J. Math. Soc. Japan
Vol. 29, No. 4, 1977

A second order theory of ordinal numbers with Ackermann-type reflection schema

By Masazumi HANAZAWA

(Received Aug. 20, 1976)

§ 1. Introduction.

The underlying logic of the ordinal number theory OA given in [3] is a weakened second order logic. Adopting the standard second order logic, we can obtain a stronger theory. We shall denote it by OA^+. In this paper we first show the consistency of OA^+ by interpreting it in ZF. In fact, OA^+ is interpretable in various theories which are much weaker than ZF. Roughly speaking, OA^+ is interpretable in those theories that have the first uncountable ordinal ω_1 and all subsets of $\omega_1 \times \omega_1$. I do not know whether OA^+ is strictly weaker than those theories. Next, we give a theory which is somewhat simple and whose strength is equal to that of OA^+.

§ 2. The theory OA^+.

2.1. The language of OA^+ (denoted by L_0).

(a) Individual variables: $x_0, x_1, \cdots$.

(b) Predicate variables: $P_0, P_1, \cdots$.

(c) Predicate constants: $*=*, *<*, O*$.

(d) Logical symbols: $\neg, \wedge, \exists$.

2.2. The axioms and the inferences of OA^+.

(a) The axioms and the inferences of the standard second order logic and the equality axiom: $a=b \leftrightarrow \forall P[Pa \rightarrow Pb]$.

(b) The following four:

$$Oa \wedge \forall x[x<a \leftrightarrow x<b] \rightarrow a=b\,;$$

$$Oa \wedge x<a \wedge y<x \rightarrow y<a\,;$$

$$\forall P[\forall x[Ox \rightarrow [(\forall y<x)Py \rightarrow Px]] \rightarrow \forall x[Ox \rightarrow Px]]\,;$$

$$Oa_1 \wedge \cdots \wedge Oa_n \wedge \forall x[A(x) \rightarrow Ox \wedge (\forall y<x)A(y)] \rightarrow \exists y[Oy \wedge \forall z[z<y \leftrightarrow A(z)]]\,,$$

where $A(x)$ contains neither the predicate constant O nor free variables except $a_1 \cdots a_n, x$.

§ 3. An interpretation of OA^+ in ZF.

There is a direct interpretation of OA^+ in ZF, but we show an indirect one to imply that OA^+ is interpretable in theories that are weaker than ZF.

By L_{ZF}, we shall denote the language of ZF.

First, we add the new constant symbol α, β to L_{ZF} and add the following axioms to ZF:

(a) $\mathrm{Ord}(\alpha) \wedge \mathrm{Ord}(\beta) \wedge \alpha < \beta$,

(b) $(\forall a_1 \cdots a_n < \alpha)[(\exists x < \beta)A(x, a_1 \cdots a_n, \beta) \to (\exists x < \alpha)A(x, a_1 \cdots a_n, \beta)]$

for every $A(x, a_1 \cdots a_n, \beta)$, where the constant symbol α does not occur in $A(x, a_1 \cdots a_n, \beta)$ and all free variables are indicated.

Denote this theory by ZF'.

LEMMA 1. *ZF' is a conservative extension of ZF.*

PROOF. Let B be a sentence of L_{ZF} and $ZF' \vdash B$. Note that for any finite set $A_1, \cdots, A_n$ of formulas of L_{ZF}, there exists a formula A of L_{ZF} such that

$$ZF \vdash (\forall ab \in On)[(\forall a_1 \cdots a_n < a)[(\exists x < b)A(xa_1 \cdots a_n b) \to (\exists x < a)A(xa_1 \cdots a_n b)]$$
$$\to \wedge_{i=1 \cdots n}(\forall a_1 \cdots a_n < a)[(\exists x < b)A_i(xa_1 \cdots a_n b) \to (\exists x < a)A_i(xa_1 \cdots a_n b)]],$$

where $x, a_1 \cdots a_n, b$ are all of the variables occurring free in $A_1, \cdots, A_n$: E. g., let $A(xa_1 \cdots a_n b)$ be

$$\wedge_{i=1 \cdots n}[(\exists y < b)A_i(ya_1 \cdots a_n b) \to (\exists y \leqq x)A_i(ya_1 \cdots a_n b)].$$

So, there exists a formula A of L_{ZF} such that

$$ZF \vdash (\exists a, b \in On)[0 < a < b \wedge (\forall a_1 \cdots a_n < a)[(\exists x < b)A(xa_1 \cdots a_n b)$$
$$\to (\exists x < a)A(xa_1 \cdots a_n b)]] \to B.$$

