J. Math. Soc. Japan
Vol. 30, No. 3, 1978

On a duality for C^*-crossed products by a locally compact group

By Shō IMAI and Hiroshi TAKAI

(Received March 12, 1977)

Abstract.

Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system, and $C_r^*(\mathfrak{A}; \alpha)$ the reduced C^*-crossed product of $\mathfrak{A}$ by α. We construct a "dual" C^*-crossed product $C_d^*(C_r^*(\mathfrak{A}; \alpha); \beta)$ of $C_r^*(\mathfrak{A}; \alpha)$ by an isomorphism β from $C_r^*(\mathfrak{A}; \alpha)$ into the full operator algebra $\mathcal{L}(\mathfrak{K})$ on a Hilbert space $\mathfrak{K}$. Then, it is isomorphic to the C^*-tensor product $\mathfrak{A}\hat{\otimes}_*\mathcal{C}(L^2(G))$ of $\mathfrak{A}$ and the C^*-algebra $\mathcal{C}(L^2(G))$ of all compact operators on $L^2(G)$.

In the abelian case, there exists a continuous action $\hat{\alpha}$ of the dual group $\hat{G}$ of G on the C^*-crossed product $C^*(\mathfrak{A}; \alpha)$ of $\mathfrak{A}$ by α such that the C^*-crossed $C^*(C^*(\mathfrak{A}; \alpha); \hat{\alpha})$ of $C^*(\mathfrak{A}; \alpha)$ by $\hat{\alpha}$ is isomorphic to $C_d^*(C_r^*(\mathfrak{A}; \alpha); \beta)$.

§ 1. Introduction.

In [4], the second author showed a C^*-algebra version of Takesaki's duality theorem for crossed products of von Neumann algebras. In other words, given a C^*-dynamical system $(\mathfrak{A}, G, \alpha)$ based on a locally compact abelian group G, there exists a continuous action $\hat{\alpha}$ of the dual group $\hat{G}$ of G on the C^*-crossed product $C^*(\mathfrak{A}, \alpha)$ of $\mathfrak{A}$ by α such that the C^*-dynamical system $(C^*(C^*(\mathfrak{A}; \alpha); \hat{\alpha}), G, \hat{\hat{\alpha}})$ is equivalent to the C^*-dynamical system $(\mathfrak{A}\hat{\otimes}_*\mathcal{C}(L^2(G)), G, \alpha\otimes \mathrm{Ad}(\lambda))$, where $\mathcal{C}(L^2(G))$ is the C^*-algebra of all compact operators on $L^2(G)$, and λ is the regular representation of G on $L^2(G)$.

Recently, Y. Nakagami [3] generalized Takesaki's duality theorem based on abelian groups to non-abelian groups using the method on Hopf-von Neumann algebras. (Also see [2].)

In this paper, we study a non-abelian duality for C^*-crossed products referring to Nakagami's construction in von Neumann algebras. Actually, we obtain that for a C^*-dynamical system $(\mathfrak{A}, G, \alpha)$, there exists an isomorphism β of the reduced C^*-crossed product $C_r^*(\mathfrak{A}; \alpha)$ of $\mathfrak{A}$ by α into the full operator algebra $\mathfrak{L}(L^2(G\times G; \mathfrak{H}))$ on the Hilbert space $L^2(G\times G; \mathfrak{H})$ such that the "dual" C^*-crossed product $C_d^*(C_r^*(\mathfrak{A}; \alpha); \beta)$ is isomorphic to the tensor product $\mathfrak{A}\hat{\otimes}_*\mathcal{C}(L^2(G))$.

In the abelian case, it is verified that the construction of "dual" C^*-crossed products is exactly that of second C^*-crossed products via the Fourier transform.

§ 2. Notation and preliminaries.

Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system in the sense of [5]. Then the twised group algebra $L^1_\alpha(G; \mathfrak{A})$ is defined as the set of all Bochner integrable $\mathfrak{A}$-valued functions on G with the following Banach $*$-algebra structure:

$$(2.1)\qquad \begin{aligned} (xy)(g) &= \int_G x(h)\alpha_h[y(h^{-1}g)]dh \\ x^*(g) &= \Delta(g)^{-1}\alpha_g[x(g^{-1})]^* \\ \|x\|_1 &= \int_G \|x(g)\|dg, \end{aligned}$$

where dg is the left Haar measure and $\Delta(g)$ the associated modular function of G.

Let Rep $\mathfrak{A}$ be the set of all non-degenerate representations of $\mathfrak{A}$. For $\rho\in$ Rep $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}_\rho$, we denote by Ind ρ the representation of $L^1_\alpha(G; \mathfrak{A})$ corresponding to the covariant representation $(\bar{\rho}, \bar{\lambda})$ of $\mathfrak{A}$ as follows:

$$(2.2)\qquad \begin{aligned} (\bar{\rho}(a)\xi)(g) &= \rho\circ\alpha_g^{-1}(a)\xi(g) \\ (\bar{\lambda}(h)\xi)(g) &= \xi(h^{-1}g) \end{aligned}$$

for every $a\in\mathfrak{A}$, $g, h\in G$, and $\xi\in L^2(G; \mathfrak{H}_\rho)$, where $L^2(G; \mathfrak{H}_\rho)$ is the Hilbert space consisting of all square integrable $\mathfrak{H}_\rho$-valued functions on G.

