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The Cauchy problem for some class of hyperbolic differential operators with variable multiple characteristics

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1. Introduction.

In this paper we shall study some class of hyperbolic differential operators with variable multiple characteristics. Under a generalized condition of Lax type, we shall establish the well posedness of the Cauchy problem for our class of hyperbolic operators. Reducing a higher order single equation to a first order system and showing an energy inequality, we prove the well posedness for the considered operators.

E. E. Levi [**9**] first investigated the solvability of the Cauchy problem for non-linear hyperbolic differential operators on R^2 with constant multiple characteristic roots, in which he imposed some conditions on the lower order terms. This condition was called Levi condition by Mizohata-Ohya [**11**]. For a differential operator on R^n the relation between a well posedness and the Levi condition was studied by many authors [**1**], [**2**], [**8**] and [**11**]. In [**2**] Flaschka and Strang found some necessary conditions for the well posedness of the Cauchy problem for operators of constant multiple characteristics and Chazarain [**5**] proved that their conditions are equivalent to the Levi condition under the hypothesis of the existence of an influence domain and that it is also a sufficient condition for the well posedness of the Cauchy problem for hyperbolic differential operators with constant multiple characteristics.

Recently many authors studied hyperbolic differential operators with variable multiple characteristics. Using cleverly nonnegative characteristic form, Oleinik [**13**] studied the well posedness of the Cauchy problem for second order hyperbolic differential operators. In her paper she did not assume that the characteristic roots are smooth. On the other hand in [**7**], [**10**], [**12**] and [**15**] they assumed that the characteristic roots are smooth. Following the idea by E. E. Levi and Mizohata-Ohya [**11**], they proved the well posedness for hyperbolic differential operators whose characteristic roots have the multiplicities not greater than 3.

In Theorem 1 of this paper we shall prove an analogous result for a rather general class of hyperbolic operators. Our conditions on the lower order terms seem natural extension of the condition of Lax type (c. f. [16]).

If the Poisson bracket of two characteristic roots satisfies some condition, then Theorem 1 is valid for a more general class of hyperbolic operators. In [14] and [18] they proved the well posedness of the Cauchy problem for some class of hyperbolic operators. In Theorem 2 of this paper we shall state their results under a more delicate condition. For some special example of hyperbolic differential operators our condition is necessary and sufficient. Of course Theorem 2 is a generalized result of Theorem 1, if the characteristic roots satisfy some conditions with respect to the Poisson bracket. This fact is verified in this paper.

In section 2 we shall state our assumptions and results precisely. In section 3 we state the equivalent condition to our generalized condition of Lax type. Using this equivalent condition, in section 4 we shall reduce our operator to a first order system and show the existence of a symmetrizer with a singularity at $t=0$. Showing energy inequalities for our operators, we shall prove Theorems 1 and 2 in Sections 5 and 6, respectively.

In section 7 we shall state some necessary conditions of the well posedness of the Cauchy problem. By the theorem of [6], for some special example of hyperbolic differential operators of R^2 our generalized condition of Lax type is necessary and sufficient condition of the well posedness of the Cauchy problem in the sense of [6].

2. Statements of the results.

We shall consider the following differential operator;

$$P(t, x, D_t, D_x)=\sum_{|\alpha|\leq m} a_\alpha(t, x)D_t^{\alpha_0}D_x^{\alpha'}, \tag{2.1}$$

where $(t, x)\in\Omega=[0, T]\times R^n$ $(0<T<\infty)$, $D_t=-i\partial/\partial t$, $D_{x_j}=-i\partial/\partial x_j$, and $D_t^{\alpha_0}D_x^{\alpha'}=D_t^{\alpha_0}D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$. We assume that $a_{m,0,\cdots,0}(t, x)=1$ and $a_\alpha(t, x)$ belongs to $\mathscr{B}(\Omega)$, which consists of all functions whose arbitrary derivatives are bounded in Ω. Let (τ, ξ) be the covariable of (t, x). Then we define the following symbols;

$$p_k(t, x, \tau, \xi)=\sum_{|\alpha|=k} a_\alpha(t, x)\tau^{\alpha_0}\xi^{\alpha'}, \ (k=0, \cdots, m).$$

Throughout this paper except t^{-l} $(l>0)$ all functions on $\Omega\times R^{n+1}$ or $\Omega\times R^n$ are elements of $\mathscr{B}(\Omega\times S^n)$ or $\mathscr{B}(\Omega\times S^{n-1})$ if $(\tau, \xi)\in S^n$ or $\xi\in S^{n-1}$ respectively. Here S^n is the unit sphere of R^{n+1}. For functions $a(t, x, \xi)$ and $b(t, x, \xi)$, $a\equiv 0 \mod t^{-l}b$ means that there exists a function $c(t, x, \xi)$ such that $a=c(t^{-l}b)$. All pseudo-differential operators $Q(t, x, D_t, D_x)$ in this paper are differential operators with respect to t. Therefore we say that $Q(t, x, D_t, D_x)$ is of order m_0 if Q have the following form

$$Q(t, x, D_t, D_x)=\sum_{j=0}^{m_0} A_j(t, x, D_x)D_t^{m_0-j},$$

where A_j is of order j.

First we shall consider the differential operator (2.1) satisfying the following three conditions;

(A.1) The principal symbol of P is denoted by

$$p_m(t, x, \tau, \xi)=\sum_{j=1}^{s}((\tau-\lambda_j)^{m_j}(\tau-\lambda_{s+j}))\prod_{j=2s+1}^{m-N+s}(\tau-\lambda_j),$$

where the positive constant integers m_j satisfy that $m_1\geqq\cdots\geqq m_s$, $N=\sum_{j=1}^{s} m_j$ and $\lambda_j(t, x, \xi)$ $(j=1, \cdots, m-N+s)$ is real homogeneous of degree 1 with respect to ξ and belongs to $\mathcal{B}(\Omega\times S^{n-1})$ if $\xi\in S^{n-1}$,

(A.2) For any couple $(i, j)\neq(k, s+k)$ $(k=1, \cdots, s)$ we suppose the following;

$$|(\lambda_i-\lambda_j)(t, x, \xi)|>\delta \qquad (t, x, \xi)\in\Omega\times S^{n-1},$$

where δ is a positive constant.

For the lower order terms of $P(t, x, D_t, D_x)$ we assume the following;

(A.3) For any k $(k=1, \cdots, s)$ we assume $P(t, x, D_t, D_x)$ has the following form;

$$P(t, x, D_t, D_x)=\sum_{l=0}^{m_k} Q_{k,l}(\Lambda_k)^{m_k-l}(t, x, D_t, D_x), \tag{2.2}$$

where Λ_k is the pseudo-differential operator defined by the symbol $\tau-\lambda_k(t, x, \xi)$ and $Q_{k,l}(t, x, D_t, D_x)$ $(l=0, \cdots, m_k)$ is a pseudo-differential operator of degree $m-m_k$. Furthermore the principal symbol $q_{k,l}(t, x, \tau, \xi)$ of $Q_{k,l}$ has the following property;

$$q_{k,l|\tau=\lambda_k}\equiv 0 \qquad \mathrm{mod}\, t^{-l}(\lambda_k-\lambda_{s+k}). \tag{2.3}$$

In the following theorem we use the function space $C^\infty([0, T]; H_\infty(R^n))$. A function $u(t, x)$ belongs to $C^\infty([0, T]; H_\infty(R^n))$ if $D_t^j u(t)$ $(j=0, 1, \cdots)$ exists in $H_\infty(R^n)=\bigcap_{s=0}^{\infty} H_s(R^n)$ and is continuous in the topology of $H_\infty(R^n)$ in $[0, T]$, where $H_s(R^n)$ is the Sobolev space.

THEOREM 1. *Let $P(t, x, D_t, D_x)$ be a differential operator with the form* (2.1). *If P satisfies the assumptions* (A.1), (A.2) *and* (A.3), *then the Cauchy problem $Pu=f$ in Ω, $D_t^j u_{|t=0}=g_j$ $(j=0, \cdots, m-1)$ is well posed in $C^\infty([0, T]; H_\infty(R^n))$, i.e., for any data $f(t, x)\in C^\infty([0, T]; H_\infty(R^n))$ and $g_j(x)\in H_\infty(R^n)$ $(j=0, \cdots, m-1)$ there exists a unique solution $u(t, x)\in C^\infty([0, T]; H_\infty(R^n))$.*

REMARK 1.1. In the assumption (A.3) we do not assume that $(\lambda_k-\lambda_{s+k})t^{-l}$ is smooth at $t=0$. Of course if $q_{k,l|\tau=\lambda_k}\equiv 0 \mod (\lambda_k-\lambda_{s+k})$, then (2.3) holds clearly. Thus Theorem 1 is a generalization of [**17**]. If $\lambda_k(t, x, \xi)=\lambda_{s+k}(t, x, \xi)$, then (2.2) becomes to the following;

$$(2.4) \qquad P(t, x, D_t, D_x)=\sum_{l=0}^{m_k+1} R_{k,l}(\Lambda_k)^{m_k+1-l}(t, x, D_t, D_x),$$

where $R_{k,l}(t, x, D_t, D_x)$ is of order $m-m_k-1$. This is the condition of Lax type. When $m_j=1$ $(j=1, \cdots, s)$, our theorem coincides formally with those of [**10**] and [**12**].

In the assumptions (A.1) and (A.3) if the commutator $[\Lambda_k, \Lambda_{s+k}]$ satisfies the condition (2.5) below, then (A.2) is relaxed. Thus Theorem 1 holds for a more general class of hyperbolic operators.

