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Scattering theory for Schrödinger equations with potentials periodic in time

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§ 1. Introduction.

Let $\mathscr{H}$ be the Hilbert space $L^2(\boldsymbol{R}^n)(n\geqq 3)$ and let $\{H(t)=H_0+V(t),\ t\in\boldsymbol{R}^1\}$ be a family of Schrödinger operators in $\mathscr{H}$ with a time-dependent perturbation $V(t)$, where $H_0=-\Delta$ is the negative Laplacian in $\boldsymbol{R}^n$. We suppose that $\{-iH(t)\,;\ t\in\boldsymbol{R}^1\}$ generates a unitary evolution group $\{U(t,s)\,;\ -\infty<t,s<\infty\}$. Fundamental problems in scattering theory under these circumstances are as follows. (1) When do the strong limits

$$W_{\pm}(s)f=\lim_{t\to\pm\infty}U(t,s)^{-1}e^{-i(t-s)H_0}f$$

exist for every $f\in\mathscr{H}$ and every $s\in\boldsymbol{R}^1$? (2) If the above limits exist, how can we characterize their ranges $R(W_{\pm}(s))$, in particular, do their ranges coincide (completeness of wave operators)?

The study of the problems has begun in recent years. However, most works appeared so far are concerned with the problems under the assumption that the perturbation $V(t)$ vanishes sufficiently rapidly as $|t|\to\infty$ so that $W_{\pm}(s)$ turn out to be unitary. In the case that the perturbation $V(t)$ is periodic in time, on the other hand, the wave operators $W_{\pm}(s)$ are not unitary in general. This case was first taken up by Schmidt [**19**] who proved, among other things, the existence and the completenese of $W_{\pm}(s)$, determining their ranges precisely. In [**19**] this result was proved in the situation that $V(t)$ is an operator of trace class for each t; and for $V(t)$ given by a potential $v(t,x)$, it was conjectured that if $v(t,\cdot)\in L^2(\boldsymbol{R}^3)\cap L^1(\boldsymbol{R}^3)$ the existence and the completeness of $W_{\pm}(s)$ would hold.

The purpose of the present paper is to study the problems in the case that the perturbation $V(t)$ is periodic in time and given by a time-dependent potential $v(t,x)$. We shall prove that the conjecture of Schmidt holds good under weaker conditions. Namely, we first assume the following Assumption (A.1).

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ASSUMPTION (A.1). The function $v(t, x)$ is a real valued function defined on $\boldsymbol{R}^1\times\boldsymbol{R}^n$ and there exists a positive constant ω such that

$$v(t+\omega, x)=v(t, x), \qquad (t, x)\in\boldsymbol{R}^1\times\boldsymbol{R}^n.$$

We further assume one of the following two assumptions.

ASSUMPTION (A.2). There exist constants p and q such that $1\leqq p<n/2<q\leqq\infty$ and such that the functions $t\to v(t, \cdot)$ is an $L^p(\boldsymbol{R}^n)\cap L^q(\boldsymbol{R}^n)$-valued absolutely continuous function. If $n=3$ and $q<2$, we further assume that the function $t\to v(t, \cdot)$ is an $L^p(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$-valued continuously differentiable function.

ASSUMPTION (A.3). There exist a constant $\delta>1$ and an $L^\infty(\boldsymbol{R}^n)$-valued absolutely continuous function $w(t, x)$ such that $v(t, x)=(1+|x|^2)^{-\delta/2}w(t, x)$.

Under Assumptions (A.1) and (A.2) with $q\geqq 2$ or under Assumptions (A.1) and (A.3), $H(t)=H_0+v(t, x)$ defined on $C_0^\infty(\boldsymbol{R}^n)$ is essentially selfadjoint and the selfadjoint extension $H(t)$ has the domain $H^2(\boldsymbol{R}^n)$. Under Assumptions (A.1) and (A.2) with $q<2$, the sesquilinear form h defined by $h(f, g)=(H_0f, g)+(v(t, x)f, g)$ for $f, g\in C_0^\infty(\boldsymbol{R}^n)$ determines uniquely a selfadjoint operator $H(t)$ on $\mathcal{H}$ and $D(|H(t)|^{1/2})=H^1(\boldsymbol{R}^n)$. Furthermore under these assumptions the family of operators $\{-iH(t);\ t\in\boldsymbol{R}^1\}$ generates a unitary evolution group $\{U(t, s);\ -\infty<t, s<\infty\}$ (see Theorem 2.1 and Theorem 2.1′ below).

Our main theorem is as follows.

THEOREM 1.1. *Let Assumptions* (A.1) *and* (A.2) *or Assumptions* (A.1) *and* (A.3) *be satisfied. Then the strong limits*

$$W_\pm(s)f=\lim_{t\to\pm\infty}U(t, s)^{-1}e^{-i(t-s)H_0}f$$

exist for every $f\in\mathcal{H}$ and $s\in\boldsymbol{R}^1$. Furthermore we have

$$R(W_\pm(s))=\mathcal{H}_{ac}(U(s+\omega, s)).$$

In particular, the completeness of wave operators $R(W_+(s))=R(W_-(s))$ holds. Here $\mathcal{H}_{ac}(U)$ stands for the absolutely continuous subspace of $\mathcal{H}$ with respect to the unitary operator U.

REMARK 1.2. Roughly speaking, (A.2) implies that $v(t, x)=O(|x|^{-2-\varepsilon})$ as $|x|\to\infty$ but some singularities are allowed. (A-3) implies $v(t, x)=O(|x|^{-1-\varepsilon})$ as $|x|\to\infty$. The assumption on the decay rate of $v(t, x)$ as $|x|\to\infty$ in Assumption (A.3) can not be weakened even in the t-independent case (see Dollard [5]).

The composition of the paper is as follows. § 2 is preparatory in nature and two theorems concerning the existence of the evolution groups are given. In § 3, some lemmas which will be needed for the proof of the theorem are collected. § 4 is devoted to proving the theorem. § 5 is an appendix and a sufficient condition for the existence of wave operators for Schrödinger operators with general time-dependent potentials will be given.