Let $C^*_r(\mathfrak{A}; \alpha)$ be the completion of $L^1_\alpha(G; \mathfrak{A})$ with respect to the reduced norm $\|\cdot\|_r$ defined as

$$(2.3)\qquad \|x\|_r = \sup\{\|(\mathrm{Ind}\,\rho)(x)\| : \rho\in \mathrm{Rep}\,\mathfrak{A}\}.$$

Then it can be considered as the quotient C^*-algebra of the enveloping C^*-algebra $C^*(\mathfrak{A}; \alpha)$ of $L^1_\alpha(G; \mathfrak{A})$ by the ideal $\bigcap_{\rho\in\mathrm{Rep}\,\mathfrak{A}}(\mathrm{Ind}\,\rho)^{-1}(0)$.

In what follows, we construct a "dual" C^*-crossed product based on $C^*_r(\mathfrak{A}; \alpha)$, which will be isomorphic to the tensor product $\mathfrak{A}\hat{\otimes}_*\mathcal{C}(L^2(G))$ of $\mathfrak{A}$ and the C^*-algebra $\mathcal{C}(L^2(G))$ consisting of all compact operators on $(L^2(G))$ in later.

Taking the uiversal representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$, we may assume by [4] that $C^*_r(\mathfrak{A}; \alpha)$ is acting on $L^2(G; \mathfrak{H})$ in such a way that

$$(2.4)\qquad (x\xi)(g) = \int_G \alpha_g^{-1}[x(h)]\xi(h^{-1}g)dh$$

for every $x\in L^1_\alpha(G; \mathfrak{A})$ and $\xi\in L^2(G; \mathfrak{H})$.

Let β be an isomorphism of $C^*_r(\mathfrak{A}; \alpha)''$ into the full operator algebra $\mathfrak{L}(L^2(G\times G; \mathfrak{H}))$ of all bounded linear operators on $L^2(G\times G; \mathfrak{H})$ such that

(2.5)
$$\beta[\bar{l}(a)]=\bar{l}(a)\otimes 1_{L^2(G)}$$
$$\beta[\bar{\lambda}(h)]=\bar{\lambda}(h)\otimes\lambda(h)$$

for every $a\in\mathfrak{A}$ and $h\in G$, where l is the identity representation of $\mathfrak{A}$ on $\mathfrak{H}$ and λ is the left regular representation of G on $L^2(G)$. Actually, according to [3], the mapping β cited above can be guaranteed at least one. Then we have by (2.4) and (2.5) that

(2.6)
$$\beta(x)=\int_G \bar{l}[x(g)]\bar{\lambda}(g)\otimes\lambda(g)dg$$

for every $x\in L_\alpha^1(G;\mathfrak{A})$.

Let $C_0(G)$ be the C^*-algebra of all complex valued continuous functions on G vanishing at infinity.

For each $x\in C_r^*(\mathfrak{A};\alpha)$ and $f\in C_0(G)$, we denote by $f*x$ the operator $(1_{L^2(G;\mathfrak{H})}\otimes L_f)\beta(x)$ on $L^2(G\times G;\mathfrak{H})$ where L is the natural representation of $C_0(g)$ on $L^2(G)$.

Now we define a dual C^*-crossed product $C_d^*(C_r^*(\mathfrak{A};\alpha);\beta)$ of $C_r^*(\mathfrak{A};\alpha)$ by β as the C^*-algebra generated by $f*x$, $x\in C_r^*(\mathfrak{A};\alpha)$, $f\in C_0(G)$. This definition can be considered as a C^*-algebra version of Nakagami's model in von Neumann algebras ([3]). In the next section, we shall show the algebra defined above is isomorphic to the tensor product $\mathfrak{A}\hat{\otimes}_*\mathcal{C}(L^2(G))$ of $\mathfrak{A}$ and the C^*-algebra $\mathcal{C}(L^2(G))$ of all compact operators on $L^2(G)$.

§ 3. Non abelian duality for C^*-crossed products.

Throughout this section, we use the same symbols as in the previous section. Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system and $C_r^*(\mathfrak{A};\alpha)$ be the reduced C^*-crossed product of $\mathfrak{A}$ by α acting on $L^2(G;\mathfrak{H})$ in such a way as (2.4). By definition, the dual C^*-crossed product $C_d^*(C_r^*(\mathfrak{A};\alpha);\beta)$ of $C_r^*(\mathfrak{A};\alpha)$ by β is acting on $L^2(G\times G;\mathfrak{H})$.