Now we shall study the differential operator (2.1) which satisfies the following assumptions;

(H.1) The principal symbol $p_m(t, x, \xi, \tau)$ does not depend on x if $|x|$ is sufficiently large. $p_m(t, x, \xi, \tau)$ is factorized by

$$p_m(t, x, \tau, \xi)=\prod_{j=1}^{m}(\tau-\lambda_j(t, x, \xi)),$$

where $\lambda_j(t, x, \xi)$ $(j=1, \cdots, m)$ is infinitely differentiable, real and positively homogeneous of degree 1.

(H.2) The maximal multiplicity of the characteristic roots $\{\lambda_j(t, x, \xi)\}_{j=1,\cdots,m}$ over $\Omega\times S^{n-1}$ is equal to r. For any couple (i, j) $(i, j=1, \cdots, m)$ we assume

$$(2.5) \qquad \{\tau-\lambda_i, \tau-\lambda_j\}(t, x, \xi)\equiv 0 \qquad \mod t^{-1}(\lambda_i-\lambda_j),$$

where $\{\ ,\ \}$ is the Poisson bracket.

(H.3) The operator $P(t, x, D_t, D_x)$ takes the following form;

$$(2.6) \qquad P(t, x, D_t, D_x)=\Lambda_1\cdots\Lambda_m+\sum_{0\leq k<m} t^{k-m}\gamma_{i_1\cdots i_k}\Lambda_{i_1}\cdots\Lambda_{i_k} + P_{m-r}(t, x, D_t, D_x),$$

where $\{i_1, \cdots, i_k\}$ is a subset of $\{1, \cdots, m\}$, $\gamma_{i_1\cdots i_k}(t, x, D_x)$ is a pseudo-differential operator of order 0 and p_{m-r} is of order $m-r$. For simplicity if $\{i_1, \cdots, i_k\}=\emptyset$, then we put $k=0$ and $\Lambda_{i_1}\cdots\Lambda_{i_k}=1$.

REMARK 1.2. The differential operator satisfying the conditions (A.1), (A.2) and (A.3) is written in the form (2.6). This is proved in Corollary 6.3. Therefore the following Theorem 2 is a generalization of Theorem 1 if $\lambda_j(t, x, \xi)$ and $\lambda_{s+j}(t, x, \xi)$ $(j=1, \cdots, s)$ satisfies the condition (2.5).

Under the above three assumptions we get the following;

THEOREM 2. *Let $P(t, x, D_t, D_x)$ be a differential operator of* (2.1) *and satisfy the assumption* (H.1), (H.2) *and* (H.3). *Then the Cauchy problem $Pu=f$ in Ω, $D_t^j u_{|t=0}=g_j$ $(j=0, \cdots, m-1)$ is well posed in $C^\infty([0, T]; H_\infty(R^n))$.*

REMARK 1.3. We shall state a few remarks with respect to (H.1), (H.2) and (H.3).

i) For simplicity we assume $p_m(t, x, \tau, \xi)=p_m(t, \tau, \xi)$ when $|x|$ is sufficiently large. Taking care of coincidences of the characteristic roots λ_j at infinite points of x, we can relax this condition. In that case r in (H.3) may be greater than the maximal multiplicity of the characteristic roots.

ii) In the condition (2.5) we do not assume $t^{-1}(\lambda_i-\lambda_j)\in C^\infty(\Omega\times(R^n\backslash 0))$. Thus Theorem 2 is a generalization of [**14**] and [**18**]. If $|\lambda_i-\lambda_j|>\delta|\xi|$ then (2.5) is clearly valid.

iii) In the summation terms of the right hand side of (2.6), each term may be singular at $t=0$, however we assume that the sum of each term has a meaning at $t=0$. In condition (H.3) we do not impose any condition in $p_j(t, x, \tau, \xi)$ $(j\leqq m-r)$.

The final part of this paper, when $\lambda_j(t, x, D_t, D_x)$ is a special differential operator on R^2, we show that (A.3) and (H.3) is necessary for the well posedness using the Theorem 4.1 of [**6**]. Thus our condition is necessary and sufficient for some special example of hyperbolic operators in R^2.

3. The equivalent conditions of (A.3).

In this section we shall state a few equivalence of (A.3).

Taking care of multiplicity of the roots λ_j $(j=1, \cdots, s)$, we denote $p_m(t, x, \tau, \xi)$ by

$$(\tau-\lambda_{m-N+s})\cdots(\tau-\lambda_{s+1})\varphi_\mu^{n_\mu}\cdots\varphi_1^{n_1},$$

where $\varphi_\nu(t, x, \tau, \xi)$ $(\nu=1, \cdots, \mu)$ is equal to $\prod_{j=1}^{s_\nu}(\tau-\lambda_j(t, x, \xi))$ and $m_{s_{\nu-1}+1}=\cdots=m_{s_\nu}$ $(\nu=1, \cdots, \mu)$. Remark that $s_1<s_2<\cdots<s_\mu=s$ and $m_1=\sum_{\nu=1}^{\mu} n_\nu$ and denote N_ν by $s_\nu n_\nu$, where $\sum_{\nu=1}^{\mu} N_\nu=N$. We introduce a product pseudo-differential operator $\Phi_\nu(t, x, D_t, D_x)=(\Lambda_{s_\nu}\cdots\Lambda_1)(t, x, D_t, D_x)$. Then we denote $\Delta_j(t, x, D_t, D_x)$ of order j $(j=0, \cdots, m)$ by

$$\Delta_0=1,\ \Delta_1=\Lambda_1,\ \cdots,\ \Delta_j,\ \cdots,\ \Delta_N=\Phi_\mu^{n_\mu}\cdots\Phi_1^{n_1},\ \cdots$$

$$\Delta_{N+k}=\Lambda_{s+k}\cdots\Lambda_s\Delta_N,\ \cdots,\ \Delta_m=\Lambda_{m-N+s}\cdots\Lambda_{s+1}\Delta_N$$

where $\Delta_j=\Lambda_\delta\cdots\Lambda_1\Phi_\nu^\sigma\Phi_{\nu-1}^{n_{\nu-1}}\cdots\Phi_1^{n_1}$ if $j=N_1+\cdots+N_{\nu-1}+\sigma s_\nu+\delta$ $(0\leqq\sigma\leqq n_\nu-1,\ 0\leqq\delta\leqq s_\nu-1)$. Moreover we define a pseudo-differential operator $(\Delta_j/\Delta_i)(t, x, D_t, D_x)$, where $j=N_1+\cdots+N_{\nu-1}+\sigma s_\nu+\delta>i=N_1+\cdots+N_{\nu'-1}+\sigma' s_{\nu'}+\delta'$, by

$$\Lambda_\delta \cdots \Lambda_1 \Phi_\nu^\sigma \Phi_{\nu-1}^{n_{\nu-1}} \cdots \Phi_{\nu'+1}^{n_{\nu'+1}} \Phi_{\nu'}^{n_{\nu'}-\sigma'-1} \Lambda_{s_{\nu'}} \cdots \Lambda_{\delta'+1}.$$

Clearly we see that $(\Delta_j/\Delta_i)\Delta_i=\Delta_j$.

First we shall state a few lemmas with respect to a commutator of pseudo-differential operators.

LEMMA 3.1. *Let* $A(t, x, D_t, D_x)$ *and* $B(t, x, D_t, D_x)$ *be pseudo-differential operators of order* 1 *and* b *respectively. Then we have*

$$A^m B^m=(AB)^m+\sum_{j=0}^{m-1} C_j B^j, \tag{3.1}$$

$$=(AB)^m+\sum_{j=0}^{m-1} D_j A^j, \tag{3.2}$$

where $C_j(t, x, D_t, D_x)$ *and* $D_j(t, x, D_t, D_x)$ *are of order* $m+(b-1)(m-j)$ *and* mb *respectively.*

PROOF. (3.1) and the following (3.3) are proved by the double induction with respect to m at the same time.

$$[A, (AB)^m]=\sum_{j=0}^{m-1} \tilde{C}_j B^j, \tag{3.3}$$

$$=\sum_{j=0}^{m-1} \tilde{D}_j A^j, \tag{3.4}$$

where $\tilde{C}_j(t, x, D_t, D_x)$ and $\tilde{D}_j(t, x, D_t, D_x)$ are of order $m+(b-1)(m-j)+1$ and mb respectively. (3.2) and (3.4) is also proved at the same time by the double induction with respect to m.

The following two lemmas are obvious.

LEMMA 3.2. *Let* $A(t, x, D_t, D_x)$ *and* $B(t, x, D_t, D_x)$ *be a pseudo-differential operators of order* a *and* b *respectively. Then we have*

$$[A, B^m]=\sum_{j=0}^{m-1} C_j B^j,$$

where $C_j(t, x, D_t, D_x)$ *is of order* $a+(b-1)(m-j)$.

LEMMA 3.3. *Let* $A(t, x, D_t, D_x)$, $B_j(t, x, D_t, D_x)$ $(j=1, \cdots, m)$ *and* $C(t, x, D_t, D_x)$ *be pseudo-differential operators of order* 1, b_j *and* c *respectively. Then we have*

$$\prod_{j=1}^{m} (B_j A C)=\sum_{j=0}^{m} D_j A^j,$$

where $D_j(t, x, D_t, D_x)$ *is of order* $(b_1+\cdots+b_m)+mc$.

