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Transfinite type theory and provability of second order formulas

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Introduction

When there is a set of objects, we can consider predicates over the domain of objects as new objects. To distinguish new objects from old ones, we call the former, *objects of type* (ι), and the latter, *objects of type* ι. This method of introducing new objects can be repeated as follows: if objects of type τ are introduced at the n-th stage, then predicates over the domain of objects of type τ are introduced at the $(n+1)$-th stage. These new objects are called *objects of type* (τ).

Simple type theory of finite type is a formalization of this idea. In it, types and objects are introduced inductively as follows: 1. Elements of an individual domain are objects of type ι. 2. If objects of type τ have been introduced, then predicates over the domain of objects of type τ can be introduced as objects of type (τ).

At the stage when the above process has been completed (we call this stage the ω-th stage), we can introduce as new objects: predicates over the domain of all objects which are introduced by this stage. These new objects are called *objects of type* (ω). The systems introduced in Andrews [**1**] and Uesu [8] are formalizations of this idea. We note that the systems have variables which range over the domain of types of simple type theory of finite type, in addition to variables which range over the domain of objects of each type.

The above method of introducing new objects can be repeated untill the μ-th stage for any ordinal μ in the following manner: 1. (0-th stage) Elements of the individual domain are called *objects of type* ι. 2. ($(\nu+1)$-th stage) If objects of type τ are introduced at the ν-th stage, then predicates over the domain of objects of type τ are introduced as objects of type (τ) at the $(\nu+1)$-th stage. 3. (λ-th stage (λ is a limit ordinal)) At this stage, predicates over the domain of all objects which are introduced at the ν-th stage with $\nu<\lambda$ are introduced as objects of type (λ).

We can consider a system which is a formalization of this idea. In this system, types, objects, and variables which range over domains of types are introduced as follows: 1. Elements of an individual domain are objects of type ι, which is a type of level 0. 2. If objects of type τ have been introduced where τ is a type of level ν with $\nu<\mu$, then predicates over the domain of objects of type τ can be introduced as objects of type (τ) where (τ) is a type of level $\nu+1$. 3. If λ is a limit ordinal with $\lambda<\mu$, predicates over the domain of objects of types of levels less than λ are introduced as objects of type (λ), where (λ) is a type of level λ. We also introduce variables which range over the domain of types of levels less than λ.

For the system which has just been described, we can enlarge the set of provable second order formulas by increasing the value of μ from ν to $\nu+1$ for $\nu<\omega^2$ but not for $\nu>\omega^2$.

The purpose of this paper is to define transfinite type theory which makes it possible for us to enlarge further the set of provable second order formulas. Roughly speaking, for the system which we will be considering, we can enlarge the set by increasing the value of μ from ν to $\nu+1$ for any $\nu<\omega^{\omega}$.

In this paper, an index constant, index variables, symbols for type variables, as well as the symbols of simple type theory of finite type, are introduced as primitive symbols. We use these symbols for defining indices, orders, and type variables. Roughly speaking, an index stands for a natural number. An order, which is defined as a finite sequence of indices, represents a limit ordinal in the same way in which a finite sequence $\langle m_0, \cdots, m_k\rangle$ of natural numbers represents a limit ordinal $\omega^{k+1}\cdot m_k+\cdots+\omega\cdot m_0$. We use orders as follows: 1. If $\mathfrak{m}$ is an order, then $(\mathfrak{m})$ is a type of level $\mathfrak{m}$. 2. If $\mathfrak{m}$ is an order and α is a symbol for a type variable, then $\alpha(\mathfrak{m})$ is a type variable which ranges over the domain of types of levels less than $\mathfrak{m}$.

There are four sections in this paper. We give the formation rules of the systems in § 1. We prove some syntactical properties of the systems in § 2. In § 3 we define the concept "general model", and state some of its semantical properties. In § 2 and § 4 we prove three theorems which relate to the sets of provable second order formulas.

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§ 1. Systems for transfinite type theory

In this section we define a logical system $H_{f,\mu,n}$ for natural numbers k and n, an element f of ω^{k+1} and an ordinal μ. We use the following primitive symbols.

Logical symbols: $\vee$(or), $\neg$(not), $\exists$(there exists). Index constant: o. Type constant: ι. Bound index variables: $p, q, \cdots$. Free index variables: $r, s, \cdots$.

Symbols for bound type variable: $\xi, \eta, \cdots$. Symbols for free type variable: $\alpha, \beta, \cdots$. Symbols for bound variable: $X, Y, \cdots$. Symbols for free variable: $A, B, \cdots$. Additional symbols: $\lambda, \Lambda, \in, (,), \rightarrow$.

The system may contain as primitive symbols function symbols for individuals $g, h, \cdots$.

Quasi-indices are defined inductively as follows. 1. The index constant o, free index variables and bound index variables are quasi-indices. 2. If I is a quasi-index then so is (I). We use symbols $I, J, \cdots$ to denote quasi-indices. A function $\mathrm{V}^{\mathrm{i}}(I)$ is defined inductively as follows. 1. $\mathrm{V}^{\mathrm{i}}(o)=\emptyset$. 2. If p is a free index variable or bound index variable then $\mathrm{V}^{\mathrm{i}}(p)=\{p\}$. 3. $\mathrm{V}^{\mathrm{i}}((I))=\mathrm{V}^{\mathrm{i}}(I)$. A quasi-index I is called an *index* if $\mathrm{V}^{\mathrm{i}}(I)$ contains no bound index variable. A function $\mathrm{Ps}(I)$ is defined inductively as follows. 1. If I is o, free index variable or bound index variable then $\mathrm{Ps}(I)=I$. 2. $\mathrm{Ps}((I))=\mathrm{Ps}(I)$. A relation $I<J$ is defined inductively as follows. 1. $o<(J)$ for any quasi-index J. 2. If p is a free index variable or bound index variable and $\mathrm{Ps}(J)=p$ then $p<(J)$. 3. If $I<J$ then $(I)<(J)$. Similarly a relation $I\leqq J$ is defined as follows. 1. $o\leqq J$ for any quasi-index J. 2. If p is a free index variable or bound index variable and $\mathrm{Ps}(J)=p$ then $p\leqq J$. 3. If $I\leqq J$ then $(I)\leqq(J)$. Indices $o^{(m)}$ are defined inductively as follows. 1. $o^{(0)}=o$. 2. $o^{(m+1)}=(o^{(m)})$.

Quasi-orders are defined as follows. 1. If $I_0, \cdots, I_k$ are quasi-indices and $\nu<\mu$ then

$$\langle I_0, \cdots, I_k, \nu\rangle$$

is a quasi-order. 2. If $I_0, \cdots, I_{i-l}$ are quasi-indices, $i\leqq k$ and $m<f(i)$ then

$$\langle I_0, \cdots, I_{i-l}, o^{(m)}, o^{(f(i+1))}, \cdots, o^{(f(k))}, \mu\rangle$$

is a quasi-order. 3.

$$\langle o^{(f(0))}, \cdots, o^{(f(k))}, \mu\rangle$$

is a quasi-order (*the maximum order*). We use symbols $\mathfrak{m}, \mathfrak{n}, \mathfrak{o}, \cdots$ to denote quasi-orders. A relation $<$ on quasi-orders is defined as follows. If (i) $\kappa<\nu$ or (ii) for some i $(0\leqq i\leqq k)$, $\kappa=\nu$, $I_k\leqq J_k, \cdots, I_{i+1}\leqq J_{i+1}$ and $I_i<J_i$, then

$$\langle I_0, \cdots, I_k, \kappa\rangle<\langle J_0, \cdots, J_k, \nu\rangle.$$

A relation $\leqq$ on quasi-orders is defined as follows. If (i) $\langle I_0, \cdots, I_k, \kappa\rangle<\langle J_0, \cdots, J_k, \nu\rangle$ or (ii) $\kappa=\nu$, $I_k\leqq J_k, \cdots, I_1\leqq J_1$ and $I_0\leqq J_0$, then

$$\langle I_0, \cdots, I_k, \kappa\rangle\leqq\langle J_0, \cdots, J_k, \nu\rangle.$$

We define

$$\mathrm{V}^{\mathrm{i}}(\langle I_0, \cdots, I_k, \nu\rangle)=\mathrm{V}^{\mathrm{i}}(I_0)\cup\cdots\cup\mathrm{V}^{\mathrm{i}}(I_k).$$

A quasi-order $\mathfrak{m}$ is called an *order* if $V^i(\mathfrak{m})$ contains no bound index variable.

Quasi-types are defined inductively as follows. 1. The type constant ι is a quasi-type. 2. If $\mathfrak{m}$ is a quasi-order and ξ is a symbol for bound type variable then $\xi(\mathfrak{m})$ is a quasi-type (*bound type variable*). 3. If $\mathfrak{m}$ is an order and α is a symbol for free type variable then $\alpha(\mathfrak{m})$ is a quasi-type (*free type variable*). 4. If $\mathfrak{m}$ is a quasi-order then $(\mathfrak{m})$ is a quasi-type. 5. If τ is a quasi-type then (τ) is a quasi-type. We use symbols $\sigma, \tau, \nu, \cdots$ to denote quasi-types. A function $V^t(\tau)$ is defined inductively as follows. 1. $V^t(\iota)=\emptyset$. 2. $V^t(\xi(m))=\{\xi(\mathfrak{m})\}$. 3. $V^t(\alpha(\mathfrak{m}))=\{\alpha(\mathfrak{m})\}$. 4. If $\mathfrak{m}$ is a quasi-order then $V^t((\mathfrak{m}))=\emptyset$. 5. If τ is a quasi-type then $V^t((\tau))=V^t(\tau)$. The function V^i is extended as follows. 1. $V^i(\iota)=\emptyset$. 2. $V^i(\alpha(\mathfrak{m}))=V^i(\xi(\mathfrak{m}))=V^i(\mathfrak{m})$. 3. $V^i((\mathfrak{m}))=V^i(\mathfrak{m})$. 4. $V^i((\tau))=V^i(\tau)$. A quasi-type τ is called a *type* if $V^i(\tau)$ contains no bound index variable and $V^t(\tau)$ contains no bound type variable. Functions $\mathrm{Ord}^1(\tau)$ and $\mathrm{Ord}^2(\tau)$ are defined inductively as follows. 1. $\mathrm{Ord}^1(\iota)=\mathrm{Ord}^2(\iota)=\emptyset$. 2. $\mathrm{Ord}^1(\xi(\mathfrak{m}))=\mathrm{Ord}^1(\alpha(\mathfrak{m}))=\{\mathfrak{m}\}$. 3. $\mathrm{Ord}^2(\xi(\mathfrak{m}))=\mathrm{Ord}^2(\alpha(\mathfrak{m}))=\emptyset$. 4. If $\mathfrak{m}$ is a quasi-order then $\mathrm{Ord}^1((\mathfrak{m}))=\emptyset$ and $\mathrm{Ord}^2((\mathfrak{m}))=\{\mathfrak{m}\}$. 5. If τ is a quasi-type then $\mathrm{Ord}^1((\tau))=\mathrm{Ord}^1(\tau)$ and $\mathrm{Ord}^2((\tau))=\mathrm{Ord}^2(\tau)$. We write $\mathrm{Ord}^1(\tau)<\mathfrak{m}$ to denote the fact that $\mathfrak{n}<\mathfrak{m}$ for any element $\mathfrak{n}$ of $\mathrm{Ord}^1(\tau)$. Similarly we use symbols $\mathrm{Ord}^2(\tau)<\mathfrak{m}$, $\mathrm{Ord}^1(\tau)\leqq\mathfrak{m}$ and $\mathrm{Ord}^2(\tau)\leqq\mathfrak{m}$.

A function $\deg(\tau)$ is defined inductively as follows. 1. $\deg(\iota)=0$. 2. If $\mathfrak{m}$ is a quasi-order but not the maximun order then $\deg(\xi(\mathfrak{m}))=\deg(\alpha(\mathfrak{m}))=\deg((\mathfrak{m}))=0$. 3. If $\mathfrak{m}$ is the maximum order then $\deg(\xi(\mathfrak{m}))=\deg(\alpha(\mathfrak{m}))=\deg((\mathfrak{m}))=1$. 4. If τ is a quasi-type with $\deg(\tau)=0$ then $\deg((\tau))=0$. 5. If τ is a quasi-type with $\deg(\tau)\neq 0$ and $\mathrm{Ord}^1(\tau)=\emptyset$ then $\deg((\tau))=\deg(\tau)+1$. 6. If τ is a quasi-type with $\deg(\tau)\neq 0$ and $\mathrm{Ord}^1(\tau)\neq\emptyset$ then $\deg((\tau))=\deg(\tau)$.

