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On the Schur indices of $GL(n, q)$ and $SL(2n+1, q)$

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Introduction.

In this paper we determine the Schur indices of irreducible (complex) characters of the finite general linear group $GL(n, q)$ and of the odd-dimensional finite special linear group $SL(2n+1, q)$, both defined over the finite field $GF(q)$ of $q=p^f$ elements.

MAIN THEOREM. *Let G denote the group $GL(n, q)$ or the group $SL(2n+1, q)$. Then if $p\neq 2$, the Schur index of any irreducible character of G with respect to the rational number field $\boldsymbol{Q}$ is* 1.

This is a consequence of the following three theorems.

THEOREM A (Gow). *Let G be as in the Main Theorem. Then the Schur index of any irreducible character of G with respect to $\boldsymbol{Q}$ divides* 2.

THEOREM B. *Let G be as above. Then the value $X(u)$ of any irreducible character X of G at a unipotent element u of G is a rational integer and the Schur index of X with respect to $\boldsymbol{Q}$ divides $X(u)$.*

THEOREM C. *For any irreducible character X of $G=GL(n, q)$, there exists a unipotent element u of G such that $|X(u)|$ is equal to the p-part of the degree of X.*

Theorem A is proved in [2] and Theorems B, C will be proved in sections 1, 2, respectively. For $G=GL(n, q)$, Main Theorem follows immediately from these theorems. But for $G=SL(2n+1, q)$, Main Theorem is not clear. So this case will be dealt with in section 3. The methods used in sections 1, 3 depend on [2]. In section 4 we will discuss some special cases.

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NOTATION. $\boldsymbol{Q}$ is the field of rational numbers. A character always means an ordinary complex one. $m_{\boldsymbol{Q}}(X)$ is the Schur index of an irreducible character X of a finite group with respect to $\boldsymbol{Q}$. A *rational* character of a finite group G is a character afforded by some $\boldsymbol{Q}[G]$-module, i.e., a character which can be realized in $\boldsymbol{Q}$ (see [1], p 279). For a positive integer r, ζ_r is a primitive r-th root of unity in the field of complex numbers. If K/k is a normal and separable extension, $\mathrm{Gal}(K/k)$ is its Galois group.

§ 1. Proof of Theorem B.

(1.1) LEMMA. *Let x be an element of a finite group G and suppose that for each integer h coprime to the order of x, x and x^h are conjugate in G. Then all characters of G take rational integral values on x.*

PROOF. Let b be the order of x. Then if X is a character of G of degree d, $X(x)=\zeta_b^{a_1}+\cdots+\zeta_b^{a_d}$ for some positive integers $a_1, \cdots, a_d$. Let τ denote a non-identity automorphism in Gal $(\boldsymbol{Q}(\zeta_b)/\boldsymbol{Q})$. Then $X(x)^\tau=\zeta_b^{ia_1}+\cdots+\zeta_b^{ia_d}$ for some positive integer i coprime to b. Put $k=b+i$. Then $ka_j\equiv ia_j$ (mod. b), $j=1, \cdots, d$, and $X(x)^\tau=X(x^k)$. Since x and x^k are conjugate in G, $X(x)^\tau=X(x)$. This holds for any automorphism τ in Gal $(\boldsymbol{Q}(\zeta_b)/\boldsymbol{Q})$. Then $X(x)$ is a rational number. But characteristic values are algebraic integers and hence $X(x)$ is a rational integer. This completes the proof of (1.1).

(1.2) COROLLARY. *Let G denote the group $GL(n, q)$ or the group $SL(2n+1, q)$. Then all characters of G take rational integral values on unipotent elements of G.*

PROOF. Firstly, let $G=SL(2n+1, q)$ and let u denote a unipotent element of G. We may assume that u is of the (lower triangular) Jordan canonical form. Then for each integer b coprime to p, we can choose an element m of G of the form, for instance,

$$\begin{bmatrix} a^{-n} & & & & & & 0 \\ & \ddots & & & & & \\ & & a^{-1} & & & & \\ & & & 1 & & & \\ & & & & a & & \\ & & & & & \ddots & \\ * & & & & & & a^n \end{bmatrix} \qquad \begin{aligned} &b\equiv a \pmod p, \\ &0<a<p, \end{aligned}$$

such that mum^{-1} is equal to u^b. Then the assertion follows from (1.1). If $G=GL(n, q)$, m can be chosen of the form

$$\begin{bmatrix} 1 & & & & 0 \\ & a & & & \\ & & a^2 & & \\ & & & \ddots & \\ * & & & & a^{n-1} \end{bmatrix}$$

such that $mum^{-1}=u^b$. This completes the proof of (1.2).

Now for a partition $\mu=(n_1, n_2, \cdots, n_k)$ of n (if $G=GL(n, q)$) or of $2n+1$ (if $G=SL(2n+1, q)$), put $P_\mu=P_1\times\cdots\times P_k$, where for each i, $i=1, \cdots, k$, P_i denotes a Sylow p-subgroup of $GL(n_i, q)$ which consists of all those lower triangular matrices whose entries on the main diagonal are 1. Next lemma is a key point in the proof of Theorem B.

