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On the irreducible characters of the finite unitary groups

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Introduction.

Let k be a finite field of q elements, and k_2 the quadratic extension of k. Let σ be the automorphism of the finite general linear group $GL_n(k_2)$ defined by

$$(x_{ij})^{\sigma}=(x_{ji}^{q})^{-1}$$

for any element $(x_{ij})_{1\leqq i,j\leqq n}$ of $GL_n(k_2)$. The group $U_n(k_2)$ of σ-fixed elements of $GL_n(k_2)$ is called the finite unitary group over k_2. So far, the irreducible complex characters of $U_n(k_2)$ have been determined only for small n (see Ernola [4] and Nozawa [8], [9]), while those of $GL_n(k_2)$ have been determined completely by J. A. Green [7]. The purpose of the present paper is to give a method by which one can construct the irreducible complex characters of $U_n(k_2)$ using those of $GL_n(k_2)$, at least if the characteristic of k is not 2. As an application, we also obtain a parametrization of the irreducible characters of $U_n(k_2)$ which is dual to a known parametrization of the conjugacy classes.

Let χ be an irreducible character of $GL_n(k_2)$ which is fixed by σ, i.e. satisfies $\chi(x)=\chi(x^{\sigma})$ for all $x\in GL_n(k_2)$. Then, by a well-known elementary lemma, χ can be extended to an irreducible character $\tilde{\chi}$ of the semi-direct product $AGL_n(k_2)$ of $GL(k_2)$ with the group $A=\{1,\sigma\}$. Our main theorem is:

Assume that char$(k)\neq 2$. *Let* χ *be a* σ*-fixed irreducible character of* $GL_n(k_2)$, *and* $\tilde{\chi}$ *an extension of* χ *to an irreducible character of* $AGL_n(k_2)$. *Then, there exists a unique irreducible character* ψ_{χ} *of* $U_n(k_2)$ *which depends only on* χ *and satisfies*

$$\tilde{\chi}(\sigma x)=\varepsilon(\tilde{\chi})\psi_{\chi}(n(x)) \qquad (x\in GL_n(k_2)),$$

where $\varepsilon(\tilde{\chi})=\pm 1$ *and* $n(x)$ *is an arbitrary element of* $U_n(k_2)$ *conjugate to* $x^{\sigma}x$ *in* $GL_n(k_2)$. *Moreover, the correspondence* $\chi\rightarrow\psi_{\chi}$ *is a bijection between the set of* σ*-fixed irreducible characters of* $GL_n(k_2)$ *and the set of irreducible characters of* $U_n(k_2)$.

This paper consists of five sections. § 1 is a recollection of some known

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results on linear representations of finite groups. § 2 concerns finite groups realized as groups of fixed points of surjective endomorphisms of connected linear algebraic groups. As a special case of a fairly general lemma proved there, we can see that there is a close relation between the conjugacy classes of $AGL_n(k_2)$ and those of $U_n(k_2)$. § 3 is devoted to prove an analogue of the main theorem for the irreducible Brauer characters of finite Chevalley groups. In § 4, we prove the main theorem. The formulation given there is slightly more general than the one stated above. In the last § 5, combining the main theorem with Green's results [7], we obtain a parametrization of the irreducible characters of $U_n(k_2)$ ($\mathrm{char}(k)\neq 2$).

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A short summary of the results of this paper has appeared in [**16**].

Notation.

Let S be a set. If σ is a transformation of S, S_σ denotes the set of σ-fixed elements of S. Let f be a mapping from S into another set, and T is a subset of S. Then $f|T$ denotes the restriction of f to T. If S is a finite set, $|S|$ means the number of its elements. For a group G and an element x of G, $Z_G(x)$ and $\mathfrak{C}_G(x)$ denote the centralizer group and the conjugacy class of x. If K is a field, K^* is the multiplicative group of K. We denote by $\boldsymbol{C}$ and $\boldsymbol{Z}$ the field of complex numbers and the ring of rational integers respectively.

§ 1. Preliminaries on representations of finite groups.

Let G be a finite group, and A a finite cyclic group of order m with a fixed generator σ. Suppose that A acts on G. In such situations we shall often assume that G and A are embedded in their semi-direct product AG; the multiplication rule in AG is defined by

$$x^\delta = \delta^{-1}x\delta \qquad (x\in G,\ \delta\in A)\,.$$

Let K be an algebraically closed field of characteristic p. Assume that m is not divisible by p. The following lemma is well-known.