Let us compute the operator $f*x$ for $x\in L_\alpha^1(G;\mathfrak{A})$ and $f\in C_0(G)$. By (2.6), we see that

(3.1)
$$\begin{aligned} f*x&=(1_{L^2(G;\mathfrak{H})}\otimes L_f)\beta(x)\\ &=\int_G(1_{L^2(G;\mathfrak{H})}\otimes L_f)(\bar{l}[x(g)]\bar{\lambda}(g)\otimes\lambda(g))dg\\ &=\int_G\bar{l}[x(g)]\bar{\lambda}(g)\otimes L_f\lambda(g)dg \end{aligned}$$

on $L^2(G\times G;\mathfrak{H})$ for every $x\in L_\alpha^1(G;\mathfrak{A})$ and $f\in C_0(G)$.

On the other hand, we consider the C^*-dynamical system $(\mathfrak{A}\hat{\otimes}_*C_0(G), G, \alpha\otimes\tau)$ where τ is the shift action of G on $C_0(G)$. Then one can associate the reduced

C^*-crossed product $C_r^*(\mathfrak{A}\hat{\otimes}_*C_0(G)\,;\,\alpha\otimes\tau)$ of $\mathfrak{A}\hat{\otimes}_*C_0(G)$ by $\alpha\otimes\tau$. Let $\operatorname{Ind}(l\otimes L)$ be the induced representation of $C_r^*(\mathfrak{A}\hat{\otimes}_*C_0(G)\,;\,\alpha\otimes\tau)$ on $L^2(G\times G\,;\,\mathfrak{H})$ associated to $l\otimes L$. Since $l\otimes L$ is faithful on $\mathfrak{A}\hat{\otimes}_*C_0(G)$ (cf: [6]), it follows by [4] that $\operatorname{Ind}(l\otimes L)$ is faithful.

For $x\in L^1_{\alpha\otimes\tau}(G\,;\,\mathfrak{A}\otimes C_0(G))$, we compute $\operatorname{Ind}(l\otimes L)(x)$ as follows:

$$\begin{aligned}[\operatorname{Ind}(l\otimes L)(x)\xi](g,\,h) &= \int_G [\overline{l\otimes L}(x(k))\bar{\lambda}(k)\xi](g,\,h)dk \\ &= \int_G [\alpha_g^{-1}\otimes L\circ\tau_g^{-1})(x(k))\xi(k^{-1}g,\,\cdot)](h)dk\end{aligned}\tag{3.2}$$

for every $\xi\in L^2(G\times G\,;\,\mathfrak{H})$. Let $K(G)$ be the set of all complex valued continuous functions on G with compact support. Taking $x=f_1\otimes a\otimes f_2$, $\xi=\xi_1\otimes\eta\otimes\xi_2$ in (3.2) ($a\in\mathfrak{A}$, $f_i\in K(G)$, $\eta\in\mathfrak{H}$, $\xi_i\in L^2(G)$), we have that

$$\begin{aligned}&\operatorname{Ind}(l\otimes L)(f_1\otimes a\otimes f_2)(\xi_1\otimes\eta\otimes\xi_2)(g,\,h)\\ &= \int_G [f_1(k)\alpha_g^{-1}\otimes L\circ\tau_g^{-1}(a\otimes f_2)\xi_1(k^{-1}g)(\eta\otimes\xi_2)](h)dk\\ &= \int_G [f_1(k)\xi_1(k^{-1}g)(\alpha_g^{-1}(a)\eta\otimes L_{\tau_g^{-1}(f_2)}\xi_2)](h)dk\\ &= \int_G f_1(k)\xi_1(k^{-1}g)(L_{\tau_g^{-1}(f_2)}\xi_2)(h)\alpha_g^{-1}(a)\eta dk\\ &= \int_G f_1(k)\xi_1(k^{-1}g)f_2(gh)\xi_2(h)\alpha_g^{-1}(a)\eta dk\\ &= \int_G f_1(k)[a'f_2'\lambda(k)'(\xi_1\otimes\eta\otimes\xi_2)](g,\,h)dk\,,\end{aligned}$$

where

$$\begin{cases}(a'\zeta)(g,\,h)=\alpha_g^{-1}(a)\zeta(g,\,h)\\ (f'\zeta)(g,\,h)=f(gh)\zeta(g,\,h)\\ (\lambda'(k)\zeta)(g,\,h)=\zeta(k^{-1}g,\,h)\end{cases}$$

for every $a\in\mathfrak{A}$, $f\in K(G)$, and $\zeta\in L^2(G\times G\,;\,\mathfrak{H})$.