(1.3) LEMMA. *Let G be the group $GL(n, q)$ or the group $SL(2n+1, q)$ and let $P=P_{\mu}$ be as above. Then if L is a linear character of P, L^{G} is a rational character.*

PROOF. Firstly, let $G=SL(2n+1, q)$. Let σ denote an element of order $p-1$ in $GF(p)$ and put $m=\mathrm{diag}(\sigma^{-n}, \cdots, \sigma^{-1}, 1, \sigma, \cdots, \sigma^{n})$. Then m lies in G. As is easily seen, for any element x in P, mxm^{-1} is equal to x^{σ} modulo P'. Then if M is the subgroup of G generated by m and P, each non-identity element of P/P' is conjugate in M/P' to its $p-1$ non-identity powers. Then (1.1) implies that each character of M/P' takes rational values on P/P'. It is easy to see that if L is a non-trivial linear character of P, L^{M} is a rational-valued irreducible character. Moreover, $L^{M}(1)=p-1$ and $L^{M}(m^{a})=0$ (if $m^{a}\neq 1$). This shows that $(L^{M})_{\langle m\rangle}$ is a character of the regular representation of $\langle m\rangle$. Then by reciprocity, we see that the multiplicity of L^{M} in $(1_{\langle m\rangle})^{M}$ is one and by the property of Schur indices we have $m_{\boldsymbol{Q}}(L^{M})=1$. Since L^{M} is rational-valued, it can be realized in $\boldsymbol{Q}$. Hence $L^{G}=(L^{M})^{G}$ is a rational character. In the case of $G=GL(n, q)$, take σ as above and put $m=\mathrm{diag}(1, \sigma, \sigma^{2}, \cdots, \sigma^{n-1})$. Then the proof can be done similarly.

REMARK. In [2] Gow proved the special case of (1.3) with P being a Sylow p-subgroup of G, i.e., $\mu=(n)$ or $=(2n+1)$ according as $G=GL(n, q)$ or as $=SL(2n+1, q)$, respectively. Our proof here is an analogue of Gow's one.

(1.4) LEMMA. *Let u denote a regular unipotent element of $G=GL(m, q)$ of the form*

$$\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & 1 & 1 \end{bmatrix}$$

and let P denote a Sylow p-subgroup of G which consists of all those lower triangular matrices whose entries on the main diagonal are 1. *Then if I is a non-linear irreducible character of P, $I(u)=0$.*

PROOF. It is easy to see that the order of the centralizer group $C_{P}(u)$ of u in P is q^{m-1}. In fact, firstly, by applying to u the formula below Lemma 2.1 of [3], we see that the order of the centralizer group $C_{G}(u)$ of u in G is $q^{m-1}(q-1)$. Secondly, we can prove that any normalizer of u is a lower triangular matrix. Thirdly, by combining these results, we see that the order of $C_{P}(u)=C_{G}(u)\cap P$ is q^{m-1}. Then by the orthogonality, we have the expression $\sum I(u)I(u^{-1})=q^{m-1}$, where the summation is over all irreducible characters I of P. Since the order of the derived factor group P/P' is q^{m-1}, P has exactly q^{m-1} linear characters. This shows that only the linear characters contribute to the above summation.

Then the assertion is clear.

Now we prove Theorem B. The following proof is an analogue of the proof of Theorem 2(a) in [2]. Firstly, let $G=GL(n, q)$ and let X denote an arbitrary irreducible character of G. Since X is a class function, we may assume that u is of the form

$$\begin{pmatrix} \overbrace{\begin{matrix}1 & & & & \\ 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 1\end{matrix}}^{n_1} & & & \\ & \overbrace{\begin{matrix}1 & & & & \\ 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 1\end{matrix}}^{n_2} & & \\ & & \ddots & \\ & & & \overbrace{\begin{matrix}1 & & & & \\ 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 1\end{matrix}}^{n_k} \end{pmatrix}$$

In this condition we can put $u=u_{\mu}$, where $\mu=(n_1, n_2, \cdots, n_k)$ is the partition of n corresponding to u. Let P_i denote a Sylow p-subgroup of $GL(n_i, q)$ of the form as in (1.3), $i=1, \cdots, k$, and put $P=P_1\times \cdots \times P_k$. The restriction X_P of X to P can be expressed as $X_P=\sum_L a_L L+\sum_I b_I I$, where the first summation is over all linear characters L of P, the second summation is over all non-linear irreducible characters I of P, and the a_L, b_I are some non-negative integers. Then by (1.4), we have the expression $X_P(u_{\mu})=\sum_L a_L L(u_{\mu})$. Since $a_L=(X_P, L)_P$ $=(X, L^G)_G$ is the multiplicity of X in a rational character L^G, by the property of Schur index, $m_{\mathbf{Q}}(X)$ divides a_L. Moreover, in the expression $X_P(u_{\mu})/m_{\mathbf{Q}}(X)$ $=\sum_L (a_L/m_{\mathbf{Q}}(X))L(u)$, the left hand side is a rational number (by (1.1)) and the right hand side is an algebraic integer. Hence $m_{\mathbf{Q}}(X)$ divides $X_P(u)=X(u)$. This completes the proof of Theorem B.

§ 2. Proof of Theorem C.

The purpose of this section is to prove (2.15) from which Theorem C follows as a corollary.

Let $\{\lambda\}=\{\lambda_1, \cdots, \lambda_p\}$ $(\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_p > 0)$ and $\{\mu\}=\{\mu_1, \cdots, \mu_q\}$ $(\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_q > 0)$ denote two Schur functions associated with a series

$$f(x)=\prod_{i=1}^{m} 1/(1-\alpha_i x)=1+\sum_{r=1}^{\infty} q_r x^r , \tag{2.1}$$

i. e., $\{\lambda\}=|q_{\lambda_s-s+t}|$ and $\{\mu\}=|q_{\mu_s-s+t}|$ (detailed discussions about Schur functions can be seen in [5]). Here p, q are some positive integers. Then the product $\{\lambda\}\{\mu\}$ can be expressed as an integral linear combination of Schur functions:

$$\{\lambda_1, \cdots, \lambda_p\}\{\mu_1, \cdots, \mu_q\}=\sum_{\delta} c_\delta\{\delta\} , \tag{2.2}$$

where the summation is over all partitions δ of $|\lambda|+|\mu|$ (if $\rho=(\rho_1, \cdots, \rho_r)$ is a partition of $\rho_1+\cdots+\rho_r$, $|\rho|$ is defined to be $\rho_1+\cdots+\rho_r$), and the c_δ are some integers. The multiplicity c_δ of each δ can completely be determined by the next lemma.