LEMMA 1.1. (a) *Let $\tilde{T}$ be an irreducible representation of AG over K, and T its restriction to G. If T is still irreducible, then two representations T and $T\circ\sigma$ of G are equivalent to each other.*

(b) *Conversely, if an irreducible representation T of G is equivalent to $T\circ\sigma$, then there exist m mutually inequivalent irreducible representations of AG whose restrictions to G are equivalent to T. If $\tilde{T}$ is one of them, any other one is equivalent to $\xi\otimes\tilde{T}$ for a suitable character ξ of AG/G.*

We may assume that there exists an injective homomorphism ϕ of K^* into $\boldsymbol{C}^*$. For a representation R of a finite group H over K, we denote by $\beta_\phi[R]$ the $\boldsymbol{C}$-valued function on H defined by

$$\beta_\phi[R](h)=\sum_i \phi(r_i(h)) \qquad (h\in H), \tag{1.1}$$

where $r_i(h)$ $(i=1, 2, \cdots, \dim R)$ are the characteristic roots of $R(h)$.

LEMMA 1.2. *Let $\tilde{T}$ be an irreducible representation of AG over K whose restriction to G is reducible. Then,*

$$\beta_\phi[\tilde{T}](\sigma x)=0 \qquad (x\in G).$$

PROOF. By a theory of Clifford [2], the matrix representation of $\tilde{T}(\sigma x)$ for a suitable base is written as

$$\begin{pmatrix} B_{11}(x), \cdots, B_{1l}(x) \\ \cdots\cdots\cdots\cdots \\ B_{l1}(x), \cdots, B_{ll}(x) \end{pmatrix}$$

where l is a divisor of m, $B_{ij}(x)(1\leqq i, j\leqq l)$ are square matrices of the same size, and $B_{ij}(x)=0$ if $j-i\not\equiv 1 \pmod l$. Hence the assertion follows from

LEMMA 1.3. *Let l be a positive integer which is not divisible by p, and*

$$B=\begin{pmatrix} B_{11}, \cdots, B_{1l} \\ \cdots\cdots\cdots \\ B_{l1}, \cdots, B_{ll} \end{pmatrix}$$

a square matrix of (N, N)-type over K, where $B_{ij}(1\leqq i, i\leqq l)$ are square matrices of $(N/l, N/l)$-type, and $B_{ij}=0$ if $j-i\not\equiv 1 \pmod l$.

(a) *The characteristic polynomial* $\det(zE_N-B)$ *(E_N is the unit matrix of (N, N)-type) is a polynomial in z^l.*

(b) *Let $\alpha_1, \alpha_2, \cdots, \alpha_N$ be the characteristic roots of B. Then* $\sum_{i=1}^{N}\phi(\alpha_i)=0$.

PROOF. (a) It is sufficient to show that

$$\det(zE_N-B)=\det(\eta zE_N-B) \tag{1.2}$$

for an arbitrary l-th root η of unity in K. Let $B'_{ij}(1\leqq i, j\leqq l)$ be the (i, j)-blocks of the matrix zE_N-B, i.e.

$$B'_{ij}=\begin{cases} zE_{N/l} & \text{if } i=j\,, \\ -B_{ij} & \text{if } j-i\equiv 1 \pmod l\,, \\ 0 & \text{otherwise}. \end{cases}$$

Multiply η^i to $B'_{i1}, B'_{i2}, \cdots, B'_{il}$ $(1\leqq i\leqq l)$, and η^{1-j} to $B'_{1j}, B'_{2j}, \cdots, B'_{lj}$ $(1\leqq j\leqq l)$. Then the resultant matrix is zE_N-B. The equality (1.2) follows from this fact.

(b) Since p does not devide l, there are l distinct l-th roots $\eta_1, \eta_2, \cdots, \eta_l$ of unity in K. By the injectivity of ϕ, $\phi(\eta_1)$, $\phi(\eta_2)$, $\cdots$, $\phi(\eta_l)$ are the l distinct roots of unity in $\boldsymbol{C}$. Hence $\sum_{i=1}^{l}\phi(\eta_i)=0$. Now the assertion follows from part (a).