Quasi-varieties, quasi-formulas, function V^p and extensions of V^i and V^t are defined by a simultaneous induction as follows.

1. If X is a symbol for bound variable and τ is a quasi-type with $\deg(\tau)\leqq n$, then X^τ is a quasi-variety of a quasi-type τ (bound variable of a quasi-type τ), $V^i(X^\tau)=V^i(\tau)$, $V^t(X^\tau)=V^t(\tau)$ and $V^p(X^\tau)=\{X^\tau\}$.

2. If A is a symbol for free variable and τ is a type with $\deg(\tau)\leqq n$, then A^τ is a quasi-variety of a type τ (free variable of a type τ), $V^i(A^\tau)=V^i(\tau)$, $V^t(A^\tau)=V^t(\tau)$ and $V^p(A^\tau)=\{A^\tau\}$.

3. If $t_1, \cdots, t_m$ are quasi-varieties of the type ι and g is a function symbol for individuals then $g(t_1, \cdots, t_m)$ is a quasi-variety of the type ι,

$$V^i(g(t_1, \cdots, t_m))=V^i(t_1)\cup\cdots\cup V^i(t_m),$$

$$V^t(g(t_1, \cdots, t_m))=V^t(t_1)\cup\cdots\cup V^t(t_m),$$

and

$$V^p(g(t_1, \cdots, t_m))=V^p(t_1)\cup\cdots\cup V^p(t_m).$$

4. $(E \in F)$ is a quasi-formula, $V^i((E\in F))=V^i(E)\cup V^i(F)$, $V^t((E\in F))=V^t(E)\cup V^t(F)$ and $V^p((E\in F))=V^p(E)\cup V^p(F)$ if E and F are quasi-varieties of quasi-types τ and (τ), respectively with the following properties (a) and (b): (a) For any symbol for free or bound type variable ξ and any quasi-orders $\mathfrak{m}$ and $\mathfrak{n}$ if $\xi(\mathfrak{m})$ and $\xi(\mathfrak{n})$ are elements of $V^t(E)$ or of $V^t(F)$ then $\mathfrak{m}=\mathfrak{n}$. (b) For any symbol for free or bound variable X and any quasi-types σ and ν if X^σ and X^ν are elements of $V^p(E)$ or of $V^p(F)$ then $\sigma=\nu$.

5. If F is a quasi-variety of a quasi-type $(\mathfrak{m})$ ($\mathfrak{m}$ is a quasi-order), E is a quasi-variety of a quasi-type τ with $\mathrm{Ord}^1(\tau)\leqq\mathfrak{m}$, $\mathrm{Ord}^2(\tau)<\mathfrak{m}$ and the properties (a) and (b) in 4, then $(E\in F)$ is a quasi-formula, $V^i((E\in F))=V^i(E)\cup V^i(F)$, $V^t((E\in F))=V^t(E)\cup V^t(F)$ and $V^p((E\in F))=V^p(E)\cup V^p(F)$.

6-7. If $\mathfrak{A}$ and $\mathfrak{B}$ are quasi-formulas with the properties (a) and (b) in 4 (with "$\mathfrak{A}$", "$\mathfrak{B}$" in place of "E", "F", respectively), $\neg(\mathfrak{A})$ and $(\mathfrak{A})\vee(\mathfrak{B})$ are quasi-formulas, $V^i(\neg(\mathfrak{A}))=V^i(\mathfrak{A})$, $V^t(\neg(\mathfrak{A}))=V^t(\mathfrak{A})$, $V^p(\neg(\mathfrak{A}))=V^p(\mathfrak{A})$, $V^i((\mathfrak{A})\vee(\mathfrak{B}))=V^i(\mathfrak{A})\cup V^i(\mathfrak{B})$, $V^t((\mathfrak{A})\vee(\mathfrak{B}))=V^t(\mathfrak{A})\cup V^t(\mathfrak{B})$ and $V^p((\mathfrak{A})\vee(\mathfrak{B}))=V^p(\mathfrak{A})\cup V^p(\mathfrak{B})$.

8. If X^τ is a bound variable of a quasi-type τ and $\mathfrak{A}$ is a quasi-formula with the property that $V^p(\mathfrak{A})$ contains X^τ or $V^p(\mathfrak{A})$ contains no bound variable of the form X^σ, then $\exists X^\tau(\mathfrak{A})$ is a quasi-formula, $V^i(\exists X^\tau(\mathfrak{A}))=V^i(\tau)\cup V^i(\mathfrak{A})$, $V^t(\exists X^\tau(\mathfrak{A}))=V^t(\tau)\cup V^t(\mathfrak{A})$ and $V^p(\exists X^\tau(\mathfrak{A}))=V^p(\mathfrak{A})-\{X^\tau\}$.

9. $\exists\xi(\mathfrak{m})(\mathfrak{A})$ is a quasi-formula, $V^i(\exists\xi(\mathfrak{m})(\mathfrak{A}))=V^i(\mathfrak{m})\cup V^i(\mathfrak{A})$, $V^t(\exists\xi(\mathfrak{m})(\mathfrak{A}))=V^t(\mathfrak{A})-\{\xi(\mathfrak{m})\}$ and $V^p(\exists\xi(\mathfrak{m})(\mathfrak{A}))=V^p(\mathfrak{A})$ if $\xi(\mathfrak{m})$ is a bound type variable and $\mathfrak{A}$ is a quasi-formula with the following properties (a) and (b): (a) $V^t(\mathfrak{A})$ contains $\xi(\mathfrak{m})$ or $V^t(\mathfrak{A})$ contains no bound type variable of the form $\xi(\mathfrak{n})$. (b) For any element Y^σ of $V^p(\mathfrak{A})$, $V^t(\sigma)$ does not contain $\xi(\mathfrak{m})$.

10. $\exists p(\mathfrak{A})$ is a quasi-formula, $V^i(\exists p(\mathfrak{A}))=V^i(\mathfrak{A})-\{p\}$, $V^t(\exists p(\mathfrak{A}))=V^t(\mathfrak{A})$ and $V^p(\exists p(\mathfrak{A}))=V^p(\mathfrak{A})$ if p is a bound index variable and $\mathfrak{A}$ is a quasi-formula with the following properties (a) and (b): (a) For any element $\xi(\mathfrak{m})$ of $V^t(\mathfrak{A})$, $V^i(\mathfrak{m})$ does not contain p. (b) For any element X^σ of $V^p(\mathfrak{A})$, $V^i(\sigma)$ does not contain p.

11. If X^τ is a bound variable of a quasi-type τ with $\deg((\tau))\leqq n$ and $\mathfrak{A}$ is a quasi-formula with the property that $V^p(\mathfrak{A})$ contains X^τ or $V^p(\mathfrak{A})$ contains no bound variable of the form X^σ, then $\lambda X^\tau(\mathfrak{A})$ is a quasi-variety of a quasi-type (τ), $V^i(\lambda X^\tau(\mathfrak{A}))=V^i(\tau)\cup V^i(\mathfrak{A})$, $V^t(\lambda X^\tau(\mathfrak{A}))=V^t(\tau)\cup V^t(\mathfrak{A})$ and $V^p(\lambda X^\tau(\mathfrak{A}))=V^p(\mathfrak{A})-\{X^\tau\}$.

12. $\Lambda X^{\xi(m)}(\mathfrak{A})$ is a quasi-variety of a quasi-type $(\mathfrak{m})$, $V^i(\Lambda X^{\xi(m)}(\mathfrak{A}))=V^i(\mathfrak{m})\cup V^i(\mathfrak{A})$, $V^t(\Lambda X^{\xi(m)}(\mathfrak{A}))=V^t(\mathfrak{A})-\{\xi(\mathfrak{m})\}$ and $V^p(\Lambda X^{\xi(m)}(\mathfrak{A}))=V^p(\mathfrak{A})-\{X^{\xi(m)}\}$ if $\xi(\mathfrak{m})$ is a bound type variable with $\deg((\mathfrak{m}))\leqq n$, $X^{\xi(m)}$ is a bound variable of the quasi-type $\xi(\mathfrak{m})$ and $\mathfrak{A}$ is a quasi-formula with the following properties (a), (b) and (c): (a) $V^p(\mathfrak{A})$ contains $X^{\xi(m)}$ or $V^p(\mathfrak{A})$ contains no bound variable of the form X^σ. (b) $V^t(\mathfrak{A})$ contains $\xi(\mathfrak{m})$ or $V^t(\mathfrak{A})$ contains no bound type

variable of the form $\xi(\mathfrak{n})$. (c) For any element Y^{σ} of $\mathrm{V^p}(\mathfrak{A})$ which is distinct from $X^{\xi(m)}$ $\mathrm{V^t}(\sigma)$ does not contain $\xi(\mathfrak{m})$.

A quasi-variety E of a quasi-type τ is called a *variety of a type* τ if $\mathrm{V^i}(E)$ contains no bound index variable, $\mathrm{V^t}(E)$ contains no bound type variable and $\mathrm{V^p}(E)$ contains no bound variable (note that if E is a quasi-variety of a quasi-type τ then $\mathrm{V^i}(\tau)\subseteqq\mathrm{V^i}(E)$ and $\mathrm{V^t}(\tau)\subseteqq\mathrm{V^t}(E)$). Similarly a quasi-formula $\mathfrak{A}$ is called a *formula* if $\mathrm{V^i}(\mathfrak{A})$ contains no bound index variable, $\mathrm{V^t}(\mathfrak{A})$ contains no bound type variable and $\mathrm{V^p}(\mathfrak{A})$ contains no bound variable. We can prove by the induction on E that if X^{τ} is an element of $\mathrm{V^p}(E)$ then $\mathrm{V^i}(\tau)\subseteqq\mathrm{V^i}(E)$ and $\mathrm{V^t}(\tau)\subseteqq\mathrm{V^t}(E)$ and that if $\xi(\mathfrak{m})$ is an element of $\mathrm{V^t}(E)$ then $\mathrm{V^i}(\mathfrak{m})\subseteqq\mathrm{V^i}(E)$. $\mathrm{V^i}(E)$ is the set of free and bound index variables which occur free in E, $\mathrm{V^t}(E)$ is the set of free and bound type variables which occur free in E and $\mathrm{V^p}(E)$ is the set of free and bound variables which occur free in E.

We write $E\binom{p}{I}$ to denote the result obtained from E by replacing every free occurrences of a bound index variable p by an index I when E is a quasi-index, quasi-order, quasi-type, quasi-variety or quasi-formula. We write $E\binom{\xi(\mathfrak{m})}{\tau}$ to denote the result obtained from E by replacing every free occurrences of a bound type variable $\xi(\mathfrak{m})$ by a type τ when E is a quasi-type, quasi-variety or quasi-formula and $\mathrm{Ord}^1(\tau)\leqq\mathfrak{m}$ and $\mathrm{Ord}^2(\tau)<\mathfrak{m}$. Similarly we write $E\binom{X^{\tau}}{F}$ to denote the result obtained from E by replacing every free occurrences of a bound variable X^{τ} of a type τ by a variety F of the type τ when E is a quasi-variety or quasi-formula. We can prove by the induction on σ that if $\mathrm{Ord}^1(\sigma)\leqq\mathfrak{n}$ and $\mathrm{Ord}^2(\sigma)<\mathfrak{n}$ then

$$\mathrm{Ord}^1\Big(\sigma\binom{p}{I}\Big)\leqq\mathfrak{n}\binom{p}{I},\qquad \mathrm{Ord}^2\Big(\sigma\binom{p}{I}\Big)<\mathfrak{n}\binom{p}{I},$$

$$\mathrm{Ord}^1\Big(\sigma\binom{\xi(\mathfrak{m})}{\tau}\Big)\leqq\mathfrak{n}\quad\text{and}\quad\mathrm{Ord}^2\Big(\sigma\binom{\xi(\mathfrak{m})}{\tau}\Big)<\mathfrak{n}.$$

