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Ergodic theorems for semigroups of positive operators

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1. Introduction.

Let $\Gamma=\{T_t;\ t>0\}$ be a strongly continuous semigroup of bounded linear operators on L_1 of a σ-finite measure space. In [5], Dunford-Schwartz proved that if all the T_t are contractions on L_1 and satisfy $\|T_tf\|_\infty\leqq\|f\|_\infty$ for every $f\in L_1\cap L_\infty$, then the limit

(1) $$\lim_{b\to\infty}\frac{1}{b}\int_0^b T_tf\,dt$$

exists and is finite a. e. for any $f\in L_1$. In [2], Berk proved that if all the T_t are positive contractions on L_1, then the limit

(2) $$\lim_{b\to\infty}\left(\int_0^b T_tf\,dt\right)\Big/\left(\int_0^b T_tg\,dt\right)$$

exists and is finite a. e. on the set $\bigcup_{b>0}\left\{\int_0^b T_tg\,dt>0\right\}$ for any $f, g\in L_1$ with $g\geqq 0$; this extends the Chacon-Ornstein theorem [3] to the continuous case and was also proved, by different methods, by Akcoglu-Cunsolo [1] and Fong-Sucheston [7]. Only assuming that all the T_t are contractions on L_1, generalizations of these results are discussed in Kubokawa [11], Tsurumi [17], and Hasegawa-Sato [9].

In this paper, we shall assume that all the T_t are positive and that Γ satisfies $\sup_{b>0}\left\|\frac{1}{b}\int_0^b T_t\,dt\right\|_1<\infty$ in the sense of *strong integral*, i. e., for each $f\in L_1$ the vector valued function $t\to T_tf$ is Bochner integrable with respect to Lebesgue measure on every finite interval $(0, b)$, and there exists a constant $M\geqq 0$ such that

$$\sup_{b>0}\left\|\frac{1}{b}\int_0^b T_tf\,dt\right\|_1\leqq M\|f\|_1$$

for all $f\in L_1$. Under these conditions on Γ, we investigate the almost everywhere and strong convergence of the average $\frac{1}{b}\int_0^b T_tf\,dt$ as $b\to\infty$. In particular we observe that, under these conditions on Γ, if there exists a strictly positive function $h\in L_1$ such that $T_t h/h\in L_\infty$ for all $t>0$ and also such that $\sup_{b>0}\left\|\left(\frac{1}{b}\int_0^b T_t h\,dt\right)/h\right\|_\infty<\infty$, then the limit (1) exists and is finite a.e. for any $f\in L_1$ with $f/h\in L_\infty$ (cf. Theorem 6). This extends a result due to Derriennic-Lin ([4], Theorem 4.2) to the continuous case.

The main tools employed below are the continuous version of the Chacon-Ornstein theorem and the decomposition theorem given in [15].

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2. Definitions and notation.

Let $(X, \mathcal{M}, m)$ be a probability space and let $L_p(X)=L_p(X, \mathcal{M}, m)$, $1\leqq p\leqq\infty$, be the (complex) Banach spaces defined as usual with respect to $(X, \mathcal{M}, m)$. All sets and functions introduced below are assumed to be measurable; all relations are assumed to hold modulo sets of measure zero. If A is a subset of X, then 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish on $X-A$. Also, $L_p^+(A)$ denotes the positive cone of $L_p(A)$ consisting of nonnegative $L_p(A)$-functions. A linear operator T on $L_p(X)$ is called *positive* if $T(L_p^+(X))\subset L_p^+(X)$ and a *contraction* if $\|T\|_p\leqq 1$. It is well-known that if T is positive then $\|T\|_p<\infty$. The adjoint of T is denoted by T^*.

Let $\Gamma=\{T_t;\ t>0\}$ be a semigroup of positive linear operators on $L_1(X)$, i.e., all the T_t are positive linear operators on $L_1(X)$ and $T_t T_{t'}=T_{t+t'}$ for all $t, t'>0$. In this paper we assume that Γ is strongly continuous on $(0, \infty)$, i.e., for each $f\in L_1(X)$ and each $t_0>0$ we have $\lim_{t\to t_0}\|T_t f-T_{t_0}f\|_1=0$, and that Γ satisfies the following condition:

$$(*)\qquad \sup_{b>0}\left\|\frac{1}{b}\int_0^b T_t\,dt\right\|_1<\infty$$

in the sense of strong integral.

It is then known (cf. [6], VIII. 7) that for any $f\in L_1(X)$ there exists a scalar function $T_t f(x)$ on $(0, \infty)\times X$, measurable with respect to the product of Lebesgue measure and m, such that for almost all $t>0$, $T_t f(x)$ belongs, as a function of x, to the equivalence class of $T_t f$. Moreover there exists a set $N(f)\subset X$ with $m(N(f))=0$, dependent on f but independent of t, such that if $x\notin N(f)$ then the function $t\to T_t f(x)$ is Lebesgue integrable over every finite interval $(a, b)\subset(0, \infty)$ and the integral $\int_a^b T_t f(x)\,dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t f\,dt$ $(\in L_1(X))$.

If μ is a σ-finite measure on $(X, \mathcal{M})$ equivalent to m, then $L_1(X, \mathcal{M}, \mu)$ and

$L_1(X, \mathcal{M}, m)$ are isometric by the Radon-Nikodym theorem, and thus a semigroup $\{T_t;\ t>0\}$ on L_1 of a σ-finite measure space can be represented as a semigroup $\{S_t;\ t>0\}$ on L_1 of a finite measure space, which preserves also pointwise convergence.

3. Some known results.

Throughout this section and the remainder of the paper, $\Gamma=\{T_t;\ t>0\}$ will be a fixed semigroup of positive linear operators on $L_1(X)$ which is strongly continuous on $(0, \infty)$ and satisfies condition (*).