(2.3) LEMMA [5, p. 94]. *The Schur functions appearing in the product* (2.2) *are those which correspond to the Young tableaux that can be built by adding to a Young tableau correspond to $\{\lambda\}$, μ_1 identical symbols α_1, μ_2 identical symbols α_2, μ_3 identical symbols α_3, etc., subject to two conditions:*

Firstly, after the addition of each set of identical symbols we must have a regular Young tableau with no two identical symbols in the same column.

Secondly, if the total set of added symbols are read from right to left in the consecutive rows of the final tableau, we obtain a lattice permutation of $\alpha_1^{\mu_1}\alpha_2^{\mu_2}\alpha_3^{\mu_3}\cdots$.

REMARK. By a *regular* Young tableau we mean a Young tableau in which "the number of the symbols in the first row" $\geqq$ "the number of the symbols in the second row" $\geqq$ "the number of the symbols in the third row" $\geqq \cdots$. Next, a permutation of symbols $x_1^{r_1}x_2^{r_2}x_3^{r_3}\cdots$ will be called a *lattice* permutation if for each positive integer k, in the sequence of first k symbols (when the symbols are read from left to right) of the permutation "the number of x_1" $\geqq$ "the number of x_2" $\geqq$ "the number of x_3" $\cdots$. For example, all the lattice permutation of $x_1^2x_2^2x_3$ are

$$x_1^2x_2^2x_3 \quad x_1^2x_2x_3x_2 \quad x_1x_2x_1x_2x_3 \quad x_1x_2x_1x_3x_2 \quad x_1x_2x_3x_1x_2 .$$

EXAMPLE. Let $\{\lambda\}=(421)$ and $\{\mu\}=(21)$. Then all the Young tableaux built according as the procedure described in (2.3) are as follows:

$$\begin{pmatrix} 0000\alpha\alpha \\ 00\beta \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0000\alpha\alpha \\ 00 \\ 0\beta \end{pmatrix} \quad \begin{pmatrix} 0000\alpha\alpha \\ 00\alpha\beta \\ 0 \\ \beta \end{pmatrix} \quad \begin{pmatrix} 0000\alpha \\ 00 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0000\alpha \\ 00\alpha \\ 0\beta \end{pmatrix} \quad \begin{pmatrix} 0000\alpha \\ 00\alpha \\ 0 \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} 0000\alpha \\ 00\beta \\ 0\alpha \end{pmatrix} \begin{pmatrix} 0000\alpha \\ 00 \\ 0\alpha \\ \beta \end{pmatrix} \begin{pmatrix} 0000\alpha \\ 00\beta \\ 0 \\ \alpha \end{pmatrix} \begin{pmatrix} 0000\alpha \\ 00 \\ 0\beta \\ \alpha \end{pmatrix} \begin{pmatrix} 0000\alpha \\ 00 \\ 0 \\ \alpha \\ \beta \end{pmatrix} \begin{pmatrix} 0000 \\ 00\alpha\alpha \\ 0\beta \end{pmatrix}$$

$$\begin{pmatrix} 0000 \\ 00\alpha\alpha \\ 0 \\ \beta \end{pmatrix} \begin{pmatrix} 0000 \\ 00\alpha \\ 0\alpha\beta \end{pmatrix} \begin{pmatrix} 0000 \\ 00\alpha \\ 0\beta \\ \alpha \end{pmatrix} \begin{pmatrix} 0000 \\ 00\alpha \\ 0\alpha \\ \beta \end{pmatrix} \begin{pmatrix} 0000 \\ 00\alpha \\ 0 \\ \alpha \\ \beta \end{pmatrix} \begin{pmatrix} 0000 \\ 00 \\ 0\alpha \\ \alpha \\ \beta \end{pmatrix}.$$

Hence by (2.3), we have

$$\{421\}\{21\}=\{631\}+\{62^2\}+\{621^2\}+\{541\}+2\{532\}+2\{531^2\}+2\{52^21\}$$
$$+\{521^3\}+\{4^22\}+\{4^21^2\}+\{43^2\}+2\{4321\}+\{431^3\}+\{42^21^2\}\,.$$

Now let us define some notations. If $\rho=(\rho_1, \cdots, \rho_r)$ $(\rho_1\geqq\rho_2\geqq\cdots\geqq\rho_r>0)$ is a partition of $|\rho|$, the *conjugate* partition of ρ which we shall denote by $\tilde{\rho}$ is defined to be the partition $(r^{\rho_r}(r-1)^{\rho_{r-1}-\rho_r}\cdots 1^{\rho_1-\rho_2})$ of $|\rho|$. If $\rho=(1^{r_1}2^{r_2}\cdots n^{r_n})$ and $\sigma=(1^{s_1}2^{s_2}\cdots n^{s_n})$ are two partitions of n, we shall denote by $\rho+\sigma$ the partition $(1^{r_1+s_1}2^{r_2+s_2}\cdots n^{r_n+s_n})$ of $2n$.

(2.4) COROLLARY. (i) *The largest partition (according as lexicographical ordering) that appears in the product* (2.2) *is* $(\lambda_1+\mu_1, \lambda_2+\mu_2, \cdots)$, *and its multiplicity is one. Moreover,* $(\lambda_1+\mu_1, \lambda_2+\mu_2, \cdots)$ *is equal to* $\widetilde{\tilde{\lambda}+\tilde{\mu}}$.

(ii) *The smallest partition that appears in* (2.2) *is* $(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q)$, *and its multiplicity is one.*

PROOF. By (2.3), the first assertion of (i) is clear. Then it only needs check up the relation with its conjugate partition. Changing λ and μ if necessary, we may assume that $p\geqq q$. But by the definition of conjugate partitions, we see that $\tilde{\lambda}+\tilde{\mu}=(p^{\lambda_p}\cdots(q+1)^{\lambda_{q+1}-\lambda_{q+2}}q^{(\lambda_q+\mu_q)-\lambda_{q+1}}(q-1)^{(\lambda_{q-1}+\mu_{q-1})-(\lambda_q+\mu_q)}\cdots 1^{(\lambda_1+\mu_1)-(\lambda_2+\mu_2)})$. This is clearly the conjugate partition of $(\lambda_1+\mu_1, \lambda_2+\mu_2, \cdots)$.

