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The nullity of compact Kähler submanifolds in a complex projective space

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Introducing the notions of index and nullity for minimal submanifolds, J. Simons [8] has developed the theory of these submanifolds. It is well known that any compact complex submanifold in a Kähler manifold is a minimal submanifold and, according to a theorem of J. Simons, the index of this submanifold is zero and the nullity of it is equal to the real dimension of the space formed by certain vector fields which are normal to the submanifold. In this paper, applying this theorem we shall obtain detailed results about the nullity of compact Kähler submanifolds in a complex projective space.

The paper is divided into three sections. § 1 is devoted to recall basic notions and results concerning minimal submanifolds. We shall also define the Killing nullity and the analytic nullity for complex submanifolds of a Kähler manifold. Then we restate the above theorem of Simons in the following form. The nullity of a Kähler submanifold coincides with the real dimension of the space of holomorphic sections of a holomorphic vector bundle, which we call normal bundle, over the submanifold.

In § 2, we shall consider Kähler C-spaces and especially compact Hermitian symmetric spaces imbedded in a complex projective space $P_N(\boldsymbol{C})$. Let M be a compact Hermitian symmetric space and put $M=G/U$ where G is a complex semi-simple Lie group and U a parabolic subgroup of G. By a recent result of H. Nakagawa and R. Takagi [7], we know that every imbedding of M in $P_N(\boldsymbol{C})$ is defined in a canonical way, by a holomorphic linear representation of G. By virtue of this result, we see that the normal bundle $N(M)$ over M imbedded in $P_N(\boldsymbol{C})$ is a holomorphic vector bundle associated to the principal bundle G over M by a representation of the group U. Thus we may apply the generalized Borel-Weil theorem to calculate the dimension of the space of holomorphic sections of $N(M)$. Studying in detail the representations of G and U which appear, we determine in this way the nullity of M in $P_N(C)$ for this case (Theorem 2).

We shall discuss in § 3 the Killing nullity and the analytic nullity of compact Kähler submanifolds in a complex projective space $P_N(C)$. We apply the results to estimate the minimal value for the nullities of these submanifolds. Also we see that, for a compact Hermitian symmetric space M imbedded in $P_N(\boldsymbol{C})$, any Jacobi field along M in $P_N(\boldsymbol{C})$ is defined by a 1-parameter family of Kähler imbeddings of M in $P_N(\boldsymbol{C})$. The author does not know whether the same conclusion holds in general or not.

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§ 1. Index and nullity of compact minimal submanifolds.

1.1. Minimal submanifolds. Let M be an r dimensional Riemannian manifold without boundary. Assume that M is imbedded isometrically in a Riemannian manifold $\bar{M}$. Let $\mathfrak{X}(M)$ be the set of all vector fields on M, and $\mathfrak{X}(M)^{\perp}$ the set of all vector fields on $\bar{M}$ defined along M and normal to M. Let g (resp. $\bar{g}$) denote the Riemannian metric of M (resp. $\bar{M}$), and ∇ (resp. $\bar{\nabla}$) the Riemannian connection of M (resp. $\bar{M}$). Let B denote the second fundamental form of M imbedded in $\bar{M}$. Then

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y) \qquad \text{for} \quad X, Y \in \mathfrak{X}(M),$$

$$\bar{\nabla}_X \xi = -A^{\xi}(X) + D_X \xi \qquad \text{for} \quad X \in \mathfrak{X}(M),\ \xi \in \mathfrak{X}(M)^{\perp},$$

where $A^{\xi}(X)$ is defined by $g(A^{\xi}(X), Y) = \bar{g}(B(X, Y), \xi)$ and $D_X \xi = (\bar{\nabla}_X \xi)^N$, $(\ )^N$ denoting the orthogonal projection to the component normal to M (S. Kobayashi and K. Nomizu [**2**]).

A submanifold M in $\bar{M}$ is said to be minimal, if $\sum_{i=1}^{r} B(e_i, e_i) = 0$ for each point $m \in M$, where $\{e_i\}_{i=1}^{r}$ is an orthonormal frame of $T_m(M)$, the tangent space at m of M.

1.2. Index and nullity. From now on we assume that M is a compact, minimal and oriented submanifold of $\bar{M}$.

For $\xi \in \mathfrak{X}(M)^{\perp}$, we define $\nabla^2 \xi \in \mathfrak{X}(M)^{\perp}$ by

$$\nabla^2 \xi(m) = \sum_{i=1}^{r} D_{e_i} D_{E_i} \xi \qquad \text{for} \quad m \in M,$$

where $\{e_i\}_{i=1}^{r}$ is an orthonormal frame of $T_m(M)$ and $\{E_i\}$ is a local orthonormal frame field such that $(E_i)_m = e_i$, and $\nabla_{e_j} E_i = 0$.

Let $T_m(M)^{\perp}$ denote the orthogonal complement of $T_m(M)$ in $T_m(\bar{M})$. We define $\tilde{A}(v)$ and $\bar{R}'(v) \in T_m(M)$ for $v \in T_m(M)^{\perp}$ by

$$\bar{g}(\tilde{A}(v), w) = \sum_{i=1}^{r} g(A^v(e_i), A^w(e_i))$$

for all $w \in T_m(M)^\perp$ and

$$\bar{R}'(v)=\sum_{i=1}^{r}(\bar{R}(e_i, v)e_i)^N,$$

where $\{e_i\}_{i=1}^r$ is an orthonormal frame of $T_m(M)$ and $\bar{R}$ is the curvature tensor of $\bar{M}$.

Now, we define a bilinear form I on $\mathfrak{X}(M)^\perp$ by

$$I(\xi, \eta)=\int_M \bar{g}(-\nabla^2\xi+\bar{R}'(\xi)-\tilde{A}(\xi), \eta)v_0$$

for $\xi, \eta \in \mathfrak{X}(M)^\perp$, where v_0 denotes the volume form on M determined by the metric g. It is known that I is a symmetric form on $\mathfrak{X}(M)^\perp$ (J. Simons [8]).

DEFINITION. The *index* of M in $\bar{M}$ is the dimension of the maximal subspace of $\mathfrak{X}(M)^\perp$ on which I is negative definite, and the *nullity* of M in $\bar{M}$ is $\dim_R\{\xi \in \mathfrak{X}(M)^\perp \mid I(\xi, \eta)=0,\ {}^\forall\eta \in \mathfrak{X}(M)^\perp\}$.

The index of M in $\bar{M}$ and the nullity of M in $\bar{M}$ are finite (J. Simons [8]). We denote the nullity of M in $\bar{M}$ by $n(M, \bar{M})$ or simply by $n(M)$.

An element $\xi \in \mathfrak{X}(M)^\perp$ is called a *Jacobi field* on M if it satisfies $\nabla^2\xi=\bar{R}'(\xi)-\tilde{A}(\xi)$. Then it is easy to see that

$$n(M)=\dim_R\{\xi \in \mathfrak{X}(M)^\perp \mid \xi \text{ is a Jacobi field on } M\}.$$

Let $\{f_t\}$ be a 1-parameter family of immersions of M into $\bar{M}$ such that f_0 is the immersion f of M into $\bar{M}$. If there exists a C^∞-map $F: M\times(-\varepsilon, \varepsilon)\to\bar{M}$ such that $f_t(m)=F(m, t)$, then we call $\{f_t\}$ a variation of f.

A variation $\{f_t\}$ of f defines a vector field E on $\bar{M}$ defined along M in the following way: let $\partial/\partial t$ be the standard vector field in $M\times(-\varepsilon, \varepsilon)$ and put

$$E(m)=dF(\partial/\partial t(m, 0))$$

for $m \in M$. We call E the variation field of $\{f_t\}$.

Let $A(t)$ be the volume of $f_t(M)$ that is $A(t)=\int_M v_t$ where v_t denotes the volume form on M determined by the induced metric $f_t^*\bar{g}$. Then it is known that $A'(0)=0$ and $A''(0)=I(E^N, E^N)$ (J. Simons [8]). Moreover, we know that, if $\{f_t\}$ are minimal immersions for all $t\in(-\varepsilon, \varepsilon)$, E^N is a Jacobi field on M.

By these observations, it follows in particular that if Z is a Killing vector field on $\bar{M}$, then Z^N is a Jacobi field on M.

The *Killing nullity* of M in $\bar{M}$ is now defined to be

$$\dim_R\{Z^N \in \mathfrak{X}(M)^\perp \mid Z \text{ is a Killing vector field on } \bar{M}\},$$

which we denote by $n_k(M, \bar{M})$ or simply $n_k(M)$. It is obvious that

$$n(M)\geqq n_k(M). \tag{1.1}$$

1.3. Kähler submanifolds. From now on, we assume that $\bar{M}$ is a Kähler manifold and M is a compact complex submanifold of $\bar{M}$. Then M is a Kähler submanifold of $\bar{M}$. Let J be the complex structure of $\bar{M}$. We also write J for the complex structure of M.