For $0 \leqq a<b<\infty$, the integral $\int_a^b T_t^* f\, dt$ $(\in L_\infty(X))$ for $f \in L_\infty(X)$ is defined by the relation:

$$\left\langle v, \int_a^b T_t^* f\, dt\right\rangle=\left\langle\int_a^b T_t v\, dt, f\right\rangle \qquad (v \in L_1(X)).$$

The following lemma is used to obtain a decomposition of the space X.

LEMMA A ([**15**], Lemma 1). *For any $f \in L_\infty(X)$ there exists a scalar function $T_t^* f(x)$ on $(0, \infty) \times X$, measurable with respect to the product of Lebesgue measure and m, and a set $N(f) \subset X$ with $m(N(f))=0$, dependent on f but independent of t, such that if $x \notin N(f)$ then the function $t \rightarrow T_t^* f(x)$ is Lebesgue integrable over every finite interval $(a, b) \subset (0, \infty)$ and the integral $\int_a^b T_t^* f(x)\, dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t^* f\, dt$.*

SKETCH OF PROOF. Without loss of generality we may assume that f is nonnegative. Let $I=(c, d]$, where $0<c<d<\infty$. Then, since $\sup\{\|T_t f\|_1 \mid c<t \leqq d\}<\infty$ for all $f \in L_1(X)$, the uniform boundedness principle (cf. [6], Corollary II. 3.21) implies that

$$\sup_{c<t \leqq d}\|T_t\|_1=M<\infty .$$

Define, for α a Lebesgue measurable subset of I and $A \in \mathcal{M}$,

$$\lambda(\alpha \times A)=\int_\alpha\langle T_t 1_A, f\rangle\, dt .$$

Then it may be readily seen that λ can be extended to a finite measure on the product space $I \times X$. Moreover, since

$$\begin{aligned}\lambda(\alpha \times A) & \leqq \int_\alpha\|T_t 1_A\|_1\|f\|_\infty\, dt \\ & \leqq M m(A)\|f\|_\infty \int_\alpha 1\, dt,\end{aligned}$$

λ is absolutely continuous with respect to the product of Lebesgue measure (on I) and m. Let $g(t, x)$ be the Radon-Nikodym derivative of λ with respect to this product measure. Fix an $A\in\mathcal{M}$. Then, for any α a Lebesgue measurable subset of I, we have, by Fubini's theorem,

$$\int_\alpha \langle T_t 1_A, f\rangle dt=\lambda(\alpha\times A)=\int_\alpha\int_A g(t, x)dm\, dt .$$

This shows that, for almost all $t\in I$, $\langle T_t 1_A, f\rangle=\int_A g(t, x)dm$.

Since $(0, \infty)$ is a disjoint union of countably many such intervals I, it follows that there exists a nonnegative function $g(t, x)$ on $(0, \infty)\times X$, measurable with respect to the product of Lebesgue measure and m, such that if $A\in\mathcal{M}$ then

$$\langle T_t 1_A, f\rangle=\int_A g(t, x)dm$$

for almost all $t\in(0, \infty)$. Let $0\leqq a<b<\infty$. Then we have, again by Fubini's theorem,

$$\left\langle 1_A, \int_a^b g(t, x)dt\right\rangle=\int_a^b\int_A g(t, x)dm\, dt=\int_a^b\langle T_t 1_A, f\rangle dt$$

$$=\left\langle \int_a^b T_t 1_A\, dt, f\right\rangle=\left\langle 1_A, \int_a^b T_t{}^*f\, dt\right\rangle .$$

Since this holds for any $A\in\mathcal{M}$, a standard approximation argument shows that, for all $v\in L_1(X)$,

$$\left\langle v, \int_a^b g(t, x)dt\right\rangle=\left\langle v, \int_a^b T_t{}^*f\, dt\right\rangle .$$

Thus the lemma is proved.

We note that the function $T_t{}^*f(x)$ in Lemma A is uniquely determined up to equivalence modulo sets of the product measure zero.

Next, using Lemma A, let us set

$$u(x)=\limsup_{b\to\infty}\frac{1}{b}\int_0^b T_t{}^*1(x)dt \qquad (x\notin N(1)) . \tag{3}$$

Since the function $b\to\frac{1}{b}\int_0^b T_t{}^*1(x)dt$ is continuous on $(0, \infty)$ for each $x\notin N(1)$, if D denotes the set of all positive rationals, then we have

$$u(x)=\limsup_{b\to\infty, b\in D}\frac{1}{b}\int_0^b T_t{}^*1(x)dt \qquad (x\notin N(1)) .$$

Hence we observe that the function $u(x)$ belongs to the equivalence class of

$$\limsup_{b\to\infty}\frac{1}{b}\int_0^b T_t{}^*1\,dt \qquad (\in L_\infty(X)).$$

Now, fix $t>0$ arbitrarily, and let $f\in L_1^+(X)$. Then

$$\begin{aligned}\langle f, T_t{}^*u\rangle &= \lim_{a\to\infty}\int(T_tf)\Big(\sup_{b>a}\frac{1}{b}\int_0^b T_s{}^*1\,ds\Big)dm\\ &\geqq \lim_{a\to\infty}\int f\Big(\sup_{b>a}\frac{1}{b}\int_t^{b+t} T_s{}^*1\,ds\Big)dm\\ &=\int fu\,dm=\langle f, u\rangle,\end{aligned}$$

and so it follows that $T_t{}^*u\geqq u$. Therefore, by Fubini's theorem and Lemma A, we can choose a set N, with $N(u)\subset N$ and $m(N)=0$, such that if $x\notin N$ and $0<b<b'<\infty$ then

$$\frac{1}{b}\int_0^b T_t{}^*u(x)\,dt\leqq\frac{1}{b'}\int_0^{b'} T_t{}^*u(x)dt.$$

