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Homomorphisms of measure algebras on the unit circle

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1. Introduction.

Let $\boldsymbol{T}$ be the unit circle. Let $L(\boldsymbol{T})$ be the Lebesgue space and $M(\boldsymbol{T})$ the set of all bounded regular Borel measures on $\boldsymbol{T}$. $M(\boldsymbol{T})$ is a commutative Banach algebra with the convolution product and the norm of total variation, and contains $L(\boldsymbol{T})$ as a closed ideal. The object of this paper is to investigate the homomorphisms of $M(\boldsymbol{T})$ which are different from the type given by W. Rudin [6].

W. Rudin characterized the homomorphisms of $L(\boldsymbol{T})$ into $M(\boldsymbol{T})$ in the following way. Let Ψ be a homomorphism of $L(\boldsymbol{T})$ into $M(\boldsymbol{T})$. Then for every integer n the mapping $f \rightarrow (\Psi f)\hat{\ }(n)$ defines a multiplicative linear functional on $L(\boldsymbol{T})$, where $\hat{\ }$ denotes the Fourier-Stieltjes transform. Thus there exists a mapping ψ of $\boldsymbol{Z}$ into $\boldsymbol{Z}\cup\{\infty\}$ such that $(\Psi f)\hat{\ }(n)=\hat{f}(\psi(n))$, $n\in\boldsymbol{Z}$, where $\boldsymbol{Z}$ is the set of integers and ∞ means the trivial functional, that is, $\hat{f}(\infty)=0$ for all f in $L(\boldsymbol{T})$.

THEOREM A (W. Rudin [6], cf. [7; p. 95]). *Let ψ be a mapping of $\boldsymbol{Z}$ into $\boldsymbol{Z}\cup\{\infty\}$. The mapping ψ induces a homomorphism Ψ of $L(\boldsymbol{T})$ into $M(\boldsymbol{T})$ satisfying $(\Psi f)\hat{\ }=\hat{f}\circ\psi$ if and only if*

(i) *the set $P=\{n; \psi(n)\neq\infty\}$ belongs to the smallest ring of subsets of $\boldsymbol{Z}$ containing all cosets in $\boldsymbol{Z}$;*

(ii) *there exists a mapping ϕ of $\boldsymbol{Z}$ into $\boldsymbol{Z}$ and $q\in\boldsymbol{Z}$ such that $\psi(n)=\phi(n)$ for $n\in P$ except possibly a finite number of n's and*

$$\phi(n+q)+\phi(n-q)=2\phi(n) \qquad \text{for all} \quad n\in\boldsymbol{Z}.$$

This theorem is extended by P. J. Cohen [3] to the homomorphisms of $L(G_1)$ into $M(G_2)$, where G_1 and G_2 are locally compact abelian groups. On the other hand J. Inoue [5] proved that P. J. Cohen's characterization holds good if we replace $L(G_1)$ by the smallest closed subalgebra of $M(G_1)$ containing all $L(G_1^{\tau})$, where G_1^{τ} denotes the group G_1 with a locally compact topological group topology τ stronger than the original one of G_1 or equal to that of G_1.

Let Ψ be a homomorphism of $L(\boldsymbol{T})$ into $M(\boldsymbol{T})$. Then it is extended to a homomorphism of $M(\boldsymbol{T})$ into $M(\boldsymbol{T})$. In fact let ϕ be the mapping of $\boldsymbol{Z}$ into $\boldsymbol{Z}\cup\{\infty\}$ such that $(\Psi f)\hat{}=\hat{f}\circ\phi$ for f in $L(\boldsymbol{T})$ and define the mapping $\tilde{\Psi}$ of $M(\boldsymbol{T})$ into $M(\boldsymbol{T})$ by $(\tilde{\Psi}\mu)\hat{}(n)=\hat{\mu}(\phi(n))$ if $\phi(n)\neq\infty$ and $=0$ otherwise. Then it is a homomorphism of $M(\boldsymbol{T})$ into $M(\boldsymbol{T})$ and $\tilde{\Psi}=\Psi$ on $L(\boldsymbol{T})$ (see [6]). But the extension of the homomorphism Ψ is not unique. We shall show in § 2 that there exists a non trivial homomorphism of $M(\boldsymbol{T})$ into $M(\boldsymbol{T})$ which vanishes on $L(\boldsymbol{T})$ (cf. [4] and [7]).