Using these lemmas, we shall prove the following

PROPOSITION 3.4. *Let* $P(t, x, D_t, D_x)$ *be a differential operator which satisfies the conditions* (A.1) *and* (A.2). *Then the condition* (A.3) *is equivalent to the following statement; we can write* P *by*

(3.5) $$P(t, x, D_t, D_x)=\sum_{i=0}^{m_1} Q_i(t, x, D_t, D_x)\Lambda_{\alpha(i)},$$

where if $i=n_\mu+\cdots+n_{\nu+1}+\sigma$ $(0\leqq\sigma\leqq n_\nu)$, *then* $\alpha(i)=N_1+\cdots+N_\nu-s_\nu\sigma$ *and* Q_i $(i=0, \cdots, m_1)$ *is a pseudo-differential operator of order* $M_i=m-i-\alpha(i)$ *and is a differential operator with respect to t. Moreover the principal symbol* $q_i(t, x, \tau, \xi)$ *of* Q_i *satisfies the following condition;*

(3.6) $$q_{i|\tau=\lambda_k}\equiv 0 \mod t^{-i}(\lambda_k-\lambda_{s+k}), \qquad \text{for } 1\leqq k\leqq s_\nu.$$

PROOF. First we shall show that (A.3) implies the condition (3.5) and (3.6). From (2.2) and (2.3) we shall show that

(3.7) $$P(t, x, D_t, D_x)=\sum_{l=0}^{m_2} R_l(\Lambda_2\Lambda_1)^{m_2-l}(\Lambda_1)^{m_1-m_2} + \sum_{l=m_2+1}^{m_1} S_l(\Lambda_1)^{m_1-l},$$

where $R_l(t, x, D_t, D_x)$ and $S_l(t, x, D_t, D_x)$ is of order $m-m_1-m_2+l$ and $m-m_1$ respectively. Moreover the principal symbols $r_l(t, x, \tau, \xi)$ and $s_l(t, x, \tau, \xi)$ of R_l and S_l respectively, have the property (2.3) putting $q_{k,l}=r_l$ and $k=1, 2$ and $q_{k,l}=s_l$ and $k=1$, respectively. Comparing the principal symbols of (2.2) as $k=1$ with as $k=2$, we have

(3.8) $$q_{1,0}(\Lambda_1)^{m_1}(t, x, \tau, \xi)=q_{2,0}(\Lambda_2)^{m_2}(t, x, \tau, \xi).$$

By (A.2), (2.3) and (3.8) we can write

(3.9) $$q_{1,0}=r_0(\Lambda_2)^{m_2}, \qquad q_{2,0}=r_0(\Lambda_1)^{m_1},$$

where $r_0(t, x, \tau, \xi)$ is positively homogeneous of degree $m-m_1-m_2$ and satisfies the condition (2.3) putting $q_{k,l}=r_0$ and $k=1, 2$. Let $R_0(t, x, D_t, D_x)$ be a pseudo-differential operator with the principal symbol $r_0(t, x, \tau, \xi)$. By (3.1) we see that

$$P=R_0(\Lambda_2\Lambda_1)^{m_2}(\Lambda_1)^{m_1-m_2}+\sum_{l=1}^{m_1}\tilde{Q}_{1,l}(\Lambda_1)^{m_1-l},$$

where $\tilde{Q}_{1,l}(t, x, D_t, D_x)$ has the same properties as $Q_{1,l}$. By Lemma 3.2 and (3.2) we have

$$P=R_0(\Lambda_2\Lambda_1)^{m_2}(\Lambda_1)^{m_1-m_2}+\sum_{l=1}^{m_2}\tilde{Q}_{2,l}(\Lambda_2)^{m_2-l},$$

where $\tilde{Q}_{2,l}(t, x, D_t, D_x)$ has the same properties as $Q_{2,l}$. Thus repeating the same arguments for

$$P-R_0(\Lambda_2\Lambda_1)^{m_2}(\Lambda_1)^{m_1-m_2},\ P-R_0(\Lambda_2\Lambda_1)^{m_2}(\Lambda_1)^{m_1-m_2}-R_1(\Lambda_2\Lambda_1)^{m_2-1}(\Lambda_1)^{m_1-m_2}, \cdots,$$

we have (3.7). Adding $\sum_{l=0}^{m_3} Q_{3,l}(\Lambda_3)^{m_3-l}$ to (3.7), we have

$$P=\sum_{l=0}^{m_3} T_l(\Lambda_3\Lambda_2\Lambda_1)^{m_3-l}(\Lambda_2\Lambda_1)^{m_2-m_3}(\Lambda_1)^{m_1-m_2}$$

$$+\sum_{l=m_3+1}^{m_2} V_l(\Lambda_2\Lambda_1)^{m_2-l}(\Lambda_1)^{m_1-m_2}+\sum_{l=m_2+1}^{m_1} W_l(\Lambda_1)^{m_1-l},$$

where $T_l(t, x, D_t, D_x)$, $V_l(t, x, D_t, D_x)$ and $W_l(t, x, D_t, D_x)$ are of order $m-m_1$ m_2-m_3+2l, $m-m_1-m_2+l$ and $m-m_1$ pespectively and have the property (3.6) putting $k=1, 2, 3$, $k=1, 2$ and $k=1$ respectively. Repeating the same arguments, we have the desired properties (3.5) and (3.6).

Conversely if P satisfies the condition (3.5) and (3.6), then P has the condition (A.3). This fact is easily derived from Lemma 3.2 and 3.3. This completes the proof of Proposition 3.4.

For any pseudo-differential operator $Q(t, x, D_t, D_x)$ of order $m-k$, which is a differential operator of t, we can express Q by

$$Q(t, x, D_t, D_x)=\sum_{j=0}^{m-k} Q_j(t, x, D_x)(\Delta_{j+k}/\Delta_k),$$

where Q_j is of order $m-k-j$. Thus we have the following;

PROPOSITION 3.5. *Let P be a differential operator satisfying the condition* (A.1) *and* (A.2). *Then the condition* (A.3) *is equivalent to the following;*

$$P(t, x, D_t, D_x)=\sum_{i=0}^{m_1}\sum_{j=0}^{M_i} R_{i,j}(t, x, D_x)\Delta_{\alpha(i)+j}, \tag{3.10}$$

where $R_{i,j}$ is of order M_i-j, whose principal symbols $r_{i,j}(t, x, \xi)$ have the following property;

$$\sum_{j=0}^{k-1} r_{i,j}\Lambda_j \cdots \Lambda_{1|\tau=\lambda_k}\equiv 0 \quad \mathrm{mod}\, t^{-i}(\lambda_k-\lambda_{s+k}). \tag{3.11}$$

Here $k\leqq s_\nu$ if $i=n_\mu+\cdots+n_{\nu+1}+\sigma$ $(1\leqq\sigma\leqq n_\nu)$.

We can express (A.3) by the conditions with respect to p_k when $m_k=1, 2$. For simplicity we identify (t, τ) with (x_0, ξ_0) in the following Remark.

REMARK (see Proposition 4.1 in [17]). We have the following;

i) When $m_k=1$, the condition (A.3) is equivalent to the following;

$$p^s_{m-1}+(i/2)r_0\{\Lambda_{s+k}, \Lambda_k\}_{|\tau=\lambda_k}\equiv 0 \ \mathrm{mod}\, t^{-1}(\lambda_k-\lambda_{s+k}),$$

where $p^s_{m-1}(t, x, \tau, \xi)$ is the subprincipal symbol of P and $r_0(t, x, \tau, \xi)=p_m/(\tau-\lambda_k)(\tau-\lambda_{s+k})$.

ii) When $m_k=2$, if the following three conditions hold, then (A.3) is valid;

$$p^s_{m-1}(t, x, \lambda_k(t, x, \xi), \xi)=0,$$

$$\partial p^s_{m-1}/\partial\tau_{|\tau=\lambda_k}\equiv\{\Lambda_{s+k}, \Lambda_k\}_{|\tau=\lambda_k}\equiv 0 \ \mathrm{mod}\, t^{-1}(\lambda_k-\lambda_{s+k}),$$

$$p_{m-2}-\sum_{j=0}^{n}p_{m-1,j}^{(j)}/2+\sum_{l,j=0}^{n}p_{m,lj}^{(lj)}/8$$

$$-\sum_{j=0}^{n}r_0(\{\Lambda_{s+k}^{(j)},\ \Lambda_k\}\Lambda_{k,j}-\{\Lambda_{s+k,j},\ \Lambda_k\}\Lambda_k^{(j)})/4$$

$$\equiv 0 \bmod t^{-2}(\lambda_k-\lambda_{s+k}),$$

where $f_{\beta}^{(\alpha)}(x, \xi)=((iD_{\xi})^{\alpha}D_x^{\beta}f)(x, \xi)$ and $r_0(t, x, \tau, \xi)=p_m/(\tau-\lambda_k)^2(\tau-\lambda_{s+k})$.

4. Reduction to a first order system and existence of the symmetrizer.

In this section using conditions (3.10) and (3.11), we show that P is reduced to a first order system whose principal symbol has a symmetrizer with a singularity at $t=0$.