Similarly we can prove

$$\mathrm{V^t}\Big(E\binom{p}{I}\Big)=\Big\{\xi\Big(\mathfrak{m}\binom{p}{I}\Big):\ \xi(\mathfrak{m})\text{ is an element of }\mathrm{V^t}(E)\Big\},$$

$$\mathrm{V^p}\Big(E\binom{p}{I}\Big)=\Big\{X^{\sigma}:\text{ for some quasi-type }\tau\ X^{\tau}\text{ is an element of }\mathrm{V^p}(E)\text{ and }\tau\binom{p}{I}=\sigma\Big\},$$

$$\mathrm{V^t}\Big(E\binom{\xi(\mathfrak{m})}{\tau}\Big)\subset(\mathrm{V^t}(\tau)\cup\mathrm{V^t}(E))-\{\xi(\mathfrak{m})\}$$

and

$$V^p\left(E\binom{\xi(\mathfrak{m})}{\tau}\right)=\left\{X^\sigma\text{: for some quasi-type } \nu \ X^\nu \text{ is an element of } V^p(E) \text{ and } \nu\binom{\xi(\mathfrak{m})}{\tau}=\sigma\right\}.$$

$\mathfrak{A}_1, \cdots, \mathfrak{A}_u \to \mathfrak{B}_1, \cdots, \mathfrak{B}_v$ is called a *sequent* if $\mathfrak{A}_i$ and $\mathfrak{B}_j$ are formulas with the following properties (a) and (b): (a) For any symbol for free type variable α and any orders $\mathfrak{m}$ and $\mathfrak{n}$ if $\alpha(\mathfrak{m})$ and $\alpha(\mathfrak{n})$ are elements of $V^t(\mathfrak{A}_1)$ or $\cdots$ or of $V^t(\mathfrak{B}_v)$ then $\mathfrak{m}=\mathfrak{n}$. (b) For any symbol for free variable A and any types τ and σ if A^τ and A^σ are elements of $V^p(\mathfrak{A}_1)$ or $\cdots$ or of $V^p(\mathfrak{B}_v)$ then $\tau=\sigma$. We use symbols $\Gamma, \Delta, \cdots$ to denote finite or empty sequences of formulas with the preceding properties (a) and (b). $V^i(\Gamma)$, $V^t(\Gamma)$ and $V^p(\Gamma)$ denote the unions of $V^i(\mathfrak{A})$, $V^t(\mathfrak{A})$ and $V^p(\mathfrak{A})$, respectively for all $\mathfrak{A}$ in Γ. We say that a free index variable r is an *eigen-variable* for a variety or formula E (for a sequent $\Gamma\to\Delta$) if $V^i(E)(V^i(\Gamma)\cup V^i(\Delta))$ does not contain r. We say that a free type variable $\alpha(\mathfrak{m})$ is an eigen-variable for E (for $\Gamma\to\Delta$) if for any order $\mathfrak{n}$ $V^t(E)(V^t(\Gamma)\cup V^t(\Delta))$ does not contain $\alpha(\mathfrak{n})$. Similarly a free variable A^τ is an eigen-variable for E (for $\Gamma\to\Delta$) if for any type σ $V^p(E)(V^p(\Gamma)\cup V^p(\Delta))$ does not contain A^σ.

A sequent of the form $\mathfrak{A}\to\mathfrak{A}$ is an *axiom*. We make a list of the inference rules in the following and then we can define as usual the notions: "proof", "provable", "proof without cut" and "provable without cut".

Structural inference rules

Thinning $\dfrac{\Gamma\to\Delta}{\mathfrak{A}, \Gamma\to\Delta}$ $\qquad$ $\dfrac{\Gamma\to\Delta}{\Gamma\to\Delta, \mathfrak{A}}$

Contraction $\dfrac{\mathfrak{A}, \mathfrak{A}, \Gamma\to\Delta}{\mathfrak{A}, \Gamma\to\Delta}$ $\qquad$ $\dfrac{\Gamma\to\Delta, \mathfrak{A}, \mathfrak{A}}{\Gamma\to\Delta, \mathfrak{A}}$

Interchange $\dfrac{\Gamma, \mathfrak{A}, \mathfrak{B}, \Delta\to\Theta}{\Gamma, \mathfrak{B}, \mathfrak{A}, \Delta\to\Theta}$ $\qquad$ $\dfrac{\Gamma\to\Delta, \mathfrak{A}, \mathfrak{B}, \Theta}{\Gamma\to\Delta, \mathfrak{B}, \mathfrak{A}, \Theta}$

Cut $\dfrac{\Gamma\to\Delta, \mathfrak{A} \quad \mathfrak{A}, \Theta\to\Xi}{\Gamma, \Theta\to\Delta, \Xi}$

Logical inference rules

$\dfrac{\Gamma\to\Delta, \mathfrak{A}}{\neg\mathfrak{A}, \Gamma\to\Delta}$ $\qquad$ $\dfrac{\mathfrak{A}, \Gamma\to\Delta}{\Gamma\to\Delta, \neg\mathfrak{A}}$

$\dfrac{\mathfrak{A}, \Gamma\to\Delta \quad \mathfrak{B}, \Gamma\to\Delta}{\mathfrak{A}\vee\mathfrak{B}, \Gamma\to\Delta}$ $\qquad$ $\dfrac{\Gamma\to\Delta, \mathfrak{A}}{\Gamma\to\Delta, \mathfrak{A}\vee\mathfrak{B}}$ $\qquad$ $\dfrac{\Gamma\to\Delta, \mathfrak{B}}{\Gamma\to\Delta, \mathfrak{A}\vee\mathfrak{B}}$

$\exists$ for variable of each type

$$\frac{\mathfrak{A}\binom{X^\tau}{A^\tau}, \Gamma\to\Delta}{\exists X^\tau\mathfrak{A}, \Gamma\to\Delta}$$

A^τ is a free variable of a type τ and an eigen-variable for the lower sequent.

$$\frac{\Gamma\to\Delta, \mathfrak{A}\binom{X^\tau}{E}}{\Gamma\to\Delta, \exists X^\tau\mathfrak{A}}$$

E is a variety of a type τ.

$\exists$ for type variable

$$\frac{\mathfrak{A}\binom{\xi(\mathfrak{m})}{\alpha(\mathfrak{m})}, \Gamma\to\Delta}{\exists\xi(\mathfrak{m})\mathfrak{A}, \Gamma\to\Delta}$$

$\alpha(\mathfrak{m})$ is a free type variable and an eigen-variable for the lower sequent.

$$\frac{\Gamma\to\Delta, \mathfrak{A}\binom{\xi(\mathfrak{m})}{\tau}}{\Gamma\to\Delta, \exists\xi(\mathfrak{m})\mathfrak{A}}$$

τ is a type with $\mathrm{Ord}^1(\tau)\leqq\mathfrak{m}$ and $\mathrm{Ord}^2(\tau)<\mathfrak{m}$.

$\exists$ for index variable

$$\frac{\mathfrak{A}\binom{p}{r}, \Gamma\to\Delta}{\exists p\mathfrak{A}, \Gamma\to\Delta}$$

r is a free index variable and an eigen-variable for the lower sequent.

$$\frac{\Gamma\to\Delta, \mathfrak{A}\binom{p}{I}}{\Gamma\to\Delta, \exists p\mathfrak{A}}$$

I is an index.

Addditional inference rules

Comprehension

$$\frac{\mathfrak{A}\binom{X^\tau}{E}, \Gamma\to\Delta}{(E\in\lambda X^\tau\mathfrak{A}), \Gamma\to\Delta} \qquad \frac{\Gamma\to\Delta, \mathfrak{A}\binom{X^\tau}{E}}{\Gamma\to\Delta, (E\in\lambda X^\tau A)}$$

$$\frac{\mathfrak{A}\binom{\xi(\mathfrak{m})}{\tau}\binom{X^\tau}{E}, \Gamma\to\Delta}{(E\in\Lambda X^{\xi(\mathfrak{m})}\mathfrak{A}), \Gamma\to\Delta} \qquad \frac{\Gamma\to\Delta, \mathfrak{A}\binom{\xi(\mathfrak{m})}{\tau}\binom{X^\tau}{E}}{\Gamma\to\Delta, (E\in\Lambda X^{\xi(\mathfrak{m})}\mathfrak{A})}$$

E is a variety of a type τ with $\mathrm{Ord}^1(\tau)\leqq\mathfrak{m}$ and $\mathrm{Ord}^2(\tau)<\mathfrak{m}$.

Extensionality

$$\frac{(A^\tau\in E), \Gamma\to\Delta, (A^\tau\in F) \quad (A^\tau\in F), \Gamma\to\Delta, (A^\tau\in E)}{(E\in B^\sigma), \Gamma\to\Delta, (F\in B^\sigma)}$$

A^τ is a variable of a type τ and an eigen-variable for the lower sequent, B^σ is a free variable of an appropriate type σ and E and F are varieties of the type (τ).

$$\frac{(A^{\alpha(\mathfrak{m})}\in E), \Gamma\to\Delta, (A^{\alpha(\mathfrak{m})}\in F) \quad (A^{\alpha(\mathfrak{m})}\in F), \Gamma\to\Delta, (A^{\alpha(\mathfrak{m})}\in E)}{(E\in B^\tau), \Gamma\to\Delta, (F\in B^\tau)}$$

A free type variable $\alpha(\mathfrak{m})$ and a free variable $A^{\alpha(\mathfrak{m})}$ of the type $\alpha(\mathfrak{m})$ are eigen-variables for the lower sequent, B^τ is a free variable of an appropriate type τ and E and F are varieties of the type $(\mathfrak{m})$.

When Σ is a set of formulas in $H_{f,\mu,n}$ $H_{f,\mu,n}(\Sigma)$ denotes the system obtained from $H_{f,\mu,n}$ by adding the formulas in Σ as axioms. $\Sigma_{f,\mu,n}$ is the set of formulas in $H_{f,\mu,n}$ of the following forms (A-1)-(A-6). $\Pi_{f,\mu,n}$ is the set of formulas in $H_{f,\mu,n}$ of the following forms (A-1)-(A-7).

(A-1) $\forall X^{(\tau)}\exists Y^{(\mathfrak{m})}\forall Z^{\tau}[(Z^{\tau}\in X^{(\tau)})\equiv(Z^{\tau}\in Y^{(\mathfrak{m})})]$,

where τ is a type with $\mathrm{Ord}^1(\tau)\leqq\mathfrak{m}$ and $\mathrm{Ord}^2(\tau)<\mathfrak{m}$.

(A-2) $\forall X^{(\mathfrak{m})}\exists Y^{(\mathfrak{n})}\forall\xi(\mathfrak{m})\forall Z^{\xi(\mathfrak{m})}[(Z^{\xi(\mathfrak{m})}\in X^{(\mathfrak{m})})\equiv(Z^{\xi(\mathfrak{m})}\in Y^{(\mathfrak{n})})]$,

where $\mathfrak{m}$ and $\mathfrak{n}$ are orders with $\mathfrak{m}\leqq\mathfrak{n}$.

(A-3) $\exists X^{(\mathfrak{m})}\forall\xi(\mathfrak{m})\forall Y^{(\xi(\mathfrak{m}))}[(Y^{(\xi(\mathfrak{m}))}\in X^{(\mathfrak{m})})\equiv\mathfrak{A}]$,

where $\exists\xi(\mathfrak{m})\mathfrak{A}$ is a formula.

(A-4) $\exists X^{(\mathfrak{m})}[\forall p\forall Y^{(\mathfrak{n})}((Y^{(\mathfrak{n})}\in X^{(\mathfrak{m})})\equiv\mathfrak{A})\wedge\forall p\forall Y^{(\mathfrak{o})}((Y^{(\mathfrak{o})}\in X^{(\mathfrak{m})})\equiv\mathfrak{A})]$,

where $\exists p\mathfrak{A}$ is a formula and $\mathfrak{n}$ and $\mathfrak{o}$ are quasi-orders of the forms $\langle I_0,\cdots,I_{i-1},p,I_{i+1},\cdots,I_k,\kappa\rangle$ and $\langle J_0,\cdots,J_{i-1},p,J_{i+1},\cdots,J_k,\nu\rangle$, respectively with $\langle 0,\cdots,0,(I_{i+1}),I_{i+2},\cdots,I_k,\kappa\rangle\leqq\mathfrak{m}$ and $\langle 0,\cdots 0,(J_{i+1}),J_{i+2},\cdots,J_k,\nu\rangle\leqq\mathfrak{m}$ (when $i=k$, $\langle 0,\cdots,0,\kappa+1\rangle\leqq\mathfrak{m}$ and $\langle 0,\cdots 0,\nu+1\rangle\leqq\mathfrak{m}$). ($I_0,\cdots,I_k,J_0,\cdots,J_k$ are indices.)