In this paper we shall obtain a sufficient condition for a mapping Ψ of $M(\boldsymbol{T})$ into $M(\boldsymbol{T})$ to be a homomorphism. It coincides with the Rudin's conditions (i) and (ii) of Theorem A when we restrict the domain of mappings to $L(\boldsymbol{T})$. We shall also prove that our condition on Ψ in Theorem 2 is necessary in a sense when it is applied to a certain class of L-subalgebras of $M(\boldsymbol{T})$, which consist of singular measures (see Theorem 3). Our theorems enable us to treat homomorphisms of a subalgebra of $M(\boldsymbol{T})$ into $M(\boldsymbol{T})$ which is essentially different from the algebra considered by J. Inoue [5] (see Remark in § 3).

2. A sufficient condition.

DEFINITION. A subset N of $M(\boldsymbol{T})$ is called an L-subalgebra if it has the following properties:

(i) N is a closed subspace.

(ii) $\mu*\nu\in N$ for every μ and ν in N, where $*$ denotes the convolution of μ and ν.

(iii) $\mu\in N$ and $\nu\ll\mu$, that is, ν is absolutely continuous with respect to μ, imply $\nu\in N$.

We use the following representation of the maximal ideal space of an L-subalgebra.

DEFINITION. Let N be an L-subalgebra of $M(\boldsymbol{T})$. A system $\phi=\{\phi_\mu\,;\,\mu\in N\}$ of functions is called a generalized character if

(i) $\phi_\mu\in L^\infty(d|\mu|)$ and $\sup_\mu \mu\text{-ess}\sup_t|\phi_\mu(t)|>0$;

(ii) $\phi_\mu=\phi_\nu$ ν-a.e. if $\nu\ll\mu$;

(iii) $\phi_{\mu*\nu}(s+t)=\phi_\mu(s)\phi_\nu(t)$ for $\mu\times\nu$-a.a. (s,t).

Let $\Delta(N)$ be the set of non-trivial multiplicative linear functional on N. Then the set of generalized characters is identified with $\Delta(N)$ by the bijection θ;

$$(\theta\phi)(\nu)=\int_{\boldsymbol{T}}\phi_\nu(t)d\nu(t)\,,\quad \phi=\{\phi_\mu\},\quad \nu\in N.$$

Thus we may use the notation $\Delta(N)$ for the set of generalized characters and denote $(\theta\psi)(\nu)=\hat{\nu}(\psi)$ without confusion.

For $\phi=\{\phi_\mu\}$ and $\psi=\{\psi_\mu\}$ in $\Delta(N)$ we define systems $\phi\psi$, $\bar{\phi}$ and $|\phi|$ by $(\phi\psi)_\mu=\phi_\mu\psi_\mu$, $(\bar{\phi})_\mu=\bar{\phi}_\mu$ and $|\phi|_\mu=|\phi_\mu|$, where these operations are defined pointwise in $L^\infty(d|\mu|)$ for each $\mu\in N$. These operations yield new elements of $\Delta(N)$. We denote the trivial linear functional by 0 (cf. Yu. A. Šreider [8]).

When $N=L(\boldsymbol{T})$, the maximal ideal space of $L(\boldsymbol{T})$ is identified with $\boldsymbol{Z}$ and embedded in $\Delta(M(\boldsymbol{T}))$. We remark that if $\psi\in\Delta(N)-\boldsymbol{Z}$, then $\hat{f}(\psi)=0$ for all $f\in L(\boldsymbol{T})$ (cf. J. L. Taylor [**9**]).