Let T be an irreducible representation of G over the complex number field $\boldsymbol{C}$, and χ its character. If χ is fixed by σ, i. e. satisfies $\chi(x)=\chi(x^\sigma)$ for all $x\in G$, then χ can be extended to an irreducible character $\tilde{\chi}$ of AG by Lemma 1.1(b).

LEMMA 1.4. *Let χ_1 and χ_2 be σ-fixed irreducible complex characters of G, and $\tilde{\chi}_1$ and $\tilde{\chi}_2$ irreducible characters of AG such that $\tilde{\chi}_1|G=\chi_1$ and $\tilde{\chi}_2|G=\chi_2$. Then, for $l=0, 1, 2, \cdots, m-1$,*

$$|G|^{-1}\sum_{x\in G}\tilde{\chi}_1(\sigma^l x)\tilde{\chi}_2(\sigma^l x)$$

equals $\xi(\sigma^l)$ if $\chi_1=\chi_2$ and $\tilde{\chi}_1=\xi\tilde{\chi}_2$ with an irreducible character ξ of AG/G, and equals 0 if $\chi_1\neq\chi_2$.

PROOF. This is proved in Glauberman [6] and Shintani [10]. Here we follow Glauberman's proof. Let $\Phi_i (i=1, 2)$ be the class functions on AG defined by

$$\Phi_i=\sum_{\xi\in\Xi}\xi(\sigma^{-l})\xi\tilde{\chi}_i\,,$$

where Ξ is the set of irreducible characters of AG/G. Clearly, $\Phi_i(\sigma^n x)=0$ $(x\in G)$ if $n\neq l$, and $\Phi_i(\sigma^l x)=m\tilde{\chi}_1(\sigma^l x)$. Therefore

$$|G|^{-1}\sum_{x\in G}\chi_1(\sigma^l x)\overline{\chi_2(\sigma^l x)}=|G|^{-1}m^{-2}\sum_{x\in G}\sum_{\delta\in A}\Phi_1(\delta x)\overline{\Phi_2(\delta x)}$$

$$=m^{-1}\sum_{\xi,\xi'\in\Xi}\{\xi(\sigma^{-l})\overline{\xi'(\sigma^{-l})}|AG|^{-1}\sum_{x\in G}\sum_{\delta\in A}(\xi\tilde{\chi}_1)(\delta x)\overline{(\xi'\tilde{\chi}_2)(\delta x)}\}\,.$$

By Lemma 1.1(b), $\xi\tilde{\chi}_i$ are irreducible characters for all $\xi\in\Xi$. Hence, using orthogonality relations of irreducible characters, we obtain the required result.

LEMMA 1.5. *For a positive integer m, put $\zeta_m=\exp(2\pi i/m)$. Let ψ be a complex valued class function on G. Assume that ψ satisfies the following two conditions:*

(1) *The restriction $\psi|E$ of ψ to an arbitrary elementary subgroup E of G is a $\boldsymbol{Z}[\zeta_m]$-linear combination of irreducible characters of E.*

(2) $|G|^{-1}\sum_{x\in G}|\psi(x)|^2=1\,.$

Then there exists an irreducible character χ of G, an integer a, and a sign ε such that

$$\psi(x)=\varepsilon\zeta_m^a\chi(x)\qquad(x\in G)\,.$$

PROOF. By a version [**5**; § 15] of Brauer's characterization of characters, the condition (1) implies that ψ can be written as

$$\psi=\sum_i c_i\chi_i\,,$$

where χ_i are the irreducible characters of G, and c_i are elements of $\boldsymbol{Z}[\zeta_m]$. Using the condition (2), we see that

$$\sum_i c_i\bar{c_i}=1\,.$$

Denote by Γ the Galois group of $\boldsymbol{Q}(\zeta_m)$ with respect to $\boldsymbol{Q}$. Since the complex conjugation is an element of Γ and since Γ is abelian, we have

$$\sum_i c_i^\gamma\overline{c_i^\gamma}=1$$

for all $\gamma\in\Gamma$. Setting $d=|\Gamma|$ we have

$$\sum_i\sum_{\gamma\in\Gamma} c_i^\gamma\overline{c_i^\gamma}=d\,.$$

Since $c_i\in\boldsymbol{Z}[\zeta_m]$, if $c_i\neq 0$,

$$\sum_{\gamma\in\Gamma} c_i^\gamma\overline{c_i^\gamma}\geqq d|\prod_{\gamma\in\Gamma} c_i^\gamma|^{2/d}\geqq d$$

and the equality holds if and only if $|c_i^\gamma|=1$ for all $\gamma\in\Gamma$. Hence $c_i=0$ except for a single index i_0, and $c_{i0}=\pm\zeta_m^a$ for some integer a. This proves the lemma.