We shall call a vector field Z on $\bar{M}$ an analytic one if the Lie derivative of J with respect to Z vanishes. Since $\bar{M}$ is Kählerian, a vector field Z on $\bar{M}$ is analytic if and only if it satisfies $\bar{\nabla}_{JW}Z=J\bar{\nabla}_W Z$ for every vector field W on $\bar{M}$. We shall also call $\xi\in\mathfrak{X}(M)^{\perp}$ an analytic one if it satisfies $D_{JX}\xi=JD_X\xi$ for every $X\in\mathfrak{X}(M)$.

It is known that a compact Kähler submanifold M is minimal and that the index of M is equal to 0. Furthermore, an element $\xi\in\mathfrak{X}(M)^{\perp}$ is a Jacobi field on M if and only if ξ is analytic, and therefore

$$n(M)=\dim_{\boldsymbol{R}}\{\xi\in\mathfrak{X}(M)^{\perp} \mid \xi \text{ analytic}\} \tag{1.2}$$

(J. Simons [8]).

PROPOSITION 1.1. *Let Z be an analytic vector field on $\bar{M}$. Then Z^N is a Jacobi field on M.*

PROOF. For any $X\in\mathfrak{X}(M)$,

$$0=(\bar{\nabla}_{JX}Z-J\bar{\nabla}_X Z)^N=B(JX, Z^T)-JB(X, Z^T)+D_{JX}Z^N-JD_X Z^N\,,$$

where $(\ )^T$ denotes the tangential projection of $T_m(\bar{M})$ onto $T_m(M)$. Since $\bar{M}$ is a Kähler manifold, $\bar{\nabla}J=0$, and it follows that $B(JX, Z^T)=JB(X, Z^T)$. Thus $D_{JX}Z^N=JD_X Z^N$, and the proposition follows from what we have recalled above. q. e. d.

DEFINITION. Let $\mathfrak{a}(\bar{M})$ denote the vector space of all analytic vector fields on $\bar{M}$. We define the analytic nullity of M in $\bar{M}$ by $\dim_{\boldsymbol{R}}\{Z^N\in\mathfrak{X}(M)^{\perp} \mid Z\in\mathfrak{a}(\bar{M})\}$ and denote it by $n_a(M, \bar{M})$ or simply $n_a(M)$.

By Proposition 1.1, we get immediately

COROLLARY 1.1.

$$n(M)\geqq n_a(M)\,.$$

In particular, if the ambient space $\bar{M}$ is compact, then a Killing vector field on $\bar{M}$ becomes analytic, and therefore we have

$$n(M)\geqq n_a(M)\geqq n_k(M)\,.$$

1.4. Holomorphic sections of the normal bundle. Let $T(M)$ (resp. $T(\bar{M})$) be the holomorphic tangent vector bundle of M (resp. $\bar{M}$). The quotient bundle $T(\bar{M})|_M/T(M)$ is called the normal bundle of M in $\bar{M}$ and is denoted by $N(M)$. Obviously $N(M)$ is a holomorphic vector bundle of M. Let $\Gamma(N(M))$ denote the vector space of all holomorphic sections of $N(M)$.

PROPOSITION 1.2. *The vector space $\Gamma(N(M))$ is canonically isomorphic to the space formed by analytic elements $\xi\in\mathfrak{X}(M)^{\perp}$.*

PROOF. We put $\dim M=n$ and $\dim \bar{M}=n+p$. For any point $m_0\in M$, we choose a neighborhood $\bar{U}$ of m_0 and a local coordinate system $(z^1, \cdots, z^n, z^{n+1}, \cdots, z^{n+p})$, $z^j=x^j+\sqrt{-1}y^j$ such that $(z^1|_U, \cdots, z^n|_U)$ is a local coordinate system in $U=M\cap\bar{U}$.

Let $T_m^{1,0}(\bar{M})$ (resp. $T_m^{1,0}(M)$) denote the space consisting of all complex tangent vector of type $(1,0)$ at m of $\bar{M}$ (resp. M). Let $(\ )^{\perp}$ denote the natural projection from $T_m^{1,0}(\bar{M})$ to $N_m(M)=T_m^{1,0}(\bar{M})/T_m^{1,0}(M)$. Let π be the projection from $N(M)$ to M. Every element $v\in N(M)$, $\pi(v)\in U$, can be expressed uniquely in the form $v=\sum_i \lambda^i\Big(\frac{\partial}{\partial z^{n+i}}\Big)^{\perp}$, and this gives a local triviality of $N(M)$ on U.

Since $\bar{M}$ is Kählerian, $\bar{\nabla}$ satisfies the following formulas.

$$\bar{\nabla}_{\partial/\partial z^k}\frac{\partial}{\partial \bar{z}^l}=0 \quad\text{and}\quad \bar{\nabla}_{\partial/\partial \bar{z}^k}\frac{\partial}{\partial z^l}=0$$

for $k, l=1, \cdots, n+p$. From these formulas, we can see easily

$$D_{\partial/\partial z^i}\Big(\frac{\partial}{\partial \bar{z}^{n+j}}\Big)^N=0 \quad\text{and}\quad D_{\partial/\partial \bar{z}^i}\Big(\frac{\partial}{\partial z^{n+j}}\Big)^N=0 \tag{1.3}$$

for $1\leqq i\leqq n$, $1\leqq j\leqq p$. Since $\bar{\nabla}J=0$, we see that

$$\begin{cases} D_{\partial/\partial z^i}\Big(\dfrac{\partial}{\partial z^{n+j}}\Big)^N \text{ is of type } (1,0), \\ D_{\partial/\partial \bar{z}^i}\Big(\dfrac{\partial}{\partial \bar{z}^{n+j}}\Big)^N \text{ is of type } (0,1). \end{cases} \tag{1.4}$$

For a holomorphic section $V=\sum_{j=1}^{p} f_j\Big(\frac{\partial}{\partial z^{n+j}}\Big)^{\perp}\in\Gamma(N(M))$, where f_j are holomorphic, we define $\xi(V)\in\mathfrak{X}(M)^{\perp}$ by

$$\xi(V)=\sum_{j=1}^{p}\Big\{f_j\Big(\frac{\partial}{\partial z^{n+j}}\Big)^N+\bar{f}_j\Big(\frac{\partial}{\partial \bar{z}^{n+j}}\Big)^N\Big\}.$$

Then we get

$$D_{\partial/\partial z^i}\xi(V)=\sum_{j=1}^{p}\Big\{\frac{\partial f_j}{\partial z^i}\Big(\frac{\partial}{\partial z^{n+j}}\Big)^N+f_j D_{\partial/\partial z^i}\Big(\frac{\partial}{\partial z^{n+j}}\Big)^N\Big\}.$$

Hence by (1.4)

$$D_{J(\partial/\partial z^i)}\xi(V)=JD_{\partial/\partial z^i}\xi(V).$$

Similarly we have

$$D_{J(\partial/\partial \bar{z}^i)}\xi(V)=JD_{\partial/\partial \bar{z}^i}\xi(V).$$

Thus $\xi(V)$ is analytic.

Conversely suppose that $\eta\in\mathfrak{X}(M)^{\perp}$ is analytic. Then we can write locally

$\eta=\sum_{j=1}^{p}\left\{a_j\left(\frac{\partial}{\partial x^{n+j}}\right)^N+b_j\left(\frac{\partial}{\partial y^{n+j}}\right)^N\right\}$. If we put $f_i=a_j+\sqrt{-1}b_j$ $(1\leqq j\leqq p)$, $\eta=\sum_{j=1}^{p}\left\{f_j\left(\frac{\partial}{\partial z^{n+j}}\right)^N+\bar{f}_j\left(\frac{\partial}{\partial \bar{z}^{n+j}}\right)^N\right\}$. Then we have

$$D_{\partial/\partial\bar{z}^i}\eta=\sum_{j=1}^{p}\left\{\frac{\partial f_j}{\partial\bar{z}^i}\left(\frac{\partial}{\partial z^{n+j}}\right)^N+\bar{f}_jD_{\partial/\partial\bar{z}^i}\left(\frac{\partial}{\partial\bar{z}^{n+j}}\right)^N+\frac{\partial\bar{f}_j}{\partial\bar{z}^i}\left(\frac{\partial}{\partial\bar{z}^{n+j}}\right)^N\right\}.$$

Since η is analytic, $D_{\partial/\partial\bar{z}^i}\eta$ is of type (0, 1). Therefore f_j are holomorphic. Putting $V=\sum_j f_j\left(\frac{\partial}{\partial z^{n+j}}\right)^{\perp}\in\Gamma(N(M))$, we have $\xi(V)=\eta$. Thus we get a linear isomorphism $\xi:\Gamma(N(M))\to\{\xi\in\mathfrak{X}(M)^{\perp}\mid\xi \text{ is analytic}\}$. q. e. d.