DEFINITION. Let N be an L-subalgebra of $M(\boldsymbol{T})$. A mapping $\psi(\cdot)$ of $\boldsymbol{Z}$ into $\Delta(N)\cup\{0\}$ is said to satisfy the condition (C), $C>0$, if

$$\lambda_\nu(t, \theta)=\sum_{n=-\infty}^{\infty}\psi(n)_\nu(t)e^{in\theta}$$

is a Fourier-Stieltjes series in θ for ν-a. a. t and

$$\nu\text{-ess}\sup_t\|\lambda_\nu(t, \cdot)\|_{M(\boldsymbol{T})}\leqq C \qquad \text{for all} \quad \nu\in N.$$

THEOREM 1. *Let N be an L-subalgebra of $M(\boldsymbol{T})$. Then a mapping Ψ of N into $M(\boldsymbol{T})$ is a homomorphism if and only if there exists a mapping $\psi(\cdot)$ of $\boldsymbol{Z}$ into $\Delta(N)\cup\{0\}$ and $C>0$ such that*

(i) $(\Psi\nu)\hat{}(n)=\hat{\nu}(\psi(n))$ *for every* $n\in\boldsymbol{Z}$;

(ii) $\{\psi(n)\}$ *satisfies the condition* (C).

PROOF. Let Ψ be a homomorphism of N into $M(\boldsymbol{T})$. Then for every n in $\boldsymbol{Z}$ the mapping $\nu\rightarrow(\Psi\nu)\hat{}(n)$ defines a multiplicative linear functional. Thus there exists $\psi(n)\in\Delta(N)\cup\{0\}$ such that $(\Psi\nu)\hat{}(n)=\hat{\nu}(\psi(n))$. Let $p(\theta)=\sum a_n e^{in\theta}$ be a polynomial. Then

$$\left|\int_{\boldsymbol{T}}\sum a_n\psi(n)_\nu(t)f(t)d\nu(t)\right|=\left|\sum a_n\Psi(fd\nu)\hat{}(n)\right|$$
$$\leqq\|p\|_\infty\|\Psi(fd\nu)\|_{M(\boldsymbol{T})}\leqq\|\Psi\|\|p\|_\infty\|\nu\|_{M(\boldsymbol{T})}$$

for every $f\in L(d|\nu|)$ such that $\int|f|d|\nu|=1$. Thus taking the supremum over f, we have

$$\nu\text{-ess}\sup_t\left|\sum a_n\psi(n)_\nu(t)\right|\leqq\|\Psi\|\|p\|_\infty$$

for every polynomial p. Thus for ν-a. a. t $\sum\psi(n)_\nu(t)e^{in\theta}$ is a Fourier-Stieltjes series of a measure with norm$\leqq\|\Psi\|$ (cf. [7; p. 32]). Thus $\psi(\cdot)$ satisfies the condition $(\|\Psi\|)$.

From the above argument the if part of the theorem is obvious.

DEFINITION. Let N be an L-subalgebra of $M(\boldsymbol{T})$ and $\psi(\cdot)$ be a mapping of $\boldsymbol{Z}$ into $\Delta(N)\cup\{0\}$. Suppose that there exist

(i) positive integers l and m, and a set $R=\{n_{m+1}, n_{m+2}, \cdots, n_l\}$ of $l-m$ integers;

(ii) $\phi_j \in \Delta(N)\cup\{0\}$ $(j=1, 2, \cdots, l)$;

(iii) $\pi_j \in \Delta(N)\cup\{0\}$ $(j=1, 2, \cdots, m)$ such that $|\pi_j|^2=|\pi_j|$;

(iv) mappings $\rho_j(\cdot)$ of $\boldsymbol{Z}$ into $\Delta(N)\cup\{0\}$ $(j=1, 2, \cdots, m)$ and a positive constant $C>0$ such that $\rho_j(\cdot)$ satisfies the condition (C) for each j and $\rho_j(n)=|\rho_j(n)|$ for $j=1, 2, \cdots, m$ and $n\in\boldsymbol{Z}$; and that ψ has the following expression

$$\psi(n)_\nu(t)=\sum_{j=1}^{m}\pi_{j\nu}(t)^k\phi_{j\nu}(t)\rho_j(n)_\nu(t)C_{m\boldsymbol{Z}+j}(n) \quad (\nu\in N)$$

for $n\notin R$ and $\psi(n)=\phi_j$ for $n=n_j\in R$, where $k=[n/m]$ denotes the integral part of n/m and C_E the characteristic function of the set E.

Then we call ψ an *almost piecewise affine* mapping from $\boldsymbol{Z}$ into $\Delta(N)\cup\{0\}$ or simply an almost piecewise affine mapping. Furthermore, if $\rho_j(n)=\{1\}$, the constant systems, we call ψ a *picewise affine* mapping from $\boldsymbol{Z}$ into $\Delta(N)\cup\{0\}$ or simply a piecewise affine mapping.