Therefore, it follows that

$$\operatorname{Ind}(l\otimes L)(f_1\otimes a\otimes f_2)=\int_G f_1(k)a'f_2'\lambda(k)'dk\tag{3.3}$$

for every $a\in\mathfrak{A}$, and $f_i\in K(G)$. Let W be the unitary operator on $L^2(G\times G\,;\,\mathfrak{H})$ so that $(W\zeta)(g,\,h)=\zeta(g,\,g^{-1}h)$ for all $\zeta\in L^2(G\times G\,;\,\mathfrak{H})$. Then we have by (3.3) that

$$\text{(3.4)} \qquad \mathrm{Ad}(W)\circ \mathrm{Ind}(l\otimes L)(f_1\otimes a\otimes f_2)$$
$$=\int_G f_1(k)a'(1_{L^2(G;\mathfrak{H})}\otimes L_{f_2})(\bar{\lambda}(k)\otimes\lambda(k))dk$$
$$=\int_G(\bar{l}[(f_1\otimes a)(k)]\otimes 1_{L^2(G)})(\bar{\lambda}(k)\otimes L_{f_2}\lambda(k))dk$$
$$=\int_G \bar{l}[(f_1\otimes a)(k)]\bar{\lambda}(k)\otimes L_{f_2}\lambda(k)dk\,.$$

By (3.1) and (3,4), we deduce that

$$f_2*(f_1\otimes a)=\mathrm{Ad}(W)\circ\mathrm{Ind}(l\otimes L)(f_1\otimes a\otimes f_2)$$

for every $a\in\mathfrak{A}$ and $f_i\in K(G)$. Therefore, we obtain by definition that

$$C_d^*(C_r^*(\mathfrak{A}\,;\,\alpha)\,;\,\beta)=\mathrm{Ad}(W)\circ\mathrm{Ind}(l\otimes L)[C_r^*(\mathfrak{A}\hat{\otimes}_*C_0(G)\,;\,\alpha\otimes\tau)]\,,$$

which implies the following proposition:

PROPOSITION 3.1. *Let* $(\mathfrak{A}, G, \alpha)$ *be a* C^**-dynamical system. Then the dual* C^**-crossed product* $C_d^*((C_r^*(\mathfrak{A}\,;\,\alpha)\,;\,\beta)$ *of* $C_r^*(\mathfrak{A}\,;\,\alpha)$ *by* β *is isomorphic to the reduced* C^**-crossed product* $C_r^*(\mathfrak{A}\hat{\otimes}_*C_0(G)\,;\,\alpha\otimes\tau)$ *of* $\mathfrak{A}\hat{\otimes}_*C_0(G)$ *by* $\alpha\otimes\tau$*, where* $\tau_g(f)(h)=f(g^{-1}h)$ *for every* $f\in C_0(G)$.

As we have seen in the abelian case, using the automorphism Φ of $\mathfrak{A}\hat{\otimes}_*C_0(G)$ such that

$$\text{(3.5)} \qquad \Phi(x)(g)=\alpha_g^{-1}[x(g)] \qquad ([4])\,,$$

we deduce the following proposition.

PROPOSITION 3.2. *The* C^**-algebra* $C_r^*(\mathfrak{A}\hat{\otimes}_*C_0(G)\,;\,\alpha\otimes\tau)$ *is isomorphic to the reduced* C^**-crossed product* $C_r^*(\mathfrak{A}\hat{\otimes}_*C_0(G)\,;\,l\otimes\tau)$ *of* $\mathfrak{A}\hat{\otimes}_*C_0(G)$ *by* $l\otimes\tau$.

PROOF. By (3.5), it is easy to see that Φ is an automorphism of $\mathfrak{A}\hat{\otimes}_*C_0(G)$. Since

$$\Phi\circ\alpha_g\otimes\tau_g\circ\Phi^{-1}=l\otimes\tau_g$$

for every $g\in G$, we have the conclusion. Q.E.D.

Now we show the compatibility between reduced crossed products and tensor products with respect to $\|\cdot\|_*$-cross norm which was announced in [4].

PROPOSITION 3.3. *Let* $(\mathfrak{A}, G, \alpha)$, $(\mathfrak{B}, H, \beta)$ *be two* C^**-dynamical systems. Then for the* C^**-dynamical system* $(\mathfrak{A}\hat{\otimes}_*\mathfrak{B}, G\times H, \alpha\otimes\beta)$*, the reduced* C^**-crossed product* $C_r^*(\mathfrak{A}\hat{\otimes}_*\mathfrak{B}\,;\,\alpha\otimes\beta)$ *of* $\mathfrak{A}\hat{\otimes}_*\mathfrak{B}$ *by* $\alpha\otimes\beta$ *is isomorphic to the tensor product* $C_r^*(\mathfrak{A}\,;\,\alpha)\hat{\otimes}_*C_r^*(\mathfrak{B}\,;\,\beta)$ *of* $C_r^*(\mathfrak{A}\,;\,\alpha)$ *and* $C_r^*(\mathfrak{B}\,;\,\beta)$.