Finally we shall list here some notations which will be used throughout the paper. $L_\gamma^2(\boldsymbol{R}^n)$ $(\gamma\in\boldsymbol{R}^1)$ denotes the class of all functions $f(x)$ on $\boldsymbol{R}^n$ such that $(1+|x|^2)^{\gamma/2}f(x)$ is square integrable on $\boldsymbol{R}^n$. The inner product $(\cdot,\cdot)_{2,\gamma}$ and the norm $\|\cdot\|_{2,\gamma}$ are defined by

$$(f, g)_{2,\gamma}=\int_{\boldsymbol{R}^n}(1+|x|^2)^\gamma f(x)\overline{g(x)}dx\,,$$

$$\|f\|_{2,\gamma}=(f,f)_{2,\gamma}^{1/2}\,.$$

When $\gamma=0$, we shall write $L_0^2(\boldsymbol{R}^n)=L^2(\boldsymbol{R}^n)=\mathcal{H}$ and $(\cdot,\cdot)_{2,0}=(\cdot,\cdot)$ etc. $H^m(\boldsymbol{R}^n)$ is the Sobolev space of order m on $\boldsymbol{R}^n$ with the usual inner product and the norm. For any open domain $G\subset\boldsymbol{R}^n$, $C_0^\infty(G)$ denotes the set of all infinitely differentiable functions on G with support compact in G. For any separable Hilbert space X and any measure space $(M, \mathcal{B}(M), dm)$, $L^2(M, X, dm)$ denotes the Hilbert space of all strongly measurable X-valued functions f with the norm

$$\|f\|_{L^2(M,X)}=\left\{\int_M\|f(m)\|_X^2dm\right\}^{1/2}$$

For any pair of Banach spaces X and Y, $B(X, Y)$ denotes the set of all bounded operators from X to Y. We write $B(X)=B(X, X)$. For any operator T from Banach space X to Y, $D(T)$ and $R(T)$ stand for the domain and the range of T, respectively.

$\mathcal{F}_{x\to\xi}$ stands for the Fourier transform from $\boldsymbol{R}_x^n$-space to $\boldsymbol{R}_\xi^n$-space and is defined by

$$(\mathcal{F}_{x\to\xi}f)(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{\boldsymbol{R}^n}e^{ix\cdot\xi}f(x)dx\,.$$

$\mathcal{F}_{t\to n}$ stands for the Fourier transform from $[0, \omega]$ to $\boldsymbol{Z}$ (the set of all integers) and is defined by

$$(\mathcal{F}_{t\to n}f)(n)=\frac{1}{\sqrt{\omega}}\int_0^\omega e^{2\pi nti/\omega}f(t)dt\,,\qquad n\in\boldsymbol{Z}\,.$$

As is well known $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}_{t\to n}$ are unitary operators from $L^2(\boldsymbol{R}_x^n)$ to $L^2(\boldsymbol{R}_\xi^n)$ and from $L^2([0, \omega])$ to $l^2(\boldsymbol{Z})$, respectively. $\boldsymbol{C}^1$ is the set of all complex numbers and $\Pi^\pm=\{\zeta\in\boldsymbol{C}^1;\ \mathrm{Im}\,\zeta\gtrless0\}$.

§ 2. Preliminaries.

In this section we shall give two theorems which guarantee the existence of the unitary evolution group and make some definitions which will be needed in the following sections.

THEOREM 2.1. *Let Assumptions* (A.1) *and* (A.2) *with* $2\leqq q$ *or Assumptions*

(A.1) *and* (A.3) *be satisfied. Then there exists a family of unitary operators* $\{U(t, s), -\infty<t, s<\infty\}$ *with the following properties:*

(2.1) $U(t, s)$ *is strongly continuous from* $\boldsymbol{R}^1\times\boldsymbol{R}^1$ *to* $B(\mathcal{H})\cap B(H^2(\boldsymbol{R}^n))$;

$$U(t, r)U(r, s)=U(t, s), \qquad -\infty<t, s, r<\infty; \tag{2.2}$$

$$\frac{dU(t, s)f}{dt}=-iH(t)U(t, s)f \quad \text{and} \quad \frac{dU(t, s)f}{ds}=iU(t, s)H(s)f, \tag{2.3}$$

$$f\in H^2(\boldsymbol{R}^n),$$

where the derivatives in these formulas are taken in the sense of strong derivatives of $\mathcal{H}$*-valued functions;*

$$U(t+\omega, s+\omega)=U(t, s), \qquad -\infty<t, s<\infty. \tag{2.4}$$

THEOREM 2.1′. *Let Assumptions* (A.1) *and* (A.2) *with* $n=3$ *and* $3/2<q<2$ *be satisfied. Then there exists a family of unitary operators* $\{U(t, s), -\infty<t, s<\infty\}$ *with properties* (2.2), (2.4) *and*

(2.1)′ $U(t, s)$ *is strongly continuous from* $\boldsymbol{R}^1\times\boldsymbol{R}^1$ *to* $B(\mathcal{H})$ *and is weakly continuous from* $\boldsymbol{R}^1\times\boldsymbol{R}^1$ *to* $B(H^1(\boldsymbol{R}^n))$;

$$\frac{dU(t, s)f}{dt}=-iH(t)U(t, s)f \quad \text{and} \quad \frac{dU(t, s)f}{ds}=iU(t, s)H(s)f, \tag{2.3$'$}$$

$$f\in H^1(\boldsymbol{R}^n),$$

where the derivatives in these formulas are taken in the sense of strong derivatives of $H^{-1}(\boldsymbol{R}^n)$*-valued functions.*

Theorem 2.1 is an immediate consequence of Theorem 1 of Kato [**13**]. Theorem 2.1′ is Theorem II.27 of Simon [**20**] (see also Kisyński [**15**]).