(A-5) $\exists X^{(\mathfrak{m})}[\forall p\forall Y^{(\mathfrak{n})}((Y^{(\mathfrak{n})}\in X^{(\mathfrak{m})})\equiv\mathfrak{A})\wedge\forall p\forall Y^{(\mathfrak{o})}((Y^{(\mathfrak{o})}\in X^{(\mathfrak{m})})\equiv\mathfrak{A})]$,

where $\exists p\mathfrak{A}$ is a formula and $\mathfrak{n}$ and $\mathfrak{o}$ are quasi-orders of the forms $\langle I_0,\cdots,I_{i-1},(p),I_{i+1},\cdots,I_k,\kappa\rangle$ and $\langle J_0,\cdots,J_{i-1},(p),J_{i+1},\cdots,J_k,\nu\rangle$, respectively with $\langle 0,\cdots,0,(I_{i+1}),I_{i+2},\cdots,I_k,\kappa\rangle\leqq\mathfrak{m}$ and $\langle 0,\cdots,0,(J_{i+1}),J_{i+2},\cdots,J_k,\nu\rangle\leqq\mathfrak{m}$ (when $i=k$, $\langle 0,\cdots,0,\kappa+1\rangle\leqq\mathfrak{m}$ and $\langle 0,\cdots,0,\nu+1\rangle\leqq\mathfrak{m}$). ($I_0,\cdots,I_k,J_0,\cdots,J_k$ are indices.)

(A-6) $\forall X^{\tau}\forall Y^{\sigma}\exists Z^{(\mathfrak{m})}[(X^{\tau}\in Z^{(\mathfrak{m})})\wedge\neg(Y^{\sigma}\in Z^{(\mathfrak{m})})]$,

where $\mathrm{Ord}^1(\tau)\leqq\mathfrak{m}$, $\mathrm{Ord}^1(\sigma)\leqq\mathfrak{m}$, $\mathrm{Ord}^2(\tau)<\mathfrak{m}$, $\mathrm{Ord}^2(\sigma)<\mathfrak{m}$ and one of the following holds: (a) $\tau=\iota$ and σ is of the form (ν) or of the form $(\mathfrak{n})$. (b) τ and σ are of the forms $(\mathfrak{n})$, (ν), respectively ($\mathfrak{n}$ is an order and ν is a type). (c) τ and σ are of the forms $(\mathfrak{n})$, $(\mathfrak{o})$, respectively with $\mathfrak{n}<\mathfrak{o}$. (d) τ is a free type variable $\alpha(\mathfrak{n})$ with $\mathfrak{n}\leqq\mathfrak{o}$ for some element $\mathfrak{o}$ of $\mathrm{Ord}^2(\sigma)$.

(A-7) (Axiom of choice for each type)

$$\begin{aligned}\exists X^{(((\tau)))}[\forall Y^{((\tau))}[(Y^{((\tau))}\in X^{(((\tau)))})\supset\\ \exists Z^{(\tau)}\exists W^{\tau}((W^{\tau}\in Z^{(\tau)})\wedge Y^{((\tau))}=\{Z^{(\tau)},\{W^{\tau}\}\})]\\ \wedge\forall Y^{(\tau)}[\exists Z^{\tau}(Z^{\tau}\in Y^{(\tau)})\supset\exists Z^{\tau}(\{Y^{(\tau)},\{Z^{\tau}\}\}\in X^{(((\tau)))})]\\ \wedge\forall Y^{(\tau)}\forall Z^{\tau}\forall W^{\tau}[((\{Y^{(\tau)},\{Z^{\tau}\}\}\in X^{(((\tau)))})\wedge\\ (\{Y^{(\tau)},\{W^{\tau}\}\}\in X^{(((\tau)))}))\supset Z^{\tau}=W^{\tau}]],\end{aligned}$$

where $A^{\sigma}=B^{\sigma}$ is an abbreviation for

$$\forall X^{(\sigma)}((A^{\sigma}\in X^{(\sigma)})\supset(B^{\sigma}\in X^{(\sigma)}))$$

and $\{A^{\sigma}\}$ and $\{A^{\sigma}, B^{\sigma}\}$ are abbreviations for

$$\lambda X^{\sigma}(X^{\sigma}=A^{\sigma}) \quad \text{and} \quad \lambda X^{\sigma}(X^{\sigma}=A^{\sigma}\vee X^{\sigma}=B^{\sigma})$$

respectively.

§ 2. Two syntactical properties

THEOREM 1. *Assume that* $\Gamma\to\Delta$ *is a sequent of* $H_{0,\omega,0}$ *and provable in* $H_{0,\mu,0}$ *for some* μ. *Then* $\Gamma\to\Delta$ *is provable in* $H_{\mathbf{0},\omega,0}$($\mathbf{0}$ *is the constant* 0 *function with the domain* $k+1$). *Similar results hold, reading* $H_{0,\omega,0}(\Sigma_{0,\omega,0})$, $H_{0,\mu,0}(\Sigma_{0,\mu,0})$ *or* $H_{0,\omega,0}(\Pi_{0,\omega,0})$, $H_{0,\mu,0}(\Pi_{0,\mu,0})$ *in place of* $H_{0,\omega,0}$, $H_{0,\mu,0}$, *respectively.*

PROOF. Let $\nu_1, \cdots, \nu_{i+j}$ be all the ordinals that occur in a proof of $\Gamma\to\Delta$ in $H_{0,\mu,0}$ with $\nu_1<\nu_2<\cdots<\nu_i(<\omega\leqq)\nu_{i+1}<\cdots<\nu_{i+j}$. We substitute $\nu_1, \cdots, \nu_i, \nu_i+1, \cdots, \nu_i+j$ for $\nu_1, \cdots, \nu_i, \nu_{i+1}, \cdots, \nu_{i+j}$, respectively throughout the proof. Then we get a proof of $\Gamma\to\Delta$ in $H_{0,\omega,0}$.

To state the next theorem we must make some preparations. The theorem holds in general for $H_{f,\mu,n}(\Sigma_{f,\mu,n})$ and for $H_{f,\mu,n}(\Pi_{f,\mu,n})$ with $\mu<\omega$ and $n\neq 0$. But, to simplify the notation, we assume that $k=0$, $f(0)=0$, $\mu=1$ and $n=1$. For the same purpose we use $\alpha(r,0)$, $\alpha(0,1)$ instead of $\alpha(\langle r,0\rangle)$, $\alpha(\langle 0,1\rangle)$, respectively.

We write $A^{\alpha(0,1)}=B^{\beta(0,1)}$, $\alpha(0,1)=\beta(0,1)$, $r=s$ to denote the following formulas respectively

$$\forall X^{(<0,1>)}[(A^{\alpha(0,1)}\in X^{(<0,1>)})\supset(B^{\beta(0,1)}\in X^{(<0,1>)})],$$

$$\exists X^{\alpha(0,1)}\exists Y^{\beta(0,1)}(X^{\alpha(0,1)}=Y^{\beta(0,1)}),$$

$$\exists X^{(<r,0>)}\exists Y^{(<s,0>)}(X^{(<r,0>)}=Y^{(<s,0>)}).$$

By the axioms (A-1), (A-3) and (A-5) we can prove

(3.1) $A^{\alpha(0,1)}=B^{\alpha(0,1)}$

$$\equiv\forall Y^{(\alpha(0,1))}[(A^{\alpha(0,1)}\in Y^{(\alpha(0,1))})\supset(B^{\alpha(0,1)}\in Y^{(\alpha(0,1))})],$$

(3.2) $(\alpha(0,1))=(\beta(0,1))\supset\alpha(0,1)=\beta(0,1)$,

(3.3) $(r)=(s)\supset r=s$.

By the axioms (A-6) we can prove

(3.4) $\iota\neq(\alpha(0,1))$,

(3.5) $\iota \neq (\langle r, 0\rangle)$,

(3.6) $(\alpha(0, 1)) \neq (\langle r, 0\rangle)$.

We can prove $\exists\xi((r), 0)((\langle 0, 0\rangle) = \xi((r), 0))$ and, by the axioms (A-6), $\neg\exists\xi(0, 0)((\langle 0, 0\rangle) = \xi(0, 0))$. Hence, by the axioms (A-4), we can prove

(3.7) $0 \neq (r)$.

The symbols $(r \in \mathfrak{J})$ and $(r \prec s)$ denote the following formulas

$$\forall X^{(\langle 0,1\rangle)}\{[\forall Y^{(\langle 0,0\rangle)}(Y^{(\langle 0,0\rangle)} \in X^{(\langle 0,1\rangle)}) \wedge$$
$$\forall p \forall Y^{(\langle p,0\rangle)} \forall Z^{(\langle (p),0\rangle)}((Y^{(\langle p,0\rangle)} \in X^{(\langle 0,1\rangle)}) \supset$$
$$(Z^{(\langle (p),0\rangle)} \in X^{(\langle 0,1\rangle)}))] \supset \forall Y^{(\langle r,0\rangle)}(Y^{(\langle r,0\rangle)} \in X^{(\langle 0,1\rangle)})\}$$

and

$$\forall X^{(\langle 0,1\rangle)}\{[\forall Y^{(\langle (r),0\rangle)}(Y^{(\langle (r),0\rangle)} \in X^{(\langle 0,1\rangle)}) \wedge$$
$$\forall p \forall Y^{(\langle p,0\rangle)} \forall Z^{(\langle (p),0\rangle)}((Y^{(\langle p,0\rangle)} \in X^{(\langle 0,1\rangle)}) \supset$$
$$(Z^{(\langle (p),0\rangle)} \in X^{(\langle 0,1\rangle)}))] \supset \forall Y^{(\langle s,0\rangle)}(Y^{(\langle s,0\rangle)} \in X^{(\langle 0,1\rangle)})\},$$

respectively. The formula $\forall p(p \in \mathfrak{J})$ is said to be the *axiom of induction on index*. By the axioms (A-4) we can prove

(3.8) $r \prec s \rightarrow \forall\xi(r, 0)\exists\eta(s, 0)(\xi(r, 0) = \eta(s, 0))$

and

(3.9) $r \prec s \rightarrow \exists\xi(s, 0)((\langle r, 0\rangle) = \xi(s, 0))$.

By the axioms (A-6) we can prove $\forall\xi(r, 0)\neg(\xi(r, 0) = (\langle r, 0\rangle))$ and so we can prove

(3.10) $r \prec s \rightarrow \forall\xi(r, 0)\neg(\xi(r, 0) = (\langle s, 0\rangle))$.

The symbols $(\alpha(0, 1) \in \mathfrak{T}(I, 0))$ (I is any index) and $(\alpha(0, 1) \in \mathfrak{T}(0, 1))$ denote the following formulas

$$\forall X^{(\langle 0,1\rangle)}\{[\forall Y^{\iota}(Y^{\iota} \in X^{(\langle 0,1\rangle)}) \wedge \forall p(p \prec I \supset$$
$$\forall Y^{(\langle p,0\rangle)}(Y^{(\langle p,0\rangle)} \in X^{(\langle 0,1\rangle)})) \wedge \forall\xi(I, 0)\forall Y^{\xi(I,0)}$$
$$\forall Z^{(\xi(I,0))}((Y^{\xi(I,0)} \in X^{(\langle 0,1\rangle)}) \supset (Z^{(\xi(I,0))} \in X^{(\langle 0,1\rangle)}))]$$
$$\supset \forall Y^{\alpha(0,1)}(Y^{\alpha(0,1)} \in X^{(\langle 0,1\rangle)})\}$$

and

$$\forall X^{(<0,1>)}\{[\forall Y^{\iota}(Y^{\iota}\in X^{(<0,1>)})\wedge\forall p((p\in\mathfrak{J})\supset$$

$$\forall Y^{(<p,0>)}(Y^{(<p,0>)}\in X^{(<0,1>)}))\wedge\forall\xi(0,1)\forall Y^{\xi(0,1)}$$

$$\forall Z^{(\xi(0,1))}((Y^{\xi(0,1)}\in X^{(<0,1>)})\supset(Z^{(\xi(0,1))}\in X^{(<0,1>)}))]$$

$$\supset\forall Y^{\alpha(0,1)}(Y^{\alpha(0,1)}\in X^{(<0,1>)})\}\,,$$

respectively. The formulas $\forall\xi(I,0)(\xi(I,0)\in\mathfrak{T}(I,0))$ and $\forall\xi(0,1)(\xi(0,1)\in\mathfrak{T}(0,1))$ are said to be the *axioms of induction on type.*

THEOREM 2. *Assume that* $\mu<\omega$, $n\neq 0$ *and* $\Gamma\to\Delta$ *is a sequent of* $H_{0,0,0}(\mathbf{0}$ *is the constant* 0 *function with the domain* $k+1$). *Assume that* Θ *is a finite sequence of the axioms of induction on index or on type. If* $\Theta,\ \Gamma\to\Delta$ *is provable in* $H_{f,\mu,n}(\Sigma_{f,\mu,n})$ *then* $\Gamma\to\Delta$ *is provable in the same system. Similar result holds, reading* $H_{f,\mu,n}(\Pi_{f,\mu,n})$ *in place of* $H_{f,\mu,n}(\Sigma_{f,\mu,n})$.