PROOF. Let π (resp. ρ) be a faithful representation of $\mathfrak{A}$ (resp. $\mathfrak{B}$) on $\mathfrak{H}_\pi$ (resp. $\mathfrak{H}_\rho$). By [6], the tensor product $\pi\otimes\rho$ is faithful on $\mathfrak{A}\hat{\otimes}_*\mathfrak{B}$. Consider the induced representation $\mathrm{Ind}(\pi\otimes\rho)$ of $C_r^*(\mathfrak{A}\hat{\otimes}_*\mathfrak{B}\,;\,\alpha\otimes\beta)$ by $\pi\otimes\rho$. Then it is faithful ([4]). For $x\in\mathfrak{A}\hat{\otimes}_*\mathfrak{B}$, and $f\in K(G\times H)$, we compute that

$$[\mathrm{Ind}\,(\pi\otimes\rho)(f\otimes x)\xi_1\otimes\xi_2](g_0,\ h_0) = \iint_{G\times H}(\overline{\pi\otimes\rho}[f(g,\ h)x]\bar{\lambda}(g,\ h)\xi_1\otimes\xi_2)(g_0,\ h_0)dgdh \tag{3.6}$$

for every $\xi_1\in L^2(G\,;\,\mathfrak{H}_\pi)$ and $\xi_2\in L^2(H\,;\,\mathfrak{H}_\rho)$.

Put $x=a\otimes b\ (a\in\mathfrak{A},\ b\in\mathfrak{B})$, and $f=f_1\otimes f_2\ (f_1\in K(G),\ f_2\in K(H))$ in (3.6), then it follows that

$$\begin{aligned}
&[\mathrm{Ind}\,(\pi\otimes\rho)(f_1\otimes f_2\otimes a\otimes b)(\xi_1\otimes\xi_2)](g_0,\ h_0)\\
&=\iint_{G\times H} f_1(g)f_2(h)[\overline{\pi\otimes\rho}(a\otimes b)\bar{\lambda}(g,\ h)(\xi_1\otimes\xi_2)](g_0,\ h_0)dg\,dh\\
&=\iint_{G\times H} f_1(g)f_2(h)\pi\circ\alpha_{g_0}^{-1}(a)\xi_1(g^{-1}g_0)\otimes\rho\circ\beta_{h_0}^{-1}(b)\xi_2(h^{-1}h_0)dg\,dh\\
&=\left(\int_G f_1(g)\pi\circ\alpha_{g_0}^{-1}(a)\xi_1(g^{-1}g_0)dg\right)\otimes\left(\int_H f_2(h)\rho\circ\beta_{h_0}^{-1}(b)\xi_2(h^{-1}h_0)dh\right)\\
&=[(\mathrm{Ind}\,\pi)(f_1\otimes a)\xi_1](g_0)\otimes[(\mathrm{Ind}\,\rho)(f_2\otimes b)\xi_2](h_0)\\
&=[(\mathrm{Ind}\,\pi)(f_1\otimes a)\otimes(\mathrm{Ind}\,\rho)(f_2\otimes b))(\xi_1\otimes\xi_2)](g_0,\ h_0)
\end{aligned}$$

for every $\xi_1\in L^2(G\,;\,\mathfrak{H}_\pi)$ and $\xi_2\in L^2(H\,;\,\mathfrak{H}_\rho)$ when we identify $L^2(G\times H\,;\,\mathfrak{H}_\pi\otimes\mathfrak{H}_\rho)$ as $L^2(G\,;\,\mathfrak{H}_\pi)\otimes L^2(H\,;\,\mathfrak{H}_\rho)$ by the natural mapping. Therefore, we have that

$$\begin{aligned}\mathrm{Ind}\,(\pi\otimes\rho)(f_1\otimes f_2\otimes a\otimes b)&=(\mathrm{Ind}\,\pi)(f_1\otimes a)\otimes(\mathrm{Ind}\,\rho)(f_2\otimes b)\\&=[(\mathrm{Ind}\,\pi)\otimes(\mathrm{Ind}\,\rho)][(f_1\otimes a)\otimes(f_2\otimes b)]\end{aligned} \tag{3.7}$$

for every $a\in\mathfrak{A}$, $b\in\mathfrak{B}$, $f_1\in K(G)$, and $f_2\in K(H)$.

By (3.7), we easily see that

$$\begin{aligned}&\mathrm{Ind}\,(\pi\otimes\rho)[C_r^*(\mathfrak{A}\hat{\otimes}_*\mathfrak{B}\,;\,\alpha\otimes\beta)]\\&\qquad=(\mathrm{Ind}\,\pi)\otimes(\mathrm{Ind}\,\rho)[C_r^*(\mathfrak{A}\,;\,\alpha)\hat{\otimes}_*C_r^*(\mathfrak{B}\,;\,\beta)]\,.\end{aligned}$$

Since π (resp. ρ) is faithful on $\mathfrak{A}$ (resp. $\mathfrak{B}$), Ind π (resp. Ind ρ) is faithful on $C_r^*(\mathfrak{A}\,;\,\alpha)$ (resp. $C_r^*(\mathfrak{B}\,;\,\beta)$) (cf: [4]). So, $(\mathrm{Ind}\,\pi)\otimes(\mathrm{Ind}\,\rho)$ is faithful on $C_r^*(\mathfrak{A}\,;\,\alpha)\hat{\otimes}_*C_r^*(\mathfrak{B}\,;\,\beta)$. This completes the proof. Q. E. D.