Therefore we can define

$$s(x)=\lim_{b\to\infty}\frac{1}{b}\int_0^b T_t{}^*u(x)dt \qquad (x\notin N). \tag{4}$$

The obtained function $s(x)$ has the following useful properties:

THEOREM B ([**15**], Theorem 1). *$s\in L_\infty^+(X)$ and $T_t^*s=s$ for all $t>0$. If we denote $Y=\{x\,|\,s(x)>0\}$ and $Z=X-Y$, then $T_t(L_1(Z))\subset L_1(Z)$ for all $t>0$ and*

$$\lim_{b\to\infty}\Big\|\frac{1}{b}\int_0^b T_tf\,dt\Big\|_1=0 \tag{5}$$

for all $f\in L_1(Z)$.

The following example shows that there exists a strongly continuous semigroup $\Gamma=\{T_t\,;\,t>0\}$ of positive linear operators on L_1 of a σ-finite measure space which is not bounded, i.e., $\sup_{t>0}\|T_t\|_1=\infty$, but satisfies condition (*).

EXAMPLE. Set $a_0=1$, $a_1=2$, $a_n=4a_{n-1}$ $(n\geqq 2)$; $b_n=\sum_{i=0}^n a_i$ $(n\geqq 0)$; $c_n=\sum_{i=0}^n b_i$ $(n\geqq 0)$. Define (h_n) a sequence of functions on $(0,\infty)$ as follows:

$$h_0(x)=\begin{cases}1 & \text{if } x\in(-\infty, c_0]\\ 0 & \text{if } x\in(c_0,\infty),\end{cases}$$

$$h_n(x)=\begin{cases}h_{n-1}(x) & \text{if } x\in(-\infty, c_{n-1}]\\ 2^{-n} & \text{if } x\in(c_{n-1}, c_{n-1}+a_n]\\ h_{n-1}(x-b_n) & \text{if } x\in(c_{n-1}+a_n, c_n]\\ 0 & \text{if } x\in(c_n,\infty)\end{cases} \qquad (n\geqq 1).$$

Then, clearly, $0 \leqq h_0 \leqq h_1 \leqq \cdots \leqq 1$, and thus we can define

$$h(x) = \lim_{n\to\infty} h_n(x) \qquad (x \in (-\infty, \infty)).$$

It is direct to see that h satisfies

(i) for each $b>0$

$$\sup \{h(t+x)/h(t) \mid -\infty < t < \infty,\ 0 < x < b\} = M(b) < \infty,$$

but

$$\lim_{b\to\infty} M(b) = \infty;$$

(ii) for all $b>0$ and all $-\infty<t<\infty$

$$\frac{1}{b}\int_0^b h(t+x)dx < 4h(t).$$

Hence if we set $L_1(h\,dx) = \left\{f \mid \int_{-\infty}^{\infty} |f|h\,dx < \infty\right\}$ and, for $f \in L_1(h\,dx)$ and $t>0$,

$$(T_t f)(x) = f(x-t) \qquad (-\infty < x < \infty),$$

then $\Gamma = \{T_t\,;\, t>0\}$ is a semigroup of positive linear operators on $L_1(h\,dx)$ and satisfies $\sup_{0<t<b} \|T_t\|_1 = M(b) < \infty$ for each $b>0$. Thus we have $\sup_{t>0} \|T_t\|_1 = \infty$. On the other hand, an easy approximation argument implies that $\lim_{t\to+0} \|T_t f - f\|_1 = 0$ for all $f \in L_1(h\,dx)$. Therefore we see that Γ is strongly continuous on $(0, \infty)$. Using Fubini's theorem and (ii), it also follows that

$$\sup_{b>0} \left\| \frac{1}{b}\int_0^b T_t f\,dt \right\|_1 \leqq 4\|f\|_1 \qquad (f \in L_1(h\,dx)).$$

Hence Γ satisfies condition (*).

4. Mean ergodic theorem.

In this section we investigate the strong convergence properties of $\frac{1}{b} \times \int_0^b T_t f\,dt$ as $b\to\infty$. The first theorem gives a sufficient (and obviously necessary) condition for the strong convergence of the average as $b\to\infty$.

THEOREM 1. *Let $\Gamma = \{T_t\,;\, t>0\}$ be a strongly continuous semigroup of positive linear operators on $L_1(X)$ which satisfies condition (*). Let $f \in L_1(X)$, and assume that there exists a strictly increasing sequence (b_n) of positive reals, with $\lim_{n\to\infty} b_n = \infty$, such that the sequence $\left(\frac{1}{b_n}\int_0^{b_n} T_t f\,dt\right)$ converges weakly in $L_1(X)$.*

Then the average $\frac{1}{b}\int_0^b T_t f\,dt$ *converges strongly as* $b\to\infty$ *to some* $f_\infty\in L_1(X)$ *with* $T_t f_\infty = f_\infty$ *for all* $t>0$.

For the proof of this theorem we need two lemmas. The first one is a continuous extension of the Banach space mean ergodic theorem given in [**16**]; essentially the same idea has been used by Yosida-Kakutani [**19**] (see Yosida [**18**], pp. 213-214) to prove a mean ergodic theorem for power bounded linear operators in Banach space.