PROOF. First we define an operation E^* inductively as follows.

1. $(Y^{\tau})^*=Y^{\tau}$.

2. $(g(t_1,\cdots,t_m))^*=g(t_1{}^*,\cdots,t_m{}^*)$.

3-5. $(E\in F)^*=E^*\in F^*$, $(\neg\mathfrak{A})^*=\neg(\mathfrak{A}^*)$ and $(\mathfrak{A}\vee\mathfrak{B})^*=\mathfrak{A}^*\vee\mathfrak{B}^*$.

6. $(\exists Y^{\tau}\mathfrak{A})^*=\exists Y^{\tau}((Y^{\tau}\in\mathfrak{F})\wedge\mathfrak{A}^*)$.

7. $(\exists\xi(I,0)\mathfrak{A})^*=\exists\xi(I,0)((\xi(I,0)\in\mathfrak{T}(I,0))\wedge\mathfrak{A}^*)$,

   $(\exists\xi(0,1)\mathfrak{A})^*=\exists\xi(0,1)((\xi(0,1)\in\mathfrak{T}(0,1))\wedge\mathfrak{A}^*)$.

8. $(\exists q\mathfrak{A})^*=\exists q((q\in\mathfrak{J})\wedge\mathfrak{A}^*)$.

9. $(\lambda Y^{\tau}\mathfrak{A})^*=\lambda Y^{\tau}((Y^{\tau}\in\mathfrak{F})\wedge\mathfrak{A}^*)$.

10. $(\Lambda Y^{\xi(I,0)}\mathfrak{A})^*=\Lambda Y^{\xi(I,0)}((\xi(I,0)\in\mathfrak{T}(I,0))\wedge(Y^{\xi(I,0)}\in\mathfrak{F})\wedge\mathfrak{A}^*)$.

    $(\Lambda Y^{\xi(0,1)}\mathfrak{A})^*=\Lambda Y^{\xi(0,1)}((\xi(0,1)\in\mathfrak{T}(0,1))\wedge(Y^{\xi(0,1)}\in\mathfrak{F})\wedge\mathfrak{A}^*)$.

In the preceding definition $(A^{\tau}\in\mathfrak{F})$ (where $\mathrm{Ord}^1(\tau)\leqq\langle 0,1\rangle$ and $\mathrm{Ord}^2(\tau)<\langle 0,1\rangle$) is the abbreviation for

$$\forall X^{(<0,1>)}\{[\forall Y^{\iota}(Y^{\iota}\in X^{(<0,1>)})\wedge\forall\xi(0,1)\forall Y^{(\xi(0,1))}$$

$$[[(\xi(0,1)\in\mathfrak{T}(0,1))\wedge\forall Z^{\xi(0,1)}((Z^{\xi(0,1)}\in Y^{(\xi(0,1))})\supset$$

$$(Z^{\xi(0,1)}\in X^{(<0,1>)}))]\supset(Y^{(\xi(0,1))}\in X^{(<0,1>)})]\wedge\forall q$$

$$\forall Y^{(<q,0>)}[[(q\in\mathfrak{J})\wedge\forall\xi(q,0)\forall Z^{\xi(q,0)}[(Z^{\xi(q,0)}\in Y^{(<q,0>)})$$

$$\supset((\xi(q,0)\in\mathfrak{T}(q,0))\wedge(Z^{\xi(q,0)}\in X^{(<0,1>)}))]]\supset$$

$$(Y^{(<q,0>)} \in X^{(<0,1>)})]] \supset (A^{\tau} \in X^{(<0,1>)})\}$$

and $(A^{(<0,1>)} \in \mathfrak{F})$ is the abbreviation for

$$\forall \xi(0,1) \forall X^{\xi(0,1)}((X^{\xi(0,1)} \in A^{(<0,1>)}) \supset (X^{\xi(0,1)} \in \mathfrak{F})).$$

We can prove the following lemma by the induction on E.

LEMMA 1. (I) $\left(E\binom{p}{I}\right)^* = E^*\binom{p}{I}$.

(II) $\left(E\binom{\xi(\mathfrak{m})}{\tau}\right)^* = E^*\binom{\xi(\mathfrak{m})}{\tau}$.

(III) $\left(E\binom{Y^{\tau}}{F}\right)^* = E^*\binom{Y^{\tau}}{F^*}$.

The following lemma is easily proved by the induction on E and $\mathfrak{A}$.

LEMMA 2. *If E is a variety of $H_{0,0,0}$ then $E^* = E$ is provable. If $\mathfrak{A}$ is a formula of $H_{0,0,0}$ then $\mathfrak{A}^* \equiv \mathfrak{A}$ is provable.*

The desired theorem can be derived from Lemma 2 with the next two lemmas. To prove the lemmas we must make some preparations.

Let F denote the following variety of the type $(\langle 0,1 \rangle)$.

$$\Lambda X^{\xi(0,1)}\{\exists Y^{\iota}(X^{\xi(0,1)} = Y^{\iota}) \vee \exists \eta(0,1) \exists Y^{(\eta(0,1))}$$

$$[X^{\xi(0,1)} = Y^{(\eta(0,1))} \wedge (\eta(0,1) \in \mathfrak{T}(0,1)) \wedge \forall Z^{\eta(0,1)}$$

$$((Z^{\eta(0,1)} \in Y^{(\eta(0,1))}) \supset (Z^{\eta(0,1)} \in \mathfrak{F}))] \vee \exists q \exists Y^{(<q,0>)}$$

$$[X^{\xi(0,1)} = Y^{(<q,0>)} \wedge (q \in \mathfrak{J}) \wedge \forall \eta(q,0) \forall Z^{\eta(q,0)}$$

$$[(Z^{\eta(q,0)} \in Y^{(<q,0>)}) \supset ((\eta(q,0) \in \mathfrak{T}(q,0)) \wedge (Z^{\eta(q,0)} \in \mathfrak{F}))]]\}.$$

Then we can prove $\forall Y^{\iota}(Y^{\iota} \in F)$ and, by the definition of $\mathfrak{F}$,

$$\forall \xi(0,1) \forall X^{\xi(0,1)}[(X^{\xi(0,1)} \in F) \supset (X^{\xi(0,1)} \in \mathfrak{F})].$$

Therefore we can prove

$$[(\alpha(0,1) \in \mathfrak{T}(0,1)) \wedge \forall Y^{\alpha(0,1)}((Y^{\alpha(0,1)} \in A^{(\alpha(0,1))})$$

$$\supset (Y^{\alpha(0,1)} \in F)))] \supset (A^{(\alpha(0,1))} \in F)$$

and

$$[(r \in \mathfrak{J}) \wedge \forall \xi(r,0) \forall X^{\xi(r,0)}[(X^{\xi(r,0)} \in A^{(<r,0>)})$$

$$\supset ((\xi(r,0) \in \mathfrak{T}(r,0)) \wedge (X^{\xi(r,0)} \in F))]] \supset (A^{(<r,0>)} \in F).$$

Therefore, by the definition of $\mathfrak{F}$, we can prove

$$(A^{\alpha(0,1)} \in \mathfrak{F}) \supset (A^{\alpha(0,1)} \in F),$$

and hence, by the axioms (A-6), (A-3) and (A-4), we can prove

$$(3.11)\quad (A^{(\alpha(0,1))} \in \mathfrak{F}) \to (\alpha(0,1) \in \mathfrak{T}(0,1)) \wedge \forall X^{\alpha(0,1)}((X^{\alpha(0,1)} \in A^{(\alpha(0,1))}) \supset (X^{\alpha(0,1)} \in \mathfrak{F}))$$

and

$$(3.12)\quad (A^{(<r,0>)} \in \mathfrak{F}) \to (r \in \mathfrak{J}) \wedge \forall \xi(r,0) \forall X^{\xi(r,0)} [(X^{\xi(r,0)} \in A^{(<r,0>)}) \supset ((\xi(r,0) \in \mathfrak{T}(r,0)) \wedge (X^{\xi(r,0)} \in \mathfrak{F}))].$$

Let G denote the following variety of the type $(\langle 0,1\rangle)$.

$$\Lambda X^{\xi(0,1)}(\exists Y^{\xi(0,1)}(Y^{\xi(0,1)} \in \mathfrak{F}) \wedge (\xi(0,1) \in \mathfrak{T}(0,1))).$$

Then, by the definition of $\mathfrak{T}(0,1)$, we can prove $\forall X^{\iota}(X^{\iota} \in G)$,

$$\forall p((p \in \mathfrak{J}) \supset \forall X^{(<p,0>)}(X^{(<p,0>)} \in G))$$

and

$$\forall \xi(0,1) \forall X^{\xi(0,1)} \forall Y^{(\xi(0,1))}((X^{\xi(0,1)} \in G) \supset (Y^{(\xi(0,1))} \in G)).$$

Therefore, by the definition of $\mathfrak{T}(0,1)$, we can prove

$$(3.13)\quad (\alpha(0,1) \in \mathfrak{T}(0,1)) \to \exists X^{\alpha(0,1)}(X^{\alpha(0,1)} \in \mathfrak{F}).$$

By the axioms (A-4) we can assume

$$\forall p \forall X^{(<p,0>)}[(X^{(<p,0>)} \in A^{(<0,1>)}) \equiv (p=0 \vee \exists q(p=(q) \wedge (q \in \mathfrak{J})))].$$

Then we can prove $\forall Y^{(<0,0>)}(Y^{(<0,0>)} \in A^{(<0,1>)})$ and

$$\forall p \forall Y^{(<p,0>)} \forall Z^{(<(p),0>)}[(Y^{(<p,0>)} \in A^{(<0,1>)}) \supset (Z^{(<(p),0>)} \in A^{(<0,1>)})].$$

Therefore we can prove

$$((r) \in \mathfrak{J}) \to \forall Y^{(<(r),0>)}(Y^{(<(r),0>)} \in A^{(<0,1>)})$$

and hence, by the axioms (A-4) and (A-5), we can prove

$$(3.14)\quad ((r) \in \mathfrak{J}) \to (r \in \mathfrak{J}).$$

By the axioms (A-4) we can assume

$$\forall p \forall X^{(<p,0>)}[(X^{(<p,0>)} \in B^{(<0,1>)}) \equiv ((r \prec p) \wedge \neg(p \in \mathfrak{J}))].$$

Then we can prove

$$\neg(r \in \mathfrak{J}) \to \forall Y^{(<(r),0>)}(Y^{(<(r),0>)} \in B^{(<0,1>)})$$

and

$$\forall p \forall Y^{(<p,0>)} \forall Z^{(<(p),0>)}[(Y^{(<p,0>)} \in B^{(<0,1>)}) \supset (Z^{(<(p),0>)} \in B^{(<0,1>)})] .$$

Therefore we can prove

$$(\neg(r \in \mathfrak{J}) \wedge r \prec s) \rightarrow \forall Y^{(<s,0>)}(Y^{(<s,0>)} \in B^{(<0,1>)})$$

and hence we can prove

(3.15) $\neg(r \in \mathfrak{J}) \wedge r \prec s \rightarrow \neg(s \in \mathfrak{J})$.

Therefore we can prove

(3.16) $(r \prec s) \wedge (s \in \mathfrak{J}) \rightarrow (r \in \mathfrak{J})$.

Similarly we can prove

(3.17) $(r \prec s) \wedge (r \in \mathfrak{J}) \rightarrow (r \prec s)^*$.

Using (3.11), (3.12), (3.13) and Lemma 1, we can prove the following lemma by the induction on the length of a proof.

LEMMA 3. *If a sequent $\Xi \rightarrow \Omega$ is provable in $H_{f,\mu,n}(\Sigma_{f,\mu,n})$ then*

$$(r \in \mathfrak{J}), \cdots, (\alpha(\mathfrak{m}) \in \mathfrak{T}(\mathfrak{m})), \cdots, (A^{\tau} \in \mathfrak{F}), \cdots, \Xi^* \rightarrow \Omega^*$$

is provable in the same system, where $r, \cdots$ are the elements of $\mathrm{V}^{\mathrm{i}}(\Xi) \cup \mathrm{V}^{\mathrm{i}}(\Omega)$, $\alpha(\mathfrak{m}), \cdots$ are the elements of $\mathrm{V}^{\mathrm{t}}(\Xi) \cup \mathrm{V}^{\mathrm{t}}(\Omega)$, $A^{\tau}, \cdots$ are the elements of $\mathrm{V}^{\mathrm{p}}(\Xi) \cup \mathrm{V}^{\mathrm{p}}(\Omega)$. Similar result holds, reading $H_{f,\mu,n}(\Pi_{f,\mu,n})$ in place of $H_{f,\mu,n}(\Sigma_{f,\mu,n})$.

