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Variational inequalities and complementarity problems

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1. Introduction.

We shall consider variational inequalities for multivalued mappings to unify mathematical programming problems and extended fixed point problems. Let X and Y be two real separated topological vector spaces with a given bilinear form $\langle\cdot, \cdot\rangle$ of $Y\times X$ into the reals R. Let T be a multivalued mapping from its domain $D(T)\subset X$ to subsets of Y, f a function from X to R. Under these conditions a solution of a variational inequality is the following; $x_0\in D(T)$ and $w_0\in T(x_0)$ such that $\langle w_0, x-x_0\rangle\geqq f(x_0)-f(x)$ for all $x\in D(T)$. When $D(T)$ is a cone, variational inequalities are related to complementarity problems. Variational inequalities in infinite dimensional spaces were studied by Browder [1], Karamardian [5] and others. Karamardian [5] also considered complementarity problems, for which we also refer to Moré [6]. In this paper we shall give two existence theorems. Using them, we shall solve variational inequalities for multivalued mappings on closed convex subsets in topological vector spaces. Then the results are applied to complementarity problems.

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2. Basic existence theorems.

Let X and Y be two topological spaces. Denote by 2^Y the family of all subsets of Y. A mapping $T: X\to 2^Y$ is said to be upper semicontinuous if $T^{-1}(F)=\{x\in X: T(x)\cap F\neq\emptyset\}$ is closed in X for any closed subset F of Y. The following result was given by Fan [3]. We shall present an elementary proof using Brouwer's fixed point theorem. In the rest of this paper let X and Y be topological vector spaces.

THEOREM 2.1. *Let K be a nonempty compact convex subset of X. Let A be a subset of $K\times K$ for which the following conditions hold*:

(i) *For each $y\in K$, the set $\{x\in K: (x, y)\in A\}$ is closed.*

(ii) $(x, x) \in A$ *for every* $x \in K$.

(iii) *For each* $x \in K$, *the set* $\{y \in K : (x, y) \notin A\}$ *is convex or empty.*

Then, there exists $x_0 \in K$ *such that* $\{x_0\} \times K \subset A$.

PROOF. Suppose that for any $x \in K$, there exists $y \in K$ such that $(x, y) \notin A$. For each $y \in K$, let $A(y) = \{x \in K : (x, y) \notin A\}$, then we have $K = \cup_{y \in K} A(y)$. By (i), $A(y)$ is open in K for all $y \in K$. Since K is compact, there exists a finite number of points $\{y_1, \cdots, y_n\}$ of K such that $K = \bigcup_{i=1}^{n} A(y_i)$. Let $\{\beta_1, \cdots, \beta_n\}$ be a partition of unity corresponding to this covering, i.e., each β_i is a continuous mapping of K into $[0, 1]$ which vanishes outside of $A(y_i)$, while $\sum_{i=1}^{n} \beta_i(x) = 1$ for all $x \in K$. We define a mapping $p : K \to K$ by $p(x) = \sum_{i=1}^{n} \beta_i(x) y_i$. Then p maps the simplex S spanned by the finite set $\{y_1, \cdots, y_n\}$ into itself. Since S is homeomorphic to an Euclidean sphere, p has a fixed point $z \in S$ by Brouwer's fixed point theorem. If $\beta_i(z) > 0$, then we have $(z, y_i) \notin A$. Thus, by (iii) we obtain $(z, p(z)) = (z, \sum_{i=1}^{n} \beta_i(z) y_i) \notin A$. On the other hand, $(z, p(z)) = (z, z) \in A$ by (ii). This is a contradiction. Therefore, there exists $x_0 \in K$ such that $\{x_0\} \times K \subset A$.

For a distinct pair of topological vector spaces, we have the following analogous result.

THEOREM 2.2. *Let* K_1 *be a nonempty compact convex subset of a locally convex space* X *and* K_2 *a nonempty closed convex subset of* Y. *Let* A *be a subset of* $K_1 \times K_2$ *having the following properties:*

(i) A *is closed.*

(ii) *For any* $y \in K_2$, *the set* $\{x \in K_1 : (x, y) \in A\}$ *is nonempty and convex.*

(iii) *For any* $x \in K_1$, *the set* $\{y \in K_2 : (x, y) \notin A\}$ *is convex or empty.*

Then there exists $x_0 \in K_1$ *such that* $\{x_0\} \times K_2 \subset A$.