§ 2. Preliminaries on algebraic groups.

In this section we denote by $\mathfrak{G}$ a connected linear algebraic group, and by σ a surjective endomorphism of $\mathfrak{G}$ such that $\mathfrak{G}_\sigma$ is finite. In such situation the following theorem is of fundamental importance.

THEOREM 2.1 (Steinberg [**15**; 10.1]). *The mapping f: $x\to x^{-\sigma}x$ of $\mathfrak{G}$ into $\mathfrak{G}$ is surjective.*

Let m be a fixed positive integer such that $\mathfrak{G}_{\sigma m}$ is finite. Put $G=\mathfrak{G}_{\sigma m}$. Let A be a finite cyclic group of order m with a generator σ'. We suppose that A acts on G by

$$x^{\sigma'}=x^\sigma \qquad (x\in G)\,.$$

In the following we write σ for σ', because there is no fear of confusion.

LEMMA 2.2. (a) *Let $\mathfrak{C}$ be an AG-conjugacy class of the set $\{\sigma\}\times G$, and σx an arbitrary element of $\mathfrak{C}$. Take an element α_x of $f^{-1}(x)$ (see Theorem 2.1), and put $N(x)=x^{\sigma^{m-1}}x^{\sigma^{m-2}}\cdots x^\sigma x$. Then $\alpha_xN(x)\alpha_x^{-1}$ is an element of G_σ; its G_σ-conjugacy class is determined by $\mathfrak{C}$.*

(b) *For all $x\in G$,*

$$|\mathfrak{C}_{AG}(\sigma x)||G|^{-1}=|\mathfrak{C}_{G_\sigma}(\alpha_xN(x)\alpha_x^{-1})||G_\sigma|^{-1}\,.$$

(c) *The correspondence $\mathfrak{N}$ from the set of AG-conjugacy classes of $\{\sigma\}\times G$ into the set of conjugacy classes of G_σ defined by*

$$\mathfrak{N}(\mathfrak{C}_{AG}(\sigma x))=\mathfrak{C}_{G_\sigma}(\alpha_x N(x)\alpha_x^{-1}) \qquad (x\in G)$$

is bijective.

PROOF. (a) Since $xN(x)x^{-1}=N(x)^\sigma$, we have $\alpha_x N(x)\alpha_x^{-1}=(\alpha_x N(x)\alpha_x^{-1})^\sigma$, i.e. $\alpha_x N(x)\alpha_x^{-1}\in G_\sigma$. Let β be another element of $f^{-1}(x)$. Then $\alpha_x^{-\sigma}\alpha_x=\beta^{-\sigma}\beta$. Hence $\beta\alpha_x^{-1}\in G_\sigma$. Next, let y be an element of G such that σy is AG-conjugate to σx. Then there is an element z of G such that $y=z^\sigma xz^{-1}$. Hence $\alpha_x z^{-1}$ is an element of $f^{-1}(y)$. Moreover, since $z^{\sigma m}=z$, we have $N(y)=zN(x)z^{-1}$. Therefore

$$(\alpha_x z^{-1})N(y)(\alpha_x z^{-1})^{-1}=\alpha_x N(x)\alpha_x^{-1}.$$

This proves part (a).