Now let $\alpha_0 = 1$ and $\alpha_{m+1} = \sup[\xi(a_1 \cdots a_n) | a_1 \cdots a_n < \alpha_m]$, where $\xi(a_1 \cdots a_n)$ means the least ordinal ξ such that $(\exists x < \omega_1)A(xa_1 \cdots a_n \omega_1) \to A(\xi a_1 \cdots a_n \omega_1)$, and put $\alpha = \sup \alpha_m$. Then $0 < \alpha < \omega_1$ and $(\forall a_1 \cdots a_n < \alpha)[(\exists x < \omega_1)A(xa_1 \cdots a_n \omega_1) \to (\exists x < \alpha) A(xa_1 \cdots a_n \omega_1)]$. Hence we have $ZF \vdash B$, q. e. d.

Now, we shall interpret OA^+ in ZF'.

For each formula A of L_0, we define its interpretation $I(A)$ in ZF' recursively as follows:

$I(A)$ is A if A is $a < b$ or $a = b$,

$I(Pa_1 \cdots a_n)$ is $\langle a_1 \cdots a_n \rangle \in P$,

$I(Oa)$ is $\mathrm{Ord}(a) \wedge a < \alpha$,

$I(\neg A)$ and $I(A\wedge B)$ are $\neg I(A)$ and $I(A)\wedge I(B)$ respectively,

$I(\exists xA)$ is $\exists x[\mathrm{Ord}(x)\wedge x<\beta\wedge I(A)]$,

$I((\exists P)A)$ is $(\exists P)I(A)$,

where every symbol which is contained in L_0 as an individual variable or a predicate variable is assumed to be contained also in L_{ZF} as a variable.

THEOREM 1. *$OA^+\vdash A \Rightarrow ZF'\vdash I(A)$, for every sentence A of L_0.*

PROOF. We shall show this for the main case that A is the axiom

$$\forall a[Oa\wedge\forall x[B(xa)\rightarrow Ox\wedge(\forall y<x)B(ya)]\rightarrow\exists u[Ou\wedge\forall x[x<u\leftrightarrow B(xa)]]],$$

where B does not contain the predicate constant O.

Suppose $a<\alpha\wedge(\forall x<\beta)[I(B(xa))\rightarrow x<\alpha\wedge(\forall y<x)I(B(ya))]$. Then we have $(\exists z<\beta)(\forall x<\beta)[I(B(xa))\rightarrow x<z]$ since $\alpha<\beta$. This formula does not contain the constant symbol α, since the symbol O does not appear in $B(xa)$. Hence we have $(\exists z<\alpha)(\forall x<\beta)[I(B(xa))\rightarrow x<z]$, which implies

$$(\exists u<\alpha)(\forall x<\beta)[x<u\leftrightarrow I(B(xa))].$$

Thus we have $ZF'\vdash I(A)$.

§ 4. The theory O_2.

DEFINITION of O_2. The underlying logic of the theory O_2 is the standard second order logic with an individual constant α and predicate constants $=$ and $<$. (We shall write L_α to denote this language.) The axioms are the following:

(a) "$<$ is a well-ordering,"

(b) $(\forall a_1\cdots a_n<\alpha)[(\exists x<\alpha)A(xa_1\cdots a_n)\leftrightarrow\exists xA(xa_1\cdots a_n)]$, where $A(xa_1\cdots a_n)$ contains neither the constant α nor free variables except the indicated.

Now, the assertion in § 3 is divided into the following two: O_2 is consistent and OA^+ is interpretable in O_2.

We used ω_1 to prove the consistency of O_2 in § 3. The author has no answer to the following question now:

QUESTION. Is ω_1 necessary to prove the consistency of O_2?; e.g., "Is the sentence $\forall P[(\forall x<\omega)(\exists! y)Pxy\rightarrow\exists z(\forall x<\omega)\neg Pxz]$ consistent with O_2 relative to O_2?"

In the rest, we show that OA^+ is almost equal to O_2 in strength.

For this purpose, we shall provide some metamathematical notions on OA^+.

O-formulas are defined recursively as follows:

(a) $a<b$, $a=b$, $Pa_1\cdots a_n$ are O-formulas;

(b) If A and B are O-formulas, then so are $\exists x[Ox \wedge A]$, $(\exists P)A$, $A \wedge B$ and $\neg A$.

A P-formula means a formula in which the predicate constant O does not occur.