By Proposition 1.2 and (1.2), we get $n(M)=\dim_{\boldsymbol{R}}\Gamma(N(M))$. In other words, $SN(M)$ being the sheaf of local holomorphic sections of $N(M)$, we have

$$n(M)=\dim_{\boldsymbol{R}}H^0(M, SN(M))\,. \tag{1.5}$$

§ 2. The nullity of compact Hermitian symmetric spaces in a complex projective space.

2.1. Kähler C-spaces. A simply connected compact Kähler homogeneous manifold is called a Kähler C-space. Let G be a simply connected complex semi-simple Lie group. A complex Lie subgroup U of G is called a parabolic subgroup if U contains a maximal solvable Lie subgroup of G. The quotient manifold $M=G/U$ is a Kähler C-space and every Kähler C-space is obtained in this way (H. C. Wang [**9**]).

Let $\mathfrak{g}$ be the Lie algebra of G, (,) the Killing form of $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote the dual space of $\mathfrak{h}$ by $\mathfrak{h}^*$, and we shall identify an element $\lambda\in\mathfrak{h}^*$ with the element $H_\lambda\in\mathfrak{h}$, which is defined by $\lambda(H)=(H_\lambda, H)$ for any $H\in\mathfrak{h}$. We denote by Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ be the fundamental root system and Π_1 be a subsystem of Π, where l is the rank of $\mathfrak{g}$. We may assume that Π is the system of simple roots with respect to a linear order in the real part $\mathfrak{h}_0=\{\alpha\mid\alpha\in\Delta\}_{\boldsymbol{R}}$ of $\mathfrak{h}$.

Let $Z\in\mathfrak{h}$ be the element defined by $(Z, \alpha_j)=0$ for $\alpha_j\in\Pi_1$ and $(Z, \alpha_k)=1$ for $\alpha_k\notin\Pi_1$. Let $\mathfrak{g}_1$ (resp. $\mathfrak{n}^+$) denote the 0-eigenspace (resp. the sum of positive eigenspaces) of $\operatorname{ad} Z$. Then $\mathfrak{u}=\mathfrak{g}_1+\mathfrak{n}^+$ contains a maximal solvable subalgebra of $\mathfrak{g}$. Note that $\mathfrak{u}$ is the normalizer of $\mathfrak{n}^+$ in $\mathfrak{g}$. The normalizer U of $\mathfrak{n}^+$ in G is a parabolic subgroup of G which corresponds to the Lie subalgebra $\mathfrak{u}$. Conversely it is known that every parabolic subgroup of G is obtained in this way from a subsystem Π_1 of Π. Let G_1 (resp. N^+) be a connected Lie subgroup with the Lie algebra $\mathfrak{g}_1$ (resp. $\mathfrak{n}^+$). Then $U=G_1\cdot N^+$ (semi-direct).

In the following, we fix a Cartan subalgebra $\mathfrak{h}$, a linear order in $\mathfrak{h}_0$ and Π_1. Take a compact real form $\mathfrak{g}_u$ such that $\mathfrak{g}_u\cap\mathfrak{h}=\sqrt{-1}\mathfrak{h}_0$. The connected subgroup G_u corresponding to the Lie algebra $\mathfrak{g}_u$ is a maximal compact subgroup of G. If we put $K=G_u \cap U$, then $M=G/U=G_u/K$ (as C^∞-manifold).

We put $\lambda^*=\frac{2\lambda}{(\lambda, \lambda)}$ for $\lambda \in \mathfrak{h}$. Let Λ be an integral form strongly associated with Π_1, namely such that $(\Lambda, \alpha_i^*)=0$ for $\alpha_i \in \Pi_1$ and $(\Lambda, \alpha_i^*)>0$ for $\alpha_i \notin \Pi_1$. We shall denote by (ρ_Λ, V) the irreducible representation of G with the highest weight Λ, and let $P(V)$ be the complex projective space consisting of all 1-dimensional subspaces of V. Since the dimension of the weight space (v) in V for the highest weight Λ is equal to 1, (v) is an element of $P(V)$. Moreover G acts canonically on $P(V)$ via the representation (ρ_Λ, V), and it is known that U coincides with the isotropy subgroup of G at (v). Therefore we get a G-equivariant imbedding $f_\Lambda: M \to P(V)$.

REMARK. An imbedding $f: M \to P(V)$ is said to be full if there exists no totally geodesic submanifold of $P(V)$ containing $f(M)$. Then f_Λ is a full imbedding, and conversely every full Kähler imbedding of a Kähler C-space in $P_N(\boldsymbol{C})$ is obtained in this way (see H. Nakagawa and R. Takagi [**7**]).

2.2. The generalized Borel-Weil theorem. In view of (1.5), to determine the nullity of M it suffices to know the dimension of $H^0(M, SN(M))$. In our case, we shall calculate this by applying Bott's generalized Borel-Weil theorem. Let us first recall Bott's results (c. f. R. Bott [**1**]).

Let D (resp. D_1) be the set of all dominant integral forms of $\mathfrak{g}$ (resp. $\mathfrak{g}_1$) with respect to the Cartan subalgebra $\mathfrak{h}$. Choose an irreducible representation $(\rho^1_{-\xi}, W_{-\xi})$ of G_1 with the lowest weight $-\xi$, $\xi \in D_1$. We extend it to the representation of U whose restriction to N^+ is trivial. We shall denote by $E_{W_{-\xi}}$ the homogenenous vector bundle over $M=G/U$ associated by $\rho^1_{-\xi}$ with the principal bundle $G \to M$ with group U. Put $D_1^0=\{\xi \in D_1 \mid \xi+\delta$ is regular$\}$, where δ is the half of sum of all positive roots. Let W be the Weyl group of $\mathfrak{g}$, W^1 is the set of $\sigma \in W$ such that $\sigma(D) \subset D_1$, $n(\sigma)$ the index of σ and $(\rho_{-\lambda}, V_{-\lambda})$ the irreducible representation of G with the lowest weight $-\lambda$.

THEOREM OF BOTT [**1**], (c. f. B. Kostant [**3**]). *The notation being as above,*

(1) *if* $\xi \notin D_1^0$, *then* $H^j(M, SE_{W_{-\xi}})=(0)$ *for all* $j=0, 1, \cdots$;

(2) *if* $\xi \in D_1^0$, ξ *is expressed uniquely as* $\xi=\sigma(\delta+\lambda)-\delta$, *where* $\lambda \in D$ *and* $\sigma \in W^1$, *and*

$$H^j(M, SE_{W_{-\xi}})=(0) \qquad \text{for} \quad j \neq n(\sigma),$$

$$\dim H^{n(\sigma)}(M, SE_{W_{-\xi}})=\dim V_{-\lambda}.$$

In particular, if $\xi \in D$, $H^j(M, SE_{W_{-\xi}})=(0)$ $(j \geqq 1)$ *and* $\dim H^0(M, \mathfrak{S}E_{W_{-\xi}})=\dim V_{-\xi}$.

2.3. Reduction to the case of full imbeddings.

THEOREM 1. *Let M be a Kähler C-space imbedded in $P_N(\boldsymbol{C})$. Furthermore assume that M is fully imbedded in a totally geodesic submanifold $P_n(\boldsymbol{C})$ of $P_N(\boldsymbol{C})$. Then*

$$n(M, P_N(\boldsymbol{C}))=n(M, P_n(\boldsymbol{C}))+2(n+1)(N-n).$$

From this theorem, we have immediately:

COROLLARY 2.1. *If $P_n(\boldsymbol{C})$ is imbedded in $P_N(\boldsymbol{C})$ as a totally geodesic submanifold, then*

$$n(P_n(\boldsymbol{C}))=2(n+1)(N-n).$$

PROOF OF THEOREM 1. Put $P_N(\boldsymbol{C})=\hat{G}/\hat{U}$, where $\hat{G}=SL(N+1, \boldsymbol{C})$ and $\hat{U}$ is the complex Lie subgroup of $\hat{G}$ defined by

$$\hat{U}=\left\{A\in\hat{G}\,;\ A=\left(\begin{array}{c|c} a & c \\ \hline 0 & B\end{array}\right),\ a\in\boldsymbol{C}^*,\ B\in GL(N, \boldsymbol{C})\right\},$$

where $\boldsymbol{C}^*=\boldsymbol{C}-\{0\}$. We may assume $P_n(\boldsymbol{C})=\tilde{G}/\tilde{U}$, where

$$\tilde{G}=\left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & E\end{array}\right)\in\hat{G}\,;\ A\in SL(n+1, \boldsymbol{C}),\ E=\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}\right\}$$

and $\tilde{U}=\tilde{G}\cap\hat{U}$.