Applying the above proposition to the C^*-dynamical systems $(\mathfrak{A},\ \{e\},\ l)$ and $(C_0(G),\ G,\ \tau)$, we have by Proposition 3.1 and Proposition 3.2 the following:

PROPOSITION 3.4. *The dual C^*-crossed product $C_d^*(C_r^*(\mathfrak{A}\,;\,\alpha)\,;\,\beta)$ of $C_r^*(\mathfrak{A}\,;\,\alpha)$ by β is isomorphic to the tensor product $\mathfrak{A}\hat{\otimes}_*C_r^*(C_0(G)\,;\,\tau)$ of $\mathfrak{A}$ and $C_r^*(C_0(g)\,;\,\tau)$.*

In what follows, we shall show that $C_r^*(C_0(G)\,;\,\tau)$ is isomorphic to the C^*-algebra $C(L^2(G))$ of all compact operators on $L^2(G)$.

Let $\delta(f)=f(e)$ for every $f\in C_0(G)$. Then the direct sum $\bigoplus_{g\in G}\tau_g\circ\delta$ of $\tau_g\circ\delta$ is faithful on $C_0(G)$. Therefore, it follows by [4] that Ind δ is faithful on

$C_r^*(C_0(G)\,;\,\tau)$. Computing the operator $(\mathrm{Ind}\,\delta)(x)$ for $x\in K(G\times G)$, we see that

$$\begin{aligned}[(\mathrm{Ind}\,\delta)(x)\xi](g)&=\int_G x(g,\ h)\xi(h^{-1}g)dh\\&=\int_G x(g,\ gh^{-1})\Delta(h)^{-1}\xi(h)dh\\&=\int_G y(g,\ h)\xi(h)dh\end{aligned}$$

for every $\xi\in L^2(G)$, where $y(g,\ h)=x(g,\ gh^{-1})\Delta(h)^{-1}$. Since $x\in K(G\times G)$, $y\in L^2(G\times G)$. So $(\mathrm{Ind}\,\delta)(x)$ is an operator of Hilbert-Schmidt class. Hence $(\mathrm{Ind}\,\delta)(x)\in \mathcal{C}(L^2(G))$. Since $K(G\times G)$ is dense in $C_r^*(C_0(G)\,;\,\tau)$, we have that

$$(\mathrm{Ind}\,\delta)[C_r^*(C_0(G)\,;\,\tau)]\subset\mathcal{C}(L^2(G))\,.$$

Moreover, since δ is a character of $C_0(G)$ and τ is free on $C_0(G)^\wedge$, it implies by [5] that $\mathrm{Ind}\,\delta$ is an irreducible representation of $C_r^*(C_0(G)\,;\tau)$ on $L^2(G)$. Therefore, we deduce by [1] that

$$(\mathrm{Ind}\,\delta)[C_r^*(C_0(G)\,;\,\tau)]=\mathcal{C}(L^2(G))\,.$$

We now state the above argument as follows:

PROPOSITION 3.5. *The reduced C*-crossed product* $C_r^*(C_0(G)\,;\,\tau)$ *of* $C_0(G)$ *by* τ *is isomorphic to the C*-algebra* $\mathcal{C}(L^2(G))$ *of all compact operators on* $L^2(G)$.

Combining all the propositions that we obtained, we have the following theorem:

THEOREM 3.6. *Let* $(\mathfrak{A},\,G,\,\alpha)$ *be a C*-dynamical system, and* $C_r^*(\mathfrak{A}:\alpha)$ *be the reduced C*-crossed product of* $\mathfrak{A}$ *by* α *acting on* $L^2(G\,;\,\mathfrak{H})$ *in such a way as* (2.4). *Then there exists an isomorphism* β *of* $C_r^*(\mathfrak{A}\,;\,\alpha)$ *into the full operator algebra* $\mathfrak{L}(L^2(G\times G\,;\,\mathfrak{H}))$ *on* $L^2(G\times G\,;\,\mathfrak{H})$ *such that the dual C*-crossed product* $C_d^*(C_r^*(\mathfrak{A}\,;\alpha)\,;\beta)$ *of* $C_r^*(\mathfrak{A}:\alpha)$ *by* β *is isomorphic to the tensor product* $\mathfrak{A}\hat{\otimes}_*\mathcal{C}(L^2(G))$ *of* $\mathfrak{A}$ *and the C*-algebra* $\mathcal{C}(L^2(G))$ *of all compact operators on* $L^2(G)$.

§ 4. Duality in the abelian case.