PROOF. Suppose that the assertion of Theorem 2.2 is false. Then for each $x \in K_1$, there is $y \in K_2$ such that $(x, y) \notin A$. Denote $A(y) = \{x \in K_1 : (x, y) \notin A\}$ for any $y \in K_2$, then there exist a finite covering $\{A(y_1), \cdots, A(y_n)\}$ of K_1 and a partition of unity $\{\beta_1, \cdots, \beta_n\}$ corresponding to this finite covering. Set $p(x) = \sum_{i=1}^{n} \beta_i(x) y_i$ for any $x \in K_1$. Then p is a continuous mapping of K_1 into K_2. Define a mapping $T : K_1 \to 2^{K_1}$ by $T(x) = \{u \in K_1 : (u, p(x)) \in A\}$, then by (i) and (ii) $T(x)$ is nonempty and compact for every $x \in K_1$. Since p is continuous and A is closed, T is upper semicontinuous. Hence, T has a fixed point $z \in K_1$ by Glicksberg and Fan's fixed point theorem [**2, 4**]. Thus, $(z, p(z)) \in A$. On the other hand, by (iii) $(z, p(z)) \notin A$ as in the proof of Theorem 2.1. This contradiction proves the theorem.

3. Variational inequalities.

In this section, we shall solve nonlinear variational inequalities. Let H

and K be nonempty subsets of a topological space, then we shall denote $B_H K=\bar{K}\cap\overline{H-K}$ and $I_H K=K\cap(B_H K)^c$, where $\bar{A}$ is the closure of A and A^c is the complement of A. The set $B_H K$ is called the boundary of K relative to H and $I_H K$ is called the interior of K relative to H. When Y is a topological vector space, let $CK(Y)$ be the family of all nonempty compact convex subsets of Y. For any pair of topological vector spaces X and Y, we denote by $\langle\cdot,\cdot\rangle$ a bilinear form of $Y\times X$ into the reals R.

THEOREM 3.1. *Let H be a nonempty closed convex subset of X and Y locally convex. Let $T: H\to CK(Y)$ be an upper semicontinuous mapping and $f: H\to R$ a lower semicontinuous convex function. Suppose that there exists a nonempty compact convex subset K of H with $I_H K\neq\emptyset$ such that $\langle\cdot,\cdot\rangle$ is jointly continuous on $Y\times K$ and for each $z\in B_H K$, there is $u\in I_H K$ for which*

$$\inf_{w\in T(z)}\langle w, z-u\rangle\geqq f(u)-f(z).$$

Then there exist $x_0\in K$ and $w_0\in T(x_0)$ such that $\langle w_0, x-x_0\rangle\geqq f(x_0)-f(x)$ for all $x\in H$.

PROOF. Define $A=\{(x, y)\in K\times K: \sup_{w\in T(x)}\langle w, y-x\rangle\geqq f(x)-f(y)\}$, then the set A satisfies conditions (i), (ii) and (iii) of Theorem 2.1. Thus, there exists $x_0\in K$ such that $\{x_0\}\times K\subset A$, i.e.,

$$\sup_{w\in T(x_0)}\langle w, y-x_0\rangle\geqq f(x_0)-f(y)$$

for all $y\in K$. Now, define

$$B=\{(w, x)\in T(x_0)\times K: \langle w, x-x_0\rangle\geqq f(x_0)-f(x)\}.$$

Then the set B has three properties (i), (ii) and (iii) of Theorem 2.2. Hence, there exists $w_0\in T(x_0)$ such that $\{w_0\}\times K\subset B$, i.e., $\langle w_0, y-x_0\rangle\geqq f(x_0)-f(y)$ for any $y\in K$. We first assume that $x_0\in I_H K$. For each $x\in H$, we can choose λ $(0<\lambda<1)$ small enough so that $y=\lambda x+(1-\lambda)x_0$ lies in K. Hence