By the remark at the end of 2.1, we may assume that the imbedding f: $M\to P_n(\boldsymbol{C})$ is induced by an irreducible representation $\rho_A: G\to GL(n+1, \boldsymbol{C})$. Since G is semi-simple, $\rho_A(G)\subset SL(n+1, \boldsymbol{C})$. Therefore ρ_A induces a homomorphism $G\to\tilde{G}$, which we denote also by ρ_A.

We denote the Lie algebra of $\hat{G}$, $\hat{U}$, $\tilde{G}$ and $\tilde{U}$ by $\hat{\mathfrak{g}}$, $\hat{\mathfrak{u}}$, $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{u}}$, respectively. Since $\hat{\mathfrak{u}}$ (resp. $\tilde{\mathfrak{u}}$) is invariant under the adjoint action of $\hat{U}$ (resp. $\tilde{U}$) on $\hat{\mathfrak{g}}$ (resp. $\tilde{\mathfrak{g}}$), a representation of $\hat{U}$ (resp. $\tilde{U}$) on $\hat{\mathfrak{g}}/\hat{\mathfrak{u}}$ (resp. $\tilde{\mathfrak{g}}/\tilde{\mathfrak{u}}$) is defined, which we denote also by ad. Then we see that $T(P_N(\boldsymbol{C}))|_M$ (resp. $T(P_n(\boldsymbol{C}))|_M$) is the vector bundle associated with the bundle $G\to M=G/U$ by the representation $(ad\circ\rho_A, \hat{\mathfrak{g}}/\hat{\mathfrak{u}})$ (resp. $(ad\circ\rho_A, \tilde{\mathfrak{g}}/\tilde{\mathfrak{u}})$). Since

$$\rho_A(x)=\left(\begin{array}{c|c|c} a(x) & * & 0 \\ \hline 0 & \rho_1(x) & 0 \\ \hline 0 & 0 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\end{array}\right)\in\tilde{U}$$

for $x\in U$, where $a(x)\in\boldsymbol{C}^*$ and $\rho_1(x)\in GL(n, \boldsymbol{C})$, we see that $ad\circ\rho_A(x): \hat{\mathfrak{g}}/\hat{\mathfrak{u}}\to\hat{\mathfrak{g}}/\hat{\mathfrak{u}}$ (resp. $\tilde{\mathfrak{g}}/\tilde{\mathfrak{u}}\to\tilde{\mathfrak{g}}/\tilde{\mathfrak{u}}$) is given by

$$\hat{\mathfrak{g}}/\hat{\mathfrak{u}}\ni\left\{\begin{array}{l} 1\,\{ \\ n\,\{ \\ N-n\,\{\end{array}\left(\begin{array}{c|c} \overbrace{*}^{1} & \\ \hline \xi & * \\ \hline \eta & \end{array}\right)\right\}\longrightarrow\left\{\left(\begin{array}{c|c} * & \\ \hline \rho_1(x)\xi a(x)^{-1} & * \\ \hline \eta a(x)^{-1} & \end{array}\right)\right\}\in\hat{\mathfrak{g}}/\hat{\mathfrak{u}}$$

$$\left(\text{resp. } \mathfrak{g}/\tilde{\mathfrak{u}} \ni \left\{ \begin{pmatrix} \begin{array}{c|c} \overbrace{*}^{1} & * \\ \hline \xi & \end{array} \end{pmatrix} \begin{array}{l} \}1 \\ \}n \end{array} \right\} \longrightarrow \left\{ \begin{pmatrix} \begin{array}{c|c} * & * \\ \hline \rho_1(x)\xi a(x)^{-1} & \end{array} \end{pmatrix} \right\} \in \mathfrak{g}/\tilde{\mathfrak{u}} \right).$$

This implies that $T(P_N(\boldsymbol{C}))|_M = T(P_n(\boldsymbol{C}))|_M \oplus \underbrace{E \oplus \cdots \oplus E}_{N-n}$ where E is the line bundle associated with the principal bundle $G \to M$ by the representation a^{-1}: $U \ni x \to a(x)^{-1} \in \boldsymbol{C}^*$. Therefore we see that as holomorphic vector bundles

$$N(M, P_N(\boldsymbol{C})) \cong N(M, P_n(\boldsymbol{C})) \oplus \underbrace{E \oplus \cdots \oplus E}_{N-n}$$

where $N(M, \bar{M})$ is the normal bundle of M in $\bar{M}$ ($=P_N(\boldsymbol{C})$ or $P_n(\boldsymbol{C})$). Since the lowest weight of a^{-1} is $-\Lambda$,

$$\dim_{\boldsymbol{C}} H^0(M, SE) = \dim_{\boldsymbol{C}} V_{-\Lambda} = n+1$$

by Theorem of Bott, which proves our theorem.

2.4. Result on compact Hermitian symmetric spaces. Let $M=G/U$ be a compact Hermitian symmetric space fully imbedded in $P_n(\boldsymbol{C})$. It is well known that M is a product $M=M_1 \times \cdots \times M_k$, where $M_s=G_s/U_s$ are compact irreducible Hermitian symmetric spaces ($1 \leqq s \leqq k$). For each s, let Π^s be a fundamental root system of the Lie algebra of G_s and $\Pi_1^s = \Pi^s - \{\alpha_{i_s}\}$ be a subsystem of Π^s which defines the parabolic subgroup U_s of G_s. We may assume that $\Pi = \Pi^1 \cup \cdots \cup \Pi^k$ and Π_1, which defines the parabolic subgroup U of G, is $\Pi - \{\alpha_{i_1}, \cdots, \alpha_{i_k}\}$. Let $\{\omega_1, \cdots, \omega_l\}$ be the fundamental weights with respect to $\Pi = \{\alpha_1, \cdots, \alpha_l\}$. A weight Λ strongly associated with Π_1 is then of the form $\Lambda = \sum_{s=1}^{k} p_s \omega_{i_s}$ ($p_s > 0$).

We denote by (ρ_Λ^*, V^*) the representation of G contragredient to (ρ_Λ, V). Since the highest weight space (v) of V is U-invariant, ρ_Λ defines a representation of U on (v), which we denote by $(h, (v))$. Then $V^* \otimes V$ is a G-module via the representation $\rho_\Lambda^* \otimes \rho_\Lambda$ and $V^* \otimes (v)$ is a U-module via $\rho_\Lambda^*|_U \otimes h$, and these spaces are related as follows.

PROPOSITION 2.1. *Suppose that the G-module $V^* \otimes V$ and the U-module $V^* \otimes (v)$ are decomposed into direct sums:*

$$V^* \otimes V = V^{\lambda_1} \oplus \cdots \oplus V^{\lambda_l} \qquad (\lambda_1 \geqq \cdots \geqq \lambda_l),$$

$$V^* \otimes (v) = W^{\mu_1} \oplus \cdots \oplus W^{\mu_m} \qquad (\mu_1 \geqq \cdots \geqq \mu_m) \quad (as\ G_1\text{-}module),$$

where V^{λ_i} (resp. W^{μ_j}) is an irreducible G-module (resp. G_1-module) with highest weight λ_i (resp. μ_j). Then, we have $l=m$ and $\lambda_i = \mu_i$ for all $i=1, \cdots, l$.

We prepare some lemmas to prove this proposition. Let Δ^λ denote the set of weights of the irreducible representation of G with the highest weight $\lambda\in D$.

LEMMA 2.1. *For any* $\mu\in\Delta^{\lambda}$,

$$|\lambda|\geqq|\mu|$$

and equality holds if and only if there exists $\sigma\in W$ *such that* $\mu=\sigma\lambda$.

PROOF. Take $\tau\in W$ so that $\tau\mu\in D$. Since $\tau\mu\in\Delta^{\lambda}$, $\lambda=\tau\mu+\sum_{i=1}^{l}m_i\alpha_i$ $(m_i\geqq 0)$. For $|\tau\mu|=|\mu|$,

$$|\lambda|^2=|\mu|^2+2\sum_{i=1}^{l}m_i(\tau\mu, \alpha_i)+|\sum_{i=1}^{l}m_i\alpha_i|^2 .$$

Since $(\tau\mu, \alpha_i)\geqq 0$ for $i=1, \cdots, l$, we get $|\lambda|\geqq|\mu|$. Moreover, if equality holds, we must have $\sum_{i=1} m_i\alpha_i=0$ and hence $\lambda=\tau\mu$. The converse is obvious. q. e. d.