We remark that the definition of the piecewise affine mappings given here is essentially same to the Rudin's one in [7] when $N=L(\boldsymbol{T})$ and the conditions (i) and (ii) on ψ in Theorem A imply that ψ is a piecewise affine mapping from $\boldsymbol{Z}$ into $\Delta(L(\boldsymbol{T}))\cup\{0\}$.

THEOREM 2. *Let N be an L-subalgebra of $M(\boldsymbol{T})$. If a mapping $\psi(\cdot)$ of $\boldsymbol{Z}$ into $\Delta(N)\cup\{0\}$ is almost piecewise affine, then the mapping Ψ defined by*

$$(\Psi\nu)^\wedge(n)=\hat{\nu}(\psi(n)) \qquad (n\in\boldsymbol{Z})$$

is a homomorphism of N into $M(\boldsymbol{T})$.

REMARK. If a mapping $\psi(\cdot)$ of $\boldsymbol{Z}$ into $\Delta(N)\cup\{0\}$ satisfies

$$\nu\text{-}\operatorname*{ess\,sup}_t\{\sum_{n=-\infty}^{\infty}|\psi(n)_\nu(t)|^2\}^{1/2}\leqq C \qquad \text{for all} \quad \nu\in N,$$

then the series $\sum\psi(n)_\nu(t)e^{in\theta}$ is a Fourier series with norm $\leqq C$ for every $\nu\in N$ and ν-a.a. t by the Riesz-Fischer theorem. Thus it satisfies the condition (C). Therefore our theorem may not be relevant in this case.

PROOF. Assume that $\psi(\cdot)$ is an almost piecewise affine mapping and use the notations in Definition. By Theorem 1 it suffices to prove that $\psi(\cdot)$ satisfies the condition (C') for some positive constant C'. We many assume that the set R is empty, since a change of finite number of $\psi(n)$'s does not affect our conclusion.

$\{C_{m\boldsymbol{Z}+j}(n): n\in\boldsymbol{Z}\}$ and $\{\pi_{j\nu}(t)^k: k\in\boldsymbol{Z}\}(j=1, 2, \cdots, m)$ are the sequences of Fourier-Stieltjes coefficients of measures with norms $\leqq 1$ for all $\nu\in N$ and ν-a.a. t. Thus by a simple computation, $\{\pi_{j\nu}(t)^{[n/m]}C_{m\boldsymbol{Z}+j}(n): n\in\boldsymbol{Z}\}$ is the sequence of Fourier-Stieltjes coefficients of a measure with norm $\leqq 1$. Thus $\psi(\cdot)$ satisfies the condition (C') with $C'=mC$.

There exist non-trivial homomorphisms of $M(\boldsymbol{T})$, which vanish on $L(\boldsymbol{T})$ (see W. Rudin [**7**; p. 78] and R.E. Edwards [**4**; p. 80]). Here we shall construct such a homomorphism of a different type. We remark also that our method is applied to get the examples cited above.

Let π, ρ and ϕ be elements of $\Delta(M(\boldsymbol{T}))$. Assume $|\pi|^2=|\pi|$ and $\rho=|\rho|$. Put $\psi(n)=\pi^n\rho^{|n|}\phi$. Then $\psi(\cdot)$ satisfies the condition (C) with $C=1$. Thus the mapping Ψ defined by (i) in Theorem 1 is a homomorphism of $M(\boldsymbol{T})$ into $M(\boldsymbol{T})$.

Let μ be a measure in $M(\boldsymbol{T})$ such that every Fourier-Stieltjes coefficient is real, that is, μ is hermitian and such that

$$\{\xi_\mu(t)\,;\ \xi=\{\xi_\nu\}\in\Delta(M(\boldsymbol{T}))\}=\{ae^{int}\,;\ a\in\boldsymbol{C},\ |a|\leqq 1,\ n\in\boldsymbol{Z}\}$$

(cf. for example G. Brown [**1**]). Let $0<r<1$ and t_0 be a real number such that t_0 divided by 2π is irrational. Choose generalized characters π, ρ and ϕ such that $\pi_\mu=e^{it_0}$, $\rho_\mu=r$ and $\phi_\mu=i$.