(b) An element g of G is contained in $Z_G(\sigma x)=\{g\in G\,|\,g(\sigma x)=(\sigma x)g\}$, if and only if it satisfies

$$xgx^{-1}=g^\sigma. \tag{2.1}$$

From (2.1) and the fact that $g^{\sigma m}=g$, we have $g\in Z_G(N(x))$. Hence

$$\alpha_x g\alpha_x^{-1}\in Z_{\mathfrak{G}}(\alpha_x N(x)\alpha_x^{-1}). \tag{2.2}$$

On the other hand, (2.1) also implies that

$$\alpha_x g\alpha_x^{-1}\in G_\sigma. \tag{2.3}$$

Therefore, from (2.2), (2.3) and part (a) we see that (2.1) is equivalent to

$$\alpha_x g\alpha_x^{-1}\in Z_{G_\sigma}(\alpha_x N(x)\alpha_x^{-1}).$$

Hence

$$|Z_G(\sigma x)|=|Z_{G_\sigma}(\alpha_x N(x)\alpha_x^{-1})|. \tag{2.4}$$

It is easy to see that

$$Z_{AG}(\sigma x)=\bigcup_{i=0}^{m-1}(\sigma x)^i Z_G(\sigma x) \qquad \text{(disjoint union)}. \tag{2.5}$$

From (2.4) and (2.5) we have

$$|Z_{AG}(\sigma x)|=m\,|Z_{G_\sigma}(\alpha_x N(x)\alpha_x^{-1})|.$$

Hence we get

$$\begin{aligned}|\mathfrak{C}_{AG}(\sigma x)|&=|AG|\,|Z_{AG}(\sigma x)|^{-1}=|G|\,|Z_{G_\sigma}(\alpha_x N(x)\alpha_x^{-1})|^{-1}\\&=|\mathfrak{C}_{G_\sigma}(\alpha_x N(x)\alpha_x^{-1})|\,|G|\,|G_\sigma|^{-1},\end{aligned}$$

as required.

(c) First we show that the correspondence $\mathfrak{N}$ is surjective. Take any $y \in G_\sigma$. Then by Theorem 2.1 and the assumption that $|\mathfrak{G}_{\sigma m}| < \infty$, there exists an element γ of $\mathfrak{G}$ such that

$$\gamma\gamma^{-\sigma m} = y \,. \tag{2.6}$$

Since $y = y^\sigma$, we have $\gamma\gamma^{-\sigma m} = \gamma^\sigma \gamma^{-\sigma m+1}$. Hence $\gamma^{-\sigma}\gamma \in G$. Put $x = \gamma^{-\sigma}\gamma$. Then

$$\gamma N(x) \gamma^{-1} = y$$

by (2.6). This proves the surjectivity of the correspondence $\mathfrak{N}$. Let $\{c_1, c_2, \cdots, c_l\}$ be the set of conjugacy classes of G_σ, and $\{C_1, C_2, \cdots, C_l\}$ AG-conjugacy classes of $\{\sigma\} \times G$ such that $\mathfrak{N}(C_i) = c_i$. Then, from part (b), we have

$$|C_i||G|^{-1} = |c_i||G_\sigma|^{-1} \qquad (1 \leqq i \leqq l) \,.$$

Hence

$$\sum_{i=1}^{l} |C_i| = \sum_{i=1}^{l} |c_i||G_\sigma|^{-1}|G| = |G| \,.$$

This implies that $\{C_1, C_2, \cdots, C_l\}$ is the set of AG-conjugacy classes of $\{\sigma\} \times G$, and that $\mathfrak{N}$ is certainly bijective.

COROLLARY 2.3. *The number of σ-fixed irreducible complex characters of G is equal to the number of irreducible complex characters of G_σ.*

PROOF. The dimension of the linear space spanned by restrictions of irreducible characters of AG to $\{\sigma\} \times G$ equals to the number of AG-conjugacy classes of $\{\sigma\} \times G$. The former is equal to the number of σ-fixed irreducible characters by Lemma 1.1, 1.2 and 1.4; the latter is, by Lemma 2.2 (c), equal to the number of conjugacy classes of G_σ, which is equal to the number of irreducible characters of G_σ. This proves the corollary.

The following result is not used in the sequel.

COROLLARY 2.4. *The number of σ-fixed conjugacy classes of G is equal to the number of conjugacy classes of G_σ.*

PROOF. Applying a theorem of Brauer ([**5**; 12.1]) to the character table of G, we see that the number of σ-invariant irreducible characters of G is equal to the number of σ-fixed conjugacy classes of G. Combining this fact with Corollary 2.3 we obtain the required result.