LEMMA 2.2. *Suppose that*

$$|\omega_j|\leqq|\omega_i| \quad \textit{for} \quad i=1, \cdots, l. \tag{2.1}$$

Then ω_j *has the following property: for each* $\mu\in\Delta^{\omega_j}$ $(\mu\neq 0)$

$$\mu=\sigma\omega_j \quad \textit{for some} \quad \sigma\in W. \tag{2.2}$$

PROOF. Take $\tau\in W$ so that $\tau\mu\in D$. Then $\tau\mu=\sum_{i=1}^{l}n_i\omega_i$ $(n_i\geqq 0)$. Note that there exists an integer k such that $n_k>0$. Since $(\omega_i, \omega_j)\geqq 0$ for any i, j,

$$|\mu|^2=|\sum_{i=1}^{l}n_i\omega_i|^2=\sum_{i,j=1}^{l}n_in_j(\omega_i, \omega_j)\geqq|\omega_k|^2 .$$

By (2.1), we get $|\mu|\geqq|\omega_j|$. Hence $|\mu|=|\omega_j|$ and $\mu=\sigma\omega_j$ for some $\sigma\in W$ by Lemma 2.1. q. e. d.

LEMMA 2.3. *Let* M *be the compact irreducible Hermitian symmetric space associated with* $\Pi_1=\Pi-\{\alpha_j\}$. *If* ω_j *has the property* (2.2), *then*

$$(\omega_j-\mu, \alpha_j^*)\geqq 0 \quad \textit{for any} \quad \mu\in\Delta^{\omega_j} .$$

PROOF. Since $(\omega_j-\mu, \alpha_j^*)\geqq 0$ if $\mu=0$, we assume $\mu=\sigma\omega_j$ for $\sigma\in W$. Then

$$(\mu, \alpha_j^*)=(\sigma\omega_j, \alpha_j^*)=(\omega_j, \sigma^{-1}\alpha_j^*)=\left(\omega_j, \frac{2\sigma^{-1}(\alpha_j)}{(\alpha_j, \alpha_j)}\right).$$

Since M is Hermitian symmetric, we can write a root $\sigma^{-1}(\alpha_j)$ as $\sigma^{-1}(\alpha_j)=\sum_{i=1}^{l}m_i\alpha_i$ with $m_j\leqq 1$. Thus

$$\left(\omega_j, \frac{2\sigma^{-1}(\alpha_j)}{(\alpha_j, \alpha_j)}\right)=\left(\omega_j, \frac{2\sum m_i\alpha_i}{(\alpha_j, \alpha_j)}\right)=m_j\leqq 1 ,$$

and hence $(\omega_j-\mu, \alpha_j^*)\geqq 0$. q. e. d.

LEMMA 2.4. *Let* M *be the compact irreducible Hermitian symmetric space* *associated with* $\Pi_1=\Pi-\{\alpha_j\}$. *Then*

$$(\omega_j-\mu, \alpha_j^*)\geqq 0 \qquad \text{for any } \mu\in\Delta^{\omega_j}.$$

PROOF. It is easy to see that a compact irreducible Hermitian symmetric space satisfies (2.1), unless it is one of the following types:

$$AIII, \quad CI, \quad DIII.$$

If M is of type $AIII$ or $DIII$, we shall check that ω_j has the property (2.2).

Assume that M is of type $AIII$. Then

$$\Pi=\{\alpha_1=\varepsilon_1-\varepsilon_2, \cdots, \alpha_l=\varepsilon_l-\varepsilon_{l+1}\}$$

where $(\varepsilon_i, \varepsilon_k)=1/2(l+1)\delta_{ik}$,

$$\Pi_1=\Pi-\{\alpha_j\} \qquad (1\leqq j\leqq l),$$

$$\omega_j=\varepsilon_1+\cdots+\varepsilon_j$$

and

$$\Delta^{\omega_j}=\{\varepsilon_{i_1}+\cdots+\varepsilon_{i_j}\,;\ i_1<\cdots<i_j\}.$$

Therefore we see easily that if $\mu\in\Delta^{\omega_j}$ is an element of D that is $(\mu, \alpha_i)\geqq 0$ for all $i=1, \cdots, l$, then $\mu=\omega_j$. Thus ω_j has the property (2.2).

Assume that M is of type $DIII$. Then

$$\Pi=\{\alpha_1=\varepsilon_1-\varepsilon_2, \cdots, \alpha_{l-1}=\varepsilon_{l-1}-\varepsilon_l, \alpha_l=\varepsilon_{l-1}+\varepsilon_l\}$$

where $(\varepsilon_i, \varepsilon_k)=1/4(l-1)\delta_{ik}$,

$$\Pi_1=\Pi-\{\alpha_{l-1}\} \quad \text{or} \quad \Pi-\{\alpha_l\},$$

$$\omega_{l-1}=\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_{l-1}-\varepsilon_l),$$

and

$$\omega_l=\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_{l-1}+\varepsilon_l).$$

We see easily that

$$\{\sigma(\omega_{l-1})\,;\ \sigma\in W\}=\left\{\frac{1}{2}(\theta_1\varepsilon_1+\cdots+\theta_l\varepsilon_l)\,;\ \theta_i=\pm 1,\ \Pi\theta_i=-1\right\};$$

$$\{\sigma(\omega_l)\,;\ \sigma\in W\}=\left\{\frac{1}{2}(\theta_1\varepsilon_1+\cdots+\theta_l\varepsilon_l)\,;\ \theta_i=\pm 1,\ \Pi\theta_i=1\right\}.$$

We know that the dimension of the representation space of $\rho_{\omega_{l-1}}$ or ρ_{ω_l} is 2^{l-1}. Since the number of elements of $\{\sigma(\omega_{l-1})\,;\ \sigma\in W\}$ or $\{\sigma(\omega_l)\,;\ \sigma\in W\}$ is 2^{l-1}, ω_j ($j=l-1$ or l) has the property (2.2).

Thus our lemma follows from Lemma 2.2 and Lemma 2.3 if M is not of type CI.

Assume that M is of type CI. We can put

$$G=Sp(l, \boldsymbol{C})=\{x\in GL(2l, \boldsymbol{C})\,;\ {}^t xJ'x=J'\}$$

where

$$J'=\begin{pmatrix} 0 & \overbrace{\begin{matrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{matrix}}^{l} \\ \overbrace{\begin{matrix} & & -1\\ & \cdot^{\cdot^{\cdot}} & \\ -1 & & \end{matrix}}^{l} & 0 \end{pmatrix}$$

Let

$$\mathfrak{h}=\left\{\begin{pmatrix} x_1 & & & & & 0\\ & \ddots & & & & \\ & & x_l & & & \\ & & & -x_l & & \\ & & & & \ddots & \\ 0 & & & & & -x_1 \end{pmatrix};\ x_i\in \boldsymbol{C}\right\}.$$

We denote by x_i the element ζ of $\mathfrak{h}$ which is defined by

$$\zeta\left(\begin{pmatrix} x_1 & & & & & 0\\ & \ddots & & & & \\ & & x_l & & & \\ & & & -x_l & & \\ & & & & \ddots & \\ 0 & & & & & -x_1 \end{pmatrix}\right)=x_i\,.$$

Then

$$\Pi=\{\alpha_1=x_1-x_2, \cdots, \alpha_{l-1}=x_{l-1}-x_l, \alpha_l=2x_l\}\,,$$

$$\Pi_1=\Pi-\{\alpha_l\}$$

and $\omega_l=x_1+\cdots+x_l$.

Since ρ_{ω_l} is an irreducible component to the l-th alternative representation of $Sp(l, \boldsymbol{C})$, a weight μ of ρ_{ω_l} is expressed by

$$\mu=(x_{i_1}+\cdots+x_{i_r})-(x_{j_1}+\cdots+x_{j_t})$$

where $i_1<\cdots<i_r$, $j_1<\cdots<j_t$ and $r+t=l$. Therefore it is clear that

$$(\omega_l-\mu, \alpha_l^*)\geqq 0\,. \qquad \text{q. e. d.}$$

LEMMA 2.5. *Let M be a compact Hermitian symmetric space. Then*

$$(\Lambda-\mu, \alpha_{i_s}^*)\geqq 0 \quad (s=1, \cdots, k) \qquad \textit{for any } \mu\in\Delta^{\Lambda}\,.$$

PROOF. Note that ρ_Λ is an irreducible component of the representation

$$\underbrace{\rho_{\omega i_1}\otimes\cdots\otimes\rho_{\omega i_1}}_{p_1}\otimes\cdots\otimes\underbrace{\rho_{\omega i_k}\otimes\cdots\otimes\rho_{\omega i_k}}_{p_k}.$$

Then a weight μ of ρ_Λ can be written as

$$\mu=\mu_1^1+\cdots+\mu_{p_1}^1+\cdots+\mu_1^k+\cdots+\mu_{p_k}^k \qquad \text{where } \mu_l^s\in\Delta^{\omega_{i_s}}.$$

But if $s\neq t$, we have

$$(\nu, \alpha_{i_s}^*)=0 \qquad \text{for any } \nu\in\Delta^{\omega_{i_t}}.$$

Thus

$$(\Lambda-\mu, \alpha_{i_s}^*)=(p_s\omega_{i_s}-(\mu_1^s+\cdots+\mu_{p_s}^s), \alpha_{i_s}^*)$$
$$=\sum_{j=1}^{p_s}(\omega_{i_s}-\mu_j^s, \alpha_{i_s}^*).$$

Since $(\omega_{i_s}-\mu_j^s, \alpha_{i_s}^*)\geqq 0$ by Lemma 2.4., we get $(\Lambda-\mu, \alpha_{i_s}^*)\geqq 0$. q. e. d.