Throughout this section we denote a column vector U by the following form;

$$U={}^t(u_0, \cdots, u_{m-1})={}^t({}^tU_0, \cdots, {}^tU_{m_1}).$$

Here if $j=n_1+\cdots+n_{\nu-1}+\sigma$ $(0\leqq\sigma<n_\nu)$, then $U_j={}^t(u_{\beta(j)}, \cdots, u_{\beta(j)+s_\nu-1})$, where $\beta(j)=N_1+\cdots+N_{\nu-1}+\sigma s_\nu$ and if $j=m_1$, then we put $U_{m_1}={}^t(u_N, \cdots, u_{m-1})$ and $\beta(m_1)=N$. Moreover for simplicity the number of components of U_j is denoted by γ_j. Let $V(t, x)$ be a vector function

$$(4.1)\qquad {}^t(\Delta_0u, \cdots, \Delta_{m-1}u)={}^t({}^tV_0, \cdots, {}^tV_{m_1}),$$

where $u(t, x)$ is smooth function on Ω, and $\Lambda(D_x)$ be a pseudo-differential operator defined by the symbol $|\xi|$. Then we define a $\gamma_j\times\gamma_j$ matrix $E_j(D_x)$ $(j=0, \cdots, m_1)$

$$E_j(D_x)=\begin{pmatrix}\Lambda^{e_j}(D_x) & & & 0\\ & \Lambda^{e_j-1}(D_x) & & \\ & & \ddots & \\ 0 & & & \Lambda^{e_j-\gamma_j+1}(D_x)\end{pmatrix}$$

where $e_j=m-m_1-1+j-\beta(j)$ if $j=n_1+\cdots+n_{\nu-1}+\sigma$ $(0\leqq\sigma<n_\nu)$ and $e_{m_1}=m-N$. We define the column vector function $U(t, x)$

$$(4.2)\qquad {}^t({}^t(E_0V_0), \cdots {}^t(E_{m_1}V_{m_1}))(t, x).$$

Then we have the following;

LEMMA 4.1. *The equation $Pu=f$ is reduced to a first order system*

$$(4.3)\qquad LU(t, x)=(D_t-A(t, x, D_x))U(t, x)=F(t, x).$$

Here $F=(0, \cdots 0, f)$ and A has the following form;

(4.4)
$$A(t, x, D_x)=\begin{pmatrix} A_0 & B_0 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & B_{m_1-1} \\ C_0 & \cdot & \cdot & \cdot & C_{m_1-1} A_{m_1} \end{pmatrix}(t, x, D_x)$$

where $A_j(t, x, D_x)$ $(j=0, \cdots, m_1)$ *is a* $\gamma_j\times\gamma_j$ *first order system,* $B_j(t, x, D_x)$ $(j=0, \cdots, m_1-1)$ *is a* $\gamma_j\times\gamma_{j-1}$ *matrix of order* 0 *and* $C_j(t, x, D_x)$ $(j=0, \cdots, m_1-1)$ *is a* $(m-N)\times\gamma_j$ *first order system. Moreover the principal symbol of* A_j, C_j *is the following;*

(4.5)
$$A_j^0(t, x, \xi)=\begin{pmatrix} \lambda_1(t, x, \xi) & |\xi| & & 0 \\ & \ddots & \ddots & \\ & & \ddots & |\xi| \\ 0 & & & \lambda_{s_\nu}(t, x, \xi) \end{pmatrix}$$

if $j=n_1+\cdots+n_{\nu-1}+\sigma$ $(0\leqq\sigma<n_\nu)$,

(4.6)
$$A_{m_1}^0(t, x, \xi)=\begin{pmatrix} \lambda_{s+1}(t, x, \xi) & |\xi| & & 0 \\ & \ddots & \ddots & \\ & & \ddots & |\xi| \\ 0 & & & \lambda_{m-N+s}(t, x, \xi) \end{pmatrix}$$

and

(4.7)
$$C_j^0(t, x, \xi)=\begin{bmatrix} & 0 & \\ -\tilde{r}_{m_1-j,0}|\xi| & \cdots & -\tilde{r}_{m_1-j,s_\nu-1}|\xi| \end{bmatrix}$$

where $\tilde{r}_{m_1-j,l}(t, x, \xi)=r_{m_1-1,l}(t, x, \xi/|\xi|)$ $(l=0, \cdots, s_\nu-1)$.

PROOF. Since $e_j-\gamma_j+1=e_{j+1}$, the order of B_j is 0. Thus we prove properties with respect to $C_j(t, x, D_x)$ and $A_{m_1}(t, x, D_x)$. Other properties are obvious. It is clear that if $j+i=m_1$, then $\beta(j)=\alpha(i)$. Therefore by (3.10) we have

(4.8)
$$Pu=\sum_{j=0}^{m_1}\sum_{k=0}^{M_{m_1-j}} R_{m_1-j,k}(t, x, D_x)\Lambda_{\beta(j)+k}u\,.$$

We shall write $j=n_1+\cdots+n_{\nu-1}+\sigma$ $(0\leqq\sigma<n_\nu)$, $k=(n_\nu-\sigma)s_\nu+N_{\nu+1}+\cdots+N_{\nu'-1}+n_{\nu'}\sigma'+\delta'$ $(0\leqq\sigma'<n_{\nu'}, 0\leqq\delta'<s_{\nu'})$ and $j'=j+(n_\nu-\sigma)+\cdots+n_{\nu'-1}+\sigma'$. Then $\beta(j)+k=\beta(j')+\delta'$. Since $\Lambda_{\beta(j)+k}u$ is the $(\delta'+1)$-th component of $U_{j'}(t, x)$,

$$R_{m_1-j,k}\Lambda_{\beta(j)+k}u=R_{m_1-j,k}\Lambda^{-(e_{j'}-\delta')}u_{\beta(j)+k}\,.$$

We shall compute the order r of $R_{m_1-j,k}\Lambda^{-(e_{j'}-\delta')}$. We have

$$r=m-(m_1-j)-\alpha(m_1-j)-k-(m-m_1-1+j'-\beta(j'))+\delta$$
$$=1+(j-j')\leqq 1\,,$$

where if $j=j'$, i.e., $k=0, \cdots, s_\nu-1$, then $r=1$. Thus $C_j(t, x, D_x)$ $(j=0, \cdots, m_1-1)$ has the desired property (4.7). Second we shall consider a case $j=m_1$ in (4.8). Since $R_{0,k}(t, x, D_x)$ is of order $m-N-k$, the order $R_{0,k}\Lambda^{-(m-N-k)}$ is 0. It implies that $A_{m_1}(t, x, D_x)$ is the desired property (4.5). This completes the proof of Lemma 4.1.

We shall show the existence of a symmetrizer of $A(t, x, D_x)$.

PROPOSITION 4.2. *There exists a* $m\times m$ *matrix* $M(t, x, \xi)$ *such that*

i) $M(t, x, \xi)$ *is denoted by* $TM^0(t, x, \xi)$, *where the components of* $M^0(t, x, \xi)$ *are positively homogeneous functions of degree* 0, $\det M^0(t, x, \xi)=1$ *and*

$$T=\begin{pmatrix} t^{m_1}I_0 & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & tI_{m_1-1} & \\ 0 & & & & I_{m_1} \end{pmatrix}$$

where I_j *is the* $\gamma_j\times\gamma_j$ *identity matrix.*

ii) *Let* $A^0(t, x, \xi)$ *be the principal symbol of* A. *Then we have the following;*

$$M^{-1}A^0M(t, x, \xi) \tag{4.9}$$

is a real diagonal matrix.

PROOF. Clearly eigen values of $\tilde{A}^0(t, x, \xi)=A^0(t, x, \xi/|\xi|)$ are $\tilde{\lambda}_1(t, x, \xi), \cdots, \lambda_s(t, x, \xi), \tilde{\lambda}_{s+1}(t, x, \xi), \cdots, \tilde{\lambda}_{m-N+s}(t, x, \xi)$ with multiplicities $m_1, \cdots, m_s, 1, \cdots, 1$ respectively, where $\tilde{\lambda}_j(t, x, \xi)=\lambda_j(t, x, \xi/|\xi|)$ $(j=1, \cdots, m-N+s)$. We shall seek an eigen vector $N={}^t(n_0, \cdots, n_{m_1})={}^t({}^tN_0, \cdots, {}^tN_{m_1})$. The equation

$$(\mu I_m-\tilde{A}^0)(t, x, \xi)N=0$$

is equivalent to the following;

$$(\mu I_0-\tilde{A}_0^0)N_0=0, \cdots, (\mu I_{m_1-1}-\tilde{A}_{m_1-1}^0)N_{m_1-1}=0\,,$$
$$-\sum_{j=0}^{m_1-1}\tilde{C}_j^0N_j+(\mu I_{m_1}-\tilde{A}_{m_1}^0)N_{m_1}=0\,,$$

where $\tilde{A}_j^0(t, x, \xi)=A_j^0(t, x, \xi/|\xi|)$ and $\tilde{C}_j^0(t, x, \xi)=C_j^0(t, x, \xi/|\xi|)$.