Then the homomorphism Ψ defined by $\psi(n)=\pi^n\rho^{|n|}\phi$ has the property that Ψ maps the singular hermitian measure μ to the absolutely continuous measure $\Psi(\mu)$ whose Fourier-Stieltjes coefficients are not real. On the other hand Ψ vanishes on $L(\boldsymbol{T})$. In fact $\psi(n)\in\Delta(M(\boldsymbol{T}))-\boldsymbol{Z}$. Thus $\hat{f}(\psi(n))=0$ for all f in $L(\boldsymbol{T})$ and n in $\boldsymbol{Z}$ (cf., for example, [**9**; p. 187]).

3. Homomorphisms of $N(\mu)$ into $M(\boldsymbol{T})$.

Let N be an L-subalgebra of $M(\boldsymbol{T})$ and Ψ be a homomorphism of N into $M(\boldsymbol{T})$. Let ψ be the mapping of $\boldsymbol{Z}$ into $\Delta(N)\cup\{0\}$ defined by $(\Psi\nu)\hat{}\,(n)=\hat{\nu}(\psi(n))$ for all ν in N and n in $\boldsymbol{Z}$. If $N=L(\boldsymbol{T})$, then $\Delta(N)$ is identified with $\{e^{int}\,;\ n\in\boldsymbol{Z}\}$. Thus if Ψ is a homomorphism of $L(\boldsymbol{T})$ into $M(\boldsymbol{T})$, then it induces a (almost) piecewise affine mapping of $\boldsymbol{Z}$ into $\Delta(L(\boldsymbol{T}))\cup\{0\}$ by Theorem A.

In this section we restrict our attention to a class of L-subalgebras which consist of singular measures and are defined later. We shall show in Theorem 3 that the converse of Theorem 2 is true in a sense, that is, the mapping ψ of $\boldsymbol{Z}$ into $\Delta(N)\cup\{0\}$ is piecewise affine under a condition for such an L-subalgebra N.

For a measure μ in $M(\boldsymbol{T})$, $N(\mu)$ will denote the smallest L-subalgebra which contains μ. We use the following properties of $\Delta(N(\mu))$.

LOCALIZATION LEMMA (cf. G. Brown and W. Moran [**2**]). *For $\mu\in M(\boldsymbol{T})$, $\Delta(N(\mu))$ is identified with*

$$S(\mu)=\{\xi_\mu\,;\ \xi=\{\xi_\nu\}\in\Delta(N(\mu))\}.$$

Let ξ, ϕ and χ be elements in $\Delta(N(\mu))$. If $\xi_\mu,\ \phi_\mu,\ \chi_\mu\in S(\mu)$ and $\xi_\mu=\phi_\mu\chi_\mu$, then $\xi=\phi\chi$ by the localization lemma. We remark also that if μ is a measure such that μ^n $(n=1, 2, \cdots)$ are mutually singular and $c\in S(\mu)$ is a constant function, then $\{c_\nu\}\in\Delta(N(\mu))$ is defined by

$$c_\nu=c^n \qquad \mu^n\text{-a. e.}$$

Now we specify the measure μ as follows. Let $\{a_n; n\geqq 1\}$ be a sequence of integers such that $a_n\geqq 2$. Let $d_n=2\pi\prod_{r=1}^{n} a_r^{-1}$ and define the Bernoulli convolution product

$$\mu=\mathop{*}_{n=1}^{\infty}\frac{1}{2}[\delta(0)+\delta(d_n)],$$

where $\delta(a)$ is the Dirac measure concentrated on $\{a\}$. We remark that the infinite product of convolution converges in the weak*-topology and it defines a positive measure with norm 1.

Denote by $\boldsymbol{B}'$ the class of the measures as is obtained above with $a_n>2$ for infinitely many n. The measures in $\boldsymbol{B}'$ are continuous and singular. Furthermore μ^n, $n=1, 2, \cdots$, are mutually singular (cf. [2]).