LEMMA 2. *Let* $\Xi=\{\xi_t\,;\,t>0\}$ *be a strongly continuous semigroup of bounded linear operators on a Banach space* $\mathfrak{B}$ *which is assumed to be strongly integrable over every finite interval, and let* (b_n) *be a strictly increasing sequence of positive reals, with* $\lim_{n\to\infty} b_n=\infty$. *Assume that* $\sup_{n\geqq 1}\left\|\frac{1}{b_n}\int_0^{b_n}\xi_t\,dt\right\|<\infty$ *in the sense of strong integral. Let* $\mathfrak{f}\in\mathfrak{B}$. *Then the sequence* $\left(\frac{1}{b_n}\int_0^{b_n}\xi_t\mathfrak{f}\,dt\right)$ *converges strongly to some* $\mathfrak{f}_\infty\in\mathfrak{B}$ *with* $\xi_t\mathfrak{f}_\infty=\mathfrak{f}_\infty$ *for all* $t>0$ *if and only if*

(i) $\lim_{n\to\infty}\left\|\frac{1}{b_n}\int_{b_n}^{a+b_n}\xi_t\mathfrak{f}\,dt\right\|=0$ *for all* $a>0$, *and*

(ii) *there exists a subsequence* (n') *of* (n) *such that*

$$\underset{n'\to\infty}{\text{weak-lim}}\frac{1}{b_{n'}}\int_0^{b_{n'}}\xi_t\mathfrak{f}\,dt$$

exists in $\mathfrak{B}$.

PROOF. Since the necessity of the conditions (i) and (ii) of the lemma is obvious, we prove here only the sufficiency of these conditions.

Since $\sup_{n\geqq 1}\left\|\frac{1}{b_n}\int_{b_n}^{a+b_n}\xi_t\,dt\right\|<\infty$ for all $a>0$, if we let

$$\mathfrak{A}=\left\{\mathfrak{b}\in\mathfrak{B}\,|\,\lim_{n\to\infty}\left\|\frac{1}{b_n}\int_{b_n}^{a+b_n}\xi_t\mathfrak{b}\,dt\right\|=0 \text{ for all } a>0\right\},$$

then $\mathfrak{A}$ is a closed subspace of $\mathfrak{B}$ containing $\mathfrak{f}$ and $\xi_t\mathfrak{A}\subset\mathfrak{A}$ for all $t>0$. Let $\mathfrak{f}_\infty\in\mathfrak{B}$ be such that

$$\mathfrak{f}_\infty=\underset{n'\to\infty}{\text{weak-lim}}\frac{1}{b_{n'}}\int_0^{b_{n'}}\xi_t\mathfrak{f}\,dt\,.$$

Then, as in [**16**], we observe that $\xi_t\mathfrak{f}_\infty=\mathfrak{f}_\infty$ for all $t>0$ and that $\mathfrak{f}-\mathfrak{f}_\infty$ belongs to the closed subspace generated by the set $\{\mathfrak{a}-\xi_t\mathfrak{a}\,|\,\mathfrak{a}\in\mathfrak{A},\,t>0\}$. Therefore, by an approximation argument, we have

$$\mathfrak{f}_\infty=\underset{n\to\infty}{\text{strong-lim}}\frac{1}{b_n}\int_0^{b_n}\xi_t\mathfrak{f}\,dt\,.$$

LEMMA 3. *Let* $\Gamma=\{T_t\,;\,t>0\}$ *be a strongly continuous semigroup of positive linear operators on* $L_1(X)$ *which satisfies condition* (*). *Then, for any* $f\in L_1(X)$

and any $a>0$, we have

$$\lim_{b\to\infty}\left\|\frac{1}{b}\int_b^{b+a}T_tf\,dt\right\|_1=0. \tag{6}$$

PROOF. This is an adaptation of the proof of Theorem 2.1 of [4]. Write $M=\sup_{b>0}\left\|\frac{1}{b}\int_0^b T_t\,dt\right\|_1(<\infty)$. Let $a>0$ be given. To prove the lemma, it is enough to consider the case where f is nonnegative. Then

$$\frac{1}{t_0}\left\|\int_b^{b+a}T_tf\,dt\right\|_1\leqq\frac{1}{t_0}\left\|\int_0^{t_0}T_t(T_{b+a-t_0}f)dt\right\|_1$$
$$\leqq M\|T_{b+a-t_0}f\|_1$$

for all $a<t_0<b$, and thus we may apply Fubini's theorem to obtain that

$$\left\|\frac{1}{b}\int_b^{b+a}T_tf\,dt\right\|_1\int_a^b\frac{1}{t}\,dt\leqq M\frac{1}{b}\int_a^b\|T_{b+a-t}f\|_1\,dt$$
$$\leqq M\frac{1}{b}\int_0^b\int_X T_tf(x)\,dm\,dt=M\frac{1}{b}\int_X\int_0^b T_tf(x)\,dt\,dm$$
$$=M\frac{1}{b}\left\|\int_0^b T_tf\,dt\right\|_1\leqq M^2\|f\|_1.$$

Hence, letting $b\to\infty$, the desired conclusion follows.

PROOF OF THEOREM 1. Let $f_\infty\in L_1(X)$ be the weak limit function of the sequence $\left(\frac{1}{b_n}\int_0^{b_n}T_tf\,dt\right)$, and let (c_n) be any strictly increasing sequence of positive reals, with $\{b_n|n\geqq1\}\subset\{c_n|n\geqq1\}$. Then, by Lemmas 2 and 3, we have $\lim_{n\to\infty}\left\|\frac{1}{c_n}\int_0^{c_n}T_tf\,dt-f_\infty\right\|_1=0$ and $T_tf_\infty=f_\infty$ for all $t>0$.

Hence the theorem is established.

The following theorem extends a result due to Fong-Sucheston ([8], Theorem 2.1) to the continuous case. See also [4].