REMARK 2.2. As a consequence of these theorems we have, for any f and $g\in H^1(\boldsymbol{R}^n)$,

$$(U(t, s)f, g)=(e^{-i(t-s)H_0}f, g)-i\int_s^t(v(r, \cdot)U(r, s)f, e^{-i(r-t)H_0}g)dr. \tag{2.5}$$

In what follows we write

$$U_1(t, s)=U(t, s) \quad \text{and} \quad U_0(t, s)=e^{-i(t-s)H_0}.$$

$\boldsymbol{T}$ is the torus $\boldsymbol{R}^1/\omega\boldsymbol{Z}$ and $(\boldsymbol{T}, \mathcal{B}(\boldsymbol{T}), dt)$ is the measure space naturally induced on $\boldsymbol{T}$ by the Lebesgue measurable sets and the Lebesgue measure on $\boldsymbol{R}^1$. We write $\mathcal{K}=L^2(\boldsymbol{T}, \mathcal{H}, dt)$ and $\mathcal{K}_\gamma=L^2(\boldsymbol{T}, L^2_\gamma(\boldsymbol{R}^n), dt)$, $\gamma\in\boldsymbol{R}^1$. Sometimes we consider an element f of $\mathcal{K}$ as $\mathcal{H}$-valued locally square integrable function on $\boldsymbol{R}^1$ with period ω.

We define two families of operators $\{\mathcal{V}_j(\sigma),\ \sigma\in \boldsymbol{R}^1\}$, $j=0,1$, on $\mathcal{K}$ as

$$(\mathcal{V}_j(\sigma)f)(t)=U_j(t, t-\sigma)f(t-\sigma)\,, \qquad f\in\mathcal{K}\,, \qquad j=0,1\,. \tag{2.6}$$

Then $\{\mathcal{V}_j(\sigma),\ -\infty<\sigma<\infty\}$ forms strongly continuous unitary group on $\mathcal{K}$. We write the generator of this group as $-iK_j$, $j=0,1$. K_j is a selfadjoint operator in $\mathcal{K}$. $R_0(\zeta)=(K_0-\zeta)^{-1}$ and $R_1(\zeta)=(K_1-\zeta)^{-1}$ ($\zeta\in\boldsymbol{C}^1$, $\operatorname{Im}\zeta\neq 0$) denote the resolvents of K_0 and K_1, respectively. A and B are the operators of multiplication by $a((t)=a(t,x)=|v(t,x)|^{1/2}$ and $b(t)=b(t,x)=\operatorname{sign} v(t,x)|v(t,x)|^{1/2}$ in $\mathcal{K}$. We put $U_0(t,0)=U_0(t)$.

§ 3. Lemmas.

In this section we collect some lemmas which will be used in the proof of the theorem.

LEMMA 3.1. *Let Assumptions* (A.1) *and* (A.2) *be satisfied. Then there exist constants* $C>0$ *and* ε $(0<\varepsilon<1)$ *such that*

$$\|AR_0(\zeta)Bf\|_{\mathcal{K}}\leqq C\Big\{\int_T\alpha(t)^2dt\Big\}^{1/2}\Big\{\int_T\beta(t)^{1+\varepsilon^{-1}}dt\Big\}^{\varepsilon/(1+\varepsilon)}\|f\|_{\mathcal{K}}\,, \tag{3.1}$$

$$f\in D(B),\ \operatorname{Im}\zeta\neq 0,\ \textit{where}\ \alpha(t)=\max\{\|a(t)\|_{L^r(\boldsymbol{R}^n)}\,;\ r=p,q\}$$

$$\textit{and}\quad \beta(t)=\max\{\|b(t)\|_{L^r(\boldsymbol{R}^n)}\,;\ r=p,q\}\,.$$

Hence $AR_0(\zeta)B$ *has a bounded closure* $Q(\zeta)\equiv[AR_0(\zeta)B]$. *Furthermore* $Q(\zeta)$ *has the following properties:*

(a) $Q(\zeta)$ *is a* $B(\mathcal{K})$*-valued analytic function of* $\zeta\in\Pi^{\pm}$;

(b) *the analytic function* $Q(\zeta)$ *on* $\Pi^{\pm}$ *can be extended to* $\overline{\Pi^{\pm}}$ *as a uniformly Hölder continuous function with exponent* $n/2p-1$;

(c) $\|Q(\zeta)\|_{B(\mathcal{K})}\to 0$ *as* $|\operatorname{Im}\zeta|\to\infty$;

(d) $Q(\zeta)$ *is a compact operator in* $\mathcal{K}$, $\zeta\in\overline{\Pi^{\pm}}$.

LEMMA 3.2. *Let Assumptions* (A.1) *and* (A.3) *be satisfied. Then* $Q(\zeta)\equiv AR_0(\zeta)B$ $(\operatorname{Im}\zeta\neq 0)$ *is a bounded operator in* $\mathcal{K}$. *Furthermore* $Q(\zeta)$ *has the following properties:*

(a) $Q(\zeta)$ *is a* $B(\mathcal{K})$*-valued analytic function of* $\zeta\in\Pi^{\pm}$;

(b) *the analytic function* $Q(\zeta)$ *on* $\Pi^{\pm}$ *can be extended to* $\Pi^{\pm}\cup(\boldsymbol{R}^1\backslash(2\pi/\omega)\boldsymbol{Z})$ *as a locally Hölder continuous function;*

(c) $\|Q(\zeta)\|_{B(\mathcal{K})}\to 0$ *as* $|\operatorname{Im}\zeta|\to\infty$;

(d) $Q(\zeta)$ *is a compact operator for every* $\zeta\in\Pi^{\pm}\cup(\boldsymbol{R}^1\backslash(2\pi/\omega)\boldsymbol{Z})$.

LEMMA 3.3. *Let Assumptions* (A.1) *and* (A.2) *or* (A.1) *and* (A.3) *be satisfied. Then for every* $\zeta\in\boldsymbol{C}^1$ *with* $\operatorname{Im}\zeta\neq 0$,

$$R_1(\zeta)=R_0(\zeta)-[BR_0(\bar{\zeta})]^*(1+Q(\zeta))^{-1}AR_0(\zeta)\,. \tag{3.2}$$

LEMMA 3.4. *Let Assumptions* (A.1) *and* (A.2) *or Assumptions* (A.1) *and* (A.3) *be satisfied. Then there exist closed null sets* $e_\pm$ *of* $\boldsymbol{R}^1$ *such that for anv* $\lambda\in\boldsymbol{R}^1\backslash e_\pm$, $\lim_{\eta\downarrow 0} R_1(\lambda\pm i\eta)=R_1(\lambda\pm i0)$ *exist in* $B(\mathcal{K}_\gamma, \mathcal{K}_{-\gamma})$, *where* $\gamma>1$ *under Assumptions* (A.1) *and* (A.2), *and* $\gamma>1/2$ *under* (A.1) *and* (A.3).