Let $L(b)$ be the conjunction of the following four:

$$\forall y[\forall x[x<b \leftrightarrow x<y] \rightarrow b=y],$$

$$\forall P[(\exists x<b)Px \rightarrow (\exists x<b)[Px \wedge (\forall y<b)[Py \rightarrow x \leqq y]]],$$

$$\forall x \forall y[x<y<b \rightarrow x<b],$$

$$\forall x[x<b \rightarrow x \neq b].$$

$O^{\#}(x)$ is the P-formula $(\forall b \leqq x)L(b)$.

We shall denote the constant $\iota x \forall y[y<x \leftrightarrow Oy]$ by Ω as in [3].

The interpretation I of OA^{+} in O_2 is defined recursively as follows: $I(a<b)$, $I(a=b)$, $I(Pa_1 \cdots a_n)$ and $I(Oa)$ are $a<b$, $a=b$, $Pa_1 \cdots a_n$ and $a<\alpha$ respectively; $I(\neg A)$, $I(A \wedge B)$, $I(\exists xA)$ and $I((\exists P)A)$ are $\neg I(A)$, $I(A) \wedge I(B)$, $\exists x I(A)$ and $(\exists P)I(A)$ respectively.

Next, for each formula $F(x)$ of L_0, we define an interpretation $R(\lambda x F(x), *)$ (or simply $R_F(*)$) of O_2 in OA^{+} recursively as follows:

$R_F(x)$ is x for every individual variable x,

$R_F(\alpha)$ is Ω,

$R_F(Xt_1 \cdots t_n)$ is $Xs_1 \cdots s_n$ where X is a predicate symbol and s_i is $R_F(t_i)$ for $i=1, \cdots, n$,

$R_F(\neg A)$ and $R_F(A \wedge B)$ are $\neg R_F(A)$ and $R_F(A) \wedge R_F(B)$ respectively,

$R_F(\exists xA)$ is $\exists x[O^{\#}(x) \wedge F(x) \wedge R_F(A)]$,

$R_F((\exists P)A)$ is $(\exists P)R_F(A)$.

We write often $R_t(A)$ for $R(\lambda x(x<t), A)$.

LEMMA 2. *If a sentence A of L_α is logically valid (i.e., provable in the second order logic with $=$), then*

$$OA^{+} \vdash F(\Omega) \rightarrow R_F(A).$$

PROOF. By induction on the length of the proof for A.

LEMMA 3. $OA^{+} \vdash A \Rightarrow O_2 \vdash I(A)$.

PROOF. Same as § 3.

LEMMA 4. $OA^{+} \vdash (\forall x<\Omega)F(x) \rightarrow [R_F(I(A)) \leftrightarrow A]$, *for every O-formula A.*

PROOF. By induction based on the recursive definition of O-formulas.

LEMMA 5. *Let A be a sentence of L_α such that $O_2 \vdash A$. Then there is a* *formula* $F(ux)$ *such that*

$$OA^{+} \vdash \exists u[(\forall x<\Omega)F(ux) \wedge R(\lambda x F(ux), A)].$$

To prove this lemma we shall provide further metamathematical notions on OA^{+}.

For any formula $A(x)$, the formula $(\exists!\, x)[A^{\#}(x) \wedge (\forall y<x) \neg A^{\#}(y)]$ is provable (in OA^{+}), where $A^{\#}(x)$ is the formula $O^{\#}(x) \wedge [\exists z[O^{\#}(z) \wedge A(z)] \rightarrow A(x)]$. We write $\mu x A(x)$ for $\iota x[A^{\#}(x) \wedge (\forall y<x) \neg A^{\#}(y)]$.

If a function f can be defined by the postulate "$y=f(x_1 \cdots x_n) \leftrightarrow A(yx_1 \cdots x_n)$" for some P-formula A, we call it a P-function.

$J(*, *)$, $K(*)$ and $L(*)$ are the P-functions defined similarly as in [3] such that for all $x, y<\Omega$, $J(K(x), L(x))=x$, $K(J(xy))=x$, $L(J(xy))=y$ and $J(xy)$, $K(x)$, $L(x)<\Omega$.

Let $L'(*, *)$ be the P-function defined by the following induction:

$L'(0, x)=x$,

$L'(k, x)=L(L'(k-1, x))$ if $0<k<\omega$,

$L'(k, x)=0$ otherwise.

We shall write $(a)_i$ for $K(L'(i, a))$, and $(a)_{ij}$ for $((a)_i)_j$.

