mu_0(s)+\int_s^1 bd\mu_0 dm(s)-(\sigma_0 b(0)-1)^{-1}\int_0^1 bd\mu_0 d\nu_0(s)\\ =(\sigma_1 b(1)+1)d\nu_0(s)-(\sigma_0 b(0)-1)d\nu_1(s)\,,\end{aligned} \tag{18}$$

and the statement follows.

If $\sigma_1 b(1)+1=0$ and $\sigma_0 b(0)-1=0$, then μ_0 satisfies the equation

$$\int_0^1 \Gamma(x,\ s)d\mu_0(x)dm(s)+d\mu_0(s)+\int_s^1 bd\mu_0 dm(s)=c_0 d\nu_0(s)+c_1 d\nu_1(s) \tag{19}$$

with some constants c_0, c_1 and the condition $\int_0^1 b\, d\mu_0=0$. Evidently, (19) implies the absolute continuity of μ_0 with respect to m_0.

By g_0 $(\in L^1(m_0))$ we denote the density of μ_0 with respect to $m_0: d\mu_0(x)=g_0(x)dm_0(x)$. The relations (18) or (19) imply an integral equation for g_0. For simplicity we shall give it only in the case $\sigma_0=\sigma_1=0$. Then the boundary conditions (16) simplify to $\int_0^1 \varphi(s)\rho(s)dm(s)=0$, and (18) becomes

$$\int_0^1 \Gamma(x,\ s)g_0(x)dm(x)+g_0(s)$$

$$=-\int_s^1 b(x)g_0(x)dm(x)+\int_0^1 bg_0dm\cdot\int_s^1(x-s)x^{-1}dq_0(x)+\rho(s),$$

a.e. with respect to m_0. Both terms on the left hand side are nonnegative and the right hand side is continuous, hence g_0 is in $L^\infty(m_0)$.

5. A relation between $\mathfrak{A}$, invariant measures and boundary conditions.

In the following we need some more properties of the operator $\mathfrak{A}$ in (1).

LEMMA 10. *The boundary problem* $\mathfrak{A}f=1$, $f(0)=f(1)=0$, *has a solution* $f\in\mathfrak{D}(\mathfrak{A})$.

PROOF. The lemma will be proved if we show that the restriction A_0 of $\mathfrak{A}$ by the boundary conditions $f(0)=f(1)=0$, defined in $\boldsymbol{C}_0:=\{f\in\boldsymbol{C}:f(0)=f(1)=0\}$ does not have the eigenvalue zero. In this case the resolvent $R_\lambda^{(0)}$ of A_0 exists at $\lambda=0$, it can be extended to all of $\boldsymbol{C}$ and $f:=R_0^{(0)}\mathbf{1}$ is the function with the stated properties.

In order to calculate $R_\lambda^{(0)}$ we consider the restriction A of $\mathfrak{A}$ by the boundary conditions

$$f'(0)-\kappa_0 f(0)=0\,,\qquad f'(1)+\kappa_1 f(1)=0\,.$$

Then the corresponding operator $\hat{A}$ is defined by the conditions $f'(0)=f'(1)=0$, and from (11) letting $\kappa_0,\ \kappa_1\to\infty$ we get for fixed $\lambda>0$ with $f_0,\ f_1$ defined in section 3:

$$R_\lambda^{(0)}f=\hat{R}_\lambda f-(\hat{R}_\lambda f)(0)f_0-(\hat{R}_\lambda f)(1)f_1\,. \tag{20}$$