Using (3.16) and (3.17) we can prove the following lemma.

LEMMA 4. *The following formulas are provable in $H_{f,\mu,n}(\Sigma_{f,\mu,n})$.*

(I) $(\forall p(p \in \mathfrak{J}))^*$.

(II) $(\forall \xi(0,1)(\xi(0,1) \in \mathfrak{T}(0,1)))^*$.

(III) $(I \in \mathfrak{J}) \supset (\forall \xi(I,0)(\xi(I,0) \in \mathfrak{T}(I,0)))^*$.

§ 3. General model

In this section we shall define a notion "general model for $H_{f,\mu,n}$".

We define a function $\mathfrak{G}(X, Y, m)$ by the induction on m as follows. 1. $\mathfrak{G}(X, Y, 0) = X$. 2. $\mathfrak{G}(X, Y, m+1) = \mathfrak{G}(X, Y, m) \cup \{(\tau); \tau$ is an element of Y or of $\mathfrak{G}(X, Y, m)\}$. We define $\mathfrak{G}(X, Y) = \bigcup_{m=0}^{\infty} \mathfrak{G}(X, Y, m)$. In this section we write $A \rightarrow B$ to denote the set of all functions whose domains are A and whose ranges are subsets of B. We write $\varphi \sim \psi(S)$ to denote the fact that φ and ψ

are functions with the same domain and $\varphi(a)=\phi(a)$ for every a which is not an element of S. Frequently we write $\varphi\sim\phi(a)$ instead of $\varphi\sim\phi(\{a\})$.

$\langle o, \mathfrak{J}_0, \iota, \mathfrak{T}_0, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}\rangle$ is a *general model for* $H_{f,\mu,n}$ if the following conditions (M-1)-(M-5) hold.

(M-1) o is an element of $\mathfrak{J}_0$.

Define $\mathfrak{J}=\mathfrak{J}_0\times\omega$. Relations $<$ and $\leqq$ on $\mathfrak{J}$ are defined as follows. 1. $(i, l)<(i, m)$ if and only if $l<m$. 2. $(o, l)<(i, m)$ if and only if $l<m$. 3. $(i, l)\leqq(i, m)$ if and only if $l\leqq m$. 4. $(o, l)\leqq(i, m)$ if and only if $l\leqq m$. 5. If $i\neq o$ and $i\neq j$ then neither $(i, l)<(j, m)$ nor $(i, l)\leqq(j, m)$ holds. We use symbols $\mathfrak{i}, \mathfrak{j}, \cdots$ to denote elements of $\mathfrak{J}$.

We define Od as the following set.

$$\{\langle\mathfrak{i}_0, \cdots, \mathfrak{i}_k, \nu\rangle ;\ \mathfrak{i}_0, \cdots, \mathfrak{j}_k \text{ are elements of } \mathfrak{J} \text{ and } \nu\leqq\mu\}\ .$$

Relations $<$ and $\leqq$ on Od are defined as follows.
$\langle\mathfrak{i}_0, \cdots, \mathfrak{i}_k, \kappa\rangle<\langle\mathfrak{j}_0, \cdots, \mathfrak{j}_k, \nu\rangle$ if and only if (i) $\kappa<\nu$ or (ii) for some i $(0\leqq i\leqq k)$, $\kappa=\nu$, $\mathfrak{i}_k\leqq\mathfrak{j}_k, \cdots, \mathfrak{i}_{i+1}\leqq\mathfrak{j}_{i+1}$ and $\mathfrak{i}_i<\mathfrak{j}_i$.
$\langle\mathfrak{i}_0, \cdots, \mathfrak{i}_k, \kappa\rangle\leqq\langle\mathfrak{j}_0, \cdots, \mathfrak{j}_k, \nu\rangle$ if and only if
(i) $\langle\mathfrak{i}_0, \cdots, \mathfrak{i}_k, \kappa\rangle<\langle\mathfrak{j}_0, \cdots, \mathfrak{j}_k, \nu\rangle$ or (ii) $\kappa=\nu, \mathfrak{i}_0\leqq\mathfrak{j}_0, \cdots, \mathfrak{i}_k\leqq\mathfrak{j}_k$. We use symbols $\mathfrak{p}$, $\mathfrak{q}, \cdots$ to denote elements of Od. Note that $<$ and $\leqq$ are well-founded relations.

(M-2) $\mathfrak{T}_0$ is a function with the following properties. 1. The domain is Od. 2. ι is an element of $\mathfrak{T}_0(\mathfrak{p})$ for any element $\mathfrak{p}$ of Od. 3. If $\mathfrak{p}\leqq\mathfrak{q}$ then $\mathfrak{T}_0(\mathfrak{p})\subseteq\mathfrak{T}_0(\mathfrak{q})$. 4. If $\mathfrak{p}\leqq\mathfrak{q}$ but neither $\mathfrak{p}=\mathfrak{q}$ nor $\mathfrak{p}<\mathfrak{q}$ then $\mathfrak{T}_0(\mathfrak{p})\neq\mathfrak{T}_0(\mathfrak{q})$. 5. If neither $\mathfrak{p}\leqq\mathfrak{q}$ nor $\mathfrak{q}\leqq\mathfrak{p}$ then $\mathfrak{T}_0(\mathfrak{p})\neq\mathfrak{T}_0(\mathfrak{q})$.

We define a function $\mathfrak{T}$ as follows. 1. The domain is Od. 2. $\mathfrak{T}(\mathfrak{p})=\mathfrak{G}(\mathfrak{T}_0(\mathfrak{p}), \{\mathfrak{q} ; \mathfrak{q}<\mathfrak{p}\})$. We define T_0 as the union of $\mathfrak{T}_0(\mathfrak{p})$ for all $\mathfrak{p}$ in Od and define $T=\mathfrak{G}(T_0, \text{Od})$. We use symbols $\mathfrak{s}, \mathfrak{t}, \cdots$ to denote elements of T.

(M-3) $\mathfrak{D}$ is a function with the following properties 1-6. 1. The domain is T. 2. $\mathfrak{D}(\mathfrak{s})\neq\emptyset$ for any element $\mathfrak{s}$ of T. 3. $\mathfrak{D}(\mathfrak{s})$ and $\mathfrak{D}(\mathfrak{t})$ are disjoint for any elements $\mathfrak{s}$ and $\mathfrak{t}$ of T_0 with $\mathfrak{s}\neq\mathfrak{t}$. 4. For any element $\mathfrak{s}$ of T_0 each element of $\mathfrak{D}(\mathfrak{s})$ is not a function. 5. For any element $\mathfrak{s}$ of T $\mathfrak{D}((\mathfrak{s}))\subseteq\mathfrak{D}(\mathfrak{s})\rightarrow\{\boldsymbol{t}, \boldsymbol{f}\}$. 6. For any element $\mathfrak{p}$ of Od $\mathfrak{D}((\mathfrak{p}))\subseteq\mathfrak{D}(\mathfrak{p})\rightarrow\{\boldsymbol{t}, \boldsymbol{f}\}$, where $\mathfrak{D}(\mathfrak{p})$ is the union of $\mathfrak{D}(\mathfrak{t})$ for all $\mathfrak{t}$ in $\mathfrak{T}(\mathfrak{p})$.

Note that $\mathfrak{D}(\mathfrak{s})$ and $\mathfrak{D}(\mathfrak{t})$ are disjoint for any elements $\mathfrak{s}$ and $\mathfrak{t}$ of T with $\mathfrak{s}\neq\mathfrak{t}$.

(M-4) $\mathfrak{F}$ is a function whose domain is the set of all pairs $\langle g, m\rangle$, where g is a function symbol and m is a natural number. And for any function symbol g and natural number m $\mathfrak{F}(g, m)$ is an element of $(\mathfrak{D}(\iota))^m\rightarrow\mathfrak{D}(\iota)$.

A function φ is said to be an assignment for index variables if the domain is the set of all free and bound index variables and the range is a subset of $\mathfrak{J}$. A function χ is said to be an assignment for type variables if it has the following properties. 1. The domain is the set of all symbols for free and bound type variables. 2. For any symbol for free or bound type variable α $\chi(\alpha)$ is a function whose domain is Od. 3. For any element $\mathfrak{p}$ of Od $\chi(\alpha)(\mathfrak{p})$ is an element of $\mathfrak{T}(\mathfrak{p})$. A function ψ is said to be an assignment for variables if it has the following properties. 1. The domain is the set of all symbols for free and bound variables. 2. For any symbol for free or bound variable A $\psi(A)$ is a function whose domain is T. 3. For any element $\mathfrak{s}$ of T $\psi(A)(\mathfrak{s})$ is an element of $\mathfrak{D}(\mathfrak{s})$.

(M-5) $\mathfrak{H}$ is a function whose domain is the set of all quadruplets $\langle E, \varphi_1, \varphi_2, \varphi_3\rangle$, where E is a quasi-index, quasi-order, quasi-type, quasi-variety or quasi-formula, φ_1 is an assignment for index variables, φ_2 is an assignment for type variables and φ_3 is an assignment for variables. And $\mathfrak{H}$ has the following properties 1.1–4.10. 1.1 $\mathfrak{H}(o, \varphi_1, \varphi_2, \varphi_3)=(o, 0)$. 1.2 $\mathfrak{H}(r, \varphi_1, \varphi_2, \varphi_3)=\varphi_1(r)$ for any free or bound index variable r. 1.3 If $\mathfrak{H}(I, \varphi_1, \varphi_2, \varphi_3)=(i, m)$ then $\mathfrak{H}((I), \varphi_1, \varphi_2, \varphi_3)=(i, m+1)$. 2.0 $\mathfrak{H}(\langle I_0, \cdots, I_k, \nu\rangle, \varphi_1, \varphi_2, \varphi_3)=\langle\mathfrak{H}(I_0, \varphi_1, \varphi_2, \varphi_3), \cdots, \mathfrak{H}(I_k, \varphi_1, \varphi_2, \varphi_3), \nu\rangle$. 3.1 $\mathfrak{H}(\iota, \varphi_1, \varphi_2, \varphi_3)=\iota$. 3.2 $\mathfrak{H}(\alpha(\mathfrak{m}), \varphi_1, \varphi_2, \varphi_3)=\varphi_2(\alpha)(\mathfrak{H}(\mathfrak{m}, \varphi_1, \varphi_2, \varphi_3))$ for any free or bound type variable $\alpha(\mathfrak{m})$. 3.3 $\mathfrak{H}((\tau), \varphi_1, \varphi_2, \varphi_3)=(\mathfrak{H}(\tau, \varphi_1, \varphi_2, \varphi_3))$. 4.0 If E is a quasi-variety of a quasi-type τ then $\mathfrak{H}(E, \varphi_1, \varphi_2, \varphi_3)$ is an element of $\mathfrak{D}(\mathfrak{H}(\tau, \varphi_1, \varphi_2, \varphi_3))$. 4.1 $\mathfrak{H}(A^\tau, \varphi_1, \varphi_2, \varphi_3)=\varphi_3(A)(\mathfrak{H}(\tau, \varphi_1, \varphi_2, \varphi_3))$ for any free or bound variable A^τ of a quasi-type τ. 4.2 $\mathfrak{H}(g(t_1, \cdots, t_m), \varphi_1, \varphi_2, \varphi_3)=\mathfrak{F}(g, m)(\mathfrak{H}(t_1, \varphi_1, \varphi_2, \varphi_3), \cdots, \mathfrak{H}(t_m, \varphi_1, \varphi_2, \varphi_3))$. 4.3 $\mathfrak{H}((E\in F), \varphi_1, \varphi_2, \varphi_3)=\mathfrak{H}(F, \varphi_1, \varphi_2, \varphi_3)(\mathfrak{H}(E, \varphi_1, \varphi_2, \varphi_3))$.