PROOF OF LEMMA 5. Suppose that a sentence A is provable in O_2. Then there is a formula $B(xa_1 \cdots a_n)$ of L_α which contains neither the constant α nor free variables except the indicated and which possesses the following property: The sentence [$<$ is a well-ordering]$\wedge 0<\alpha \wedge (\forall a_1 \cdots a_n<\alpha)[\exists x B(xa_1 \cdots a_n) \rightarrow (\exists x<\alpha) B(x, a_1 \cdots a_n)] \rightarrow A$ is logically valid. (See the proof of Lemma 1 for this reason.) Since B does not contain α, it is also a formula of L_0; besides a P-formula.

Let H be the P-function defined by the following induction:

$$H(0, a)=\mu x[R_{O^{\#}}(B)(x, (a)_{01} \cdots (a)_{0N})],$$

$$H(k, a)=\mu x[x<H(k-1, a) \wedge R_{H(k-1,a)}(B)(x, (a)_{k1} \cdots (a)_{kN})]$$

if $0<k<\omega$,

$H(k, a)=0$ otherwise,

where N means the n-th numeral.

$$\text{Put } \beta=\mu x[x \geqq \Omega \wedge (\exists k, a<\Omega)[x=H(k, a)]].$$

From the definition we easily obtain that

(a) $O^{\#}(H(k, a))$,

(b) $(\forall j \leqq k)[(a)_j=(b)_j] \rightarrow H(k, a)=H(k, b)$,

(c) $H(k, a) \neq 0 \rightarrow H(k+1, a) < H(k, a)$,

(d) $a, k < \Omega \wedge \beta = H(k, a) \neq 0 \rightarrow H(k+1, a) < \Omega$.

SUBLEMMA 1. $\beta=0 \rightarrow (\forall a_1 \cdots a_n < \Omega)[\exists x[O^{\#}(x) \wedge R_{O^{\#}}(B)(xa_1 \cdots a_n)] \rightarrow (\exists x < \Omega) R_{O^{\#}}(B)(xa_1 \cdots a_n)]]$.

PROOF. $\beta=0$ implies $H(0, a) < \Omega$ for every $a < \Omega$.

SUBLEMMA 2. $a_1 \cdots a_n < \Omega \wedge \beta \neq 0 \wedge (\exists x < \beta)[R_\beta(B)(xa_1 \cdots a_n)] \rightarrow (\exists x < \Omega)[R_\beta(B)(xa_1 \cdots a_n)]$.

PROOF. Since $\beta \neq 0$, there exist $a, k < \Omega$ such that $\beta = H(k, a)$. Since $a, a_1 \cdots a_n < \Omega$, there exists $c < \Omega$ such that $((c)_{k+1})_1 = a_1, \cdots, ((c)_{k+1})_N = a_n$ and $(\forall j \leqq k)[(a)_j = (c)_j]$. Put $d = H(k+1, c)$. Then $d = \mu x[x < \beta \wedge R_\beta(B)(xa_1 \cdots a_n)]$ since $H(k, c) = H(k, a) = \beta$. Hence $R_\beta(B)(da_1 \cdots a_n)$ since $(\exists x < \beta)[R_\beta(B)(xa_1 \cdots a_n)]$. Besides $d = H(k+1, c) < \Omega$, since $H(k, c) = \beta \neq 0$ and $k, c < \Omega$, q. e. d.

Now let $F(ux)$ be the P-formula

$$O^{\#}(x) \wedge [u=0 \vee [u \neq 0 \wedge x < H((u)_0, (u)_1)]] .$$

SUBLEMMA 3. $OA^+ \vdash (\exists u < \Omega)[(\forall x < \Omega)F(ux) \wedge R(\lambda x F(ux),$

$$(\forall a_1 \cdots a_n < \alpha)[\exists x B(xa_1 \cdots a_n) \rightarrow (\exists x < \alpha) B(xa_1 \cdots a_n)])] .$$

PROOF. Case 1: $\beta=0$. Put $u=0$. Then $F(ux) \leftrightarrow O^{\#}(x)$. Hence the desired conclusion is immediate from Sublemma 1.

Case 2: $\beta \neq 0$. There exist $a, k < \Omega$ such that $\beta = H(k, a)$. Put $u = J(k, J(a, 1))$. Then $F(ux) \leftrightarrow x < \beta$. The desired conclusion follows from $u < \Omega$ and Sublemma 2, q. e. d.