LEMMA 3.5. *The strong limits*

$$\mathcal{W}_\pm f=\operatorname*{s-lim}_{\sigma\to\pm\infty} e^{i\sigma K_1}e^{-i\sigma K_0}f \tag{3.3}$$

exist for every $f\in\mathcal{K}$. *Moreover we have*

$$R(\mathcal{W}_\pm)=\mathcal{K}_{ac}(K_1), \tag{3.4}$$

where $\mathcal{K}_{ac}(K_1)$ *is the absolutely continuous subspace of* $\mathcal{K}$ *with respect to the selfadjoint operator* K_1. *In particular,* $R(\mathcal{W}_+)=R(\mathcal{W}_-)$.

REMARK 3.6. The proof of Lemma 3.1 will show that under Assumptions (A.1) and (A.2) operators $AR_0(\zeta)A$, $BR_0(\zeta)A$ and $BR_0(\zeta)B$ have the same properties as $AR_0(\zeta)B$ stated in Lemma 3.1. Hence the proof of Theorem 5.1 of Kato [**11**] shows that the operators A and B are K_0-smooth in the sense of Kato [**11**], that is,

(a) $D(A),\ D(B)\supset D(K_0)$,

(b) there exists a constant $C>0$ such that

$$\int_{\boldsymbol{R}^1}\|AR_0(\lambda\pm i\varepsilon)f\|_{\mathcal{K}}^2 d\lambda\leqq C\|f\|_{\mathcal{K}}^2,$$

$$\int_{\boldsymbol{R}^1}\|BR_0(\lambda\pm i\varepsilon)f\|_{\mathcal{K}}^2 d\lambda\leqq C\|f\|_{\mathcal{K}}^2,\qquad \varepsilon>0,\ f\in\mathcal{K}.$$

Hence the formulas in Lemma 3.3 have a meaning under our assumptions.

In what follows we shall give the proof of lemmas. If we assume the validity of Lemmas 3.1 to 3.4, Lemma 3.5 is an immediate consequence of the abstract stationary theory of scattering (see Kato-Kuroda [**14**] or Kako-Yajima [**9**]). We shall omit the proof of Lemma 3.5. We first prove Lemma 3.4, admitting the validity of Lemmas 3.1 to 3.3.

PROOF OF LEMMA 3.4. Multiply both sides of (3.2) by $(1+|x|^2)^{-\delta/2}$ from the left and right ($\delta>1$ or $\delta>1/2$, according as the first or the second type of assumptions is assumed). Since the function $(1+|x|^2)^{-\delta/2}$ satisfies the conditions imposed on A in Lemma 3.2 (or 3.3), the statement is an immediate consequence of Lemma 6.2 of Kuroda [**16**]. (Q. E. D.)

For the proof of Lemma 3.1 the following lemma of Kato [**11**] is needed.

LEMMA 3.7 (Kato [**11**]). *Let* f *and* $g\in L^r(\boldsymbol{R}^n)$, $2\leqq r\leqq\infty$, $n\geqq 1$. *Let* F *and* G *be the operators of multiplication by* f *and* g *in* $\mathcal{H}$. *Then for every* $u\in D(G)$ *we have*

$$\|Fe^{-itH_0}Gu\| \leqq (4\pi|t|)^{-n/r}\|f\|_{L^r(R^n)}\|g\|_{L^r(R^n)}\|u\| . \tag{3.5}$$

PROOF OF LEMMA 3.1. We shall first prove (3.1) for $\mathrm{Im}\,\zeta>0$. The other case can be proved similarly. Using Lemma 3.7 and the Laplace transform, we see after a simple consideration that for any $f\in D(B)$ and for almost every $t\in[0, \omega]$

$$\begin{aligned}(A(K_0-\zeta)^{-1}Bf)(t) &= i\int_0^\infty e^{-is\zeta}a(t)U_0(s)b(t-s)f(t-s)ds \\ &= i\int_{-\infty}^t e^{-i(t-s)\zeta}a(t)U_0(t-s)b(s)f(s)ds \\ &= i\sum_{n=-\infty}^{-2}\int_{n\omega}^{(n+1)\omega} e^{-i(t-s)\zeta}a(t)U_0(t-s)b(s)f(s)ds \\ &\quad +i\int_{-\omega}^t e^{-i(t-s)\zeta}a(t)U_0(t-s)b(s)f(s)ds = I_1+I_2 .\end{aligned} \tag{3.6}$$

Using the periodicity of $a(t)$, $b(t)$ and $f(t)$, we have

$$I_1(t) = i\sum_{n=-\infty}^{-2}\int_0^\omega e^{-i(t-s-n\omega)\zeta}a(t)U_0(t-s-n\omega)b(s)f(s)ds ,$$

$$I_2(t) = i\int_{-\omega}^t e^{-i(t-s)\zeta}a(t)U_0(t-s)b(s)f(s)ds , \qquad t\in \boldsymbol{T},$$

where we write $f(t)=f(t-[t/\omega]\omega)$, $t\in \boldsymbol{R}^1$. By the use of Lemma 3.7 we have with some constant $c>0$

$$\begin{aligned}\|I_1(t)\|_{\mathscr{H}} &\leqq c\sum_{n=-\infty}^{-2}\int_0^\omega e^{\mathrm{Im}\zeta(t-s-n\omega)} \min\,(\|a(t)\|_{L^r}|t-s-n\omega|^{-n/r}\|b(s)\|_{L^r}\,;\ r=p,\,q)ds \\ &\leqq c\alpha(t)e^{\mathrm{Im}\zeta(t-\omega)}\int_0^\omega \sum_{n=-\infty}^{-2}\min(|t-s-n\omega|^{-n/r}\,;\ r=p,\,q)\beta(s)\|f(s)\|_{\mathscr{H}}ds .\end{aligned} \tag{3.7}$$