(ii) We may assume that $\lambda\geqq\mu$. Let Λ denote the Young tableau corresponding to λ. To Λ add μ_1 identical symbols α_1, μ_2 identical symbols α_2, μ_3 identical symbols α_3, etc., so that the α_1 are below the lowest node in each column from the first column to the μ_1-th column if the columns are read from left to right, that the α_2 are below α_1 in each column from the first column to the μ_2-th column, that the α_3 are below α_2 in each column from column to the μ_3-th column, $\cdots$, and that the α_q are below α_{q-1} in each column from the first column to the μ_q-th column. This procedure gives us a new regular Young tableau which we shall call Λ_q. Now let γ denote the partition corresponding to Λ_q. It is not hard to see from (2.3) and from the way of the construction of Λ_q that γ is the smallest partition that appears in (2.2). It is also clear that γ appears in (2.2) exactly once. Then it is sufficient to show that γ coincides with $(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q)$. Let γ_i denote the partition corresponding to the Young tableau which can be built by adding to Λ the symbols $\alpha_1, \cdots, \alpha_i$ as above, $i=1, \cdots, q$. Then we see from the construction of Λ_1 that if $\lambda_{i-1} \geqq \mu_1 \geqq \lambda_i$, $\tilde{\gamma}_1$ equals $((p+1)^{\lambda_p} p^{\lambda_{p-1}-\lambda_p} \cdots (i+2)^{\lambda_{i+1}-\lambda_{i+2}} (i+1)^{\mu_1-\lambda_{i+1}} i^{\lambda_i-\mu_1} (i-1)^{\lambda_{i-1}-\lambda_i} \cdots 1^{\lambda_1-\lambda_2})$, which is the conjugate partition of $(\lambda_1, \cdots, \lambda_{i-1}, \mu_1, \lambda_i, \cdots, \lambda_p)$. By repeating the same consideration for $\Lambda_2, \cdots, \Lambda_q$, we can conclude that γ coincides with $(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q)$. This completes the proof of (2.4).

REMARK. By (2.3) and by the proof of (2.4), the assertions in (2.4) can be generalized for a product of Schur functions of finite number.

Let χ_π^ν denote an irreducible character of the symmetric group S_v of order $v\,!$. As is well known, there is a natural bijection between the set of all conjugacy classes in S_v and the set of all partitions of v. Then if μ is a partition of dv, the correspondence $\pi \mapsto \chi_{d.\pi}^\mu$ can be regarded as a class function on S_v, where if $\pi=(1^{p_1}2^{p_2}3^{p_3}\cdots)$ is a partition of v, $d.\pi$ is defined to be a partition $(d^{p_1}(2d)^{p_2}(3d)^{p_3}\cdots)$ of dv. Since the irreducible characters form a basis of the space of all complex-valued class functions on S_v, this function can be expressed as

$$\chi_{d.\pi}^\mu = \sum_\xi c_\xi^\mu \chi_\pi^\xi, \tag{2.5}$$

where the summation is over all partitions ξ of v and the c_ξ^μ are some complex numbers. The informations about which partitions really appear in (2.5) and about their multiplicities play an important role for the proof of Theorem C.

(2.6) LEMMA. *If χ_π^ξ appears in $\chi_{d.\pi}^\lambda$, λ does not exceed $d.\xi$. Moreover, if λ is equal to $d.\nu$ for some partition ν of v, χ_π^ν appears in $\chi_{d.\pi}^{d.\nu}$ and its multiplicity is one.*

This follows from the next lemma.

(2.7) LEMMA [7, pp. 145-146]. *If $\lambda=(\lambda_1, \cdots, \lambda_{di})$ $(\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_{di} \geqq 0)$ is a partition of v, and the numbers of the sequence*

$$\lambda_1+di-1, \quad \lambda_2+di-2, \cdots, \lambda_{di}$$

congruent respectively to $0, 1, 2, \cdots, d-1$ modulo d are not equal, the corresponding characteristics of all classes of S_{dv} in which the order of all cycles are divisible by d are zero.

Otherwise let the numbers of the sequence which are congruent to q modulo d be

$$d(\mu_{q1}+i-1)+q\,, \quad d(\mu_{q2}+i-2)+q, \cdots, d\mu_{qi}+q\,.$$

Denote ϕ the compound character of the group S_v corresponding to the product *of the Schur functions*

$$\{\mu_{01}, \mu_{02}, \cdots, \mu_{0i}\}, \{\mu_{11}, \cdots, \mu_{1i}\}, \cdots, \{\mu_{d-1,1}, \cdots, \mu_{d-1,i}\}.$$

Then if ρ *denotes the class* $(1^{r_1}2^{r_2}3^{r_3}\cdots)$ *of* S_v, *we have*

$$\chi^{\lambda}_{d.\rho}=\theta\psi_{\rho},$$

where θ *is* $+1$ *according as the sequence*

$$\begin{array}{llll} d(\mu_{d-1,1}+i-1)+d-1, & d(\mu_{d-2,1}+i-1)+d-2, \cdots, & d(\mu_{01}+i-1), \\ d(\mu_{d-1,2}+i-2)+d-1, & \cdots & \\ & \cdots & \\ & \cdots & d\mu_{0i} \end{array}$$

is a positive or negative permutation of

$$\lambda_1+di-1,\ \lambda_2+di-2, \cdots, \lambda_{di}.$$

REMARK. If we put $\{\mu_{01}, \cdots, \mu_{0i}\}\cdots\{\mu_{d-1,1}, \cdots, \mu_{d-1,i}\}=\sum_{|\xi|=v} c_\xi\{\xi\}$ (the c_ξ being some rational integers), the compound character ψ corresponding to this product can be expressed as $\psi_\pi=\sum_{|\xi|=v} c_\xi\chi^{\xi}_{\pi}$ (see [5]). Then by (2.7) we see that all coefficients c^{λ}_{ξ} in (2.5) are integers.