Denote by $\hat{\mu}_0$ the invariant measure of the semigroup $(\hat{T}_t^*)_{t\geq 0}$. From Theorem 5 it follows $\operatorname{supp}\hat{\mu}_0=[0,\ 1]$ and Lemma 2 implies

$$(\hat{R}_\lambda g,\ \hat{\mu}_0)=\lambda^{-1}(g,\ \hat{\mu}_0)\,. \tag{21}$$

If $A_0v=0$, we have $R_\lambda^{(0)}v=\lambda^{-1}v$ and v does not change sign. Now from (20) and (21) it follows

$$(\hat{R}_\lambda v)(0)(f_0,\ \hat{\mu}_0)+(\hat{R}_\lambda v)(1)(f_1,\ \hat{\mu}_0)=0\,,$$

which is equivalent to

$$(\hat{R}_\lambda v)(0)=(\hat{R}_\lambda v)(1)=0\,.$$

Hence (20) implies $\lambda^{-1}v=R_\lambda^{(0)}v=\hat{R}_\lambda v$, that is $\hat{A}v=0$. Using Lemma 4 we find $v=c\mathbf{1}$, and from $v(0)=0$ we get finally $c=0$, $v=0$.

The function f in Lemma 10 is $-E_x\tau$, where τ denotes the first exit time of (0, 1) for the canonical Feller process corresponding to $\mathfrak{A}$ and boundary conditions (2).

Denote in this section by A always a restriction of $\mathfrak{A}$ by boundary conditions (2) satisfying the conditions 1)-3) of Lemma 4. Then $f=\mathbf{1}$ is the (unique) solution of the equation $Af=0$, hence by the relation $\|R_\lambda\|\leqq\lambda^{-1}$, $\lambda>0$, the function $\mathbf{1}$ cannot belong to $\mathfrak{R}(A)$, and Lemma 10 implies

$$\mathfrak{R}(A)\neq\mathfrak{R}(\mathfrak{A}). \tag{22}$$

For the quotient space $\mathfrak{D}(\mathfrak{A})/\mathfrak{D}(A)$ we have

$$\dim(\mathfrak{D}(\mathfrak{A})/\mathfrak{D}(A))=2 \tag{23}$$

(see e. g. [4], proof of Theorem 4). Moreover, as $\dim(\mathfrak{R}(\mathfrak{A})/\mathfrak{R}(A))\leqq\dim(\boldsymbol{C}/\mathfrak{R}(A))=1$, relation (22) implies $\dim(\mathfrak{R}(\mathfrak{A})/\mathfrak{R}(A))=1$. As a consequence we have the following result.

LEMMA 11. *Under the conditions of Lemma 4 there exists a solution* $h_0\in\mathfrak{D}(\mathfrak{A})\backslash\mathfrak{D}(A)$ *of the equation* $\mathfrak{A}h=0$. *Every solution* h *of this equation is of the form* $h=c_0h_0+c_1\mathbf{1}$ *with some constants* c_0, c_1.

In case $b=0$ we have evidently (up to scalar multiples) $h_0(x)=x$.

The equation $\mathfrak{A}h=0$ is equivalent to the integral equation

$$\varphi(x)+b(x)\int_0^x\varphi dm+\int_0^1\int_x^y(y-s)\varphi(s)dm(s)\tilde{n}_x(dy)=-b(x)h'(0), \tag{24}$$

where $h(x)=h(0)+xh'(0)+\int_0^x(x-s)\varphi(s)dm(s)$. The left hand side of (24) is of the form $(I+G)\varphi$ with some compact operator G in $\boldsymbol{C}$ (see [4], p. 247).

LEMMA 12. *The homogeneous integral equation* $(I+G)\varphi=0$ *corresponding to* (24) *has a nontrivial solution* $\varphi\neq0$ *if and only if* $h_0'(0)=0$, *where* h_0 *denotes the solution given in Lemma* 11.

PROOF. If $h_0'(0)=0$ we have $h_0(x)=h_0(0)+\int_0^x(x-s)\varphi_0(s)dm(s)$ and the function $\varphi_0\neq0$ is a solution of $(I+G)\varphi=0$. On the other hand, if $h_0'(0)\neq0$, Lemma 11 implies that there is exactly one function φ_0 satisfying $(I+G)\varphi_0=-bh_0'(0)$, that is the homogeneous equation $(I+G)\varphi=0$ has only the obvious solution $\varphi=0$.