For $\mu=\mathop{*}_{n=1}^{\infty}\frac{1}{2}[\delta(0)+\delta(d_n)]$ in $\boldsymbol{B}'$ let D be the subgroup of $\boldsymbol{T}$ generated by $\{d_n; n=1, 2, \cdots\}$ with the discrete topology. Put

$$\mu_r=\mathop{*}_{n=r+1}^{\infty}\frac{1}{2}[\delta(0)+\delta(d_n)]$$

and

$$D_r=\left\{\sum_{n=1}^{r}\varepsilon_n d_n;\ \varepsilon_n=0 \text{ or } 1\right\}.$$

We recall the following properties of the measures in $\boldsymbol{B}'$.

THEOREM B ([2]). *Let* $\mu=\mathop{*}_{n=1}^{\infty}\frac{1}{2}[\delta(0)+\delta(d_n)]$ *be a measure in* $\boldsymbol{B}'$. *Then we have*

(i) *for every* $\chi_\mu\in S(\mu)$ *and* $n=1, 2, \cdots$, *there exists a unique element* $\gamma(\chi_\mu)$ *in* $\hat{D}$, *the dual group of* D, *such that*

$$\chi_\mu(d+t)=\beta(d)\chi_\mu(t) \qquad \textit{for}\ \ \mu_n\text{-a. a. } t \ \textit{ and }\ d\in D_n \tag{1}$$

where $\beta=\gamma(\chi_\mu)$,

(ii) *the mapping* γ *of* $S(\mu)$ *to* $\hat{D}$ *defined by* (1) *is a continuous semigroup homomorphism, and*

(iii) *if* $\beta\in$ *Image of* γ, *then* $\gamma^{-1}(\beta)=\{af;\ a\in\boldsymbol{C},\ 0<|a|\leqq 1\}$, *where* f *is a member of* $S(\mu)$ *with constant unit modulus which is a pointwise limit point of the sequence* $\{\sum_{d\in D_n}\beta(d)C_n(d)\}$, $C_n(d)$ *being the characteristic function of the interval* $[d, d+d_n)$.

THEOREM 3. *Let* μ *be a measure in* $\boldsymbol{B}'$. *Let* Ψ *be a homomorphism of* $N(\mu)$ *into* $M(\boldsymbol{T})$ *and* $\phi(\cdot)$ *be the mapping of* $\boldsymbol{Z}$ *into* $\Delta(N(\mu))\cup\{0\}$ *defined by* Ψ.

Suppose that $|\phi(n)|^2=|\phi(n)|$ *for all* n. *Then the mapping* $\phi(\cdot)$ *is piecewise affine.*

PROOF. By Theorem B (iii) $|\phi_\mu(n)|=1$ μ-a. e. or 0. Put $P=\{n\in \boldsymbol{Z};\ |\phi_\mu(n)|=1\}$, and $\beta(n)=\gamma(\phi_\mu(n))$ for $n\in P$ and $=$ the unit of $\hat{D}$ otherwise, where γ is the mapping given by Theorem B. The first step of our proof is to show that the mapping $n\rightarrow\beta(n)$ of $\boldsymbol{Z}$ into $\hat{D}$ defines a homomorphism of $L(D)$ into $M(\boldsymbol{T})$.

By Theorem 1

$$\lambda(\nu\,;\,t,\,\theta)=\sum_{n=-\infty}^{\infty}\phi(n)_\nu(t)e^{in\theta} \tag{2}$$

is a Fourier-Stieltjes series for ν-a. a. t and $\|\lambda(\nu\,;\,t,\,\cdot)\|_{M(\boldsymbol{T})}\leqq\|\Psi\|$ for every $\nu\in N(\mu)$. Now put $\nu=\nu_1*\nu_2*\cdots*\nu_k$, where $\nu_j\geqq0$ and $\nu_j\in N(\mu)$ $(j=1,2,\cdots,k)$. Then, by (2)

$$\lambda(\nu\,;\,t_1+t_2+\cdots+t_k,\,\theta)=\sum_{n=-\infty}^{\infty}\phi(n)_\nu(t_1+t_2+\cdots+t_k)e^{in\theta} \tag{3}$$

is the Fourier-Stieltjes series of a measure with norm$\leqq\|\Psi\|$ for $\nu_1\times\nu_2\times\cdots\times\nu_k$-a. a. $(t_1,t_2,\cdots,t_k)$.