PROOF OF (2.6). If necessary, add some 0's, we can put $\lambda=(\lambda_1, \cdots, \lambda_p, \lambda_{p+1}, \cdots, \lambda_{di})$ so that the λ_j are arranged in descending order and that $\lambda_{(i-1)d+1}\neq 0$. It only needs consider in the case when the sequence $\lambda_1+di-1, \lambda_2+di-2, \cdots, \lambda_{di}$ satisfies the condition "Otherwise ..." in (2.7). In this sequence, for each q $(0\leqq q\leqq d-1)$, choose the numbers which are congruent to q modulo d and arrange them in descending order: $d(\mu_{q1}+i-1)+q, d(\mu_{q2}+i-2)+q, \cdots, d\mu_{qi}+q$. Then it is easy to see that $\mu_{q1}\geqq\mu_{q2}\geqq\cdots\geqq\mu_{qi}$ $(0\leqq q\leqq d-1)$. Moreover, we may assume that all the μ_{qj} are non-negative. For if some μ_{qj} is negative, the smallest part μ_{qi} of $(\mu_{q1}, \cdots, \mu_{qj}, \cdots, \mu_{qi})$ is also negative and by the property of Schur functions, we have $\{\mu_{q1}, \cdots, \mu_{qi}\}=0$ [**6**, p. 99]. Now to prove the first assertion in (2.6) it is sufficient to consider in the case when ξ is the smallest partition that appears in (2.5). Let ξ^1 be this partition. By (2.4), we know that ξ^1 equals such a partition that can be built by arranging all the μ_{qj} $(0\leqq q\leqq d-1,\ 1\leqq j\leqq i)$ in descending order. Let μ_{qj} be the part of ξ^1 corresponding to λ_1, i.e., $\lambda_1+di-1=d(\mu_{qj}+i-j)+q$. Then it is easy to check that $d\mu_{qj}\geqq\lambda_1$. If the inequality holds here, we have $d.\xi^1>\lambda$, since λ_1 is the largest part of λ, and the first assertion in (2.6) can be proved. So we may assume that $d\mu_{qj}=\lambda_1$, $j=1$, and $q=d-1$. Now generally assume that $\lambda_j=d\mu_{d-1,1}$ for $1\leqq j\leqq d-1$. Consider the expression $\lambda_{j+1}+di-(j+1)=d(\mu_{qk}+i-k)+q$ where $j+1\leqq d$ and k being 1 or 2 (this is because of the way of the construction of

μ_{qj}). If $k=2$, then $q=d-m$ for some integer $m>j$, and we have $d\mu_{q2}-\lambda_{j+1}=d+m-(j+1)\geqq m>0$, which implies that $d.\xi^1>\lambda$. So we may assume that $\lambda_{j+1}=d\mu_{d-(j+1),1}$. Thus we can continue our proof by assuming that $\lambda_j=d\mu_{d-j,1}$ $(1\leqq j\leqq d)$, for if not, the first assertion in (2.6) can be proved. Next consider the expression $\lambda_{d+1}+di-(d+1)=d(\mu_{qk}+i-k)+q$ $(q\leqq d-1)$. Since k cannot be any other number different from 2, $d\mu_{q2}\geqq\lambda_{d+1}$. So we may assume that $\lambda_{d+1}=d\mu_{d-1,2}$. Now it is clear that by repeating the same considerations we have $d.\xi^1\geqq\lambda$, and if $d.\xi^1=\lambda$, we have $\lambda_{kd+j}=d\mu_{d-j,k-1}$ $(0\leqq k\leqq i-1,\ 0\leqq j\leqq d)$. Conversely, if $\lambda=d.\nu$ for some partition $\nu=(\nu_1, \cdots, \nu_{id})$ $(\nu_1\geqq\nu_2\geqq\cdots\geqq\nu_{id}\geqq 0)$,it is easy to see that the condition "Otherwise ..." in (2.6) is satisfied, that $\mu_{d-j,k-1}=\nu_{kd+j}$ $(0\leqq k\leqq i-1,\ 0\leqq j\leqq d)$, and that $\theta=c_{d.\nu,\nu}=1$. This completes the proof of (2.6).

(2.8) COROLLARY.

$$\sum_{|\pi|=v}\frac{1}{z_\pi}\chi_\pi^\nu\chi_{d.\pi}^\lambda=\begin{cases}1 & \text{if } \lambda=d.\nu,\\ 0 & \text{if } \lambda>d.\nu,\end{cases}$$

where if $\rho=(1^{r_1}2^{r_2}3^{r_3}\cdots)$ *is a partition,* z_ρ *is defined to be* $1^{r_1}r_1!\,2^{r_2}r_2!\,3^{r_3}r_3!\cdots$.

PROOF. Since $v!/z_\pi$ is the order of the conjugacy class in S_v corresponding to π, the assertion follows from the orthogonality.

Let $Q_\rho^\lambda(q)$ denote a Green polynomial of q introduced by J. A. Green in [3] (see [3], p. 420, Definition 4.2). To prove Theorem C we need the explicit information about this polynomial (see the proof of (2.14)). But Definition 4.2 above is not satisfying for us.

Put $X_\rho^\lambda(t)=q^{-n_\lambda}Q_\rho^\lambda(q)$ $\left(t=\frac{1}{q}\right)$, where if $\tilde{\rho}=(\tilde{r}_1, \cdots, \tilde{r}_k)$ is the conjugate partition of ρ, n_ρ is defined to be $\sum_{i=1}^{k}\tilde{r}_iC_2$. By Lemma 4.3 of [3], we see that $X_\rho^\lambda(t)$ is a polynomial of t. In [7], A. O. Morris gave an effective procedure to calculate $X_\rho^\lambda(t)$. To state his results, let us consider the Schur function $\{\lambda\}=\{\lambda_1, \cdots, \lambda_k\}$ $(\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_k>0)$ associated with a new series

$$f'(x)=\prod_{i=1}^{m}(1-t\alpha_i x)/(1-\alpha_i x)=1+\sum_{r=1}^{\infty}q_r'x^r\,, \tag{2.9}$$

i. e., $\{\lambda\}=|q'_{\lambda_s-s+1}|$.