Suppose $(\mathfrak{A},\,G,\,\alpha)$ is a C^*-dynamical system based on a locally compact abelian group G. According to [4], we can construct a C^*-dynamical system $(C_r^*(\mathfrak{A}\,;\,\alpha)\,;\,\hat{G},\,\hat{\alpha})$ based on the character group $\hat{G}$ of G such that

$$\hat{\alpha}_p(x)(g)=\overline{\langle g,\,p\rangle}x(g) \tag{4.1}$$

for every $x\in L_\alpha^1(G\,;\,\mathfrak{A})$ and $p\in\hat{G}$. Let $C_r^*(C_r^*(\mathfrak{A}\,;\,\alpha)\,;\,\hat{\alpha})$ be the reduced C^*-crossed product of $C_r^*(\mathfrak{A}\,;\,\alpha)$ by $\hat{\alpha}$. In what follows, we shall show that $C_r^*(C_r^*(\mathfrak{A}\,;\,\alpha)\,;\,\hat{\alpha})$ is isomorphic to the dual C^*-crossed product $C_d^*(C_r^*(\mathfrak{A}\,;\,\alpha)\,;\,\beta)$ of $C_r^*(\mathfrak{A}\,;\,\alpha)$ by β defined in the previous section.

Since we may assume that $C_r^*(\mathfrak{A}; \alpha)$ is acting on $L^2(G; \mathfrak{H})$ by the fashion as (2.4), it follows by (4.1) that

$$\hat{\alpha}_p = \mathrm{Ad}(\hat{U}_p) \quad \text{on} \quad C_r^*(\mathfrak{A}; \alpha) \quad (p \in \hat{G}), \tag{4.2}$$

where $(\hat{U}_p\xi)(g) = \overline{\langle g, p\rangle}\xi(g)$ for every $\xi \in L^2(G; \mathfrak{H})$.

Let l_r be the identity representation of $C_r^*(\mathfrak{A}; \alpha)$ on $L^2(G; \mathfrak{H})$. Then $\mathrm{Ind}\, l_r$ is the faithful representation of $C_r^*(C_r^*(\mathfrak{A}; \alpha); \hat{\alpha})$ on $L^2(G \times \hat{G}; \mathfrak{H})$. For $x \in C_r^*(\mathfrak{A}; \alpha)$, and $f \in L^1(G)$, we see that

$$\begin{aligned}(\mathrm{Ind}\, l_r)(f \otimes x) &= \int_{\hat{G}} \bar{l}_r[(f \otimes x)(p)]\bar{\lambda}(p)dp \\ &= \int_{\hat{G}} f(p)\bar{l}_r(x)(1_{L^2(G;\mathfrak{H})} \otimes \lambda(p))dp\,.\end{aligned} \tag{4.3}$$

Let F be the isometry mapping from $L^2(G \times \hat{G}; \mathfrak{H})$ onto $L^2(G \times G; \mathfrak{H})$ such that

$$(F\xi)(g, h) = \int_{\hat{G}} \overline{\langle h, p\rangle}\xi(g, p)dp \tag{4.4}$$

for every $\xi \in K(G \times \hat{G}; \mathfrak{H})$, the set of all $\mathfrak{H}$-valued continuous functions on $G \times G$ with compact support. Then we have by (4.3) and (4.4) that

$$\begin{aligned}&\mathrm{Ad}(F) \circ (\mathrm{Ind}\, l_r)(f \otimes x) \\ &= \int_{\hat{G}} f(p)\, \mathrm{Ad}(F) \circ \bar{l}_r(x)\, \mathrm{Ad}(F) \circ (1_{L^2(G;\mathfrak{H})} \otimes \lambda(p))dp \\ &= \mathrm{Ad}(F) \circ \bar{l}_r(x)\, \mathrm{Ad}(F) \circ \Big(1_{L^2(G;\mathfrak{H})} \otimes \int_{\hat{G}} f(p)\lambda(p)dp\Big) \\ &= \mathrm{Ad}(F) \circ \bar{l}_r(x)\, (1_{L^2(G;\mathfrak{H})} \otimes L_{\hat{f}})\,,\end{aligned} \tag{4.5}$$

where $\hat{f}$ is the Fourier image of f. Moreover, it follows by (2.4) and (4.2) that for $a \in \mathfrak{A}$ and $f' \in K(G)$,

$$\begin{aligned}&[\mathrm{Ad}(F) \circ \bar{l}_r(f' \otimes a)\xi](g, h) \\ &= \int_{\hat{G}} \overline{\langle h, p\rangle}(\bar{l}_r(f' \otimes a)F^*\xi)(g, p)dp \\ &= \int_{\hat{G}} \overline{\langle h, p\rangle}[\hat{\alpha}_p^{-1}(f' \otimes a)F^*\xi(g, p)]dp \\ &= \int_{\hat{G}} \overline{\langle h, p\rangle}[U_p^*(f' \otimes a)\hat{U}_pF^*\xi](g, p)dp \\ &= \int_{\hat{G}} \overline{\langle h, p\rangle}\langle g, p\rangle \int_G \alpha_g^{-1}[(f' \otimes a)(k)](\hat{U}_pF^*\xi)(k^{-1}g, p)dk\, dp\end{aligned}$$