Let r be a positive integer. For k elements $d^1,d^2,\cdots,d^k$ in D_r put

$$\nu_j=\delta(d^j)*\mu_r\quad(j=1,2,\cdots,k)\,.$$

Then $\nu_j\ll\mu$. Thus $\nu_j\in N(\mu)$. By the property of the generalized characters and Theorem B, we have

$$\phi(n)_\mu(d^j+t_j)=\beta(n)(d^j)\,\phi(n)_\mu(t_j)\qquad \mu_r\text{-a. e. in } t_j$$

for every $n\in\boldsymbol{Z}$ and $j=1,2,\cdots,k$. Thus by (3), the multiplicative property of the generalized characters and Theorem B,

$$\sum_{n=-\infty}^{\infty}\Big[\prod_{j=1}^{k}\beta(n)(d^j)\prod_{j=1}^{k}\phi(n)_\mu(t_j)\Big]e^{in\theta} \tag{4}$$

is the Fourier-Stieltjes series of a measure with norm$\leqq\|\Psi\|$ for $\mu_r\times\mu_r\times\cdots\times\mu_r$-a. a. $(t_1,t_2,\cdots,t_k)$.

By the same way for k convolution products $\mu_r^k=\mu_r*\cdots*\mu_r$ we have

$$\lambda(\mu_r^k\,;\,t_1+t_2+\cdots+t_k,\,\theta)=\sum_{n=-\infty}^{\infty}\Big[\prod_{j=1}^{k}\phi(n)_\mu(t_j)\Big]e^{in\theta} \tag{5}$$

and $\|\lambda(\mu_r^k\,;\,t_1+t_2+\cdots+t_k,\,\cdot)\|_{M(\boldsymbol{T})}\leqq\|\Psi\|$ for $\mu_r\times\mu_r\times\cdots\times\mu_r$-a. a. $(t_1,t_2,\cdots,t_k)$. Since $|\phi(n)_\mu(t)|=1$ or 0 by our assumption, the composition of the series (4) and the series of $\bar{\lambda}(\mu_r^k\,;\,t_1+t_2+\cdots+t_k,\,-\theta)$

$$(6)\qquad \sum_{n=-\infty}^{\infty}[\prod_{j=1}^{k}\beta(n)(d^{j})]C_{P}(n)e^{in\theta}$$

is the Fourier-Stieltjes series of a measure with norm $\leqq\|\Psi\|^{2}$.

Since $d_{i}+(a_{1}a_{2}\cdots a_{i}-1)\equiv 0 \mod 2\pi$,

$$D=\{\sum_{i=1}^{\infty}n_{i}d_{i};\ n_{i}\in\boldsymbol{Z},\ n_{i}\geqq 0 \text{ and } n_{i}=0 \text{ except a finite number of } i\text{'s}\}$$

Thus by (6), $\sum\beta(n)(d)e^{in\theta}$ is the Fourier-Stieltjes series of a measure with norm $\leqq\|\Psi\|^{2}$ for every d in D. Thus the mapping

$$\Phi f(\theta)=\sum_{n=-\infty}^{\infty}[\sum_{d\in D}f(d)\beta(n)(d)]C_{P}(n)e^{in\theta}\quad \text{for}\quad f\in L(D)$$

defines a homomorphism of $L(D)$ to $M(\boldsymbol{T})$. Thus by P. J. Cohen's theorem [3], P belongs to the coset ring of $\boldsymbol{Z}$ and the mapping $n\rightarrow\beta(n)$ of $\boldsymbol{Z}$ to $\hat{D}$ is piecewise affine. Thus there exist a positive integer m, a finite subset $R=\{n_{m+1}, n_{m+2}, \cdots, n_{l}\}$ of $\boldsymbol{Z}$ and $\zeta_{j}, \eta_{j}\in\hat{D}$ $(j=1, 2, \cdots, m)$ such that

$$(7)\qquad \beta(n)=\sum_{j=1}^{m}\zeta_{j}^{k}\eta_{j}C_{mZ+j}(n)$$

for $n\in P-R$ with $k=[n/m]$ and $(P-R)\cup F$ is periodic with the period m for some finite set F.