We shall define

$$M(t, x, \xi)=(N^0(t, x, \xi), \cdots, N^{m-1}(t, x, \xi))\,,$$

where $N^j(t, x, \xi)$ is an eigen column vector of $\lambda_{\delta+1}(t, x, \xi)$ if $j=N_1+\cdots+N_{\nu-1}+\sigma s_\nu+\delta$ $(0\leqq\sigma<n_\nu, 0\leqq\delta<s_\nu)$ and of $\lambda_{s+k+1}(t, x, \xi)$ if $j=N+k$. Let $j'=n_1+\cdots$

$+n_{\nu-1}+\sigma$ if $j=N_1+\cdots+N_{\nu-1}+\sigma s_\nu+\delta$ and $j'=m_1$ if $j\geqq N$. Then we seek $N^j(t, x, \xi)$ as the vector $(N^j)_l(t, x, \xi)=0$ if $l\neq j', m_1$. First we consider the case $j=N_1+\cdots+N_{\nu-1}+\sigma s_\nu+\delta$. For simplicity we drop the above index j. The equation $(\tilde{\lambda}_{\delta+1}I_{j'}-\tilde{A}_{j'})N_{j'}=0$ is equivalent to the following;

$$(4.10)\qquad \begin{aligned}&(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_1)n_{\beta(j')}-n_{\beta(j')+1}=0, \cdots\\ &(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{s_\nu-1})n_{\beta(j')+s_\nu-2}-n_{\beta(j')+s_\nu-1}=0,\\ &(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{s_\nu})n_{\beta(j')+s_\nu-1}=0.\end{aligned}$$

By (4.6) and (4.7) $-\tilde{C}_j^0N_{j'}+(\tilde{\lambda}_{\delta+1}I_{m_1}-A_{m_1}^0)N_{m_1}=0$ becomes to the following;

$$(4.11)\qquad \begin{aligned}&(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{s+1})n_N-n_{N+1}=0, \cdots\\ &(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{m-N+s-1})n_{m-2}-n_{m-1}=0,\end{aligned}$$

$$(4.12)\qquad \sum_{k=0}^{s_\nu-1}\tilde{r}_{m_1-j',k}n_{\beta(j')+k}+(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{m-N+s})n_{m-1}=0.$$

From (4.10) we have $n_{\beta(j')+k}=0$ if $k>\delta$ and can inductively determine $n_{\beta(j')+k}$ $(k<\delta)$ as $n_{\beta(j')+\delta}=t^{m_1-j'}$. Thus (4.12) is denoted by

$$(4.13)\qquad \begin{aligned}&(\tilde{r}_{m_1-j',0}+\sum_{k=1}^{\delta}\tilde{r}_{m_1-j',k}(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_k)\cdots(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_1))n_{\beta(j')}\\ &\quad+(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{m-N+s})n_{m-1}\\ &=S_{j'}n_{\beta(j')}+(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{m-N+s})n_{m-1}=0,\end{aligned}$$

where first equality is a definition of $S_{j'}(t, x, \xi)$. By (A.2), (4.11), and (4.13) we can easily determine n_{N+k} if $k>\delta+1$. When $k=\delta+1$, we see that

$$\begin{aligned}&(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{s+\delta+1})n_{N+\delta}\\ &\quad=-S_{j'}n_{\beta(j')}(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{s+\delta+2})^{-1}\cdots(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{m-N+s})^{-1}.\end{aligned}$$

Since by (3.11) $S_{j'}n_{\beta(j')}\equiv 0 \mod(\tilde{\lambda}_{\delta+1}-\tilde{\lambda}_{s+\delta+1})$, the function $n_{N+\delta}$ is well defined. Inductively by (4.11) we can define n_{N+k} if $k<\delta+1$.

Second we shall construct an eigen vector $N^j(t, x, \xi)$ $(j=N+k)$ of $\tilde{\lambda}_{s+k+1}$. In this case we put $N_l^j=0$ if $l<m_1$. Thus we have $n_{N+i}^j=0$ if $i>k$. Putting $n_{N+k}^j=1$, we can determine inductively n_{N+i}^j $(i<k)$.

Therefore $M(t, x, \xi)$ has the following form;

$$M(t, x, \xi)=\begin{pmatrix} t^{m_1}M_0^0 & & & & 0 & \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & 0 & & & tM_{m_1-1}^0 & \\ D_0 & \cdot & \cdot & \cdot & D_{m_1-1} & M_{m_1}^0 \end{pmatrix}(t, x, \xi)$$

where M_j^0 is a $\gamma_j \times \gamma_j$ triangular matrix whose determinant is 1 and D_j is $N \times \gamma_j$ matrix. This matrix has the desired properties i) and ii). This completes the proof of Proposition 4.2.

Let $M^0(t, x, D_x)$ be a pseudo-differential operator defined by the symbol $M^0(t, x, \xi)$. Then there exists a pseudo-differential operator $(M^0)^{-1}(t, x, D_x)$ such that

$$M^0(M^0)^{-1}(t, x, D_x)=(M^0)^{-1}M^0(t, x, D_x)=I_m .$$

Put $M^{-1}(t, x, D_x)=(M^0)^{-1}T^{-1}(t, x, D_x)$. Then we have the following;

PROPOSITION 4.3. *Let* $\tilde{U}(t, x)=M^{-1}(t, x, D_x)U$ *and* $\tilde{F}(t, x)=M^{-1}(t, x, D_x)F$. *Then the first order system* $LU=F$ *becomes to*

$$\tilde{L}(t, X, D_t, D_x)\tilde{U}=(D_t-(\tilde{A}+t^{-1}\tilde{B}))\tilde{U}=\tilde{F} . \tag{4.14}$$

Here $\tilde{A}(t, x, D_x)$ *is of order* 1 *and* $(\tilde{A}-\tilde{A}^*)(t, x, D_x)$, *where* $\tilde{A}^*$ *is the adjoint of* $\tilde{A}$ *and* $\tilde{B}(t, x, D_x)$ *are of order* 0.

PROOF. Let $A^0(t, x, D_x)$ be a pseudo-differential operator defined by the symbol $A^0(t, x, \xi)$ and put $B(t, x, D_x)=(A-A^0)(t, x, D_x)$. Then

$$\tilde{L}\tilde{U}=(D_t-M^{-1}(A^0+B)M+M^{-1}(D_tM))\tilde{U}=\tilde{F} .$$

By (4.4) the components of $T^{-1}A^0T$ are smooth in $\Omega\times(R^n\backslash 0)$ and $T^{-1}BT+T^{-1}(D_tT)$ is denoted by $t^{-1}\tilde{B}'$, where the symbol of $\tilde{B}'$ is smooth in $\Omega\times(R^n\backslash 0)$. This completes the proof of Proposition 4.3.

5. Energy inequality and existence theorem.

In this section we derive the energy inequality for (4.3) from Proposition 4.3. Using the energy inequality for adjoint system of (4.3), we obtain a existence theorem of our considered operator. The operators with the analogous property to (4.14) are studied by many authors (see [**10**] and [**12**]).

For non-negative integer k and $s\in R$ we use the following norm;

$$|||u(t)|||_{k,s}^2=\sum_{j=0}^{k}\|D_t^j u(t)\|_{s+k-j}^2 ,$$

where $\|\cdot\|_{s+k-j}$ is the usual norm of $H_{s+k-j}(R^n)$. The following lemmas are easy.

LEMMA 5.1. *Let* $L(t, x, D_t, D_x)U=(D_t-A(t, x, D_x))U=F$ *be a first order system such that* $(A-A^*)$ (t, x, D_x) *is of order* 0. *Then we have*

$$|D_t(|||U(t)|||_{k,s})|\leqq C(|||U(t)|||_{k,s}+|||F(t)|||_{k,s}) , \tag{5.1}$$

where C does not depend on U and F.

LEMMA 5.2. *Let* $\Phi(t)$ *be a real valued function of* $C^1(0, T]$. *Then we have* *the following two statements.*

i) *If* $\Phi(t)$ *satisfies the conditions* $\Phi(t)=O(t^{C_0+1})$ *and*

$$t(\partial\Phi/\partial t)(t)\leqq C_0\Phi(t)+C_1t\Phi(t)+C_2\Psi(t)\,, \tag{5.2}$$

where $\Psi(t)$ *is real, then we have*

$$\Phi(t)\leqq C_3\int_0^t \tau^{-(C_0+1)}\Psi(\tau)d\tau\,. \tag{5.3}$$

ii) *If* $\Phi(t)$ *satisfies the condition;*

$$-t(\partial\Phi/\partial t)(t)\leqq C_4\Phi(t)+C_5t\Phi(t)+C_6\Psi(t)\,, \tag{5.4}$$

then we have

$$t^{C_4}\Phi(t)\leqq C_7t^{C_4}\Phi(T)+C_8\int_t^T \tau^{C_4-1}\Psi(\tau)d\tau\,. \tag{5.5}$$

Now we shall examine the energy inequality of $L(t, x, D_t, D_x)$ of (4.3).

PROPOSITION 5.3. *Let* $\tilde{L}\tilde{U}=\tilde{F}$ *be a first order system of* (4.14). *If* $\tilde{U}(t)=O(t^{N_1})$, *where* N_1 *is sufficiently large, then for any non-negative integer k and* $s\in R$ *there exists a positive constant* N_2 *depending on k and s such that*

$$|||\tilde{U}(t)|||_{k,s}^2\leqq C\int_0^t \tau^{-N_2}|||\tilde{F}(\tau)|||_{k,s}^2d\tau\,. \tag{5.6}$$

PROOF. Applying Lemma 5.1 to $L=D_t-\tilde{A}(t, x, D_x)$ as $U(t)=\tilde{U}(t)$ and $F(t)=\tilde{F}(t)+t^{-1}\tilde{B}\tilde{U}(t)$, by (5.1) we have

$$(5.7)_j \qquad \partial(|||\tilde{U}(t)|||_{j,s}^2)/\partial t$$
$$\leqq C(|||\tilde{U}(t)|||_{j,s}^2+\sum_{l=0}^{j}|||U(t)|||_{j-l,s}^2/t^{2l+1}+|||F(t)|||_{j,s}^2)\,.$$

Multiply $(5.7)_j$ by $t^{-2(k-j)}$ and add them from $j=0$ to $j=k$. Then we have

$$t(\partial\Phi_{k,s}/\partial t)(t)\leqq C_0\Phi_{k,s}(t)+C_1t\Phi_{k,s}(t) +C_2t^{-2k}|||F(t)|||_{k,s}^2 \tag{5.8}$$

where $\Phi_{k,s}(t)=\sum_{l=0}^{k}|||U(t)|||_{k-l,s}^2t^{-(2l+1)}$. Therefore by (5.3) we have (5.6) if $2(N_1-k)\geqq C_0+2$. This completes the proof of Proposition 5.3.