$$=\iint_{\hat{G}\times G}\overline{\langle h, p\rangle}\langle k, p\rangle f'(k)\alpha_g^{-1}(a)(F^*\xi)(k^{-1}g, p)dk\, dp$$

$$=\iiint_{\hat{G}\times G\times G}\overline{\langle hk^{-1}, p\rangle} f'(k)\alpha_g^{-1}(a)\langle l, p\rangle\xi(k^{-1}g, l)dl\, dk\, dp$$

$$=\int_G f'(k)dk\iint_{\hat{G}\times G}\langle h^{-1}l, p\rangle\alpha_g^{-1}(a)\xi(k^{-1}g, k^{-1}l)dl\, dp$$

$$=\int_G f'(k)\alpha_g^{-1}(a)\xi(k^{-1}g, k^{-1}h)dk$$

$$=\int_G f'(k)[(\bar{l}(a)\otimes 1_{L^2(G)})(\bar{\lambda}(k)\otimes\lambda(k))\xi](g, h)dk$$

$$=\int_G(\bar{l}[(f'\otimes a)(k)]\bar{\lambda}(k)\otimes\lambda(k)\xi)(g, h)dk$$

for every $\xi\in L^2(G\times G;\mathfrak{H})$. Therefore, we obtain by (2.6) that

$$\mathrm{Ad}(F)\circ\bar{l}_r(f'\otimes a)=\beta(f'\otimes a) \tag{4.6}$$

for every $a\in\mathfrak{A}$ and $f'\in K(G)$. By (4.5) and (4.6), we have that

$$\begin{aligned}\mathrm{Ad}(F)\circ(\mathrm{Ind}\, l_r)&(f\otimes f'\otimes a)\\ &=\mathrm{Ad}(F)\circ\bar{l}_r(f'\otimes a)(1_{L^2(G;\mathfrak{H})}\otimes L_{\hat{f}})\\ &=\beta(f'\otimes a)(1_{L^2(G;\mathfrak{H})}\otimes L_{\hat{f}})\\ &=[\bar{\hat{f}}*(\bar{f}'\otimes a^*)]^*,\end{aligned} \tag{4.7}$$

where $\bar{f}$ is the complex conjugate of f. By the definition of $C_d^*(C_r^*(\mathfrak{A};\alpha);\beta)$, we see by (4.7) that

$$\begin{aligned}\mathrm{Ad}(F)\circ(\mathrm{Ind}\, l_r)&[C_r^*(C_r^*(\mathfrak{A};\alpha);\hat{\alpha})]\\ &=C_d^*(C_r^*(\mathfrak{A};\alpha);\beta),\end{aligned}$$

which means that $C_r^*(C_r^*(\mathfrak{A};\alpha);\hat{\alpha})$ is isomorphic to $C_d^*(C_r^*(\mathfrak{A};\alpha);\beta)$. Since G is abelian, we know by [**4**] that $C_r^*(\mathfrak{A};\alpha)$ is identified with the C^*-crossed product $C^*(\mathfrak{A};\alpha)$ of $\mathfrak{A}$ by α.

Summing up the above argument, we deduce by Theorem 3.6 the following abelian version:

THEOREM 4.1. *Let $(\mathfrak{A}\ G, \alpha)$ be a C^*-dynamical system where G is abelian. Let $\hat{G}$ be the dual group of G. Then there exists a C^*-dynamical system $(C^*(\mathfrak{A};\alpha);\hat{G},\hat{\alpha})$ such that the C^*-crossed product $C^*(C^*(\mathfrak{A};\alpha);\hat{\alpha})$ is isomorphic to the tensor product $\mathfrak{A}\hat{\otimes}_*C(L^2(G))$ of $\mathfrak{A}$ and the C^*-algebra $C(L^2(G))$ of all compact operators on $L^2(G)$* ([**4**]).

REMARK. In [**2**], Landstad also discussed a duality for C^*-crossed products.

However, our dual object seems to be closer to Nakagami's one rather than Landstad's method.

References

[1] J. Dixmier, Les C^*-algèbres et leurs représentations, Gauthier-Villars, Paris, 1969.
[2] M. B. Landstad, Duality for dual covariance algebras, Preprint, Univ. of Trondheim, 1976.
[3] Y. Nakagami, Dual action of a von Neumann algebra and Takesaki's duality for a locally compact group, to appear in Publ. Res. Inst. Math. Sci. Ser. A.
[4] H. Takai, On a duality for crossed products of C^*-algebras, J. Functional Analysis. **19** (1975), 25-39.
[5] H. Takai, The Quasi-orbit space of continuous C^*-dynamical systems, Trans. Amer. Math. Soc., **216** (1976), 105-113.
[6] A. Wulfsohn, Le produit tensoriel de C^*-algèbres, Bull. Sci. Math., **87** (1963), 13-27.

Shō IMAI
Department of Mathematics
Osaka University
Toyonaka, Osaka
Japan

Hiroshi TAKAI
Department of Mathematics
Tokyo Metropolitan University
Fukazawa, Setagaya, Tokyo
Japan