THEOREM 4. *Let $\Gamma=\{T_t\,;\,t>0\}$ be a strongly continuous semigroup of positive linear operators on $L_1(X)$ which satisfies condition (*). Let Y, Z, and s be the same as in Theorem B. Let f, $g\in L_1(X)$ satisfy $\lim_{t\to\infty}\int|T_tf-g|s\,dm=0$. Then* strong-$\lim_{b\to\infty}\frac{1}{b}\int_0^b T_tg\,dt=g_\infty$ *exists, and we have*

$$\lim_{b\to\infty}\frac{1}{b}\int_0^b\|T_tf-g_\infty\|_1\,dt=0. \tag{7}$$

In particular, if $\Gamma=\{T_t\,;\,t>0\}$ satisfies $\sup_{t>0}\|T_t\|_1<\infty$, then we have

(8) $$\lim_{t\to\infty}\|T_tf-g_\infty\|_1=0.$$

PROOF. By Theorem B, we may and will assume without loss of generality that $g\in L_1(Y)$. For $t>0$ and $sf\in L_1(Y)$, where $f\in L_1(Y)$, define

$$V_t(sf)=s(T_tf).$$

Since $\{sf|f\in L_1(Y)\}$ is a dense subspace of $L_1(Y)$ in the strong topology and $\|V_t(sf)\|_1\leqq\|sf\|_1$ (cf. [**14**]), V_t may be considered to be a positive linear contraction on $L_1(Y)$. By an approximation argument, we observe that $V_tV_{t'}=V_{t+t'}$ on $L_1(Y)$ for all t, $t'>0$ and that the semigroup $\Delta=\{V_t;t>0\}$ on $L_1(Y)$ is strongly continuous on $(0,\infty)$. It follows from the hypothesis of the theorem that $\lim_{t\to\infty}\|V_t(sf)-sg\|_1=0$. Therefore we observe that $s(T_tg)=V_t(sg)=sg$ for all $t>0$. Since $g=0$ on Z, it then follows that $T_tg^+\geqq g^+$ and $T_tg^-\geqq g^-$ for all $t>0$, where $g^+(x)=\max\{g(x), 0\}$ and $g^-(x)=\max\{-g(x), 0\}$. By this and condition (*), there exist two functions h_1 and h_2 in $L_1^+(X)$ such that

$$h_1=\text{strong-}\lim_{b\to\infty}\frac{1}{b}\int_0^b T_tg^+\,dt,$$

$$h_2=\text{strong-}\lim_{b\to\infty}\frac{1}{b}\int_0^b T_tg^-\,dt.$$

If we set $g_\infty=h_1-h_2$, then it follows that

$$g_\infty=\text{strong-}\lim_{b\to\infty}\frac{1}{b}\int_0^b T_tg\,dt,$$

$$T_tg_\infty=g_\infty\qquad\text{for all}\quad t>0,$$

and

$$g_\infty=g\qquad\text{on}\quad Y.$$

Hence, in order to prove (7), it suffices to show that

(9) $$\lim_{t\to\infty}\int|T_tf|s\,dm=0\qquad\text{implies}\quad\lim_{b\to\infty}\frac{1}{b}\int_0^\infty\|T_tf\|_1dt=0.$$

To prove this, let $t_0>0$ be fixed arbitrarily. Then, by Fatou's lemma, we have

$$\limsup_{b\to\infty}\frac{1}{b}\int_0^b\|T_tf\|_1\,dt$$
$$=\limsup_{b\to\infty}\frac{1}{b}\int_{t_0}^b\|T_tf\|_1\,dt$$

$$\leqq \limsup_{b\to\infty} \frac{1}{b}\int_{t_0}^{b} \langle |T_{t_0}f|,\ T_{t-t_0}{}^*1\rangle\, dt$$

$$\leqq \limsup_{b\to\infty} \left\langle |T_{t_0}f|,\ \frac{1}{b}\int_0^b T_t{}^*1\, dt\right\rangle$$

$$\leqq \int_X |T_{t_0}f| \left(\limsup_{b\to\infty} \frac{1}{b}\int_0^b T_t{}^*1\, dt\right) dm$$

$$= \int_X |T_{t_0}f|\, u\, dm \leqq \int_X |T_{t_0}f|\, s\, dm\,,$$

since $0\leqq u\leqq s$. This establishes (9), because the right hand side of the last inequality can be arbitrarily small.

Next let us assume that (7) holds and that Γ satisfies $\sup_{t>0} \|T_t\|_1<\infty$. Then, by (7), we have $\inf_{t>0} \|T_t f-g_\infty\|_1=0$. Therefore, given an $\varepsilon>0$, we can find a $t_0>0$ such that $\|T_{t_0}f-g_\infty\|_1<\varepsilon$. Then we have, for all $t>t_0$,

$$\|T_t f-g_\infty\|_1=\|T_{t-t_0}(T_{t_0}f-g_\infty)\|_1<(\sup_{t>0} \|T_t\|_1)\varepsilon\,.$$

Consequently we have $\lim_{t\to\infty} \|T_t f-g_\infty\|_1=0$, and this completes the proof.

Let us now assume that $X=Y$ in Theorem B. It may be readily seen from Theorem B that this condition is equivalent to the following condition:

$$0\leqq f\in L_1(X) \text{ and } \|f\|_1>0 \text{ imply } \limsup_{b\to\infty} \left\|\frac{1}{b}\int_0^b T_t f\, dt\right\|_1>0\,.$$

It is then known (cf. [7] and [**15**]) that the ratio ergodic theorem holds for $\Gamma=\{T_t\,;\, t>0\}$, i.e., for any f and g in $L_1(X)$, with $g\geqq 0$, the ratio limit

$$\lim_{b\to\infty} \left(\int_0^b T_t f(x)\, dt\right)\Big/\left(\int_0^b T_t g(x)\, dt\right) \tag{10}$$

exists and is finite a.e. on the set $\left\{x \mid \int_0^\infty T_t g(x)\, dt>0\right\}$. Thus Hopf's decomposition holds, i.e., X decomposes into two sets C and D, called, respectively, the *conservative* and *dissipative* parts of the semigroup Γ, such that if $0\leqq g\in L_1(X)$, then $\int_0^\infty T_t g(x)\, dt=\infty$ or 0 a.e. on C, and $\int_0^\infty T_t g(x)\, dt<\infty$ a.e. on D. The semigroup Γ is called *conservative*, if $C=X$.