By this proposition we have the following

THEOREM 5.4. *Let* $P(t, x, D_t, D_x)$ *be our considered differential operator satisfying conditions* (A.1), (A.2) *and* (A.3). *Then for the Cauchy problem* $Pu=f$, $D_t^ju=g_j$ $(j=0, \cdots, m-1)$ *we have the following inequality; For any non-negative integer k and* $s\in R$ *there exists N depending on k and s such that*

$$(5.9)\qquad |||u(t)|||_{k+m-m_1-1,s}^2 \leqq C\Big(\sum_{j=0}^{m-1}\|g_j\|_{s+k+N+m-j}^2 + |||f(0)|||_{N,K+s}^2 + \int_0^t |||D_t^N f(\tau)|||_{k,s}^2 d\tau\Big).$$

PROOF. Let $\hat{U}(t, x)=U(t, x)-\sum_{j=0}^{N}(it)^j(D_t^j U)(0, x)/j!$ and $\hat{F}(t, x)=L\hat{U}(t, x)=F(t, x)-\sum_{j=0}^{N}L((it)^j(D_t^j U)(0, x))/j!$. Put $\tilde{U}(t, x)=M^{-1}(t, x, D_x)\hat{U}$ and $\tilde{F}(t, x)=M^{-1}(t, x, D_x)\hat{F}$. Then we have $LU(t, x)=F(t, x)$. Since $F(t, x)=O(t^N)$, by the Taylor expansion we see that

$$(5.10)\qquad \int_0^t \tau^{-N_2}|||\tilde{F}(\tau)|||_{k,s}^2 d\tau \leqq C\int_0^t |||D_t^N \hat{F}(\tau)|||_{k,s}^2 d\tau,$$

if N is sufficiently large. Thus by (5.6), (5.10) and the definition of $\hat{U}(t, x)$ and $\hat{F}(t, x)$ we obtain

$$(5.11)\qquad |||U(t)|||_{k,s}^2 \leqq C\Big(|||U(0)|||_{N+1,k+s}^2 + \int_0^t |||D_t^N F(\tau)|||_{k,s}^2 d\tau\Big).$$

On the other hand we have

$$(5.12)\qquad |||u(t)|||_{k,s} \leqq C_1|||U(t)|||_{k-m+m_1+1,s} \leqq C_2|||u(t)|||_{k+m_1,s}.$$

Thus by (5.11) and (5.12) we have the desired estimate (5.9). This completes the proof of Theorem 5.4.

REMARK. Instead of (2.3) if we assume that

$$q_{k,l|\tau=\lambda_k} \equiv 0 \qquad \bmod \lambda_k - \lambda_{s+k},$$

then we have the following estimate (see [**17**]); For an integer $k \geqq 0$

$$|||u(t)|||_{k+m-m_1-1,s}^2 \leqq C\Big(\sum_{j=0}^{m-1}\|g_j\|_{s+k+m-j-1}^2 + |||f(0)|||_{k-1,s}^2 + \int_0^t |||f(\tau)|||_{k,s}^2 d\tau\Big),$$

where if $k=0$, then $|||f(0)|||_{k-1,s}^2$ does not appear, i.e., $|||f(0)|||_{-1,s}=0$.

To prove the existence of a solution for the Cauchy problem

$$L(t, x, D_t, D_x)U(t, x)=F(t, x), \quad U(0, x)=0$$

we shall consider the adjoint Cauchy problem

$$L^*(t, x, D_t, D_x)V(t, x)=G(t, x), \quad V(T, x)=0.$$

The following lemma is derived from an integration by parts (see (33) in [**13**]).

LEMMA 5.5. *Let N be a non-negative integer. Then we have*

(5.13) $$\int_0^T |||V(t)|||_{k,s}^2 dt \leqq C\int_0^T t^{2N} |||D_t^N V(t)|||_{k,s}^2 dt .$$

Since $L^*(t, x, D_t, D_x)$ is noncharacteristic with respect to t, we have the following

LEMMA 5.6. *Let $V(t, x)$ be a solution of $(L^*V)(t, x)=G(t, x)$ and N a nonnegative integer. Then we have*

(5.14) $$|||D_t^N V(t)|||_{k,s}^2 \leqq C(|||V(t)|||_{k,s+N}^2 + |||G(t)|||_{k+N-1,s}^2) .$$

On the energy inequality for the Cauchy problem of L^* with zero data we have the following

PROPOSITION 5.7. *Let $V(t, x) \in C^\infty([0, T]; H_\infty(R^n))$ with* $\operatorname{supp} V \subset (-\infty, T] \times R^n$. *Then there exists $N>0$ depending on s and k such that*

(5.15) $$\int_0^T |||V(t)|||_{k,s-N}^2 dt \leqq C\int_0^T |||L^*V(t)|||_{k+N,s-N}^2 dt .$$

PROOF. Since L is transformed into $\tilde{L}$ by $M(t, x, D_x)$, L^* is transformed into the operator $\tilde{L}^*$, which has same properties as L, by $(M^*)^{-1}(t, x, D_x)$. Let $\tilde{V}(t, x)=M^*(t, x, D_x)V$ and $\tilde{G}(t, x)=M^*L^*V(t, x)$. Then we have $\tilde{L}^*V(t, x) =\tilde{G}(t, x)$. Applying ii) of Lemma 5.2 to the analogous inequality to (5.8), whose the left hand side is $-t(\partial\Phi_{k,s}/\partial t)(t)$, we see that

$$t^{N_1}|||\tilde{V}(t)|||_{k,s}^2 \leqq C\int_t^T \tau^{N_1-1-2k}|||\tilde{G}(\tau)|||_{k,s}^2 d\tau .$$

Our desired estimate (5.15) follows from (5.13), (5.14) and (5.16). This completes the proof of Proposition 5.7.

We put

$$\|V\|_{k,s}^2 = \int_0^T |||V(t)|||_{k,s}^2 dt .$$

Then the space completed by this norm is denoted by $H_{k,s}(\Omega)$. When $\Omega = \bar{R}_{n+1}^+$, for any k, $s \in R$ this space is also defined in Definition 2.5.1 of [3]. By (3.15) with $k=0$ we have

$$|(F, V)| = \left|\int_0^T\int_{R^n} FV dx dt\right| \leqq C\|L^*V\|_{N,s-N} ,$$

where $F \in C^\infty([0, T]; H_\infty(R^n))$, $\operatorname{supp} V \subset (-\infty, T] \times R^n$ and N depends on s. Thus by Theorem 2.5.1 of [3] there exists $U(t, x) \in H_{-N,-s+N}(\bar{R}_{n+1}^+)$ such that for all $V \in C^\infty([0, T]; H_\infty(R^n))$ with $\operatorname{supp} V \subset (-\infty, T] \times R^n$

$$(F, V) = (U, L^*V) .$$

The function U is therefore a distribution solution of $LU=F$. Since $F \in C^\infty([0, T]; H_\infty(R^n))$, by Theorem 4.3.1 of [3] we see that for any $k \in R$, $U \in H_{k,-s-k}$

$(\bar{R}^+_{n+1})$. Therefore by Green's theorem we see that $U(t, x)$ is a smooth solution of the Cauchy problem $LU=F$, $U(0, x)=0$. This completes the proof of Theorem 1.

6. The proof of Theorem 2.

In this section using lemmas in Section 5 and a few propositions, we shall prove the well posed of the Cauchy problem for the operators satisfying the conditions (H.1), (H.2) and (H.3).

We shall start from the following

PROPOSITION 6.1. *Let $I=\{i_1, \cdots, i_k\}$ be a subset of $J=\{1, \cdots, m\}$. We assume that the maximal multiplicity of $\{\lambda_j(t, x, \xi)\}_{j\in J\setminus I}$ is r_1. Let $A(t, x, D_t, D_x)$ be a pseudo-differential operator of order $m-k-r_1$ which is a differential operator with respect to t. Then we have*

$$A(t, x, D_t, D_x)=\sum_{l=0}^{m-k-r_1}\gamma_{j_1\cdots j_l}\Lambda_{j_1}\cdots\Lambda_{j_l}, \tag{6.1}$$

where $\gamma_{j_1\cdots j_l}(t, x, D_x)$ is of order 0 *and $\{j_1\cdots j_l\}\subset J\setminus I$.*

PROOF. First we shall prove the lemma when A is a pseudo-differential operator of x with a parameter t. By the partition of unity we may assume that $J\setminus I=J_1\cup\cdots\cup J_\mu$ and $\lambda_i(t, x, \xi)\neq\lambda_j(t, x, \xi)$ if $i\in J_\alpha$, $j\in J_\beta$ and $\alpha\neq\beta$ $(1\leqq\alpha, \beta\leqq\mu)$. We denote the number of $\lambda_j(t, x, \xi)$ belonging to $j\in J_\alpha$ by $|J_\alpha|$ and assume $r_1\geqq|J_1|\geqq\cdots\geqq|J_\mu|$. Let $J'=J'_1\cup\cdots\cup J'_\mu$ be a subset of $J\setminus I$, where J'_α is also a subset of J_α. If $|J'|\leqq(m-k-r_1)-1\leqq|J\setminus I|-|J_1|-1$, then there exists α, β $(1\leqq\alpha<\beta\leqq\mu)$ such that $\alpha\neq\beta$ and $J_\alpha\setminus J'_\alpha$ and $J_\beta\setminus J'_\beta$ are not empty. Therefore since $|(\lambda_i-\lambda_j)(t, x, \xi)|>\delta|\xi|$ if $i\in J_\alpha$ and $j\in J_\beta$, we have $A=A_1(\Lambda_i-\Lambda_j)+A_2$, where $A_l(t, x, D_x)$ $(l=1, 2)$ is of order $m-k-r_1-1$. Thus we have (6.1) when $A=A(t, x, D_x)$.