LEMMA 2.5. *Assume that $\mathfrak{G}$ is abelian. Let $\tilde{\chi}$ be an irreducible complex character of AG, and χ its restriction to G. Then, for $x \in G$, we have*

$$\tilde{\chi}(\sigma x) = \begin{cases} 0 & \text{if } \chi \text{ is reducible,} \\ \zeta_m^a \phi_\chi(N(x)) & \text{if } \chi \text{ is irreducible,} \end{cases}$$

where $\zeta_m = \exp(2\pi i/m)$, a is an integer, and ϕ_χ is an irreducible character of G_σ determined by χ.

PROOF. By Lemma 1.2, $\tilde{\chi}(\sigma x)=0$ if χ is reducible. Assume that χ is irreducible, i.e. one dimensional representation of G. Since $\chi(x)=\chi(x^{\sigma})$ for all $x\in G$, we have

$$\chi(x^{-\sigma}x)=1 \qquad (x\in G). \tag{2.7}$$

On the other hand, from Theorem 2.1, it is easy to see that

$$\{x\in G \mid N(x)=1\}=\{x^{-\sigma}x \mid x\in G\}. \tag{2.8}$$

By (2.7), (2.8) and the surjectivity of the mapping N from G into G_σ, we have

$$\chi=\psi_\chi\circ N \tag{2.9}$$

for a unique irreducible character ψ_χ of G_σ. By Lemma 1.1 (b), $\tilde{\chi}$ can be written as

$$\tilde{\chi}(\sigma^n x)=\zeta^n\chi(x) \qquad (x\in G,\ 0\leqq n\leqq m-1) \tag{2.10}$$

The assertion follows from (2.9) and (2.10).

THEOREM 2.6 (Springer and Steinberg [**12**; 1, 3.4]). *Let $\mathfrak{C}$ be a conjugacy class of $\mathfrak{G}$ which is fixed by σ. Assume that the centralizer $Z_G(x)$ of $x\in\mathfrak{C}$ is connected. Then $\mathfrak{C}\cap\mathfrak{G}_\sigma$ forms a single conjugacy class of $\mathfrak{G}_\sigma$.*

COROLLARY 2.7. *Let $\mathfrak{C}$ be a conjugacy class of $G(=\mathfrak{G}_{\sigma m})$ which is fixed by σ. Assume that $Z_G(x)$ is connected for $x\in\mathfrak{C}$. Then $\mathfrak{C}\cap G_\sigma$ forms a single conjugacy class of G_σ.*

PROOF. Let x be an element of $\mathfrak{C}$. Since $\mathfrak{C}=\mathfrak{C}^\sigma$, x^σ is also contained in $\mathfrak{C}$. Hence $\mathfrak{C}_{\mathfrak{G}}(x)$ is fixed by σ. Therefore, by Theorem 2.6, $\mathfrak{C}_{\mathfrak{G}}(x)\cap G$ is a single conjugacy class of G. This implies that $\mathfrak{C}=\mathfrak{C}_{\mathfrak{G}}(x)\cap G$. Using again Theorem 2.6 we see that

$$\mathfrak{C}\cap G_\sigma=(\mathfrak{C}_{\mathfrak{G}}(x)\cap G)\cap G_\sigma=\mathfrak{C}_{\mathfrak{G}}(x)\cap G_\sigma$$

is a single conjugacy class of G_σ.

COROLLARY 2.8. *Let $\mathfrak{G}=GL_{n_1}\times GL_{n_2}\times\cdots\times GL_{n_l}$ for some positive integers $n_1, n_2, \cdots, n_l$. Then, for any σ-fixed conjugacy class $\mathfrak{C}$ of G, $\mathfrak{C}\cap G_\sigma$ forms a single conjugacy class of G_σ.*

PROOF. This follows from Corollary 2.7 and the fact that $Z_{\mathfrak{G}}(x)$ is connected for all $x\in\mathfrak{G}$ (see [**12**; III, 3.22]).