$$\lambda\langle w_0, x-x_0\rangle\geqq f(x_0)-f(\lambda x+(1-\lambda)x_0).$$

Since f is convex,

$$\lambda\langle w_0, x-x_0\rangle\geqq\lambda(f(x_0)-f(x)).$$

Cancelling $\lambda>0$, we have $\langle w_0, x-x_0\rangle\geqq f(x_0)-f(x)$. Now we assume that $x_0\in B_H K$. Then, by the hypothesis there exists $y_0\in I_H K$ such that $\langle w_0, x_0-y_0\rangle\geqq f(y_0)-f(x_0)$. Since $\langle w_0, y-x_0\rangle\geqq f(x_0)-f(y)$ for all $y\in K$, it follows that $\langle w_0, y_0-x_0\rangle=f(x_0)-f(y_0)$, and $\langle w_0, y-y_0\rangle\geqq f(y_0)-f(y)$ for all $y\in K$. Since $y_0\in I_H K$, by the same way as above we have

$$\langle w_0, x-y_0\rangle\geqq f(y_0)-f(x)$$

for all $x\in H$. The above equality and this inequality together implies that

$\langle w_0, x-x_0\rangle \geqq f(x_0)-f(x)$ for all $x\in H$.

As a direct consequence of Theorem 3.1, we have the following corollary. In the sequel, we suppose that the topology of X^* (the dual of X) is the strong topology and $\langle w, x\rangle$ is the value of $w\in X^*$ at $x\in X$.

COROLLARY 3.2. *Let H be a nonempty closed convex subset of a locally convex space $X, T: H\to CK(X^*)$ an upper semicontinuous mapping and $f: H\to R$ a lower semicontinuous convex function. Suppose that there exists a nonempty compact convex subset K of H for which $I_H K\neq\emptyset$ and for each $z\in B_H K$, there is $u\in I_H K$ such that*

$$\inf_{w\in T(z)}\langle w, z-u\rangle \geqq f(u)-f(z).$$

Then there exist $x_0\in H$ and $w_0\in T(x_0)$ such that $\langle w_0, x-x_0\rangle \geqq f(x_0)-f(x)$ for all $x\in H$.

Let H be a cone in a topological vector space X, i.e., H is a nonempty closed convex subset of X such that $\alpha x+\beta y$ belongs to H for all $\alpha, \beta\geqq 0$ and $x, y\in H$. The polar H^* of H is the cone defined by $H^*=\{y\in X^*: \langle y, x\rangle\geqq 0$ for all $x\in H\}$.

Now we can solve the following multivalued nonlinear complementarity problems. R^- is the set of non positive real numbers.

THEOREM 3.3. *Let H be a cone of a locally convex space $X, T: H\to CK(X^*)$ an upper semicontinuous mapping and $f: H\to R^-$ a lower semicontinuous convex function such that $f(0)=0$ and $f(\lambda x)=\lambda f(x)$ for $\lambda\geqq 1$, $x\in H$. If there exists a nonempty compact convex subset K of H such that $I_H K\neq\emptyset$ and for each $z\in B_H K$, there is $u\in I_H K$ for which*

$$\inf_{w\in T(z)}\langle w, z-u\rangle \geqq f(u)-f(z),$$

then there exist $x_0\in K$ and $w_0\in T(x_0)$ such that $w_0\in H^*$ and $\langle w_0, x_0\rangle=-f(x_0)$.

PROOF. By Corollary 3.2, there exist $x_0\in K$ and $w_0\in T(x_0)$ for which $\langle w_0, x-x_0\rangle\geqq f(x_0)-f(x)$ for all $x\in H$. Taking $x=0$, we have $\langle w_0, x_0\rangle\leqq -f(x_0)$. Also, by taking $x=\alpha x$ with $\alpha>1$, we see that $(\alpha-1)\langle w_0, x_0\rangle\geqq(1-\alpha)f(x_0)$. Cancelling $\alpha-1$, we have $\langle w_0, x_0\rangle\geqq -f(x_0)$. Therefore, $\langle w_0, x_0\rangle=-f(x_0)$. It remains to show that $w_0\in H^*$. For any $x\in H$, we obtain that

$$\langle w_0, x\rangle+f(x)\geqq\langle w_0, x_0\rangle+f(x_0)=0,$$

and $\langle w_0, x\rangle\geqq -f(x)\geqq 0$. Hence, $w_0\in H^*$.