We define $\mu_j(t, x, \xi)$ $(j=1, \cdots, m-k)$ as the following; $\{\mu_j\}_{j=1,\cdots,m-k}$ equal to $\{\lambda_j(t, x, \xi)\}_{j\in J\setminus I}$ and if $j=|J_1|+\cdots+|J_{\alpha-1}|+\sigma$ $(1\leqq\sigma\leqq|J_\alpha|)$ then $\mu_j\in\{\lambda_j\}_{j\in J_\alpha}$. Denote a pseudo-differential operator defined by the symbol $\tau-\mu_j(t, x, \xi)$ by $\tilde{\Lambda}_j(t, x, D_t, D_x)$. Then we have

$$A(t, x, D_t, D_x)=A_0+\sum_{j=1}^{m-k-r_1}A_j\tilde{\Lambda}_1\cdots\tilde{\Lambda}_j,$$

where $A_j(t, x, D_x)$ $(j=0, \cdots, m-k-r_1)$ is of order $m-k-r_1-j$. Since $m-k-r_1-j\leqq m-k-j-\max(|J_{\alpha+1}|, |J_\alpha|-\sigma)$, by the result of the first half we obtain (6.1). This completes the proof of Proposition 6.1.

By this lemma we get the following two corollaries.

COROLLARY 6.2. *Let $P(t, x, D_t, D_x)$ be a differential operator satisfying the conditions* (H.1), (H.2) *and* (H.3). *Then we have*

$$P=\Lambda_1\cdots\Lambda_m+\sum_{0\leq k<m}t^{-(m-k)}\gamma_{i_1\cdots i_k}\Lambda_{i_1}\cdots\Lambda_{i_k},$$

where $\{i_1,\cdots,i_k\}$ *is a subset of* $\{1,\cdots,m\}$ *and* $\gamma_{i_1\cdots i_k}(t,x,D_x)$ *is of order* 0.

Now we shall prove the statement mentioned in Remark 1.2.

COROLLARY 6.3. *Let* $P(t,x,D_t,D_x)$ *be a differential operator which satisfies the conditions* (A.1), (A.2) *and* (A.3). *Then we can write* P *by the form* (2.6).

PROOF. From (A.2) we can apply Proposition 6.1 to $\lambda_j(t,x,\xi)$. We use the notation in (3.5). If $i=n_\mu+\cdots+n_{\nu+1}+\sigma$ $(\sigma\neq 0)$, by the proof of Proposition 6.1 it follows that

$$Q_i(t,x,D_t,D_x)=\sum_I Q_{i,I}\Lambda_{i_1}\cdots\Lambda_{i_{s_\nu-1}}+Q_{i,0},$$

where $I=\{i_1,\cdots,i_{s_\nu-1}\}$ is a subset of $\{1,\cdots,s_\nu\}$, $Q_{i,I}(t,x,D_t,D_x)$ is of order $m-i-\alpha(i)-s_\nu+1$ and $Q_{i,0}(t,x,D_t,D_x)$ is of order $m-i-\alpha(i)-1$. Furthermore by (3.6) we have

$$t^iQ_i(t,x,D_t,D_x)=\sum_J Q_{i,J}\Lambda_{j_1}\cdots\Lambda_{j_{s_\nu}}+Q'_{i,0},$$

Here $J=\{j_1,\cdots,j_{s_\nu}\}$ is equal to $\{1,\cdots,s_\nu\}$ or $\{s+i_{s_\nu}\}\cup I$, where $\{i_{s_\nu}\}\cup I=\{1,\cdots,s_\nu\}$, $Q_{i,J}(t,x,D_t,D_x)$ is of order $m-i-\alpha(i)-s_\nu$ and $Q'_{i,0}(t,x,D_t,D_x)$ is of order $m-i-\alpha(i)-1$. Thus by Proposition 6.1 $Q_i\Delta_{\alpha(i)}$ is denoted by the desired form. When $i=0$, we have $Q_0\Delta_{\alpha(i)}=\Delta_m+R_0\Delta_N$, where $R_0(t,x,D_t,D_x)$ is of order $m-N-1$. Using Proposition 6.1 again as $A=R_0$, we have Corollary 6.3.

About the permutation of $\Lambda_1,\cdots,\Lambda_m$ we have the following

LEMMA 6.4. *Let* $\{i_1,\cdots,i_m\}$ *be a permutation of* $\{1,\cdots,m\}$. *Then we have*

$$\Lambda_{i_1}\cdots\Lambda_{i_m}=\Lambda_1\cdots\Lambda_m+\sum_{0\leq k<m}t^{-(m-k)}\gamma_{i_1\cdots i_k}\Lambda_{i_1}\cdots\Lambda_{i_k},\tag{6.2}$$

where $\gamma_{i_1\cdots i_k}(t,x,D_x)$ *is of order* 0 *and* $\{i_1,\cdots,i_k\}$ *is a subset of* $\{1,\cdots m\}$.

PROOF. We may consider the case $\{i_1,\cdots,i_m\}=\{1,\cdots,j-1,j+1,j,j+2,\cdots,m\}$. We have

$$\begin{aligned}&\Lambda_1\cdots\Lambda_{j-1}\Lambda_{j+1}\Lambda_j\Lambda_{j+2}\cdots\Lambda_m\\&\qquad=\Lambda_1\cdots\Lambda_m+\Lambda_1\cdots\Lambda_{j-1}[\Lambda_{j+1},\Lambda_j]\Lambda_{j+2}\cdots\Lambda_m.\end{aligned}$$

By (H.2) we have $[\Lambda_{j+1},\Lambda_j]=t^{-1}\gamma_1(\Lambda_j-\Lambda_{j+1})+t^{-1}\gamma_2$, where $\gamma_l(t,x,D_x)$ $(l=1,2)$ is of order 0. Since $[\Lambda_i,t^{-k}\gamma_3]=t^{-k-1}\gamma_4$, where $\gamma_l(t,x,D_x)$ $(l=3,4)$ is of order 0, we have the desired (6.2). This completes the proof of Lemma 6.4.

Now we shall start from the proof of Theorem 2.

THEOREM. 6.5. *Let* $P(t,x,D_t,D_x)$ *be a differential operator satisfying the conditions* (H.1), (H.2) *and* (H.3). *Then for any* $s\in R$ *and a non-negative integer* k *there exists* N *such that*

$$(6.3)\qquad |||u(t)|||^2_{k+m-r,s} \leqq C\{\sum_{j=0}^{m-1} \|g_j\|^2_{s+k+m+N-j} + |||f(0)|||^2_{N-m,k+s+m} + \int_0^t |||D^{N-m} f(\tau)|||^2_{k,s} d\tau\},$$

where $Pu=f$, $D_t^j u_{|t=0}=g_j$ $(j=0, \cdots, m-1)$ *and* $u \in C^\infty([0, T]; H_\infty(R^n))$.

PROOF. Let $I=(i_1, \cdots, i_l)$ be a subset of $\{1, \cdots, m\}$ with the length $|I|=l$. Then we put

$$(A_I v)(t, x)=t^{-(m-k)} \Lambda_{i_1} \cdots \Lambda_{i_l} v(t, x),$$

where if $I=\emptyset$, then we put $|I|=0$ and $A_I v=t^{-m}v$. If $I'=(i_0, I)$, we see that

$$\Lambda_{i_0}(A_I v)(t, x)=-(m-k)t^{-1}(A_I v)+t^{-1}(A_{I'} v).$$

By the same computation in the proof of Proposition 5.3, we obtain

$$(6.4)_I\qquad t(\partial \Phi_I/\partial t)(t) \leqq C\{\Phi_I(t)+t\Phi_I(t)+\Phi_{I'}(t)\},$$

where $\Phi_I(t)=\sum_{j=0}^{k}(|||A_I v(t)|||^2{}_{k-j,s}/t^{2j+1})$. Add $(6.4)_I$ from $|I|=0$ to $|I|=m-1$. By Corollary 6.2 and Lemma 6.4 then we have

$$t(\partial \Phi/\partial t)(t) \leqq C\{\Phi(t)+t\Phi(t)+t^{-2k-1}|||Pv(t)|||^2_{k,s}\},$$

where $\Phi(t)=\sum_{|I| \leqq m-1} \Phi_I(t)$. Therefore if $v(t, x)=O(t^{N+1})$, where N is sufficiently large, by i) of Lemma 5.2 and the Taylor expansion of $Pv(t, x)$ we get

$$(6.5)\qquad \sum_{|I| \leqq m-1} |||t^{-(m-l)} \Lambda_{i_1} \cdots \Lambda_{i_l} v(t)|||^2_{k,s} \leqq C\int_0^t |||D^{N-m}(Pv)(\tau)|||^2_{k,s} d\tau.$$

Since $(1+|D_x|^2)^{(m-r-l)/2} D_t^l$ $(l=0, \cdots, m-r)$ is of order $m-r$, we get by Proposition 6.1 and (6.5)

$$(6.6)\qquad |||v(t)|||^2_{k+m-r,s} \leqq C\int_0^t |||D_t^{N-m}(Pv)(\tau)|||^2_{k,s} d\tau.$$

Put $v(t, x)=u-\sum_{j=0}^{N}(it)^j(D_t^j u)(0, x)/j!$ in (6.6). Then we get the desired (6.3). This completes the proof of Proposition 6.5.