Since $\sum_{n=-\infty}^{-2}\min(|t-s-n\omega|^{-n/r}\,;\ r=p,\,q)\leqq M$ for any $s, t\in \boldsymbol{T}$ with some constant M independent on s, t, we get

$$\begin{aligned}\int_{\boldsymbol{T}}\|I_1(t)\|^2dt &\leqq CM\int_{\boldsymbol{T}}\alpha(t)^2e^{2\mathrm{Im}\zeta(t-\omega)}dt\cdot\left(\int_{\boldsymbol{T}}\beta(t)\|f(t)\|_{\mathscr{H}}dt\right)^2 \\ &\leqq CM\int_{\boldsymbol{T}}\alpha(t)^2e^{2\mathrm{Im}\zeta(t-\omega)}dt\int_{\boldsymbol{T}}\beta(t)^2dt\cdot\|f\|^2_{\mathscr{K}} .\end{aligned} \tag{3.8}$$

As for $I_2(t)$ we proceed as follows. Since $\gamma(s)\equiv\min(s^{-n/r}\,;\ r=p,\,q)\in L^{1+\varepsilon}((0, \infty))$ for some positive constant $0<\varepsilon<1$ we have by Young's inequality and Hölder's inequality, (3.9) $$\int_T \|I_2(t)\|_{\mathcal{K}}^2 dt \leqq c\int_T \alpha(t)^2 dt \int_T dt \Big\{\int_{-\omega}^{t} e^{\mathrm{Im}\zeta(t-s)}\gamma(t-s)\beta(s)\|f(s)\| ds\Big\}^2$$

$$\leqq 2c\int_T \alpha(t)^2 dt\cdot\Big\{\int_0^\infty (e^{\mathrm{Im}\zeta\cdot s}\gamma(s))^{1+\varepsilon} ds\Big\}^{\frac{2}{1+\varepsilon}}$$

$$\times\Big\{\int_T (\beta(s)\|f(s)\|)^{\frac{2+2\varepsilon}{1+3\varepsilon}} ds\Big\}^{\frac{1+3\varepsilon}{2+2\varepsilon}\cdot 2}$$

$$\leqq 2c\int_T \alpha(t)^2 dt\cdot\Big\{\int_0^\infty e^{(1+\varepsilon)\mathrm{Im}\zeta\cdot s}\gamma(s)^{1+\varepsilon} ds\Big\}^{\frac{2}{1+\varepsilon}}$$

$$\times\Big\{\int_T \beta(s)^{\frac{1+\varepsilon}{\varepsilon}} ds\Big\}^{\frac{2\varepsilon}{1+\varepsilon}}\Big\{\int_T \|f(s)\|^2 ds\Big\}.$$

Combining (3.8) and (3.9) we obtain the desired result. Statement (a) is obvious. Statement (b) can be proved easily by the use of the inequality

$$|e^{is\zeta}-e^{it\zeta}| \leqq \min(2, |t-s|)$$

and similar calculations used in the proof of (3.1). Statement (c) is proved by (3.8), (3.9) and the use of dominated convergence theorem. Finally we prove statement (d). Let $a_k(t)\in C(\boldsymbol{T}, C_0^\infty(\boldsymbol{R}^n))(k=1, 2, \cdots)$ converges to $a(t)$ in $L^2(\boldsymbol{T}, L^p(\boldsymbol{R}^n)\cap L^q(\boldsymbol{R}^n))$ and $b_k(t)\in C(\boldsymbol{T}; C_0^\infty(\boldsymbol{R}^n))$ converges to $b(t)$ in $L^{1+\frac{1}{\varepsilon}}(\boldsymbol{T}, L^p(\boldsymbol{R}^n)\cap L^q(\boldsymbol{R}^n))$ where the constant ε is the same constant as appeared in (3.9) (we have chosen $0<\varepsilon<1$). Let $\delta_k>0$ converges to 0. Then similar calculations used for proving (3.9) and (3.8) show that the operator G_k defined by

$$(G_k f)(t) = i\int_{\delta_k}^\infty e^{-is\zeta} a_k(t) U_0(s) b_k(t-s) f(t-s) ds, \qquad f\in\mathcal{K},$$

converges to the operator $Q(\zeta)$ in operator norm topology of $B(\mathcal{K})$. Therefore it is sufficient to prove that G_k is a compact operator in $\mathcal{K}$. To prove this we note that the operator G_k is Hilbert-Schmidt type with Hilbert-Schmidt norm

$$\int_0^\omega dt\int_0^\omega ds\int_{\boldsymbol{R}^n} dx\int_{\boldsymbol{R}^n} dy$$

$$\times\Big|a_k(t, x)\Big(\sum_{n=-\infty}^{0}\frac{\exp\Big(-\dfrac{i|x-y|^2}{(2|t-s-n\omega|)^{n/2}}\Big)}{(2|t-s-n\omega|)^{n/2}}\chi_k(t-s-n\omega)\Big)b_k(s, y)\Big|^2$$

which is dominated by constant times

$$\int_T dt \int_T ds \int_{R^n} dx \int_{R^n} dy |a_k(t, x) b_k(s, y)|^2 < \infty .$$

Here $\chi_k(s)$ is the characteristic function of the interval $[\delta_k, \infty)$. This concludes the proof of the lemma. (Q. E. D.)

PROOF OF LEMMA 3.2. If $\delta>1$, the lemma is a simple consequence of Lemma 3.1. Hence we assume in what follows that $1/2<\delta\leqq 1$. Since statements (a) and (c) are obvious, we shall prove statements (b) and (d). Let $d(x)=(1+|x|)^{-\delta/2}$ and let D be the operator of multiplication by $d(x)$ in $\mathcal{K}$. Since AD^{-1} and $D^{-1}B$ are bounded operators in $\mathcal{K}$ it is sufficient to prove that the operator $P(\zeta)\equiv DR_0(\zeta)D$ is a compact operator in $\mathcal{K}$ for $\mathrm{Im}\,\zeta\neq 0$ and that $B(\mathcal{K})$-valued function $P(\zeta)$ of ζ can be extended to $\Pi^{\pm}\cup\left(\boldsymbol{R}^1\backslash\frac{2\pi}{\omega}\boldsymbol{Z}\right)$ as a locally Hölder continuous function. By the Fourier transform $\mathcal{F}_{t\to n}$ we have

$$(\mathcal{F}_{t\to n}P(\zeta)f)(n)=dr_0(\zeta+2n\pi/\omega)d(\mathcal{F}_{t\to n}f)(n), \qquad n\in\boldsymbol{Z}, \tag{3.10}$$

where d is the operator of multiplication by $d(x)$ in $\mathcal{H}$ and $r_0(\zeta)=(H_0-\zeta)^{-1}$.