To complete the proof we pull back the relation (7) to another relation involving $\{\psi(n)\}$. For each j in $[(P-R)\cup F]\cap\{1, 2, \cdots, m\}$ choose π_{j} and ϕ_{j} in $\Delta(N(\mu))$ such that $\gamma(\pi_{j})=\zeta_{j}$, $\gamma(\phi_{j})=\eta_{j}$ and $|\pi_{j}|=1$, $|\phi_{j}|=1$. Then by Theorem B (iii) there exist unitary constants c_{n} such that

$$(8)\qquad \psi_{\mu}(n)=c_{n}\sum_{j=1}^{m}\pi_{j\mu}^{k}\phi_{j\mu}C_{mZ+j}(n)$$

for $n\in P-R$ with $k=[n/m]$. For $j\notin[(P-R)\cup F]$, $1\leqq j\leqq m$, let ϕ_{j} be the zero system, that is, the trivial functional. Then (8) holds for $n\in\boldsymbol{Z}-R$.

Put $a_{n}=c_{n}$ for $n\in P-R$ and $=1$ for $n\notin P-R$. Let $\phi_{j}=\psi(n)$ for $n=n_{j}\in R$. Then we have

$$(9)\qquad \psi_{\mu}(n)=\sum_{j=1}^{m}\pi_{j\mu}^{k}a_{n}\phi_{j\mu}C_{mZ+j}(n)$$

for $n\notin R$ with $k=[n/m]$ and $\psi_{\mu}(n)=\phi_{j\mu}$ for $n=n_{j}\in R$.

We denote by $\alpha(n)$ the generalized character of $\Delta(N(\mu))$ such that $\alpha_{\mu}(n)=a_{n}$. The final step of the proof is to show that $\{\alpha(n)\}$ is expressed in the form

$$(10)\qquad \alpha(n)=\sum_{j=1}^{m'}\pi_{j}'^{k}\phi_{j}'C_{m'Z+j}(n)$$

outside a finite set R', where m' is a positive integer, $k=[n/m']$ and π'_j, $\phi'_j \in \Delta(N(\mu))$, $j=1, 2, \cdots, m'$. Then our theorem follows from (9) and (10) replacing m by mm' and R by $R\cup R'$. Furthermore, π_j and ϕ_j are replaced by the generalized characters of the form $\pi_i^p \pi'^{p'}_{i'}$ and $\phi_i^q \phi'^{q'}_{i'}$ respectively.

Put $\psi'(n)=\overline{\psi(n)}$ for $n\in R$ and

$$\psi'(n)=\sum_{j=1}^{m} \bar{\pi}_j^k \bar{\phi}_j C_{mz+j}(n)$$

for $n\notin R$ with $k=[n/m]$. Then by Theorem 2 $\{\psi'(n)\}$ defines a homomorphism. We have $\psi(n)\psi'(n)=\alpha(n)$ for all n except a finite number of n's, so that $\{\alpha(n)\}$ defines a homomorphism in the obvious way. Let $c>0$ be the norm of that homomorphism. Then $\|\sum \alpha_\nu(n)e^{in\theta}\|_{M(T)}\leqq c$ for every $\nu=\mu^k$, $k>0$. As we have mentioned in the section 2, $\alpha_\nu(n)=a_n{}^k$ for $\nu=\mu^k$, $k>0$ and $|\alpha_\nu(n)|=1$. Thus

$$\|\sum a_n^k e^{in\theta}\|_{M(T)}\leqq c$$

for all $k\in \boldsymbol{Z}$. This implies, by the theorem in [7; p. 93], that the mapping $n\rightarrow a_n$ of $\boldsymbol{Z}$ to $\boldsymbol{T}$ is piecewise affine. Thus we get (10). Thus our proof is complete.

REMARK. Let N be the smallest closed subalgebra of $M(\boldsymbol{T})$ which contains all $L(\boldsymbol{T}^\tau)$, where $\boldsymbol{T}^\tau$ is the group $\boldsymbol{T}$ with a locally compact topological group topology τ stronger than the original one or equal to that of $\boldsymbol{T}$. Since the discrete topology and the natural one are only such topologies, $N=L(\boldsymbol{T})+L(\boldsymbol{T}^d)$, where d is the discrete topology on $\boldsymbol{T}$. Thus N contains no continuous singular measures. On the other hand the algebras $N(\mu)$ in Theorem 3 consist of continuous singular measures.

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