COROLLARY 2.9. (a) *Let $\mathfrak{G}$ be semisimple and simply connected. If x is an element of G such that $N(x)$ is semisimple, then we have*

$$\mathfrak{N}(\mathfrak{C}_{GA}(\sigma x))=\mathfrak{C}_G(N(x))\cap G_\sigma.$$

(b) *Let $\mathfrak{G}=GL_{n_1}\times GL_{n_2}\times\cdots\times GL_{n_l}$ for some positive integers $n_1, n_2, \cdots, n_l$. Then we have*

$$\mathfrak{N}(\mathfrak{C}_{GA}(\sigma x))=\mathfrak{C}_G(N(x))\cap G_\sigma$$

for all $x\in G$.

PROOF. (a) By [15; 8.1], $Z_{\mathfrak{G}}(N(x))$ is connected. Hence, by Theorem 2.6, we see that $\mathfrak{C}_{\mathfrak{G}}(\alpha_x N(x)\alpha_x^{-1})\cap G$ is a single conjugacy class of G. Hence $\mathfrak{C}_G(N(x))=\mathfrak{C}_{\mathfrak{G}}(\alpha_x N(x)\alpha_x^{-1})\cap G$. Using again Theorem 2.6 we also have

$$\mathfrak{C}_{\mathfrak{G}}(\alpha_x N(x)\alpha_x^{-1})\cap G_\sigma=\mathfrak{C}_{G_\sigma}(\alpha_x N(x)\alpha_x^{-1}).$$

Hence

$$\mathfrak{N}(\mathfrak{C}_{AG}(\sigma x))=\mathfrak{C}_{G_\sigma}(\alpha_x N(x)\alpha_x^{-1})=G_{\mathfrak{G}}(\alpha_x N(x)\alpha_x^{-1})\cap G_\sigma=\mathfrak{C}_G(N(x))\cap G_\sigma.$$

(b) By [12; III, 3.22], $Z_{\mathfrak{G}}(N(x))$ is connected. Hence (b) follows by the same argument as in the proof of (a).

§ 3. Modular representations of finite Chevalley groups.

In this section we denote by $\mathfrak{G}$ a simply connected semisimple linear algebraic group. We consider $\mathfrak{G}$ as a subgroup of some $GL_l(K)$ for a fixed algebraically closed field K. Assume that $\mathfrak{G}$ has a surjective endomorphism σ such that $\mathfrak{G}_\sigma$ is finite. Then the characteristic p of K must be positive ([15; 10.5]). The main result of this section is Theorem 3.6. Before stating this, we summarize some known facts on $\mathfrak{G}$ and σ. These are mostly due to C. Chevalley and R. Steinberg ([1], [13], [14], [15]).

Let $\mathfrak{B}$ be a Borel subgroup of $\mathfrak{G}$, and $\mathfrak{H}$ a maximal torus of $\mathfrak{G}$ contained in $\mathfrak{B}$. One can choose $\mathfrak{B}$ and $\mathfrak{H}$ to be fixed by σ. Then the unipotent radical $\mathfrak{U}$ of $\mathfrak{B}$ is also fixed by σ. Let $X(\mathfrak{H})$ be the character module of $\mathfrak{H}$, and $\Sigma\subset X(\mathfrak{H})$ the root system of $\mathfrak{G}$ with respect to $\mathfrak{H}$. For each $\alpha\in\Sigma$ there is an isomorphism x_α of the additive group of K onto a closed subgroup $\mathfrak{U}_\alpha$ of G such that

$$hx_\alpha(t)h^{-1}=x_\alpha(\alpha(h)t)\qquad(h\in\mathfrak{H},\ t\in K).$$

Take an order on Σ so that $\mathfrak{U}=\prod_{\alpha>0}\mathfrak{U}_\alpha$. Let Π be the set of simple roots with respect to this order. From the construction of $\mathfrak{G}$ given in [14] we have the following

LEMMA 3.1. *One can choose $x_\alpha(\sigma\in\Sigma)$ so that the following statements hold.*

(a) *Put $h_\alpha(t)=w_\alpha(t)w_\alpha(-1)$ $(t\in K^*)$ for $\alpha\in\Sigma$, where $w_\alpha(t)=x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$. Then $h_\alpha(t)$ are multiplicative as functions of $t\in K^*$.*

(b) *Put $\mathfrak{H}_\alpha=\{h_\alpha(t)\mid t\in K^*\}$ for $\alpha\in\Sigma$. These are subgroups of $\mathfrak{H}$. Moreover, $\mathfrak{H}$ is a direct product of the subgroups $\mathfrak{H}_\alpha$ for $\alpha\in\Pi$.*

(c) *For $\alpha\in\Pi$, define the element ω_α of $X(\mathfrak{H})$ by*

$$\omega_\alpha\Big(\prod_{\beta\in\Pi}h_\beta(t_\beta)\Big)=t_\alpha.$$

Then ω_α $(\alpha\in\Pi)$ are the fundamental dominant weights ([1; 16-07]).