PROPOSITION 5. *Let $\Gamma=\{T_t\,;\, t>0\}$ be a strongly continuous semigroup of positive linear operators on $L_1(X)$ which satisfies condition* (*). *Assume that $X=Y$ in Theorem B and that Γ is conservative. Let $w\in L_\infty(X)$ satisfy $w>0$ a.e. on X and $T_t{}^*w=w$ for all $t>0$. Then, for any f and g in $L_1(X)$,*

$$\lim_{t\to\infty}\int|T_tf-g|w\,dm=0$$

implies

$$\lim_{t\to\infty}\int|T_tf-g|s\,dm=0\,. \tag{11}$$

PROOF. As in the proof of Theorem 4, we get $T_tg=g$ for all $t>0$. Therefore if we write $h=f-g$, then it follows that

$$\lim_{t\to\infty}\int|T_th|w\,dm=0\,.$$

Set $w_n(x)=\min\{s(x), nw(x)\}$ $(n\geqq 1)$. It then follows that $T_t^*w_n\leqq w_n$ for all $t>0$. Hence, for any $g'\in L_1^+(X)$ and any $t_0>0$, we have

$$\begin{aligned}0&\leqq\lim_{b\to\infty}\left\langle\int_0^b T_tg'\,dt,\ w_n-T_{t_0}^*w_n\right\rangle\\ &=\lim_{b\to\infty}\left\langle g',\int_0^{t_0}T_t^*w_ndt-\int_b^{b+t_0}T_t^*w_ndt\right\rangle\\ &\leqq\left\langle g',\int_0^{t_0}T_t^*w_ndt\right\rangle<\infty\,.\end{aligned}$$

This shows that $T_{t_0}^*w_n=w_n$, since Γ is conservative. Hence $T_t^*(s-w_n)=s-w_n$ for all $t>0$, and we have

$$\begin{aligned}&\limsup_{t\to\infty}\int|T_th|s\,dm\\ &\quad\leqq\limsup_{t\to\infty}\int|T_th|w_n\,dm+\limsup_{t\to\infty}\int|T_th|(s-w_n)dm\\ &\quad=\limsup_{t\to\infty}\int|h|T_t^*(s-w_n)dm=\int|h|(s-w_n)dm\,,\end{aligned}$$

from which the proposition follows, because $\lim_{n\to\infty}\int|h|(s-w_n)dm=0$.

5. Individual ergodic theorem.

In this section we investigate the almost everywhere convergence of the average $\frac{1}{b}\int_0^b T_tf\,dt$ as $b\to\infty$. The main result of the section is the following theorem; we refer the reader to [4] and [10] for the discrete case.

THEOREM 6. *Let $\Gamma=\{T_t\,;\,t>0\}$ be a strongly continuous semigroup of positive linear operators on $L_1(X)$ which satisfies condition* (*). *Assume that $T_t1\in L_\infty(X)$ for all $t>0$, and also that*

$$\sup_{b>0}\left\|\frac{1}{b}\int_0^b T_t 1\,dt\right\|_\infty = M < \infty\,. \tag{12}$$

Then, for any $f\in L_\infty(X)$, *the limit*

$$\lim_{b\to\infty}\frac{1}{b}\int_0^b T_t f(x)dt \tag{13}$$

exists and is finite a.e. on X.

PROOF. Let Y, Z, and s be the same as in Theorem B. For $f\in L_\infty(X)$ and $t>0$, define

$$S_t f = T_t^* f\,.$$

It follows that $\|S_t f\|_1=\int|T_t^* f|dm\leqq\int T_t^*|f|dm=\int|f|T_t 1\,dm\leqq\|f\|_1\|T_t 1\|_\infty$. Since $\|T_t 1\|_\infty<\infty$ by hypothesis and since $L_\infty(X)$ is a dense subspace of $L_1(X)$, this shows that S_t can be extended to a positive linear operator on $L_1(X)$. By an approximation argument, we see that $S_t S_{t'}=S_{t+t'}$ on $L_1(X)$ for all $t, t'>0$.

To prove that the semigroup $\Delta=\{S_t\,;\,t>0\}$ on $L_1(X)$ is strongly continuous on $(0,\infty)$, fix an $f\in L_\infty(X)$. Then, since the vector valued function $t\to S_t f$ is weakly continuous on $(0,\infty)$, it follows that this function is also strongly measurable on $(0,\infty)$. Now let $f\in L_1(X)$ be given arbitrarily. Choose (f_n) a sequence of functions in $L_\infty(X)$ satisfying $\lim_{n\to\infty}\|f-f_n\|_1=0$. Then, since

$$\lim_{n\to\infty}\|S_t f-S_t f_n\|_1=0 \qquad \text{for all} \quad t>0\,,$$

we observe that the vector valued function $t\to S_t f$ is also strongly measurable on $(0,\infty)$. Therefore $\Delta=\{S_t\,;\,t>0\}$ is strongly continuous on $(0,\infty)$, by Lemma VIII. 1.3 of [**6**].

For $f\in L_\infty(X)$ and $b>0$, we have

$$\left\|\frac{1}{b}\int_0^b S_t f\,dt\right\|_1\leqq\frac{1}{b}\int_0^b\|S_t f\|_1 dt\leqq\frac{1}{b}\int_0^b\langle|f|,\,T_t 1\rangle dt$$

$$=\left\langle|f|,\,\frac{1}{b}\int_0^b T_t 1\,dt\right\rangle\leqq M\|f\|_1\,.$$

Hence, by an approximation argument, we observe that $\Delta=\{S_t\,;\,t>0\}$ satisfies condition (*), replacing T_t by S_t.