The function h_0 can always be chosen such that $h_0(0)=0$. Then the condition of Lemma 12 holds if and only if the initial problem

$$\mathfrak{A}h=0\,,\quad h(0)=h'(0)=0$$

has a nontrivial solution. We do not know if this can really happen. It is impossible if one of the following conditions is satisfied:

1) $b(x)=0\quad(x\in[0,1])$;

2) $\sup_x |b(x)|(m(1)-m(0))+\sup_x n_x([0, 1])<1$;

3) $\operatorname{supp} n_x \supset [x, 1] \quad (x\in[0, 1])$.

Indeed, in the first case we can choose $h_0(x)\equiv x$. If condition 2) is satisfied, the homogeneous equation

$$\varphi(x)+b(x)\int_0^x \varphi dm+\int_0^1\int_x^y (y-s)\varphi(s)dm(s)\tilde{n}_x(dy)=0$$

can only have the obvious solution $\varphi=0$. If the third condition holds the statement follows as in [4], Lemma 3.

The function h_0 in Lemma 11 has evidently the property $|\Phi_0(h_0)|^2+|\Phi_1(h_0)|^2\neq 0$. We choose $h_1\in\mathfrak{D}(\mathfrak{A})$ such that

$$\mathfrak{D}(\mathfrak{A})=\text{l. s.}\{\mathfrak{D}(A),\ h_0,\ h_1\}. \tag{25}$$

Then

$$\Delta := \begin{vmatrix} \Phi_0(h_0) & \Phi_0(h_1) \\ \Phi_1(h_0) & \Phi_1(h_1) \end{vmatrix} \neq 0,$$

otherwise with some complex number γ we would have $\gamma h_0-h_1\in\mathfrak{D}(A)$, which is impossible. If μ_0 is the measure given by Theorem 5, then

$$\int_0^1 \mathfrak{A}h_1 d\mu_0\neq 0.$$

Indeed, otherwise $h_1\in\mathfrak{D}(A)$ or $\mathfrak{A}h_1=0$. But the first relation is impossible by (25) and (23), the second relation is impossible by (25) and Lemma 11.

THEOREM 13. *For arbitrary $f\in\mathfrak{D}(\mathfrak{A})$ we have*

$$\int_0^1 \mathfrak{A}f d\mu_0=\{-\Phi_0(f)\Phi_1(h_0)+\Phi_1(f)\Phi_0(h_0)\}\,\Delta^{-1}\int_0^1 \mathfrak{A}h_1 d\mu_0. \tag{26}$$

Indeed,

$$\hat{f} := f-\Delta^{-1}\{-\Phi_0(f)\Phi_1(h_0)+\Phi_1(f)\Phi_0(h_0)\}\,h_1$$
$$-\Delta^{-1}\{\Phi_0(f)\Phi_1(h_1)-\Phi_1(f)\Phi_0(h_1)\}\,h_0\in\mathfrak{D}(A),$$

and $\int_0^1 \mathfrak{A}\hat{f}d\mu_0=0$ is evidently equivalent to (26).

Choose now h_1 as the solution of the initial problem $\mathfrak{A}h_1=-1$, $h_1(0)=h_1(1)=0$. Then the maximum principle implies $h_1\geqq 0$, and we have

$$\Phi_0(h_1)\leqq 0, \qquad \Phi_1(h_1)\leqq 0.$$

With a solution $h_0 : \mathfrak{A}h_0=0$, $h_0\notin\mathfrak{D}(A)$, we normalize the functionals Φ_i by the conditions

$$\Phi_0(h_0) = -1 \text{ or } 0, \qquad \Phi_1(h_0) = 1 \text{ or } 0. \tag{27}$$

This implies $\Delta > 0$. The invariant measure $\mu_0 > 0$ can be chosen such that $\Delta^{-1}\int \mathfrak{A} h_1 d\mu_0 = -1$, and (26) simplifies to

$$\int_0^1 \mathfrak{A} f \, d\mu_0 = \Phi_0(f)\gamma_0 + \Phi_1(f)\gamma_1, \tag{28}$$

where $\gamma_0 = \Phi_1(h_0)$, $\gamma_1 = -\Phi_0(h_0)$.