In [6] D. E. Littlewood introduced a certain symmetric function $Q_\lambda(t)$ and gave an explicit formula to calculate it, i. e.,

$$Q_{(\lambda_1,\cdots,\lambda_k)}(t)=\prod_{1\leqq i<j\leqq k}(1+t\delta_{ij}+t^2\delta_{ij}{}^2+\cdots)\{\lambda_1, \cdots, \lambda_k\},$$

where $\{\lambda\}$ is a Schur function associated with the series (2.9) and δ_{ij} is an operator which transforms λ_i to λ_i+1 and λ_j to λ_j-1 (see [6], Theorem V, p. 498). For example, if $\lambda=(2^3)$,

$$Q_{(2^3)}(t)=\{2^3\}+(t+t^2)\{321\}+t^3\{3^2\}+t^3\{41^2\}$$
$$+(t^2+t^3+t^4)\{42\}+(t^4+t^5)\{51\}+t^6\{6\}\ .$$

Put

(2.10) $$Q_{(\lambda_1,\cdots,\lambda_k)}(t)=\sum_{\mu} f_{\lambda\mu}(t)\{\mu\}\ ,$$

where the $f_{\lambda\mu}(t)$ are some polynomials of t. For the above example,

$$f_{(2^3)(2^3)}(t)=1\ ,\qquad f_{(2^3)(321)}(t)=t+t^2\ ,\qquad f_{(2^3)(3^2)}(t)=t^3\ ,$$
$$f_{(2^3)(41^2)}(t)=t^3\ ,\qquad f_{(2^3)(42)}(t)=t^2+t^3+t^4\ ,$$
$$f_{(2^3)(51)}(t)=t^4+t^5\ ,\qquad f_{(2^3)(6)}(t)=t^6\ .$$

Now we can state Morris' result:

(2.11) LEMMA. *If* $Q_\lambda(t)=\sum_{\mu} f_{\lambda\mu}(t)\{\mu\}$, *where* $\{\mu\}$ *are Schur functions associated with the series* (2.9). *Then* $X_\rho^\lambda(t)=\sum_{\mu} f_{\lambda\mu}(t)\chi_\rho^\mu$.

(2.12) LEMMA. *In the expression* (2.10), *we have* $f_{\lambda\lambda}(t)=1$, *and if* $\mu<\lambda$, *we have* $f_{\lambda\mu}(t)=0$.

PROOF. We shall prove (2.12) by induction on k. If $k=1$, the assertion is clear. Now let $k>1$ and assume that (2.12) has been proved for $k-1$. We need the following lemma that was established by Morris [7]:

SUBLEMMA. *If*

$$Q_{(\lambda_2,\cdots,\lambda_k)}(t)=\sum_{\mu} g_{\lambda\mu}(t)\{\mu\}\ ,$$

then

$$Q_{(\lambda_1,\lambda_2,\cdots,\lambda_k)}(t)=\sum_{r=0}^{\infty} t^r \sum_{\mu} g_{\lambda\mu}(t) \sum_{\omega} \{\lambda_1+r,\ \omega\}\ ,$$

where $\{\lambda\}$ *is a Schur function of the series* (2.9) *and the last summation is over all partitions* ω *so that* $\{\mu\}$ *appears in the product* $\{\omega\}\{r\}$.

Now we return to the proof of (2.12). If $r=0$, $\{\omega\}\{0\}=\{\omega\}$ and ω cannot be any other partition different from ξ. Then we have

$$Q_{(\lambda_1,\lambda_2,\cdots,\lambda_k)}(t)=\sum_{|\xi|=n-\lambda_1} g_{\lambda\xi}(t)\{\lambda_1,\ \xi\}$$
$$+\sum_{r=1}^{\infty} t^r \sum_{|\xi|=n-r} g_{\lambda\xi}(t)\sum_{\omega}\{\lambda_1+r,\ \omega\}\ .$$

In this expression, any partition appearing in the second summation is of the form $(\lambda_1+r,\ \omega)$ $(r\geqq 1)$ which is larger than λ. In the first summation, by hypothesis of induction, we see that $g_{\lambda,(\lambda_2,\cdots,\lambda_k)}(t)=1$, and that if $\xi<(\lambda_2,\cdots,\lambda_k)$, $g_{\lambda\xi}(t)=0$. Then if ξ is a partition such that $g_{\lambda\xi}(t)\neq 0$ and that $\xi\neq(\lambda_2,\cdots,\lambda_k)$, then $\xi>(\lambda_2,\cdots,\lambda_k)$ and $(\lambda_1,\ \xi)>\lambda=(\lambda_1,\ \lambda_2,\cdots,\lambda_k)$. Thus we see that $f_{\lambda\lambda}(t)=$

$g_{\lambda,(\lambda_2,\cdot\cdot,\lambda_k)}(t)=1$ and that if $\mu<\lambda$,

$$f_{\lambda\mu}(t)=\begin{cases} g_{\lambda\xi}(t) & \text{if } \mu=(\lambda_1, \xi) \text{ for some partition } \xi, \\ 0 & \text{otherwise}, \end{cases}$$

$$=0.$$

This completes the proof of (2.12).

(2.13) COROLLARY.

$$\sum_{|\pi|=v} \frac{1}{z_\pi} \chi_\pi^\nu X_{d.\pi}^\lambda(t)=\begin{cases} 1 & \textit{if } \lambda=d.\nu, \\ 0 & \textit{if } \lambda>d.\nu. \end{cases}$$

PROOF. This follows immediately from (2.8), (2.11) and (2.12).

From now on we will frequently use notations in [**3**].