Here we note the following fact. Let ε be a positive constant and let $\varphi(x)$ be an infinitely differentiable function such that supp $\varphi\subset[\varepsilon, \infty]$ and $\varphi(x)=1$, $x>2\varepsilon$. Denote by Φ the operator of multiplication by φ. Define the operator U as $(Uf)(r, \cdot)=r^{(n-2)/4}f(\sqrt{r}\ \cdot)$, $f(r\omega)=f(x)$, $x=r\omega$. Then U is a unitary operator from $L^2(\boldsymbol{R}^n)$ to $L^2(\boldsymbol{R}^+, L^2(S^{n-1}), dr)$ and transform the operator H_0 to the operator of multiplication by r. Furthermore $U\Phi$ maps $H^\delta(\boldsymbol{R}^n)$ to $H_0^\delta(\boldsymbol{R}^+, L^2(S^{n-1}), dr)$ boundedly, where $H_0^\delta(\boldsymbol{R}^+, L^2(S^{n-1}), dr)$ is the usual Sobolev space with the boundary condition $f(0)=0$. As is well known elements of $H_0^\delta(\boldsymbol{R}^+, L^2(S^{n-1}), dr)$ are $L^2(S^{n-1})$-valued uniformly Hölder continuous functions.

Using the above fact, we can easily see, by a routine method of scattering theory that for any $\varepsilon>0$, $dr_0(\zeta)d$ is a $B(\mathcal{H})$-valued uniformly Hölder continuous function of ζ on $\Pi^{\pm}\cup(\boldsymbol{R}^1\backslash(-\varepsilon, \varepsilon))$. Hence statement (b) is obvious from (3.10). Compactness of $P(\zeta)$ is an immediate consequence of (3.10) and the following facts: (1) $dr_0(\zeta)d$ is a compact operator in $\mathcal{H}$: (2) for any fixed $\zeta\in\Pi^{\pm}\cup(\boldsymbol{R}^1\backslash(2\pi/\omega)\boldsymbol{Z})$, $\|dr_0(\zeta+(2\pi/\omega)n)d\|_{B(\mathcal{H})}\to 0$ as $n\to\pm\infty$. (Q. E. D.)

PROOF OF LEMMA 3.3. Let $f(t)$ and $g(t)$ be $H^1(\boldsymbol{R}^n)$-valued continuous functions on $\boldsymbol{T}$. Then by Theorem 2.1 and Theorem 2.1′ we have

$$(U_1(t, s)f(r), g(r))=(U_0(t, s)f(r), g(r))$$
$$-\int_s^t (a(\tau)U_1(\tau, s)f(r), b(\tau)U_0(\tau, t)g(r))d\tau .$$

Hence we have

$$((e^{isK_1}f)(t), g(t))_{\mathcal{K}}=(U_1(t, t-s)f(t-s), g(t))_{\mathcal{H}} \tag{3.11}$$

$$=(U_0(t, t-s)f(t-s), g(t))_{\mathcal{H}}$$

$$-\int_0^s (a(r+(t-s))U_1(r+(t-s), t-s)f(t-s),$$

$$b(r+(t-s))U_0(r+(t-s), t)g(t))_{\mathcal{H}} dr$$

$$=((e^{isK_0}f)(t), g(t))_{\mathcal{H}}$$

$$-\int_0^s ((Ae^{irK_1}f)(r+(t-s)), (Be^{i(r-s)K_0}g)(r+(t-s))_{\mathcal{H}} dr .$$

Hence, integrating (3.11) with respect to t on $\boldsymbol{T}$, we get

$$(e^{isK_1}f, g)_{\mathcal{K}}=(e^{isK_0}f, g)_{\mathcal{K}}-\int_0^s (Ae^{irK_1}f, Be^{i(r-s)K_0}g)_{\mathcal{K}} dr . \tag{3.12}$$

Since e^{isK_0} is a strongly continuous (e^{isK_1} is a weakly continuous) group on $\mathcal{L}=L^2(\boldsymbol{T}, H^1(\boldsymbol{R}^n), dt)$, there exist constants $\eta>0$ and $C>0$ such that $\|Ae^{isK_1}\| \leqq Ce^{\eta|s|}$, $\|Be^{isK_0}\| \leqq Ce^{\eta|s|}$ $(s\in \boldsymbol{R}^1)$. Let $|\mathrm{Im}\,\zeta|>\eta$. Then by the use of the Laplace transform and Fubini's theorem it follows easily from (3.12) that

$$(R_1(\zeta)f, g)_{\mathcal{K}}=(R_0(\zeta)f, g)_{\mathcal{K}}-(AR_1(\zeta)f, BR_0(\bar{\zeta})g)_{\mathcal{K}}, \qquad |\mathrm{Im}\,\zeta|>\eta .$$

Since the set of all $H^1(\boldsymbol{R}^n)$-valued continuous functions is dense in $\mathcal{K}$ we get for $|\mathrm{Im}\,\zeta|>\eta$

$$R_1(\zeta)f=R_0(\zeta)f-[BR_0(\bar{\zeta})]^*AR_1(\zeta)f, \qquad f\in C(\boldsymbol{T}, H^1(\boldsymbol{R}^n)) . \tag{3.13}$$

Multiplying both sides of (3.13) by A and noting $\|Q(\zeta)\|\to 0$ as $|\mathrm{Im}\,\zeta|\to 0$, we have $AR_1(\zeta)f=(1+Q(\zeta))^{-1}AR_0(\zeta)f$ for sufficiently large $|\mathrm{Im}\,\zeta|$. Replacing $AR_1(\zeta)f$ in the right hand side of (3.13) by this expression we get for sufficiently large $|\mathrm{Im}\,\zeta|$

$$R_1(\zeta)f=R_0(\zeta)f-[BR_0(\bar{\zeta})]^*(1+Q(\zeta))^{-1}AR_0(\zeta)f . \tag{3.14}$$

In (3.14) the left hand side is a $\mathcal{K}$-valued analytic function of $\zeta\in\Pi^{\pm}$ and the right hand side is a $\mathcal{K}$-valued meromorphic function. Therefore we see that the equation (3.14) holds for any ζ with $\mathrm{Im}\,\zeta\neq 0$. Since the operators appearing on each side of (3.14) are bounded in $\mathcal{K}$ we get the desired result.