Suppose now $\gamma_0 = \gamma_1 = 1$. Then we have

$$\mu_0(\{i\}) = \sigma_i, \qquad i = 0, 1. \tag{29}$$

Indeed, (28) implies

$$\begin{aligned} \mu_0(\{0\})(\mathfrak{A}f)(0) + \mu_0(\{1\})(\mathfrak{A}f)(1) + \int_{0+}^{1-} \mathfrak{A} f \, d\mu_0 = \sigma_0(\mathfrak{A}f)(0) + \sigma_1(\mathfrak{A}f)(1) \\ + \int_0^1 (f(0) - f(s)) s^{-1} dq_0(s) + \int_0^1 (f(1) - f(s))(1-s)^{-1} dq_1(s). \end{aligned} \tag{30}$$

Choose a sequence $(\varphi_n) \subset C$, $\varphi_n(0) = 1$, $\varphi_n(x) \geqq 0$, $\varphi_n(x) \downarrow 0$ $(n \to \infty,\ 0 < x \leqq 1)$. Putting $f(x) = f_n(x) = \int_0^x (x-s)\varphi_n(s) dm(s)$ in (30) and letting $n \to \infty$ we get $\mu_0(\{0\}) = \sigma_0$.

6. Quadratic forms connected with $\mathfrak{A}$.

In the following we have to impose two more conditions:

(d) $b(x) = 0$ $(x \in [0, 1])$.

(e) The Lebesgue measure is absolutely continuous with respect to m and the corresponding density $\nu := dx/dm$ is a continuous function.

The first condition is mainly for technical reason. It implies that we can choose e.g. $h_0(x) = x$, and the normalization (27) of the functionals Φ_i amounts to

$$\int_0^1 dq_i = 1, \qquad i = 0, 1$$

(here we suppose again $\kappa_0 = \kappa_1 = 0$). Condition (e) implies e.g.

$$|f|^2 \in \mathfrak{D}(D_m D_x) \quad \text{if} \quad f \in \mathfrak{D}(D_m D_x).$$

We now suppose that the functionals Φ_i are such that the corresponding $\hat{\Phi}_i$, $i = 0, 1$, satisfy the conditions 2) and 3) of Lemma 4. By μ_0 we denote the invariant measure of the semigroup $(\hat{T}_t^*)_{t \geq 0}$ (see Theorem 5), normalized according to the foregoing section (that is there we have to put $\hat{\Phi}_i$ instead of Φ_i, $i=0, 1$). By Theorem 8, μ_0 is a subinvariant measure of the semigroup $(T_t^*)_{t\geq 0}$, and from (28) we have

$$\int_0^1 \mathfrak{A} f\, d\mu_0 = \hat{\Phi}_0(f) + \hat{\Phi}_1(f) \qquad (f \in \mathfrak{D}(D_m D_x)). \tag{31}$$

If f, $g \in \boldsymbol{C}$ we put $[f, g] := \int_0^1 f(x)\overline{g(x)}\, d\mu_0(x)$ and shall calculate $\operatorname{Re}[Af, f]$ and $\operatorname{Im}[Af, f]$ $(f \in \mathfrak{D}(A))$.

To do this we consider for arbitrary $f \in \mathfrak{D}(D_m D_x)$ the function g: $g(x) = \int_0^x f'(s)\overline{f(s)}\, ds$. Condition (b) implies $g \in \mathfrak{D}(D_m D_x)$ and we get

$$(\mathfrak{A} g)(x) = (D_m D_x f)(x)\overline{f(x)} + |f'(x)|^2 \nu(x)$$
$$+ \int_0^1 \left[\int_x^y f'(s)\overline{f(s)}\, ds - (y-x) f'(x)\overline{f(x)}\right] \tilde{n}_x(dy).$$

From (31), $\int_0^1 \mathfrak{A} g\, d\mu_0 = \hat{\Phi}_0(g) + \hat{\Phi}_1(g)$, which is equivalent to

$$[\mathfrak{A} f, f] = -\int_0^1 |f'(x)|^2 \nu(x)\, d\mu_0(x) + \hat{\Phi}_0(g) + \hat{\Phi}_1(g)$$
$$-\int_0^1\int_0^1 \left[\int_x^y f'(s)\overline{f(s)}\, ds - (f(y)-f(x))\overline{f(x)}\right] \tilde{n}_x(dy)\, d\mu_0(x). \tag{32}$$