To complete the proof of the theorem, we now fix an f in $L_\infty^+(X)$ and define two functions $\bar{f}$ and $\underline{f}$ in $L_\infty^+(X)$ by the relations:

$$\bar{f}(x)=\limsup_{b\to\infty}\frac{1}{b}\int_0^b T_t f(x)\,dt \qquad \text{a.e.},$$

and

$$\underline{f}(x)=\liminf_{b\to\infty}\frac{1}{b}\int_0^b T_t f(x)\,dt \qquad \text{a. e.}$$

Then, since $T_t\bar{f}\geqq\bar{f}\geqq\underline{f}\geqq T_t\underline{f}$ for all $t>0$, we can define

$$f^*(x)=\lim_{b\to\infty}\frac{1}{b}\int_0^b T_t\bar{f}(x)\,dt \qquad \text{a. e.,}$$

and

$$f_*(x)=\lim_{b\to\infty}\frac{1}{b}\int_0^b T_t\underline{f}(x)\,dt \qquad \text{a. e.}$$

Clearly $f^*, f_*\in L_\infty^+(X)$, $T_tf^*=f^*$ and $T_tf_*=f_*$ for all $t>0$.

On the other hand, since $S_t^*=T_t$ on $L_\infty(X)$ for all $t>0$ and $S_ts=T_t^*s=s$ for all $t>0$, we may apply Corollary 2 of [**15**] to $\Delta=\{S_t\,;\,t>0\}$ to obtain that $\bar{f}=\underline{f}=f^*=f_*$ on $Y=\{x\,|\,s(x)>0\}$. It follows that $f^*-f_*\in L_\infty^+(Z)$ and $T_t(f^*-f_*)=f^*-f_*$ for all $t>0$. Therefore, by Theorem B,

$$\|f^*-f_*\|_1=\lim_{b\to\infty}\left\|\frac{1}{b}\int_0^b T_t(f^*-f_*)dt\right\|_1=0\,,$$

and thus $f^*-f_*=0$ on X.

Hence the theorem is established.

Let (a_n) be a sequence of functions on $(0,\infty)$ satisfying

(14) $$\int_0^\infty |a_n(t)|\,dt<\infty \qquad \text{for} \quad n=1, 2, \cdots\,;$$

(15) $$\lim_{n\to\infty}\int_0^\infty a_n(t)dt=1\,;$$

(16) $$\lim_{n'\to\infty}\int_0^\infty a_{n'}(t)b(s+t)dt=b \qquad \text{for every} \quad s>0$$

whenever $b(t)$ is a continuous bounded function on $(0,\infty)$ for which

$$\lim_{n'\to\infty}\int_0^\infty a_{n'}(t)b(t)dt=b$$

exists and is finite, where (n') is a subsequence of (n).

Under these conditions, we have the following theorem, which is a continuous extension of the individual ergodic theorem given in [**13**].

THEOREM 7. *Let $\Delta=\{S_t\,;\,t>0\}$ be a strongly continuous semigroup of positive linear contractions on $L_1(X)$. Suppose there exists a strictly positive function h in $L_1(X)$ such that the set*

$$\left\{\int_0^\infty a_n(t)T_th\,dt \;\middle|\; n\geqq 1\right\}$$

is weakly sequentially compact in $L_1(X)$. *Then there exists a function* $f_0 \in L_1^+(X)$, *with* $S_t f_0 = f_0$ *for all* $t>0$ *and* $C=\{x \mid f_0(x)>0\}$, *where* C *denotes the conservative part of* Δ; *consequently, for any* $f \in L_1(X)$, *the limit*

$$\lim_{b\to\infty} \frac{1}{b}\int_0^b S_t f(x)\, dt \tag{17}$$

exists and is finite a.e. on X.

PROOF. Choose a subsequence (n') of (n) and an $f_0 \in L_1(X)$ such that $f_0 = \text{weak-}\lim_{n'\to\infty} \int_0^\infty a_{n'}(t) S_t h\, dt$. Then, for any $w \in L_\infty(X)$ and any $s>0$, we have

$$\begin{aligned}\int f_0 w\, dm &= \lim_{n'\to\infty} \left\langle \int_0^\infty a_{n'}(t) S_t h\, dt,\, w\right\rangle \\ &= \lim_{n'\to\infty} \int_0^\infty a_{n'}(t) \langle S_t h, w\rangle dt \\ &= \lim_{n'\to\infty} \int_0^\infty a_{n'}(t) \langle S_{s+t} h, w\rangle dt \\ &= \lim_{n'\to\infty} \int_0^\infty a_{n'}(t) \langle S_t h, S_s^* w\rangle dt \\ &= \lim_{n'\to\infty} \left\langle \int_0^\infty a_{n'}(t) S_t h\, dt,\, S_s^* w\right\rangle \\ &= \langle f_0, S_s^* w\rangle = \langle S_s f_0, w\rangle = \int (S_s f_0) w\, dm\,.\end{aligned}$$

This implies that $S_s f_0 = f_0$ for any $s>0$. Next let $w \in L_\infty(X)$ satisfy $\int fw\, dm = \int (S_t f) w\, dm$ for all $f \in L_1(X)$ and all $t>0$. Then we have

$$\begin{aligned}\int f_0 w\, dm &= \lim_{n'\to\infty} \int_0^\infty a_{n'}(t) \langle S_t h, w\rangle dt \\ &= \lim_{n'\to\infty} \int_0^\infty a_{n'}(t) \langle h, w\rangle dt \\ &= \langle h, w\rangle = \int hw\, dm\,.\end{aligned}$$