Let W_1 be the subgroup of the Weyl group W generated by the reflections corresponding to $\alpha_i\in\Pi_1$ and denote by Δ_1^+ the set of positive roots $\alpha=\sum_{i=1}^{l}m_i\alpha_i$ such that $m_{i_s}=0$ for $s=1, \cdots, k$.

LEMMA 2.6. *Suppose $\mu\in\Delta^\Lambda$, $\gamma_1\in W_1$ and put $\delta_1=\frac{1}{2}\sum_{\alpha\in\Delta_1^+}\alpha$. If $\Lambda-\mu+\gamma_1\delta_1-\delta_1\in D_1$, then $\Lambda-\mu+\gamma_1\delta_1-\delta_1\in D$.*

PROOF. Since $\gamma_1\delta_1-\delta_1=-\sum_{\alpha_i\in\Pi_1}m_i\alpha_i$ with $m_i\geqq 0$, we get $(\gamma_1\delta_1-\delta_1, \alpha_{i_s}^*)\geqq 0$ for $1\leqq s\leqq k$. By Lemma 2.5 $(\Lambda-\mu, \alpha_{i_s}^*)\geqq 0$. Thus

$$(\Lambda-\mu+\gamma_1\delta_1-\delta_1, \alpha_{i_s}^*)\geqq 0 \qquad \text{for } s=1, \cdots, k.$$

q. e. d.

Now for $\lambda\in D$, let $M(\lambda)$ denote the multiplicity of ρ_λ in $\rho_\Lambda^*\otimes\rho_\Lambda$. For $\mu\in D_1$, let $M_1(\mu)$ denote the multiplicity of the irreducible representation ρ_μ^1 of G_1 with highest weight μ in $\rho_\Lambda^*|_{G_1}\otimes h|_{G_1}$.

LEMMA 2.7. *For $\lambda\in D$ and $\mu\in D_1$, we have*

$$(1) \qquad M(\lambda)=\sum_{\sigma\in W}\det\sigma\cdot m(\Lambda+\delta-\sigma(\lambda+\delta)),$$

$$(2) \qquad M_1(\mu)=\sum_{\tau\in W_1}\det\tau\cdot m(\Lambda+\delta-\tau(\mu+\delta)),$$

where, for an integral form ν, $m(\nu)$ denotes the multiplicity of ν as a weight of ρ_Λ.

PROOF. For $\lambda\in D$, let χ_λ denote the character of ρ_λ. Since there is a canonical bijection between representations of G and representations of the maximal compact Lie subgroup G_u on complex vector spaces, it is enough to prove the assertions for G_u.

Let w denote the number of elements of W, and put $H_u=G_u\cap\exp\mathfrak{h}$ and

$$\xi_\nu(x)=\sum_{\sigma\in W}\det\sigma\cdot e^{\sigma\nu(x)} \text{ for } x=\exp X\in H_u=G_u\cap\exp\mathfrak{h}.$$

By Weyl's integral formula and Weyl's characteristic formula, we have

$$
\begin{aligned}
M(\lambda) &= \int_{G_u} \chi_\Lambda \cdot \bar{\chi}_\Lambda \cdot \bar{\chi}_\lambda dg \\
&= \frac{1}{w}\int_{H_u} \chi_\Lambda \cdot \bar{\chi}_\Lambda \cdot \bar{\chi}_\lambda |\xi_\delta|^2 dx \\
&= \frac{1}{w}\int_{H_u} \xi_{\Lambda+\delta} \cdot \bar{\chi}_\Lambda \cdot \bar{\xi}_{\lambda+\delta} dx \\
&= \frac{1}{w}\sum_{\alpha,\beta\in W} \det(\alpha\beta)\cdot \int_{H_u} e^{\alpha(\Lambda+\delta)-\beta(\lambda+\delta)}\circ \bar{\chi}_\Lambda dx \\
&= \frac{1}{w}\sum_{\alpha,\beta\in W} \det(\alpha\beta)\cdot m(\alpha(\Lambda+\delta)-\beta(\lambda+\delta)) \\
&= \frac{1}{w}\sum_{\alpha,\beta\in W} \det(\alpha^{-1}\beta)\cdot m(\Lambda+\delta-\alpha^{-1}\beta(\lambda+\delta)) \\
&= \sum_{\sigma\in W} \det\sigma\cdot m(\Lambda+\delta-\sigma(\lambda+\delta))\,,
\end{aligned}
$$

where dg (resp. dx) is the normalized Haar measure on G_u (resp. H_u). Thus we have proved (1).

Since $\delta_2=\delta-\delta_1$ lies in the center of $\mathfrak{g}_1$ (B. Kostant [3]), $\tau(\delta_2)=\delta_2$ for $\tau\in W_1$. We also have $\tau(\Lambda)=\Lambda$ for $\tau\in W_1$ since Λ is a weight strongly associated with Π_1. Using these facts, we can prove (2) by the same way as for (1). q. e. d.

PROOF OF PROPOSITION 2.1. since $M_1(\mu_j)\neq 0$ for $j=1,\cdots,m$, by Lemma 2.7, we may choose $\tau\in W_1$ so that

$$\upsilon=\Lambda+\delta-\tau(\mu_j+\delta)\in\Delta^\Lambda\,.$$

Since $\tau(\delta_2)=\delta_2$, we have

$$\upsilon=\Lambda+\delta_1-\tau(\mu_j+\delta_1)\,.$$

So $\mu_j=\Lambda-\tau^{-1}\upsilon+\tau^{-1}\delta_1-\delta_1$. Therefore we get $\mu_j\in D$ for $j=1,\cdots,m$ by Lemma 2.6.

Now we note that each element $\gamma\in W$ can be written uniquely as $\gamma=\gamma_1\cdot\gamma^1$ where $\gamma_1\in W_1$ and $\gamma^1\in W^1$ (B. Kostant [3]). Put $\mu=\Lambda+\delta-\gamma(\lambda+\delta)$ for $\lambda\in D$. We prove that if $\mu\in\Delta^\Lambda$ then $\gamma\in W_1$. Putting $\mu'=\gamma_1^{-1}\mu$, we get

$$\mu'=\Lambda+\gamma_1^{-1}\delta-\gamma^1(\lambda+\delta)\in\Delta^\Lambda\,,$$

and hence

$$\Lambda-\mu'+\gamma_1^{-1}\delta_1-\delta_1=\gamma^1(\lambda+\delta)-\delta\,.$$

Since $\gamma^1(\lambda+\delta)-\delta\in D_1$ (B. Kostant [3]), $\Lambda-\mu'+\gamma_1^{-1}\delta_1-\delta_1\in D_1$. By Lemma 2.6, $\Lambda-\mu'+\gamma_1^{-1}\delta_1-\delta_1\in D$, that is, $\gamma^1(\lambda+\delta)-\delta\in D$. Hence $\gamma^1(\lambda+\delta)$ is a regular element of D. On the other hand $\lambda+\delta\in D$. Thus we see that γ^1 is the identity, and $\gamma=\gamma_1\in W_1$. We have thus proved that if $m(\Lambda+\delta-\gamma(\lambda+\delta))\neq 0$, then $\gamma\in W_1$. By Lemma 2.7, it follows that for $\lambda\in D$ $M(\lambda)=M_1(\lambda)$, and therefore each λ_i $(1\leqq i\leqq l)$ is equal to one and only one of μ_j's $(1\leqq j\leqq m)$. On the other hand, since $\mu_j\in D$, μ_j is equal to one of λ_i's $(1\leqq i\leqq l)$. Proposition 2.1 is thus proved.