Suppose now $f \in \mathfrak{D}(A)$, that is f satisfies also the boundary conditions (2). Then

$$\hat{\Phi}_0(g) = -\int_0^1 \left[|f(s)-f(0)|^2 - \int_0^s (f(t)-f(0))\overline{f'(t)}\, dt\right] s^{-1} dq_0(s)$$
$$-\overline{f(0)}[\sigma_0(\mathfrak{A} f)(0) + \kappa_0 f(0)] + \sigma_0(\mathfrak{A} g)(0)$$

and a similar expression for $\hat{\Phi}_1(g)$. Using Theorem 9 and

$$-\overline{f(0)}(\mathfrak{A} f)(0) + (\mathfrak{A} g)(0)$$
$$= |f'(0)|^2 \nu(0) + \int_0^1 \left[\int_0^y f'(s)\overline{f(s)}\, ds - \overline{f(0)}(f(y)-f(0))\right] \tilde{n}_0(dy),$$

it follows from (32)

$$[Af, f] = -\int_0^1 |f(x)|^2 g_0(x)\, dx$$
$$-\int_{0+}^{1-}\int_0^1 \left[\int_x^y f'(s)\overline{f(s)}\, ds - \overline{f(x)}(f(y)-f(x))\right] \tilde{n}_x(dy)\, d\mu_0(x)$$

$$-\int_0^1\left[|f(s)-f(0)|^2-\int_0^s(f(t)-f(0))\overline{f'(t)}dt\right]s^{-1}dq_0(s)-\kappa_0|f(0)|^2$$

$$-\int_0^1\left[|f(s)-f(1)|^2-\int_s^1(f(1)-f(t))\overline{f'(t)}dt\right](1-s)^{-1}dq_1(s)-\kappa_1|f(1)|^2.$$

With the relations

$$\mathrm{Re}\int_i^s(f(t)-f(i))\overline{f'(t)}dt=|f(i)-f(s)|^2/2, \qquad i=0,1,$$

$$\mathrm{Re}\left[\int_x^y f'(s)\overline{f(s)}ds-\overline{f(x)}(f(y)-f(x))\right]=|f(y)-f(x)|^2/2,$$

$$\mathrm{Im}\left[\int_x^y f'(s)\overline{f(s)}ds-\overline{f(x)}f(y)\right]=\mathrm{Im}\int_x^y\overline{f'(s)}\int_s^y f'(t)dt\cdot ds$$

we get finally

$$\mathrm{Re}[Af,f]$$

$$=-\int_0^1|f'(x)|^2g_0(x)dx-\frac{1}{2}\int_{0+}^{1-}\int_0^1|f(y)-f(x)|^2\tilde{n}_x(dy)d\mu_0(x)$$

$$-\frac{1}{2}\int_0^1|f(s)-f(0)|^2s^{-1}dq_0(s)-\frac{1}{2}\int_0^1|f(s)-f(1)|^2(1-s)^{-1}dq_1(s)$$

$$-\kappa_0|f(0)|^2-\kappa_1|f(1)|^2,$$

$$\mathrm{Im}[Af,f]$$

$$=\mathrm{Im}\int_{0+}^{1-}\int_0^1\int_x^y\overline{f'(s)}\int_s^y f'(t)dtds\,\tilde{n}_x(dy)d\mu_0(x)$$

$$+\mathrm{Im}\left[\int_0^1\int_0^s\int_0^t f'(u)du\overline{f'(t)}dts^{-1}dq_0(s)\right.$$

$$\left.+\int_0^1\int_s^1\int_t^1 f'(u)du\overline{f'(t)}dt(1-s)^{-1}dq_1(s)\right].$$

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H. LANGER
Sektion Mathematik
Technische Universität Dresden
8027 Dresden
DDR

W. SCHENK
Sektion Mathematik
Technische Universität Dresden
8027 Dreseden
DDR