This implies that $f_0 - h$ belongs to the closed subspace generated by the set $\{f - S_t f \mid f \in L_1(X),\, t>0\}$. Hence we have

$$\lim_{b\to\infty} \left\| \frac{1}{b}\int_0^b S_t h\, dt - f_0 \right\|_1 = 0\,,$$

and so f_0 is nonnegative. Write $A=\{x \mid f_0(x)=0\}$. Since $S_t f_0 = f_0$ and $\|S_t\|_1 \leqq 1$

for all $t>0$, it follows that $X-C\subset A$ and $S_t^*1_A\leqq 1_A$ for all $t>0$. It also follows from an argument used in the proof of Proposition 5 that $S_t^*1_A=1_A$ on $C\cap A$ for all $t>0$. Therefore

$$\int_{C\cap A} h\,dm=\langle h, 1_{C\cap A}\rangle\leqq\lim_{b\to\infty}\frac{1}{b}\int_0^b\langle h, S_t^*1_A\rangle dt$$
$$=\lim_{b\to\infty}\frac{1}{b}\int_0^b\langle S_t h, 1_A\rangle dt=\langle f_0, 1_A\rangle=0\,.$$

Since h is strictly positive, we have $m(C\cap A)=0$ and hence $A\subset X-C$. Consequently, we have $C=\{x\,|\,f_0(x)>0\}$.

Since the ratio ergodic theorem holds for the semigroup $\varDelta=\{S_t\,;\,t>0\}$ and since

$$\frac{1}{b}\int_0^b S_t f(x)dt=f_0(x)\frac{\int_0^b S_t f(x)dt}{\int_0^b S_t f_0(x)dt}\qquad \text{a.e. on } C$$

for any $f\in L_1(X)$, the remainder of the theorem is immediate.

COROLLARY 8. *Let $\varDelta=\{S_t\,;\,t>0\}$ be a strongly continuous semigroup of positive linear contractions on $L_1(X)$. Suppose there exists a strictly increasing sequence (b_n) of positive reals, with $\lim_{n\to\infty} b_n=\infty$, such that*

$$\sup_{n\geqq 1}\left\|\frac{1}{b_n}\int_0^{b_n}S_t 1\,dt\right\|_\infty<\infty\,. \tag{18}$$

Then, for any $f\in L_1(X)$, the limit (17) *exists and is finite a.e. on X.*

PROOF. For each integer $n\geqq 1$, define $a_n(t)=1/b_n$ if $t\in(0, b_n]$ and $a_n(t)=0$ if $t\in(b_n, \infty)$. Then it is direct to see that (a_n) satisfies conditions (14), (15) and (16). Moreover, since $(X, \mathcal{M}, m)$ is a probability space and since

$$\sup_{n\geqq 1}\left\|\int_0^\infty a_n(t)S_t 1\,dt\right\|_\infty<\infty$$

by (18), the set $\left\{\int_0^\infty a_n(t)S_t 1\,dt \mid n\geqq 1\right\}$ is weakly sequentially compact in $L_1(X)$.

Hence Theorem 7 completes the proof of the corollary.

References

[1] M. A. Akcoglu and J. Cunsolo, An ergodic theorem for semigroups, Proc. Amer. Math. Soc., **24** (1970), 161-170.

[2] K. N. Berk, Ergodic theory with recurrent weights, Ann. of Math. Statist., **39** (1968), 1107-1114.

[3] R. V. Chacon and D. S. Ornstein, A general ergodic theorem, Illinois J. Math., **4** (1960), 153-160.

[4] Y. Derriennic and M. Lin, On invariant measures and ergodic theorems for positive operators, J. Functional Analysis, **13** (1973), 252-267.
[5] N. Dunford and J. T. Schwartz, Convergence almost everywhere of operator averages, J. Rational Mech. Anal., **5** (1956), 129-178.
[6] N. Dunford and J. T. Schwartz, Linear operators, Part I, Interscience Publishers, New York, 1958.
[7] H. Fong and L. Sucheston, On the ratio ergodic theorem for semi-groups, Pacific J. Math., **39** (1971), 659-667.
[8] H. Fong and L. Sucheston, On unaveraged convergence of positive operators in Lebesgue space, Trans. Amer. Math. Soc., **179** (1973), 383-397.
[9] S. Hasegawa and R. Sato, A general ratio ergodic theorem for semigroups, Pacific J. Math., **62** (1976), 435-437.
[10] G. Helmberg, On the converse of Hopf's ergodic theorem, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **21** (1972), 77-80.
[11] Y. Kubokawa, Ergodic theorems for contraction semi-groups, J. Math. Soc. Japan, **27** (1975), 184-193.
[12] M. Lin, Semi-groups of Markov operators, Boll. Un. Mat. Ital., (4) **6** (1972), 20-44.
[13] R. Sato, On the individual ergodic theorem for positive operators, Proc. Amer. Math. Soc., **36** (1972), 456-458.
[14] R. Sato, Ergodic properties of bounded L_1-operators, Proc. Amer. Math. Soc., **39** (1973), 540-546.
[15] R. Sato, Invariant measures for semigroups, Studia Math., **53** (1975), 129-134.
[16] R. Sato, A mean ergodic theorem, Amer. Math. Monthly, **82** (1975), 487-488.
[17] S. Tsurumi, An ergodic theorem for a semigroup of linear contractions, Proc. Japan Acad., **49** (1973), 306-309.
[18] K. Yosida, Functional analysis, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1965.
[19] K. Yosida and S. Kakutani, Operator-theoretical treatment of Markoff's process and mean ergodic theorem, Ann. of Math., (2) **42** (1941), 188-228.

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