2.5. Determination of the nullity. We retain the same notation and assumptions introduced in 2.4.

Since the weight space (v) is U-invariant, $\rho|_U$ induces the quotient representation $(\bar{\rho}, V/(v))$ of U. We denote by $(h^*, (v^*))$ the contra-gredient representation of the representation $(h, (v))$ of U. We see easily that the restriction $T(P_n(\boldsymbol{C}))|_M$ to M of the holomorphic tangent bundle $T(P_n(\boldsymbol{C}))$ is the homogeneous vector bundle associated with the principal bundle $G\to M=G/U$ by the representation $\bar{\rho}\otimes h^*$:

$$T(P_n(\boldsymbol{C}))|_M=G\times_{\bar{\rho}\otimes h^*}(V/(v)\otimes(v^*)).$$

PROPOSITION 2.3. *Let* $\mathfrak{a}(P_n(\boldsymbol{C}))$ *denote the space of all analytic vector fields on* $P_n(\boldsymbol{C})$. *Then*

$$\dim_{\boldsymbol{R}} H^0(M, S(T(P_n(\boldsymbol{C}))|_M)=\dim_{\boldsymbol{R}}\mathfrak{a}(P_n(\boldsymbol{C})),$$

$$H^j(M, S(T(P_n(\boldsymbol{C}))|_M)=(0) \qquad \textit{for } j=1, 2, \cdots.$$

PROOF. By Proposition 2.1, the U-module $V\otimes(v^*)$ decomposes as G_1-module in the following way:

$$V\otimes(v^*)=W_{-\lambda_1}\oplus\cdots\oplus W_{-\lambda_l} \qquad (\lambda_1\geqq\cdots\geqq\lambda_l),$$

where $W_{-\lambda_i}$ is an irredudible component with the lowest weight $-\lambda_i$.

Put $U_i=W_{-\lambda_i}+\cdots+W_{-\lambda_l}$ for $i=1, \cdots, l$. Then we have

$$V\otimes(v^*)=U_1\supset U_2\supset\cdots\supset U_l\supset U_{l+1}=(0).$$

Since $N^+\cdot U_i$ is a G_1-invariant subspace whose lowest weight is higher than $-\lambda_i$, it follows that $N^+\cdot U_i\subset U_{i+1}$ for $i=1, \cdots, l$. Therefore $\rho_\Lambda|_U\otimes h^*$ induces the quotient representation g_i of U on $\widetilde{W}_i=U_i/U_{i+1}$. It is clear that $g_i|_{N^+}$ is the trivial representation and $g_i|_{G_1}$ is an irreducible representation of G_1 with the lowest weight $-\lambda_i$.

Applying Theorem of Bott to the homogeneous vector bundle $E_{\widetilde{W}_i}$ associated to the principal bundle $G\to M$ by the representation g_i, we see the followings,

$$\dim_{\boldsymbol{C}} H^0(M, SE_{\widetilde{W}_i})=\dim_{\boldsymbol{C}} V_{-\mu_i}, \tag{2.3}$$

$$H^j(M, SE_{\widetilde{W}_i})=(0) \qquad \text{for } j=1, 2, \cdots. \tag{2.4}$$

From the exact sequence of U-modules

$$0 \longrightarrow U_{i+1} \longrightarrow U_i \longrightarrow \widetilde{W}_i \longrightarrow 0,$$

we get the exact sequence of cohomology groups

$$0 \longrightarrow H^0(M, SE_{U_{i+1}}) \longrightarrow H^0(M, SE_{U_i}) \longrightarrow H^0(M, SE_{\widetilde{W}_i}) \longrightarrow H^1(M, SE_{U_{i+1}}) \longrightarrow \cdots,$$

where $E_{U_j}=G\times_U U_j$. Using (2.3) and (2.4), we get

$$\dim_C H^0(M, SE_{U_1})=\sum_{i=1}^{l}\dim_C V_{-\lambda_i} \tag{2.5}$$

$$H^j(M, SE_{U_1})=(0) \qquad \text{for } j=1, 2, \cdots. \tag{2.6}$$

Since $(v)\otimes(v^*)$ is the irreducible component of G_1 with the lowest weight 0, we have $\widetilde{W}_l=U_l=(v)\otimes(v^*)$ and $\lambda_l=0$. Let g' denote the quotient representation of U on $U_1/\widetilde{W}_l$ induced by $\rho_A|_U\otimes h^*$. Then we easily see $g'\cong\bar{\rho}\otimes h^*$. Therefore the exact sequence of U-modules

$$0 \longrightarrow \widetilde{W}_l \longrightarrow U_1 \longrightarrow U_1/\widetilde{W}_l \longrightarrow 0$$

induces the following exact sequence.

$$0 \longrightarrow H^0(M, SE_{\widetilde{W}_l}) \longrightarrow H^0(M, SE_{U_1}) \longrightarrow H^0(M, S(T(P_n(\boldsymbol{C}))|_M)) \longrightarrow H^1(M, SE_{\widetilde{W}_l}) \longrightarrow H^1(M, SE_{U_1}) \longrightarrow H^1(M, S(T(P_n(\boldsymbol{C}))|_M)) \longrightarrow \cdots.$$

By (2.3), (2.4), (2.5) and (2.6),

$$\dim_C H^0(M, S(T(P_n(\boldsymbol{C}))|_M))=\sum_{i=1}^{l-1}\dim_C V_{-\lambda_i}$$

$$H^j(M, S(T(P_n(\boldsymbol{C}))|_M))=(0) \qquad \text{for } j=1, 2, \cdots.$$

Since $V\otimes V^*=V^{-\lambda_1}\oplus\cdots\oplus V^{-\lambda_l}$ and $\dim_C V_{-\lambda_l}=1$, we get

$$\dim_C H^0(M, S(T(P_n(\boldsymbol{C}))|_M))=\dim_C V\otimes V^*-1=(n+1)^2-1.$$

On the other hand, we know

$$\dim_C \mathfrak{a}(P_n(\boldsymbol{C}))=(n+1)^2-1.$$

Therefore our proposition is proved.

Now we can prove the following theorem.

THEOREM 2. *Let M be a compact Hermitian symmetric space imbedded fully in $P_n(\boldsymbol{C})$. Then*

$$n(M)=\dim_R \mathfrak{a}(P_n(\boldsymbol{C}))-\dim_R \mathfrak{a}(M).$$

PROOF. By the exact sequence of holomorphic vector bundles

$$0 \longrightarrow T(M) \longrightarrow T(P_n(\boldsymbol{C}))|_M \longrightarrow N(M) \longrightarrow 0,$$

we get the following exact sequence

$$0 \longrightarrow H^0(M, ST(M)) \longrightarrow H^0(M, S(T(P_n(\boldsymbol{C}))|_M)) \longrightarrow H^0(M, SN(M))$$

$$\longrightarrow H^1(M, ST(M)) \longrightarrow \cdots.$$

By a theorem of R. Bott [1], $H^1(M, ST(M))=(0)$ and therefore

$$\dim_{\boldsymbol{R}} H^0(M, SN(M))=\dim_{\boldsymbol{R}} H^0(M, S(T(P_n(\boldsymbol{C}))|_M))-\dim_{\boldsymbol{R}} H^0(M, ST(M)).$$

By Proposition 2.3 and (1.5), we get our theorem. q. e. d.

§ 3. Minimal value of the nullities of compact Kähler submanifolds in a complex projective space.

3.1. Killing nullity. Let M be a compact connected Kähler submanifold of $P_N(\boldsymbol{C})$. Furthermore assume that M is fully imbedded in a totally geodesic submanifold $P_n(\boldsymbol{C})$ of $P_N(\boldsymbol{C})$.

LEMMA 3.1. *Notation being as defined in* 1.2, *we get*

$$n_k(M, P_N(\boldsymbol{C}))=\dim_{\boldsymbol{R}}\mathfrak{k}(P_N(\boldsymbol{C}))-\dim_{\boldsymbol{R}}\mathfrak{k}(P_N(\boldsymbol{C}), M),$$

where $\mathfrak{k}(P_N(\boldsymbol{C}), M)=\{Z\in\mathfrak{k}(P_N(\boldsymbol{C}))\,|\,Z_m\in T_m(M)$ *for* $m\in M\}$.

PROOF. Put $\mathfrak{k}^N=\{Z^N\in\mathfrak{X}(M)^\perp\,|\,Z\in\mathfrak{k}(P_N(\boldsymbol{C}))\}$. Then from the exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{k}(P_N(\boldsymbol{C}), M) \longrightarrow \mathfrak{k}(P_N(\boldsymbol{C})) \longrightarrow \mathfrak{k}^N \longrightarrow 0,$$

we get

$$n_k(M, P_N(\boldsymbol{C}))=\dim_{\boldsymbol{R}}\mathfrak{k}^N=\dim_{\boldsymbol{R}}\mathfrak{k}(P_N(\boldsymbol{C}))-\dim_{\boldsymbol{R}}\mathfrak{k}(P_N(\boldsymbol{C}), M),$$

which proves a lemma. q. e. d.

Let $A(M)$ denote the Lie group of all holomorphic isometries of M and $A(P_n(\boldsymbol{C}), M)$ (resp. $A(P_N(\boldsymbol{C}), M)$ denote the Lie group of all holomorphic isometries of $P_n(\boldsymbol{C})$ (resp. $P_N(\boldsymbol{C})$) which leave M invariant. Then, we have the following lemma.