(Q. E. D.)

§4. Proof of Theorem 1.1.

We first remark that under our assumptions wave operators $W_{\pm}(s)$ exist and $\mathcal{W}_{\pm}$ are the operators of multiplication by $W_{\pm}(t)$. If Assumptions (A.1) and (A.2) with $q\geqq 2$ or Assumptions (A.1) and (A.3) are satisfied, Theorem 5.1 in appendix is applicable. If Assumption (A.1) and (A.2) with $q<2$ be satisfied, the existence and the properties given in Theorem 5.1 are consequences of Lemma 3.3 of this paper and Corollary 1 of Theorem 4 and Theorem 6 of Howland [7]. Therefore it is sufficient to prove the last statement of the theorem. To prove the statement we proceed as follows. By Lemma 3.5 we have $R(\mathcal{W}_{\pm})=\mathcal{K}_{ac}(K_1)=\mathcal{K}_{ac}(e^{i\omega K_1})$. On the other hand we get by the definition of $e^{i\omega K_1}$ and property (2.4) that

$$(e^{i\omega K_1}f)(t)=U_1(t,t-\omega)f(t-\omega)=U_1(t+\omega,t)f(t)$$
$$=U_1(t,0)U_1(\omega,0)U_1(0,t)f(t)\,. \tag{3.15}$$

Writing the unitary operator of multiplication by $U_1(t,0)$ (or $U_1(\omega,0)$) as $\mathcal{U}_1$ (or $\mathcal{U}_\omega$), we have by (3.15)

$$e^{i\omega K_1}=\mathcal{U}_1\mathcal{U}_\omega\mathcal{U}_1^{-1}\,. \tag{3.16}$$

Hence we can easily see that

$$\mathcal{K}_{ac}(K_1)=\mathcal{K}_{ac}(e^{i\omega K_1})=\mathcal{U}_1\mathcal{K}_{ac}(\mathcal{U}_\omega)=\mathcal{U}_1L^2(\boldsymbol{T},\mathcal{H}_{ac}(U_1(\omega,0)),dt)\,. \tag{3.17}$$

On the other hand, since $\mathcal{V}_0^{-1}$ is unitary we have

$$R(\mathcal{W}_{\pm})=R(\mathcal{U}_1\mathcal{W}_{\pm,0}\mathcal{V}_0^{-1})=\mathcal{U}_1R(\mathcal{W}_{\pm,0})=\mathcal{U}_1L^2(\boldsymbol{T},R(W_{\pm}(0)),dt) \tag{3.18}$$

where $\mathcal{W}_{\pm,0}$ is the operator of multiplication by $W_{\pm}(0)$. Comparing (3.17) and (3.18), we get

$$R(W_{\pm}(0))=\mathcal{H}_{ac}(U_1(\omega,0))\,. \tag{3.19}$$

Therefore by (3.19), Theorem 5.1 (b) and the unitarity of $U_0(s,t)$ we get

$$\begin{aligned}R(W_{\pm}(s))&=R(U_1(s,0)W_{\pm}(0)U_0(0,s))=U_1(s,0)R(W_{\pm}(0))\\&=U_1(s,0)\mathcal{H}_{ac}(U_1(\omega,0))=U_1(s+\omega,\omega)\mathcal{H}_{ac}(U_1(\omega,0))\\&=\mathcal{H}_{ac}(U_1(s+\omega,\omega)U_1(\omega,0)U_1(\omega,s+\omega))\\&=\mathcal{H}_{ac}(U_1(s+\omega,s))\,.\end{aligned} \tag{3.23}$$

This concludes the proof of Theorem 1.1. (Q. E. D.)

§ 5. Appendix.

In this section we shall give a sufficient condition for the existence of wave operators for Schrödinger operator with time dependent potentials. The theorem obtained in this section includes the result of Hack [6] for time-independent potentials and the result of Morita (announced at the meeting of Mathematical Society of Japan, April 1974) for a "spreading-out" potential, i.e. for the potential of type $v(t, x)=\frac{1}{t^{\alpha}}q(t^{\beta}x)$, $\alpha, \beta\in \boldsymbol{R}$. For proving the theorem we shall use partial integration and the method of stationary phase which was used by Buslaev-Matveev [2] to prove the existence of wave operators for time independent long range potentials. We shall record in the theorem an immediate consequence which is usually named as intertwining property of the wave operators.

THEOREM 5.1. *Let $v_1(t, x)$ and $v_2(t, x)$ be real valued functions defined on $\boldsymbol{R}^1\times \boldsymbol{R}^n$ such that:*
(a.1) *$v_1(t, x)$ is an $L^{\infty}(\boldsymbol{R}_x^n)$-valued function of $t\in \boldsymbol{R}^1$ and there exists an integer k such that*

$$(1+|t|)^{-k}\|v_1(t, \cdot)\|_{L^{\infty}(\boldsymbol{R}_x^n)}\in L^{\infty}(\boldsymbol{R}_t^1);$$

(a.2) *$v_2(t, x)$ is $L^2(R_x^n)$-valued function of $t\in \boldsymbol{R}^1$ and there exists an integer k such that $(1+|t|)^{-k}\|v_2(t, \cdot)\|_{L^2(R_x^n)}\in L^{\infty}(\boldsymbol{R}_t^1)$.*
Suppose $v(t, x)=v_1(t, x)+v_2(t, x)$ has the following properties:
(a.3) *the operator valued function $V(t)$ defined by the multiplication by $v(t, x)$ is a $B(H^2(\boldsymbol{R}^n), \mathcal{H})$-valued absolutely continuous function;*
(a.4) *for any constants a and b such that $0<a<b<\infty$*

$$\int_{-\infty}^{\infty}\left\{\int_{a<|y|<b}|v(t, ty)|^2 dy\right\}^{1/2} dt<\infty .$$