(2.14) THEOREM. *Let $X=(g^\nu)$ denote a primary irreducible character of $G=GL(dv, q)$, where d is the degree of a simplex g and ν is a partition of v, and let u_λ denote a unipotent element of G corresponding to a partition λ of dv. Then $X(u_{d.\nu})=(-1)^{(d-1)v}q^{n d.\nu}$, and if $\lambda>d.\nu$, $X(u_\lambda)=0$.*

PROOF. By the definition ([**3**], p. 439),

$$X=(g^\nu)=(-1)^{(d-1)v}I_d^k[\nu],$$

where by Definition 7.3 of [**3**],

$$I_d^k[\nu]=\sum_{|\pi|=v}\frac{1}{z_\pi}\chi_\pi^\nu B^{d.\pi}\left(k\frac{\pi}{d}\right).$$

If $\pi=(p_1, p_2, \cdots)$ is a partition of v, by Lemma 7.1 and Theorem 9 of [**3**], we see that the value of $B_{d.\pi}$ at u_λ equals

$$\begin{aligned} B_{d.\pi}\left(k\frac{\pi}{d}:1\right)&=z_\pi U_{d.\pi}(k:1) \\ &=z_\pi \prod_e \prod_{i=1}^{p_e} T_{d,e}(k:1) \\ &=z_\pi \prod_e \prod_{i=1}^{p_e} \sum_{i=0}^{d-1} \theta^{q^i k}(1) \\ &=z_\pi d^{\sum_e p_e}. \end{aligned}$$

Then if c denotes the conjugacy class of u_λ, by Definition 4.12 of [**3**], we have

$$\begin{aligned} B^{d.\pi}\left(k\frac{\pi}{d}\right)(c)&=\sum_m Q(m, c)U_{d.\pi}(k:1) \\ &=Q(m, c)U_{d.\pi}(k:1) \\ &=\prod_{f\in F}\frac{1}{z_{\rho(m,f)}} Q_{\rho(m,f)}^{\nu_c(f)}(q^{d(f)})U_{d.\pi}(k:1) \end{aligned}$$

$$=\frac{1}{z_{d.\pi}}Q^{\lambda}_{d.\pi}(q)U_{d.\pi}(k:1)$$

$$=\frac{1}{d^{\sum_e p_e}.z_{\pi}}Q^{\lambda}_{d.\pi}(q)z_{\pi}d^{\sum_e p_e}$$

$$=Q^{\lambda}_{d.\pi}(q)\,.$$

Here the second equality follows from the fact that there is only one substitution m of $X^{d.\pi}$ into c such that $x^{d.\pi}\to 1$ for any $d.\pi$-root (or $d.\pi$-variable) $x^{d.\pi}$ in $X^{d.\pi}$. Then we have

$$X(u_{\lambda})=(-1)^{(d-1)v}I^{k}_{d}[\nu](u_{\lambda})$$

$$=(-1)^{(d-1)v}\sum_{|\pi|=v}\frac{1}{z_{\pi}}\chi^{\nu}_{\pi}Q^{\lambda}_{d.\pi}(q)\,.$$

Replacing $Q^{\lambda}_{d.\pi}(q)$ with $q^{n\lambda}X^{\lambda}_{d.\pi}(t)$, we have

$$X(u_{\lambda})=(-1)^{(d-1)v}q^{n\lambda}\sum_{|\pi|=v}\frac{1}{z_{\pi}}\chi^{\nu}_{\pi}X^{\lambda}_{d.\pi}(t)\,.$$

Then the assertion follows immediately from (2.13).

REMARK. By Theorem 14 of [3], we know that the p-part of the degree of (g^{ν}) is $q^{v(\nu_2+2\nu_3+\cdots+(k-1)\nu_k)}$ ($\nu=(\nu_1,\nu_2,\cdots,\nu_k)$, $\nu_1\geqq\nu_2\geqq\cdots\geqq\nu_k>0$). But by induction on k, we can prove that $n_{d.\nu}=v(\nu_2+2\nu_3+\cdots+(k-1)\nu_k)$. So (2,14) is a special case of Theorem C.

The object of this section is to prove the next Theorem:

(2.15) THEOREM. *Let* $X=(\cdots g^{\nu(g)}\cdots)=(g_1{}^{\nu_1}g_2{}^{\nu_2}\cdots g_N{}^{\nu_N})$ *denote an arbitrary irreducible character of* $G=GL(n,q)$, *where if* d_i *is the degree of simplex* g_i, $i=1,\cdots,N$, *the* ν_i ($i=1,\cdots,N$) *are partitions so that* $\sum_{i=1}^{N}|\nu_i|d_i=n$, *and let* λ *be the largest partition that appears in the product of Schur functions* $\{d.\nu_i\}$, $i=1,\cdots,N$ *associated with the series* (2.1). *Then if* u_{μ} *is a unipotent element of* G *corresponding to a partition* μ *of* n,

$$X(u_{\mu})=\begin{cases}\prod_{i=1}^{N}(-1)^{(d_i-1)v_i}q^{n_{d.\nu_i}} & \text{if } \mu=\lambda\,,\\ 0 & \text{if } \mu>\lambda\,.\end{cases}$$

REMARK. By Theorem 14 of [3] and the remark below the proof of (2.14), we see that the p-part of the degree of $(g_1{}^{\nu_1}g_2{}^{\nu_2}\cdots g_N{}^{\nu_N})$ equals $\prod_{i=1}^{N}q^{n_{d.\nu_i}}$. Then Theorem C is a corollary from (2.15).