Now, write W for the sentence "$<$ is a well-ordering." Write B^* for the sentence $(\forall a_1 \cdots a_n < \alpha)[\exists x B(xa_1 \cdots a_n) \rightarrow (\exists x < \alpha) B(xa_1 \cdots a_n)]$. Since $\vdash W \wedge B^* \wedge 0 < \alpha \rightarrow A$, we have, by Lemma 2,

$$OA^+ \vdash F(u, \Omega) \wedge R(\lambda x F(ux), W \wedge B^* \wedge 0 < \alpha) \rightarrow R(\lambda x F(ux), A) .$$

Since $O^{\#}$ is well-ordered by $<$, $OA^+ \vdash R(\lambda x F(ux), W)$. Since $F(ux)$ is a P-formula, $(\forall x < \Omega)F(ux) \wedge u < \Omega$ implies $F(u, \Omega)$. Hence by SubLemma 3, $OA^+ \vdash (\exists u < \Omega)[(\forall x < \Omega)F(ux) \wedge R(\lambda x F(ux), A)]$. This completes the proof of Lemma 5.

Now, we see that O_2 is a conservative extension of OA^+ in the following sense:

THEOREM 2. *$OA^+ \vdash A \Leftrightarrow O_2 \vdash I(A)$ for every O-sentence A.*

PROOF. ($\Rightarrow$) See § 3.

($\Leftarrow$) Let $O_2 \vdash I(A)$. Then by Lemma 5 there is a formula $F(ux)$ such that $OA^+ \vdash \exists u[(\forall x < \Omega)F(ux) \wedge R(\lambda x F(ux), I(A))]$. Now, suppose $(\forall x < \Omega) F(ux) \wedge$

$R(\lambda xF(ux), I(A))$ in OA^+. Then by Lemma 4, $R(\lambda xF(ux), I(A)) \leftrightarrow A$. Hence A,
q. e. d.

§ 5. A remark.

Indeed, OA^+ is stronger than OA. Because the consistency of OA is provable in OA^+. We verify this fact in this section. Since $OA^+ \vdash \text{Cons}(OA) \Leftrightarrow O_2 \vdash \text{Cons}(OA)$ by the result of the previous section, it suffices to show $O_2 \vdash \text{Cons}(OA)$.

For this purpose we shall provide some notions.

If a formula of L_α does not contain the constant α, we call it a P-formula. A term which is defined by a P-formula is called a P-term.

Consider (in O_2) the model L of the constructible sets in the similar manner in [3]. Similarly as xEy and $\langle xy\rangle^\circ$ in [3], there exist a P-formula $x \in y$ which means $\mathfrak{F}'x \in \mathfrak{F}'y$ intuitively and a P-term $\langle xy\rangle$ which means an ordered pair in L. We can easily define a P-term $\tilde{x}$ which means the x-th ordinal in L.

Now, there exists a formula $I(*, *)$ of L_α which possesses the following properties in O_2:

(a) $I(s, \ulcorner x < y \urcorner) \leftrightarrow s(\ulcorner x \urcorner) < s(\ulcorner y \urcorner)$,

(b) $I(s, \ulcorner Ox \urcorner) \leftrightarrow s(\ulcorner x \urcorner) < \alpha$,

(c) $I(s, \ulcorner Px_1 \cdots x_n \urcorner) \leftrightarrow \langle \widetilde{s(\ulcorner x_1 \urcorner)} \cdots \widetilde{s(\ulcorner x_n \urcorner)} \rangle \in s(\ulcorner P \urcorner)$,

(d) $I(s, \ulcorner A \wedge B \urcorner) \leftrightarrow I(s, \ulcorner A \urcorner) \wedge I(s, \ulcorner B \urcorner)$,

(e) $I(s, \ulcorner \neg A \urcorner) \leftrightarrow \neg I(s, \ulcorner A \urcorner)$,

(f) $I(s, \ulcorner \exists x A \urcorner) \leftrightarrow \exists a \forall s' [\forall b [b \neq \ulcorner x \urcorner \rightarrow s'(b) = s(b)] \wedge s'(\ulcorner x \urcorner) = a$
$\rightarrow I(s', \ulcorner A \urcorner)]$,

(g) $I(s, \ulcorner (\exists P) A \urcorner) \leftrightarrow \exists a \forall s' [\forall b [b \neq \ulcorner P \urcorner \rightarrow s'(b) = s(b)] \wedge s'(\ulcorner P \urcorner) = a$
$\rightarrow I(s, \ulcorner A \urcorner)]$,

where $\ulcorner X \urcorner$ means Gödel number of X and $s(x)$ means the individual assigned to the "variable" x by the assignment s.