Then the family of anti-selfadjoint operators $\{-iH(t)=-i(H_0+V(t))\}$ generates a unitary evolution group $\{U(t, s)\}$ and the strong limits

$$\underset{t\to\pm\infty}{\text{s-lim}}\, U(t, s)^{-1}e^{-i(t-s)H_0}f=W_{\pm}(s)f$$

exist for every $f\in\mathcal{H}$. Furthermore wave operators $W_{\pm}(s)$ have the following properties:

1) *$W_{\pm}(s)$ are isometric operators in $\mathcal{H}$;*
2) *(intertwining property)*

$$U(t, s)W_{\pm}(s)=W_{\pm}(t)e^{-i(t-s)H_0}.$$

PROOF. We shall prove the existence of the strong limit only for $W_+(0)$. Other cases can be proved similarly. Put $\Omega(t)=U(0, t)e^{-itH_0}$. Since $\Omega(t)$ is uniformly bounded, it is sufficient to prove the existence of the limit of $\Omega(t)f$ for every element f of some dense subset of $\mathcal{H}$. We choose this dense subset as $\mathcal{F}_{x\to\xi}^{-1}C_0^{\infty}(\boldsymbol{R}^n-\{0\})$. By the well known method of Cook [3] it is sufficient to prove that for any f in this set there exists $\sigma>0$ such that

$$\| V(t)e^{-itH_0}f\|_{\mathcal{H}} \in L^1((\sigma, \infty)) .$$

First of all we shall investigate the asymptotic behavior of $(e^{-itH_0}f)(x)$ as $t\to\infty$. Let the support of $(\mathcal{F}_{x\to p}f)(p)=\hat{f}(p)$ be contained in

$$\{p\in \boldsymbol{R}^n ;\ 0<\alpha<|p|<\beta<\infty\} .$$

Let $0<\varepsilon<\alpha/2$. We first estimate the integral

$$(e^{-itH_0}f)(x)=\frac{1}{\sqrt{2\pi}^n}\int_{R^n} e^{-ix\cdot p-it|p|^2/2}\hat{f}(p)dp \tag{5.1}$$

in the region $|x/t|<\varepsilon$. In this region the phase function $-ix\cdot p+it|p|^2/2$ is not stationary with respect to p on the support of $\hat{f}$. Hence the partial integration shows that for any positive integer j there exists a constant C_j depending only on j and f such that

$$|(e^{-itH_0}f)(x)|\leqq C_j t^{-j}, \qquad |x/t|<\varepsilon . \tag{5.2}$$

Next we estimate the integral of (5.1) in the region $|x/t|>\varepsilon$. Put $x=ty$ and $x/|x|=y/|y|=\omega\in S^{n-1}$=the unit sphere in $\boldsymbol{R}^n$. Making the change of variables $p=|y|\xi$, we have

$$(e^{itH_0}f)(ty)=\frac{1}{\sqrt{2\pi}^n}|y|^n e^{it|y|^2/2}\int_{R^n} e^{-it|y|^2(\xi+\omega)^2/2}\hat{f}(|y|\xi)d\xi .$$

Let $\eta(\xi)$ be a $C_0^\infty(\boldsymbol{R}^n\backslash\{0\})$-function such that $\eta(\xi)=1$ on some small neighbourhood of S^{n-1}, $\tilde{\eta}(\xi)=1-\eta(\xi)$. We put

$$I_1(t, ty)=\frac{1}{\sqrt{2\pi}^n}|y|^n e^{it|y|^2/2}\int_{R^n} e^{-it|y|^2(\xi+\omega)^2/2}\tilde{\eta}(\xi)\hat{f}(|y|\xi)d\xi$$

and

$$I_2(t, ty)=\frac{1}{\sqrt{2\pi}^n}|y|^n e^{it|y|^2/2}\int_{R^n} e^{-it|y|^2(\xi+\omega)^2/2}\eta(\xi)\hat{f}(|y|\xi)d\xi .$$

Then we get by the partial integration that for any positive integer j there exists a constant C_j dependent only on j and f such that

$$|I_1(t, ty)|\leqq C_j t^{-j}|y|^{n-j}, \qquad |y|>\varepsilon . \tag{5.3}$$

Next we estimate I_2. Using the polar coordinate in ξ-space we have

$$I_2(t, ty)=\frac{1}{\sqrt{2\pi}^n}|y|^n e^{it|y|^2/2}\int_0^\infty e^{-it|y|^2r^2/2}r^{n-1}\Big\{\int_{S^{n-1}}\eta(r\omega'+\omega)\hat{f}(|y|r\omega'+y)d\omega'\Big\}dr .$$

$I_2(t, ty)$ vanishes for y in the exterior of some t-independent compact subset K of $\boldsymbol{R}^n$ hence by the use of the stationary phase method we get after a simple calculation that

$$(5.4) \qquad I_2(t, ty) = \frac{1}{\sqrt{2\pi}^n}(t/2)^{-n/2} e^{it|y|^2/2} \hat{f}(y) + O(t^{-(n+1)/2}), \qquad t \to \infty,$$

where O is taken uniformly with respect to $y \in K$. By (5.1) to (5.3) and the conditions of the theorem we see that there exists $\sigma > 0$ such that

$$\|v(t, x)e^{-itH_0}f(x)\|_{L^2(|x|<t\varepsilon)} \in L^1((\sigma, \infty))$$

and

$$\|v(t, x)I_1(t, x)\|_{L^2(|x|>t\varepsilon)} \in L^1((\sigma, \infty)).$$

By (5.4) we have

$$\int_{|x|>\varepsilon t} |v(t, x)I_2(t, x)|^2 dx = t^n \int_K |v(t, ty)I_2(t, ty)|^2 dy \leqq C \int_K |v(t, ty)|^2 dy,$$

where C is a constant independent of t. Hence we obtain the desired result. Other statements can be proved by a routine method which has been used for the time independent perturbations. (Q. E. D.)

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