Since $P^*(t, x, D_t, D_x)$ satisfies the condition (H.1), (H.2) and (H.3) if $P(t, x, D_t, D_x)$ has the same those, we have the following;

PROPOSITION 6.6. *Let* $P(t, x, D_t, D_x)$ *be a differential operator satisfying the conditions* (H.1), (H.2) *and* (H.3) *and* $P^*(t, x, D_t, D_x)$ *be the adjoint operator of* P. *Then for* $k \geqq m-1$ *and* $s \in R$ *there exists* N *such that*

$$(6.7)\qquad \int_0^T |||v(t)|||^2_{k,s-N} dt \leqq C\int_0^T |||P^* v(t)|||^2_{k+N,s-N} dt.$$

where $v(t, x)\in C^{\infty}([0, T]; H_{\infty}(R^{n}))$ *with* supp $v\subset(-\infty, T]\times R^{n}$.

PROOF. We use the same notation as those in the proof of Proposition 6.5. By the same computation we have

$$-t(\partial\Phi/\partial t)(t)\leqq C\{\Phi(t)+t\Phi(t)+t^{-2k-1}\|P^{*}v(t)\|_{k,s}^{2}\}.$$

Using ii) of Lemma 5.2 instead of i), we see that

$$t^{2N}|||v(t)|||_{k,s}^{2}\leqq C\int_{0}^{T}|||P^{*}v(\tau)|||_{k,s}^{2}d\tau.$$

By Lemma 5.5 and 5.6 as $L^{*}=P^{*}$ when $k\geqq m-1$ we have the desired (6.7). This completes the proof of Proposition 6.6.

By the same argument as that in Section 5 we can show the existence of a smooth solution for the Cauchy problem $Pu=f$ in Ω, $D_{t}^{j}u_{|t=0}=0$ $(j=0, \cdots, m-1)$. This completes the proof of Theorem 2.

7. Some necessary condition and examples.

In this section we call $P(t, x, D_{t}, D_{x})$ to be uniformly well posed if the Cauchy problem $Pu=f$ in Ω_{t_0}, $D_{t}^{j}u_{|t=t_0}=g_{j}$ $(j=0, \cdots, m-1)$, where $0\leqq t_{0}<T$ and $\Omega_{t_0}=[t_{0}, T]\times R^{n}$, has the unique solution for every $t_{0}\in[0, T)$ and a local uniqueness property or $P(t, x, D_{t}, D_{x})$ satisfies the condition (E) of Definition 1 and the condition $(U_{\Gamma(\eta)})$ of Definition 2 in [6].

Necessary condition with respect to a multiplicity of λ_{k} in Theorem 1 is the following

REMARK 7.1. Let $P(t, x, D_{t}, D_{x})$ satisfy the conditions (A.1) and (A.2). Moreover if P is uniformly well posed, then there exist $Q_{k,l}(t, x, D_{t}, D_{x})$ of order $m-m_{k}$ such that

$$P(t, x, D_{t}, D_{x})=\sum_{l=0}^{m_k}Q_{k,l}(\Lambda_{k})^{m_{k}-l}.$$

Moreover if (t, x, ξ) is an interior point of $N_{k}=\{(t, x, \xi);\ \lambda_{k}=\lambda_{k+s}\}$, then $q_{k,l|\tau=\lambda_{k}}=0$, where $q_{k,l}$ is the principal symbol of $Q_{k,l}$. Since the proofs of the statement in [2] and Theorem 2.10 in [1] are done microlocally, this remark is clear when (t, x, ξ) is an interior point of N_{k} or belongs to the complementary set of N_{k}. Therefore by a limit process we obtain Remark 7.1.

Next we consider a differential operator on $[0, T]\times R^{1}$. We assume that the characteristic roots $\lambda_{k}(t, x, \xi)=f_{k}(t, x)\xi$ $(k=1, \cdots, m-N+s)$ and a difference

$$(f_{k}-f_{k+s})(t, x)=t^{j}G_{k}(t, x)^{n}A_{k}(t, x).$$

Here $A_{k}\neq 0$, $j>m_{k}$ and $G_{k}(t, x)$ is the inverse function of $x=g_{k}(t, y)$ in a neighbourhood of $t=0$, where g_{k} is the solution of the characteristic equation

$\partial g_k/\partial t=-f_k(t, g_k)$, $g_k(0, y)=y$. Then we have the following

EXAMPLE 7.2. Let $P(t, x, D_t, D_x)$ be a differential operator satisfying (A.1), (A.2) and the above conditions. If P is uniformly well posed, then (A.3) is valid.

PROOF. We consider the coordinate transform $t=s$, $x=g_k(s, y)$. Then by Remark 7.1 we have

$$P(s, y, D_s, D_y)=\sum_{l=0}^{m_k} R_{k,l}(s, y, D_s, D_y)D_s^{m_k-l}$$

where $R_{k,l}$ is of order $m-m_k$, whose principal symbol is $r_{k,l}(s, y, \sigma, \eta)$. The condition (A.3) is equivalent to

$$r_{k,l}(s, y, 0, \eta)=s^{j-l}y^n B_{k,l}(s, y)\eta^{m-m_k}, \tag{7.1}$$

where $B_{k,l}(s, y)$ is some smooth function. The Theorem 4.1 of [6] implies (7.1). Now we shall examine the condition (H.3).

EXAMPLE 7.3. We shall consider the following;

$$P(t, x, D_t, D_x)=(D_t-t^l x^n D_x)^2(D_t+t^l x^n D_x)^2$$
$$+P_3(t, x, D_t, D_x),$$

where $l>3$ and $P_3(t, x, D_t, D_x)$ is of order 3. Then by the same reason as the proof of Remark 7.1 we have

$$P_3=(aD_t+bD_x)(D_t-t^l x^n D_x)(D_t+t^l x^n D_x)$$
$$+cD_t^2+dD_tD_x+eD_x^2+fD_t+gD_x+h.$$

By Theorem 4.1 of [6] we get that $b=t^{l-1}x^n b'$, $d=t^{l-2}x^n d'$, $e=t^{2(l-1)}x^{2n}e'$ and $g=t^{l-3}x^n g'$. This is the condition (H.3).

References

[1] J. Chazarain, Opérateurs hyperboliques à charactéristiques de multiplicité constante, Ann. Inst. Fourier (Grenoble), **24** (1974), 173-202.

[2] H.Flaschka and G. Strang, The correctness of the Cauchy problem, Advances in Math., **6** (1971), 347-379.

[3] L. Hörmander, Linear partial differential operators, Springer Verlag, Berlin, 1963.

[4] L. Hörmander, The Cauchy problem for differential equations with double characteristics, J. D'analyse Math., **32** (1977), 118-196.

[5] V. Ia. Ivrii, Sufficient conditions for regular and completely regular hyperbolicity, Trudy Moskov Math. Obšč., **33** (1975), 1-65.

[6] V. Ia. Ivrii and V. M. Petkov, Necessary conditions for the correctness of the Cauchy problem for non-strictly hyperbolic equations, Uspehi Mat. Nauk, **29** (1974), 3-70.

[7] K. Kitagawa and T. Sadamatsu, Une condition suffisante pour que le problème de Cauchy faiblement hyperbolique soit bien posé, to appear in J. Math. Kyoto

Univ.
[8] A. Lax, On Cauchy's problem for partial differential equations with multiple characteristics, Comm. Pure Appl. Math., **9** (1956), 135-169.
[9] E.E. Levi, Caratteristiche multiple e problema di Cauchy, Ann. di Math., **16** (1909), 161-201.
[10] A. Menikoff, The Cauchy problem for weakly hyperbolic equations, Amer. J. Math., **97** (1975), 548-558.
[11] S. Mizohata and Y. Ohya, Sur la condition de E. E. Levi concernant des équations hyperbolique, Publ. Res. Inst. Math. Sci., Ser. A. **4** (1968), 511-526.
[12] Y. Ohya, Le probleme de Cauchy à charactéristique multiples, to appear in Ann. Scuola Norm. Sup. Pisa.
[13] O. A. Oleinik, On the Cauchy problem for weakly hyperbolic equations, Comm. Pure Appl. Math., **23** (1970), 569-586.
[14] G. Paser, Hyperbolic equations with multiple characteristics and time dependent coefficients, SIAM J.Math. Anal., **2** (1971), 402-412.
[15] V. M. Petkov, The Cauchy problem for a class of non-strictly hyperbolic equations with double characteristics, Serdica Bulg. Math. Publ., **1** (1975), 372-380.
[16] M. Yamaguti, Le problème de Cauchy et les opérateurs d'intégrale singulière, Mem. Coll. Sci. Univ. Kyoto Ser. A Math., **32** (1959), 121-151.
[17] K. Yamamoto, Cauchy problem for hyperbolic operators with variable multiple characteristics, Proc. Amer. Math. Soc., **72** (1978), 109-116.
[18] M. Zeman, The well-posedness of the Cauchy problem for partial differential equations with multiple characteristics, Comm. in Partial Diff. Equations, **2** (1977), 223-249.

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