$$4.4\ \mathfrak{H}(\neg\mathfrak{A}, \varphi_1, \varphi_2, \varphi_3)=\begin{cases} \boldsymbol{t} & \text{if } \mathfrak{H}(\mathfrak{A}, \varphi_1, \varphi_2, \varphi_3)=\boldsymbol{f}, \\ \boldsymbol{f} & \text{otherwise}. \end{cases}$$

$$4.5\ \mathfrak{H}(\mathfrak{A}\vee\mathfrak{B}, \varphi_1, \varphi_2, \varphi_3)=\begin{cases} \boldsymbol{t} & \text{if } \mathfrak{H}(\mathfrak{A}, \varphi_1, \varphi_2, \varphi_3)=\boldsymbol{t} \text{ or } \mathfrak{H}(\mathfrak{B}, \varphi_1, \varphi_2, \varphi_3)=\boldsymbol{t}, \\ \boldsymbol{f} & \text{otherwise}. \end{cases}$$

$$4.6\ \mathfrak{H}(\exists X^\tau\mathfrak{A}, \varphi_1, \varphi_2, \varphi_3)=\begin{cases} \boldsymbol{t} & \text{if } \mathfrak{H}(\mathfrak{A}, \varphi_1, \varphi_2, \psi)=\boldsymbol{t} \text{ for some assignment} \\ & \text{for variables } \psi \text{ with } \varphi_3\sim\psi(X), \\ \boldsymbol{f} & \text{otherwise}. \end{cases}$$

$$4.7\ \mathfrak{H}(\exists\xi(\mathfrak{m})\mathfrak{A}, \varphi_1, \varphi_2, \varphi_3)=\begin{cases} \boldsymbol{t} & \text{if } \mathfrak{H}(\mathfrak{A}, \varphi_1, \chi, \varphi_3)=\boldsymbol{t} \text{ for some assignment} \\ & \text{for type variables } \chi \text{ with } \varphi_2\sim\chi(\xi), \\ \boldsymbol{f} & \text{otherwise}. \end{cases}$$

$$4.8\ \mathfrak{H}(\exists p\mathfrak{A}, \varphi_1, \varphi_2, \varphi_3)=\begin{cases} \boldsymbol{t} & \text{if } \mathfrak{H}(\mathfrak{A}, \varphi, \varphi_2, \varphi_3)=\boldsymbol{t} \text{ for some assignment} \\ & \text{for index variables } \varphi \text{ with } \varphi_1\sim\varphi(p), \\ \boldsymbol{f} & \text{otherwise}. \end{cases}$$

$$4.9\ \mathfrak{H}(\lambda X^{\tau}\mathfrak{A}, \varphi_1, \varphi_2, \varphi_3)(\mathfrak{d}) = \begin{cases} \boldsymbol{t} & \text{if } \mathfrak{H}(\mathfrak{A}, \varphi_1, \varphi_2, \psi) = \boldsymbol{t} \text{ for some assignment} \\ & \text{for variables } \psi \text{ with } \varphi_3 \sim \psi(X) \text{ and} \\ & \psi(X)(\mathfrak{H}(\tau, \varphi_1, \varphi_2, \varphi_3)) = \mathfrak{d}, \\ \boldsymbol{f} & \text{otherwise.} \end{cases}$$

$4.10\ \mathfrak{H}(\Lambda X^{\xi(\mathfrak{m})}\mathfrak{A}, \varphi_1, \varphi_2, \varphi_3)(\mathfrak{d})$

$$= \begin{cases} \boldsymbol{t} & \text{if } \mathfrak{H}(\mathfrak{A}, \varphi_1, \chi, \psi) = \boldsymbol{t} \text{ for some assignments } \chi \text{ and } \psi \\ & \text{with } \varphi_2 \sim \chi(\xi), \chi(\xi)(\mathfrak{H}(\mathfrak{m}, \varphi_1, \varphi_2, \varphi_3)) = \mathfrak{s}, \varphi_3 \sim \psi(X) \text{ and} \\ & \psi(X)(\mathfrak{s}) = \mathfrak{d} (\mathfrak{s} \text{ is the type of } \mathfrak{d}, \text{ that is, } \mathfrak{s} \text{ is the unique} \\ & \text{element of } \mathfrak{T}(\mathfrak{H}(\mathfrak{m}, \varphi_1, \varphi_2, \varphi_3)) \text{ such that } \mathfrak{D}(\mathfrak{s}) \text{ contains } \mathfrak{d}), \\ \boldsymbol{f} & \text{otherwise.} \end{cases}$$

We can prove the following lemma by the induction on τ and E.

LEMMA 5. (I) *If* $\mathfrak{m}$ *is a quasi-order and* τ *is a quasi-type with* $\mathrm{Ord}^1(\tau) \leqq \mathfrak{m}$ *and* $\mathrm{Ord}^2(\tau) < \mathfrak{m}$ *then* $\mathfrak{H}(\tau, \varphi_1, \varphi_2, \varphi_3)$ *is an element of* $\mathfrak{T}(\mathfrak{H}(\mathfrak{m}, \varphi_1, \varphi_2, \varphi_3))$. (II) *If* $\varphi_1(p) = \psi_1(p)$ *for any element* p *of* $\mathrm{V}^{\mathrm{i}}(E)$, $\varphi_2(\alpha)(\mathfrak{H}(\mathfrak{m}, \varphi_1, \varphi_2, \varphi_3)) = \psi_2(\alpha)(\mathfrak{H}(\mathfrak{m}, \psi_1, \psi_2, \psi_3))$ *for any element* $\alpha(\mathfrak{m})$ *of* $\mathrm{V}^{\mathrm{t}}(E)$ *and* $\varphi_3(A)(\mathfrak{H}(\tau, \varphi_1, \varphi_2, \varphi_3)) = \psi_3(A)(\mathfrak{H}(\tau, \psi_1, \psi_2, \psi_3))$ *for any element* A^{τ} *of* $\mathrm{V}^{\mathrm{p}}(E)$ *then* $\mathfrak{H}(E, \varphi_1, \varphi_2, \varphi_3) = \mathfrak{H}(E, \psi_1, \psi_2, \psi_3)$. (III) *If* $\varphi_1 \sim \psi_1(p)$ *and* $\psi_1(p) = \mathfrak{H}(I, \varphi_1, \varphi_2, \varphi_3)$ *then* $\mathfrak{H}\left(E\binom{p}{I}, \varphi_1, \varphi_2, \varphi_3\right) = \mathfrak{H}(E, \psi_1, \varphi_2, \varphi_3)$. *If* $\varphi_2 \sim \psi_2(\xi)$, $\psi_2(\xi)(\mathfrak{H}(\mathfrak{m}, \varphi_1, \psi_2, \varphi_3)) = \mathfrak{H}(\tau, \varphi_1, \varphi_2, \varphi_3)$ *and* $\mathrm{V}^{\mathrm{t}}(E)$ *contains no bound type variable of the form* $\xi(\mathfrak{n})$ *with* $\mathfrak{m} \neq \mathfrak{n}$ *then* $\mathfrak{H}\left(E\binom{\xi(\mathfrak{m})}{\tau}, \varphi_1, \varphi_2, \varphi_3\right) = \mathfrak{H}(E, \varphi_1, \psi_2, \varphi_3)$. *If* $\varphi_3 \sim \psi_3(X)$, $\psi_3(X)(\mathfrak{H}(\tau, \varphi_1, \varphi_2, \psi_3)) = \mathfrak{H}(F, \varphi_1, \varphi_2, \varphi_3)$ *and* $\mathrm{V}^{\mathrm{p}}(E)$ *contains no bound variable of the form* X^{σ} *with* $\sigma \neq \tau$ *then*

$$\mathfrak{H}\left(E\binom{X^{\tau}}{F}, \varphi_1, \varphi_2, \varphi_3\right) = \mathfrak{H}(E, \varphi_1, \varphi_2, \psi_3).$$

Using Lemma 5 we can prove the following theorem as usual by the induction on the length of a proof.

THEOREM 3. *If a sequent* $\Gamma \rightarrow \Delta$ *is provable in* $H_{f,\mu,n}$ *then for any assignments* $\varphi_1, \varphi_2, \varphi_3$ *either* $\mathfrak{H}(\mathfrak{A}, \varphi_1, \varphi_2, \varphi_3) = \boldsymbol{t}$ *for some* $\mathfrak{A}$ *in* Δ *or* $\mathfrak{H}(\mathfrak{B}, \varphi_1, \varphi_2, \varphi_3) = \boldsymbol{f}$ *for some* $\mathfrak{B}$ *in* Γ.

Hence the systems $H_{f,\mu,n}$, $H_{f,\mu,n}(\Sigma_{f,\mu,n})$ *and* $H_{f,\mu,n}(\Pi_{f,\mu,n})$ *are consistent because we can define principal models, i.e., such general models that* $\mathfrak{Z}_0 = \{o\}$, $\mathfrak{T}_0(\mathfrak{p}) = \{\iota\}$ *for any element* $\mathfrak{p}$ *of* Od *and* $\mathfrak{D}((\tau)) = \mathfrak{D}(\tau) \rightarrow \{\boldsymbol{t}, \boldsymbol{f}\}$.

The proof of the following theorem is routine and so omitted (see Takahashi [6] and Uesu [8]).

THEOREM 4. *If* $\mu < \omega_1$ *the completeness of the general models for* $H_{f,\mu,n}$ *and the cut-elimination theorem for* $H_{f,\mu,n}$ *hold.*

§ 4. Provabilities of second order formulas

THEOREM 5. *If $\mu<\omega$ there is a second order formula which is not provable in $H_{f,\mu,n}\Sigma(_{f,\mu,n})$ but provable in $H_{f,\mu,n+1}(\Sigma_{f,\mu,n})$. The similar result holds, reading $H_{f,\mu,n}(\Pi_{f,\mu,n})$, $H_{f,\mu,n+1}(\Pi_{f,\mu,n})$ in place of $H_{f,\mu,n}(\Sigma_{f,\mu,n})$, $H_{f,\mu,n+1}(\Sigma_{f,\mu,n})$, respectively.*

PROOF. We use symbols $a, b, \cdots, x, y, \cdots$ instead of $A^{\iota}, B^{\iota}, \cdots, X^{\iota}, Y^{\iota}, \cdots$, respectively. Let $\mathfrak{N}_0$ denote the conjunction of the following formulas (N-1)-(N-7).

(N-1) $\forall x(0\neq x')$ (N-2) $\forall x\forall y(x'=y'\supset x=y)$

(N-3) $\forall x(x+0=x)$ (N-4) $\forall x\forall y(x+y'=(x+y)')$

(N-5) $\forall x(x\cdot 0=0)$ (N-6) $\forall x\forall y(x\cdot y'=x\cdot y+x)$

(N-7) $\forall X^{(\iota)}\{[(0\in X^{(\iota)})\wedge\forall x((x\in X^{(\iota)})\supset(x'\in X^{(\iota)}))]\supset\forall x(x\in X^{(\iota)})\}$,

where 0 is a particular free variable of the type ι and $'$, $+$, $\cdot$ are particular function symbols.

Let $\text{Consis}_{f,\mu,n}$ denote the second order formula which states, via the Gödel numbering, the consistency of $H_{f,\mu,n}(\Sigma_{f,\mu,n}\cup\{\mathfrak{N}_0\})$. By Gödel's second incompleteness theorem $\mathfrak{N}_0\supset\text{Consis}_{f,\mu,n}$ is not provable in $H_{f,\mu,n}(\Sigma_{f,\mu,n})$. But it is provable in $H_{f,\mu,n+1}(\Sigma_{f,\mu,n})$. In the remainder of this section we show how to formalize a main part of the semantics for $H_{f,\mu,n}(\Sigma_{f,\mu,n}\cup\{\mathfrak{N}_0\})$ in $H_{f,\mu,n+1}(\Sigma_{f,\mu,n}\cup\{\mathfrak{N}_0\})$. To save space we assume that $k=0$, $\mu=1$ and $n=0$.

By Theorem 2 we can use the axioms of induction on index and on type. We use an informal language together with the formal one to simplify the notation. We use symbols $A, B, \cdots, X, Y, \cdots$ instead of $A^{\tau}, B^{\tau}, \cdots, X^{\tau}, Y^{\tau}, \cdots$, respectively if the type τ is uniquely determined by the context.

We can define $\text{Typ}(a)$ by the induction on a as follows ($\text{Typ}(a)$ means that a is the Gödel number of a type). 1. Typ(2). 2. If $\text{Typ}(a)$ then $\text{Typ}(2^2\cdot 3^a)$. 3. If $a=2^{b+1}$ for some b then $\text{Typ}(2^3\cdot 3^a)$. We can define $g(a)$ and $h(a)$ by the induction on a as follows. 1. $g(2)=0$. 2. $g(2^2\cdot 3^a)=g(a)$. 3. $g(2^3\cdot 3^a)=(a)_0$. 4. $h(2)=0$. 5. $h(2^2\cdot 3^a)=h(a)+1$. 6. $h(2^3\cdot 3^a)=1$.