And there exists a P-formula $J(*, *)$ of L_α which possesses the properties (a), (c)-(g) in O_2.

The following is clear:

LEMMA 6.

$$O_2 \vdash \forall \ulcorner A \urcorner [\ulcorner A \urcorner \textit{ is a "P-formula"} \rightarrow \forall s [I(s, \ulcorner A \urcorner) \leftrightarrow J(s, \ulcorner A \urcorner)]].$$

The notation $\bar{b}$ (also $\bar{x}$) below means a finite sequence of variables.

LEMMA 7. *Let A be a formula of L_α which does not contain free predicate variables, and a, $\bar{b}$, $\bar{x}$, y be all of the free variables in A. Then*

$$O_2 \vdash \forall a \forall \bar{b}[\forall \bar{x} < a \exists y A \rightarrow \exists c \forall \bar{x} < a \exists y < cA].$$

PROOF. We may assume that A does not contain the constant α. For, the assertion for the general case results from the above by substituting α for one of the variables $\bar{b}$.

$$\forall a < \alpha \forall \bar{b} < \alpha \forall \bar{x} < \alpha[\exists y A \rightarrow \exists y < \alpha A]$$

is an axiom of O_2. Hence

$$\forall a < \alpha \forall \bar{b} < \alpha[\forall \bar{x} < a \exists y A \rightarrow \forall \bar{x} < a \exists y < \alpha A].$$

Hence

$$\forall a < \alpha \forall \bar{b} < \alpha[\forall \bar{x} < a \exists y A \rightarrow \exists c \forall \bar{x} < a \exists y < cA].$$

This implies

$$\forall a \forall \bar{b}[\forall \bar{x} < a \exists y A \rightarrow \exists c \forall \bar{x} < a \exists y < cA]$$

by axioms of O_2, since

$$[\forall \bar{x} < a \exists y A \rightarrow \exists c \forall \bar{x} < a \exists y < cA]$$

is a P-formula, q. e. d.

LEMMA 8. $O_2 \vdash$ "*For every formula A of L_0 and every individual a, there exists u such that*

$$\forall x_1 \cdots x_n[\langle \tilde{x}_1 \cdots \tilde{x}_n \rangle \in u \leftrightarrow x_1 \cdots x_n < a \wedge I((v_1/x_1 \cdots v_n/x_n), A)],$$

where $v_1 \cdots v_n$ is a sequence of variables in which every free variable in A appears and $(v_1/x_1 \cdots v_n/x_n)$ means the assignment that assigns x_i to v_i for each $i=1 \cdots n$."

PROOF. Note that $O_2 \vdash \forall xy \exists z \forall v < x \forall w < y[\langle vw \rangle < z]$. Use induction (in O_2) on the complexity of A with the aid of Lemma 7.

LEMMA 9. $O_2 \vdash \forall \ulcorner A \urcorner [\ulcorner A \urcorner$ *is an "axiom of OA"* $\rightarrow I(O, \ulcorner A \urcorner)]$.

PROOF. If A is an axiom of comprehension, it follows from Lemma 8. If A is an axiom of reflection (of Ackermann-type), it follows from Lemma 6. The other cases are trivial.

By Lemma 9 and the properties of $I(*, *)$, we have $O_2 \vdash \mathrm{Cons}(OA)$.

References

[1] W. Ackermann, Zur Axiomatik der Mengenlehre, Math. Ann., **131** (1956), 336-345.

[2] M. Hanazawa, On replacement schemas in Ackermann's set theory, Sci. Rep. Saitama Univ., **8** (1957), 25-28.

[3] M. Hanazawa, A theory of ordinal numbers with Ackermann's schema, to appear.

[4] J. Lake, On an Ackermann-type set theory, J. Symbolic Logic, **38** (1973), 410-412.

[5] J. Lake, Natural models and Ackermann-type set theories, J. Symbolic Logic, **40** (1975), 151-158.

[6] A. Levy, On Ackermann's set theory, J. Symbolic Logic, **24** (1959), 154-166.

[7] A. Levy and R.L. Vaught, Principles of partial reflection in the set theories of Zermelo and Ackermann, Pacific J. Math., **11** (1961), 1045-1062.

[8] W. Reinhardt, Ackermann's set theory equals ZF, Ann. Math Logic, **2** (1970), 189-249.

Masazumi HANAZAWA

Department of Mathematics
Faculty of Science
Saitama University
Urawa, Saitama
Japan