LEMMA 3.2. *Let M be a compact connected Kähler manifold imbedded fully in $P_n(\boldsymbol{C})$. Then the Lie groups $A(P_n(\boldsymbol{C}), M)$ and $A(M)$ are isomorphic in a natural way.*

This lemma is proved in the same way as Theorem 4.3 in H. Nakagawa and R. Takagi [7], and so the proof is omitted.

LEMMA 3.3.

$$\dim_{\boldsymbol{R}}\mathfrak{k}(P_N(\boldsymbol{C}), M)=\dim_{\boldsymbol{R}}\mathfrak{k}(M)+(N-n)^2.$$

PROOF. Since any Killing vector field on a compact Kähler manifold $P_N(\boldsymbol{C})$ (resp. M) belongs to $\mathfrak{a}(P_N(\boldsymbol{C}))$ (resp. $\mathfrak{a}(M)$) (Y. Matsushima [6]), the dimension of $\mathfrak{k}(P_N(\boldsymbol{C}), M)$ and $\mathfrak{k}(M)$ are respectively equal to the dimension of $A(P_N(\boldsymbol{C}), M)$ and $A(M)$.

For $g\in A(P_N(\boldsymbol{C}), M)$, we have $g(P_n(\boldsymbol{C}))=P_n(\boldsymbol{C})$, because M is fully imbedded in $P_n(\boldsymbol{C})$. Therefore it follows that the group $A(P_N(\boldsymbol{C}), M)$ is locally isomorphic to the product of $A(P_n(\boldsymbol{C}), M)$ and the projective unitary group of degree $N-n$. Thus by Lemma 3.2

$$\begin{aligned}\dim_{\boldsymbol{R}}\mathfrak{k}(P_N(\boldsymbol{C}), M)&=\dim_{\boldsymbol{R}}A(P_N(\boldsymbol{C}), M)\\&=\dim_{\boldsymbol{R}}A(P_n(\boldsymbol{C}), M)+(N-n)^2\\&=\dim_{\boldsymbol{R}}A(M)+(N-n)^2\\&=\dim_{\boldsymbol{R}}\mathfrak{k}(M)+(N-n)^2\,.\end{aligned}$$

This proves Lemma 3.3.

By Lemma 3.1 and Lemma 3.3, we get the following theorem.

THEOREM 3.

$$n_k(M, P_N(\boldsymbol{C}))=\dim_{\boldsymbol{R}}\mathfrak{k}(P_N(\boldsymbol{C}))-\dim_{\boldsymbol{R}}\mathfrak{k}(M)-(N-n)^2\,.$$

Since the Euler characteristic of a Kähler C-space X differs from zero, there exists no non-zero parallel vector field on X. Therefore, by a theorem of A. Lichnerowicz [5],

$$\mathfrak{a}(X)=\mathfrak{k}(X)+J\mathfrak{k}(X) \qquad \text{(direct sum)}\,.$$

Applying this to the cases $X=M$ and $X=P_n(\boldsymbol{C})$, from Theorem 2 and Theorem 3, we get the following theorem.

THEOREM 4. *Let M be a compact Hermitian symmetric space fully imbedded in $P_n(\boldsymbol{C})$. Then*

$$n(M)=2n_k(M)\,.$$

3.2. Minimal value for the nullities. As an application of Theorem 3, we have the following theorem which gives the minimal value of the nullities.

THEOREM 5. *Let M be a p dimensional compact Kähler submanifold of $P_N(\boldsymbol{C})$. Then*

$$n(M)\geqq 2(p+1)(N-p)\,.$$

Furthermore equality holds if and only if M is a totally geodesic submanifold $P_p(\boldsymbol{C})$ of $P_N(\boldsymbol{C})$.

PROOF. For a totally geodesic submanifold $P_p(\boldsymbol{C})$ of $P_N(\boldsymbol{C})$, we have

$$n(P_p(\boldsymbol{C}))=2(p+1)(N-p)$$

by Corollary 2.1.

Assume that M is fully imbedded in a totally geodesic submanifold $P_n(\boldsymbol{C})$ of $P_N(\boldsymbol{C})$. Let M_0 be a connected component of M, then $n(M_0)\leqq n(M)$. Therefore we can assume that M is connected. Note that, for a compact Kähler manifold M of dimension p,

$$\dim_{\boldsymbol{R}}\mathfrak{k}(M)\leqq p^2+2p$$

(A. Lichnerowicz [4]).

By (1.1), Theorem 3 and the above remark, we see

$$\begin{aligned} & n(M)-2(p+1)(N-p) \\ & \quad \geqq n_k(M)-2(p+1)(N-p) \\ & \quad =\dim_{\boldsymbol{R}}\mathfrak{k}(P_N(\boldsymbol{C}))-\dim_{\boldsymbol{R}}\mathfrak{k}(M)-(N-n)^2-2(p+1)(N-p) \\ & \quad \geqq N^2+2N-p^2-2p-(N-n)^2-2(p+1)(N-p) \\ & \quad =(N-p)^2-(N-n)^2 \\ & \quad \geqq 0 . \end{aligned}$$

Thus $n(M)\geqq 2(p+1)(N-p)$, and the equality holds if and only if $n=p$, that is, $M=P_p(\boldsymbol{C})$. q. e. d.

3.3. Analytic nullity. Let M be a compact Kähler submanifold of $P_N(\boldsymbol{C})$. In the following we consider the analytic nullity.

LEMMA 3.4. *Let M be a compact connected Kähler submanifold of $P_N(\boldsymbol{C})$. Furthermore, suppose that M is imbedded fully in a totally geodesic submanifold $P_n(\boldsymbol{C})$ of $P_N(\boldsymbol{C})$. Then*

$$n_a(M)\geqq \dim_{\boldsymbol{R}}\mathfrak{a}(P_N(\boldsymbol{C}))-\dim_{\boldsymbol{R}}\mathfrak{a}(M)-2(N+1)(N-n)\,.$$

PROOF. Let $H(M)$ denote the Lie group of all holomorphic automorphisms of M and $H(P_n(\boldsymbol{C}), M)$ the Lie group of all holomorphic automorphisms of $P_n(\boldsymbol{C})$ leaving M invariant.

Let $\pi: C^{n+1}-(0)\to P_n(\boldsymbol{C})$ be the canonical projection. Since M is imbedded fully in $P_n(\boldsymbol{C})$, $\pi^{-1}(M)$ spans the vector space C^{n+1}. Assume that the restriction to M of an element $\{f\}\in H(P_n(\boldsymbol{C}), M)$, $f\in SL(n+1, \boldsymbol{C})$, is the identity. Since every non-zero vector of $\pi^{-1}(M)$ is an eigenvector of f, f is a diagonal matrix. So we can prove that $\{f\}$ is the identity in the same way as Theorem 4.3 in H. Nakagawa and R. Takagi [7]. Therefore the restriction of $H(P_n(\boldsymbol{C}), M)$ to $H(M)$ is an injective isomorphism to Lie groups and the result follows in the similar way to in the proof of Lemma 3.3. q. e. d.

We get the following theorem.

THEOREM 6. *Let M be a compact Hermitian symmetric space imbedded in $P_N(\boldsymbol{C})$. Then we have $n(M)=n_a(M)$.*

PROOF. Assume that M is fully imbedded in a totally geodesic submanifold $P_n(\boldsymbol{C})$ of $P_N(\boldsymbol{C})$. By Theorem 1 and Theorem 2, we get

$$n(M)=2(n^2+2n)-\dim_{\boldsymbol{R}}\mathfrak{a}(M)+2(n+1)(N-n).$$

On the other hand, by Lemma 3.4 we get

$$n_a(M)\geqq 2(N^2+2N)-\dim_{\boldsymbol{R}}\mathfrak{a}(M)-2(N+1)(N-n).$$

Thus we have $n_a(M)\geqq n(M)$. Obviously, $n(M)\geqq n_a(M)$. Our theorem is proved.

COROLLARY 3.1. *Let M be a compact Hermitian symmetric space imbedded by f in $P_N(\boldsymbol{C})$. Then, for $\xi\in\mathfrak{X}(M)^{\perp}$, ξ is a Jacobi field on M if and only if there exists a variation $\{f_t\}$ of f such that*

(1) *f_t is a Kähler imbedding for each t,*

(2) *E^N is equal to ξ where E is the variation field of $\{f_t\}$.*

PROOF. Assume ξ is a Jacobi field on M. Since $n(M)=n_a(M)$ there exists an analytic vector field Z on $P_N(\boldsymbol{C})$ such that $Z^N=\xi$. For the 1-parameter group of transformations $\{g_t\}$ generated by Z, we put $f_t=g_t\circ f$. Then $\{f_t\}$ is clearly a variation of f which satisfies (1) and (2). The converse is clear.

q. e. d.

This corollary answers the following question raised by Simons [8] affirmatively in this case "Given a Jacobi field on a compact minimal submanifold, does it always arise from a 1-parameter family of minimal submanifolds?"

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