For each a we can define the least set $\mathfrak{E}(a)$ of the type $(\langle o, 1\rangle)$ with the following three properties. 1. a is an element of $\mathfrak{E}(a)$. 2. If $A^{\alpha(0,1)}$ is an element of $\mathfrak{E}(a)$ so is the set $\{A^{\alpha(0,1)}\}$ of the type $(\alpha(o, 1))$. 3. For any index r the set $\{a\}$ of the type $(\langle r, 0\rangle)$ is an element of $\mathfrak{E}(a)$. We can prove

(5.1) $a\neq b\rightarrow\mathfrak{E}(a)\cap\mathfrak{E}(b)=\emptyset$,

(5.2) $(A^{\alpha(0,1)}\in\mathfrak{E}(a))\wedge(B^{\alpha(0,1)}\in\mathfrak{E}(a))\rightarrow A^{\alpha(0,1)}=B^{\alpha(0,1)}$,

(5.3) $\forall\xi(0, 1)\exists X^{\xi(0,1)}(X^{\xi(0,1)}\in\mathfrak{E}(a))$.

We write $a^{\alpha(0,1)}$ to denote a unique element of $\mathfrak{E}(a)$ of a type $\alpha(0,1)$.

We can define the least set $\mathfrak{K}$ of the type $(\langle o,1\rangle)$ with the following two properties. 1. The set $\{0\}$ of the type $(\langle o,0\rangle)$ is an element of $\mathfrak{K}$. 2. If a set $\{a\}$ of a type $(\langle r,0\rangle)$ is an element of $\mathfrak{K}$ so is the set $\{a+1\}$ of the type $(\langle (r),0\rangle)$. We write $r=o^{(a)}$ to denote the fact that the set $\{a\}$ of a type $(\langle r,0\rangle)$ is an element of $\mathfrak{K}$. We can prove

$$(5.4)\quad \forall x\exists p(p=o^{(x)}),$$

$$(5.5)\quad r=o^{(a)}\wedge s=o^{(a)}\rightarrow r=s,$$

$$(5.6)\quad r=o^{(a)}\wedge s=o^{(a+b+1)}\rightarrow\exists\xi(s,0)(\xi(s,0)=(\langle r,0\rangle)).$$

We can define the least set $\mathfrak{D}$ of the type $(\langle o,1\rangle)$ with the following three properties. 1. For any a the set $\{2,5^{a+1}\}$ of the type (ι) is an element of $\mathfrak{D}$. 2. Suppose that Typ(a) and, for some $A^{\alpha(0,1)}$, the set $\{a^{\alpha(0,1)},A^{\alpha(0,1)}\}$ of the type $(\alpha(o,1))$ is an element of $\mathfrak{D}$. If, for any element $C^{\alpha(0,1)}$ of $B^{(\alpha(0,1))}$, the set $\{a^{\alpha(0,1)},C^{\alpha(0,1)}\}$ of the type $(\alpha(o,1))$ is an element of $\mathfrak{D}$, then so is the set $\{b^{(\alpha(0,1))},B^{(\alpha(0,1))}\}$ of the type $((\alpha(o,1)))$, where $b=2^2\cdot 3^a$. 3. If $r=o^{(a)}$, $b=2^{a+1}$ and, for any type $\alpha(r,0)$ and any element $B^{\alpha(r,0)}$ of $A^{(<r,0>)}$ of the type $\alpha(r,0)$, there exists a number c such that (i) $g(c)<a+1$, (ii) Typ(c) and (iii) the set $\{c^{\alpha(r,0)},B^{\alpha(r,0)}\}$ of the type $(\alpha(r,0))$ is an element of $\mathfrak{D}$, then the set $\{d^{(<r,0>)},A^{(<r,0>)}\}$ of the type $((\langle r,0\rangle))$ is an element of $\mathfrak{D}$, where $d=2^3\cdot 3^b$. We write $(A^{\alpha(0,1)}\in\mathfrak{D}(a))$ to mean that Typ(a) and the set $\{a^{\alpha(0,1)},A^{\alpha(0,1)}\}$ of the type $(\alpha(o,1))$ is an element of $\mathfrak{D}$. We write $\alpha(o,1)=[a]$ to mean that $(A^{\alpha(0,1)}\in\mathfrak{D}(a))$ for some set $A^{\alpha(0,1)}$ of the type $\alpha(o,1)$. We can prove

$$(5.7)\quad \alpha(o,1)=[a]\wedge\beta(o,1)=[a]\rightarrow\alpha(o,1)=\beta(o,1),$$

$$(5.8)\quad \mathrm{Typ}(a)\rightarrow\exists\xi(o,1)(\xi(o,1)=[a]),$$

$$(5.9)\quad \alpha(o,1)=[a]\wedge\alpha(o,1)=[b]\rightarrow a=b.$$

For each $A^{\alpha(0,1)}$ we can define the least set $\mathfrak{E}(A^{\alpha(0,1)})$ of the type $(\langle o,1\rangle)$ with the following three properties. 1. $A^{\alpha(0,1)}$ is an element of $\mathfrak{E}(A^{\alpha(0,1)})$. 2. If $B^{\beta(0,1)}$ is an element of $\mathfrak{E}(A^{\alpha(0,1)})$ then so is the set $\{B^{\beta(0,1)}\}$ of the type $(\beta(o,1))$. 3. If for some type $\gamma(r,0)$ $\gamma(r,0)=\alpha(o,1)$ then the set $\{A^{\alpha(0,1)}\}$ of the type $(\langle r,0\rangle)$ is an element of $\mathfrak{E}(A^{\alpha(0,1)})$. We write $A^{\alpha(0,1)}\approx B^{\beta(0,1)}$ to mean that $B^{\beta(0,1)}$ is an element of $\mathfrak{E}(A^{\alpha(0,1)})$. We can prove

$$(5.10)\quad A^{\alpha(0,1)}\neq B^{\alpha(0,1)}\wedge A^{\alpha(0,1)}\approx C^{\beta(0,1)}\wedge B^{\alpha(0,1)}\approx D^{\gamma(0,1)}\rightarrow C^{\beta(0,1)}\neq D^{\gamma(0,1)},$$

$$(5.11)\quad A^{\alpha(0,1)}\approx B^{\beta(0,1)}\wedge A^{\alpha(0,1)}\approx C^{\beta(0,1)}\rightarrow B^{\beta(0,1)}=C^{\beta(0,1)},$$

$$(5.12)\quad (a)_2=0\wedge(A^{\alpha(0,1)}\in\mathfrak{D}(b))\rightarrow\mathfrak{E}(a)\cap\mathfrak{E}(A^{\alpha(0,1)})=\emptyset,$$

(5.13) $\alpha(o, 1)=[a] \wedge \beta(o, 1)=[b] \wedge (g(a)<g(b) \vee (g(a)=g(b) \wedge$

$$h(a)<h(b))) \rightarrow \forall X^{\alpha(0,1)} \exists Y^{\beta(0,1)}(X^{\alpha(0,1)} \approx Y^{\beta(0,1)}).$$

A set $\varphi^{((\alpha(0,1)))}$ of a type $((\alpha(o, 1)))$ is said to be an assignment for variables if it has the following three properties. 1. Each element of $\varphi^{((\alpha(0,1)))}$ is a set of the form $\{a^{\alpha(0,1)}, b^{\alpha(0,1)}, A^{\alpha(0,1)}\}$ for some $a, b, c, \beta(o, 1), A^{\alpha(0,1)}$ and $B^{\beta(0,1)}$ with $a=7^{c+1}$, Typ(b), $(B^{\beta(0,1)} \in \mathfrak{D}(b))$ and $B^{\beta(0,1)} \approx A^{\alpha(0,1)}$. 2. If $a=7^{c+1}$, Typ(b) and sets $\{a^{\alpha(0,1)}, b^{\alpha(0,1)}, A^{\alpha(0,1)}\}$ and $\{a^{\alpha(0,1)}, b^{\alpha(0,1)}, B^{\alpha(0,1)}\}$ are elements of $\varphi^{((\alpha(0,1)))}$ then $A^{\alpha(0,1)}=B^{\alpha(0,1)}$. 3. There exists a number a such that for any b if Typ(b) and $b^{\alpha(0,1)}$ is an element of an element of $\varphi^{((\alpha(0,1)))}$ then $b \leqq a$. (Note that the arguments in § 3 hold good with slight modifications even if we change the definition of assignment for variables as follows. 1. The domain of φ is the set of all symbols for free and bound variable. 2. For any symbol for free or bound variable A $\varphi(A)$ is a function whose domain is a *finite* subset of T. 3. For any element $\mathfrak{s}$ of the domain of $\varphi(A)$ $\varphi(A)(\mathfrak{s})$ is an element of $\mathfrak{D}(\mathfrak{s})$).

When $\varphi^{((\alpha(0,1)))}$ is an assignment for variables and Typ(b) we write $\varphi(a, b) =A^{\beta(0,1)}$ to mean that one of the following holds. 1. For some $B^{\alpha(0,1)}$ with $A^{\beta(0,1)} \approx B^{\alpha(0,1)}$ the set $\{c^{\alpha(0,1)}, b^{\alpha(0,1)}, B^{\alpha(0,1)}\}$ is an element of $\varphi^{((\alpha(0,1)))}$ and $(A^{\beta(0,1)} \in \mathfrak{D}(b))$, where $c=7^{a+1}$. 2. $b=2$, $A^{\beta(0,1)}=0$ and for all $B^{\alpha(0,1)}$ the set $\{c^{\alpha(0,1)}, b^{\alpha(0,1)}, B^{\alpha(0,1)}\}$ is not an element of $\varphi^{((\alpha(0,1)))}$, where $c=7^{a+1}$. 3. $b \neq 2$, $A^{\beta(0,1)}$ is an empty set, $\beta(o, 1)=[b]$ and for all $B^{\alpha(0,1)}$ the set $\{c^{\alpha(0,1)}, b^{\alpha(0,1)}, B^{\alpha(0,1)}\}$ is not an element of $\varphi^{((\alpha(0,1)))}$, where $c=7^{a+1}$. When $\varphi^{((\alpha(0,1)))}$ and $\psi^{((\beta(0,1)))}$ are assignments for variables we write $\varphi \sim \psi(a)$ to mean that for any b, c and $A^{\gamma(0,1)}$ if $\varphi(b, c) =A^{\gamma(0,1)}$ and $b \neq a$ then $\psi(b, c)=A^{\gamma(0,1)}$. We can prove

(5.14) If $\varphi^{((\alpha(0,1)))}$ is an assignment for variables and $(A^{\beta(0,1)} \in \mathfrak{D}(a))$ there exists an assignment for variables $\psi^{((\gamma(0,1)))}$ with $\varphi \sim \psi(b)$ and $\psi(b, a)=A^{\beta(0,1)}$.

Similarly we can define assignments for index variables and for type variables. When $\varphi^{((\alpha(0,1)))}$, $\chi^{((\alpha(0,1)))}$ and $\psi^{((\alpha(0,1)))}$ are assignments for index variables, for type variables and for variables, respectively, a is the Gödel number of a quasi-variety E, $(A^{\beta(0,1)} \in \mathfrak{D}(b))$ and $A^{\beta(0,1)} \approx B^{\alpha(0,1)}$ the set

$$\varphi \cup \chi \cup \psi \cup \{\{c^{\alpha(0,1)}, b^{\alpha(0,1)}, B^{\alpha(0,1)}\}\}$$

can be regarded as the statement that $\mathfrak{H}(E, \varphi, \chi, \psi)=A^{\beta(0,1)}$ (in the notation of § 3), where $c=11^{a}$. Similarly when a is the Gödel number of a quasi-formula $\mathfrak{A}$ and $b=0$ or $b=1$ the set

$$\varphi \cup \chi \cup \psi \cup \{\{c^{\alpha(0,1)}, b^{\alpha(0,1)}\}\}$$

can be regarded as the statement that $\mathfrak{H}(\mathfrak{A}, \varphi, \chi, \psi)=\boldsymbol{t}$ (in the case $b=0$) or that $\mathfrak{H}(\mathfrak{A}, \varphi, \chi, \psi)=\boldsymbol{f}$ (in the case $b=1$), where $c=11^{a}$. Using quantifiers for variables of the type $(\langle o, 1\rangle)$ we can pick out the true statements by the induction on a.

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