THEOREM 3.4. *Let H be a cone of a locally convex space $X, T: H\to CK(X^*)$ an upper semicontinuous mapping and $f: H\to R^-$ a lower semicontinuous convex function such that $f(0)=0$ and for any $x, y\in H, f(x+y)\leqq f(x)$. Suppose that there exists a nonempty compact convex subset K of H for which $I_H K\neq\emptyset$ and for each $z\in B_H K$, there is $u\in I_H K$ such that*

$$\inf_{w\in T(z)}\langle w, z-u\rangle\geqq f(u)-f(z).$$

Then there exist $x_0 \in K$ *and* $w_0 \in T(x_0)$ *for which* $w_0 \in H^*$ *and* $0 \leqq \langle w_0, x_0 \rangle \leqq -f(x_0)$.

PROOF. By Corollary 3.2, there exist $x_0 \in K$ and $w_0 \in T(x_0)$ such that $\langle w_0, x-x_0 \rangle \geqq f(x_0)-f(x)$ for all $x \in H$. Letting $x=0$, we obtain that $\langle w_0, x_0 \rangle \leqq -f(x_0)$. By taking $x=2x_0$, it follows that $\langle w_0, x_0 \rangle \geqq f(x_0)-f(2x_0) \geqq 0$. Thus, $0 \leqq \langle w_0, x_0 \rangle \leqq -f(x_0)$. For each $y \in H$, let $x=y+x_0$, then $\langle w_0, y \rangle \geqq f(x_0)-f(y+x_0) \geqq 0$. Hence $w_0 \in H^*$.

If f equals to 0 everywhere, then we have the following corollary.

COROLLARY 3.5. *Let* H *be a cone of a locally convex space* X *and* $T: H \to CK(X^*)$ *an upper semicontinuous mapping. Suppose that there exists a nonempty compact convex subset* K *of* H *such that* $I_H K \neq \emptyset$ *and for each* $z \in B_H K$, *there is* $u \in I_H K$ *for which*

$$\inf_{w \in T(z)} \langle w, z-u \rangle \geqq 0 .$$

Then there exist $x_0 \in H$ *and* $w_0 \in T(x_0)$ *such that* $w_0 \in H^*$ *and* $\langle w_0, x_0 \rangle = 0$.

In the real n-dimensional space R^n, the following holds. This extends Karamardian's result [5] to multivalued mappings.

THEOREM 3.6. *Let* H *be a cone in the real n-dimensional space* R^n *and* $T: H \to CK(R^n)$ *be an upper semicontinuous mapping for which there is a constant* $c>0$ *such that* $\langle w-v, x \rangle \geqq c\|x\|^2$ *for all* $x \in H$, $w \in T(x)$ and $v \in T(0)$. *Then there exist* $x_0 \in H$ *and* $w_0 \in T(x_0)$ *such that* $w_0 \in H^*$ *and* $\langle w_0, x_0 \rangle = 0$.

PROOF. If $0 \in T(0)$, the conclusion is obvious. Hence, we may assume that $0 \notin T(0)$. Take $v_0 \in T(0)$ and let $K=\{x \in H: \|x\| \leqq \|v_0\|/c\}$, then K is a nonempty compact convex subset of H. For any $z \in B_H K$, it follows that $c\|z\|^2 = \|v_0\| \|z\|$. From the hypothesis, we have

$$\begin{aligned} \inf_{w \in T(z)} \langle w, z \rangle &\geqq \langle v_0, z \rangle + c\|z\|^2 \\ &= \langle v_0, z \rangle + \|v_0\| \|z\| \\ &\geqq 0 . \end{aligned}$$

This is the case that $u=0$ in Corollary 3.5. Thus, there exist $x_0 \in H$ and $w_0 \in T(x_0)$ for which $w_0 \in H^*$ and $\langle w_0, x_0 \rangle = 0$.

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