PROOF OF (2.15). Put $X_i=(g_i{}^{\nu_i})$, $i=1,\cdots,N$. Then by Theorem 13 of [3], we have $X=X_1\circ\cdots\circ X_N$, where the notation "$\circ$" is defined in the introduction of [3]. By Theorem 2 and Lemma 2.6 of [3], we have the expression

(2.16) $$X(u_\mu)=\sum g^{\mu}_{\lambda_1\cdots\lambda_N}(q)X_1(u_{\lambda_1})\cdots X_N(u_{\lambda_N}),$$

where the summation is over all families of partitions $(\lambda_i)_{i=1}^N$ so that $|\lambda_i|=d_i|\nu_i|$, $i=1,\cdots,N$, and the $g^{\mu}_{\lambda_1\cdots\lambda_N}(q)$ are Hall polynomials (see [3], pp. 411-412). As in (2.15), let λ be the largest partition that appears in the expression

(2.17) $$\{d_1.\nu_1\}\cdots\{d_N.\nu_N\}=\sum_{|\mu|=n}c_\mu\{\mu\}.$$

Firstly, suppose that $\mu>\lambda$. If for each i, $i=1,\cdots,N$, λ_i does not exceed $d_i.\nu_i$, μ cannot appear in the product

(2.18) $$\{\lambda_1\}\cdots\{\lambda_N\}=\sum_{|\delta|=n}c^{\delta}_{\lambda_1\cdots\lambda_N}\{\delta\},$$

since by (2.4) the largest partition that appears in (2.18) cannot exceed λ and since μ is larger than λ. By Theorem 4 of [3], $c^{\mu}_{\lambda_1\cdots\lambda_N}=0$ implies that $g^{\mu}_{\lambda_1\cdots\lambda_N}(q)\equiv0$. Then we have

(2.19) $$X(u_\mu)=\sum g^{\mu}_{\lambda_1\cdots\lambda_N}(q)X_1(u_{\lambda_1})\cdots X_N(u_{\lambda_N}),$$

where the summation is over all families of partitions $(\lambda_i)_{i=1}^N$ so that $|\lambda_i|=d_i|\nu_i|$, $i=1,\cdots,N$ and that for at least one suffix i, λ_i exceeds $d_i.\nu_i$. But if i is such a suffix, (2.14) implies that $X_i(u_{\lambda_i})=0$. Hence $X(u_\mu)=0$.

Secondly, suppose that $\mu=\lambda$. By the above consideration, we have

$$X(u_\lambda)=g^{\lambda}_{d_1.\nu_1,\cdots,d_N.\nu_N}(q)X_1(u_{d_1.\nu_1})\cdots X_N(u_{d_N.\nu_N}).$$

However, the assertion (i) in (2.14) implies that $n_\lambda-n_{d_1.\nu_1}-\cdots-n_{d_N.\nu_N}=0$ and that $c^{\lambda}_{d_1.\nu_1,\cdots,d_N.\nu_N}=1$. Then by Theorem 4 of [3], we see that $g^{\tilde{\lambda}}_{\widetilde{d_1.\nu_1},\cdots,\widetilde{d_N.\nu_N}}(q)=1$. In fact, the former equality follows from the fact $\tilde{\lambda}=\widetilde{d_1.\nu_1}+\cdots+\widetilde{d_N.\nu_N}$ and the latter from that the multiplicity of λ in (2.18) is one. Now the assertion follows from (2.14). This completes the proof of (2.15).

(2.19) COROLLARY (Gow). *If X is an irreducible character of $G=GL(n, q)$ of degree coprime to p, then $m_Q(X)=1$.*

PROOF. In this case the p-part of the degree of X is 1. Then by Theorem C, there is a unipotent element u of G such that $X(u)=\pm1$. Then the assertion follows from Theorem B.

REMARK. A unipotent element u in the proof of (2.19) can be chosen of the form as in (1.4).

(2.20) COROLLARY. *Let X denote an irreducible character of $G=GL(n, q)$ and let u denote a regular unipotent element of G. Then the degree of X is coprime to p, if and only if $X(u)=\pm1$.*

(2.21) COROLLARY. *Let X denote an arbitrary irreducible character of $G=GL(n, q)$. Then if $p\neq2$, $m_Q(X)=1$.*

§ 3. Schur indices of characters of $SL(2n+1, q)$.

(3.1) THEOREM. *Let X denote an irreducible character of $G=SL(2n+1, q)$. Then*

(i) *if the degree of X is coprime to p, $m_{Q}(X)=1$,*

(ii) *if $p\neq 2$, for any X, $m_{Q}(X)=1$.*

PROOF. The following proof is due essentially to Gow [2]. Put $G_1=GL(2n+1, q)$ and let $X=X_1, \cdots, X_r$ denote the distinct G_1-conjugates of X. By Clifford theory (see, for instance, Endliche Gruppen, by B. Huppert, Springer), there is an irreducible character I of G_1 with $I_G=X_1+\cdots+X_r$. By Theorem C, there is a unipotent element u of G_1 (which also lies in G) such that $|I(u)|$ equals the p-part of the degree of I. By noting that $m_{Q}(X_1)=m_{Q}(X_2)=\cdots=m_{Q}(X_r)$ and by Theorem B, we see that $m_{Q}(X)$ divides $I(u)$ which is a power of p. Then the assertion (ii) follows from Theorem A. If the degree of X is coprime to p, that of I is also coprime to p (note that r devides $(G_1: G)=q-1$). Then we can choose u so that $I(u)=\pm 1$ and hence the assertion (i) is clear.

§ 4. The case of $p=2$ and some other results.

If $p=2$, our method cannot determine the Schur indices. However, for small n, we have

(4.1) PROPOSITION (Gow [2], Ohmori-Yamada [8]). (i) *If $n\leqq 4$, for any irreducible character X of $GL(n, q)$, $m_{Q}(X)=1$.*

(ii) *The Schur indices of all the irreducible characters of $SL(3, q)$ are* 1.

REMARK. G. J. Janusz showed that the Schur indices of all the irreducible characters of $SL(2, 2^f)$ are 1. But he also showed that $SL(2, p^f)$ $(p\neq 2)$ has irreducible characters of Schur indices 2 (see [4]). Generally, $SL(2n, q)$ has real-valued irreducible characters of Schur indices 2 (see [2]).

(4.2) PROPOSITION. *If X is an irreducible character of $SL(2n, 2^f)$ of degree coprime to $p=2$, $m_{Q}(X)=1$.*

PROOF. By Theorem A and by the fact that $m_{Q}(X)$ divides the degree of X, the assertion is clear.

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