Henceforth, we assume that x_α ($\alpha\in\Sigma$) have been chosen as in Lemma 3.1.

LEMMA 3.2. *There exists a permutation* ρ *of* Σ *and for each* $\alpha\in\Sigma$ *a power* $q(\alpha)$ *of* p *such that the following statements hold.*

(a) Π *is stable under* ρ.

(b) $x_\alpha(t)^\sigma = x_{\rho\alpha}(c_\alpha t^{q(\alpha)})$ $(t\in K)$

for some $c_\alpha\in K^*$.

(c) *One can normalize* x_α ($\alpha\in\Sigma$) *so that*

$$x_\alpha(t)^\sigma = x_{\rho\alpha}(t^{q(\alpha)}) \qquad (t\in K)$$

for all $\alpha\in\Pi$. *Then*

$$x_{-1}(t) = x_{-\rho\alpha}(t^{q(\alpha)}) \qquad (t\in K)$$

for all $\alpha\in\Pi$.

PROOF. Part (a) and part (b) are proved in [**15**; 11.2]. Part (c) follows from part (b) and [**14**; p. 160, (2)].

For each $\lambda\in X(\mathfrak{H})$ there exists an irreducible rational representation R_λ of $\mathfrak{G}$ whose highest weight ([**14**; p. 209]) is λ; the equivalence class of R_λ is uniquely determined by λ. Since $\{\omega_\alpha \mid \alpha\in\Pi\}$(see Lemma 3.1 (d)) is a basis of $X(\mathfrak{H})$, any element λ of $X(\mathfrak{H})$ can be written as $\lambda=\sum_{\alpha\in\Pi}\lambda(\alpha)\omega_\alpha$ for some $\lambda(\alpha)\in\boldsymbol{Z}$.

THEOREM 3.3 (Steinberg [**15**; 13.1, 13.3]). *Let* $\mathfrak{R}_{\mathfrak{G}}$ *denote the set of irreducible rational representations of* $\mathfrak{G}$ *for which the highest weight* $\lambda=\sum\lambda(\alpha)\omega_\alpha$ *satisfies* $0\leqq\lambda(\alpha)\leqq q(\alpha)-1$ $(\alpha\in\Pi)$.

(a) *The elements of* $\mathfrak{R}_{\mathfrak{G}}$ *remain distinct and irreducible on restriction to* $\mathfrak{G}_\sigma$.

(b) *A complete set of irreducible representations of* $\mathfrak{G}_\sigma$ *over* K *is obtained in this way.*

(c) *The collection* $\{\bigotimes_{i=0}^{\infty} R_i\circ\sigma^i \mid R_i\in\mathfrak{R}_{\mathfrak{G}}$, *most* R_i *trivial*$\}$ *is a complete set of irreducible rational representations of* G, *each counted exactly once.*

LEMMA 3.4. (a) *Let* R *be an irreducible rational representation of* $\mathfrak{G}$ *whose highest weight is* $\sum\lambda(\alpha)\omega_\alpha$. *Then the highest weight of the irreducible representation* $R\circ\sigma$ *is* $\sum q(\alpha)\lambda(\rho\alpha)\omega_\alpha$.

(b) *Let* $R_i (i=0,1,2,\cdots)$ *be irreducible rational representations of* $\mathfrak{G}$ *whose highest weights are* $\sum_\alpha\lambda_i(\alpha)\omega_\alpha$ *respectively. Then the highest weight of the irreducible representation* $\bigotimes_{i=0}^{m-1}R_i\circ\sigma^i$ *is* $\sum_{\alpha\in\Pi}\{\sum_{i=0}^{m-1}q(\alpha)q(\rho\alpha)\cdots q(\rho^{i-1}\alpha)\lambda_i(\rho^i\alpha)\}\omega_\alpha$.

PROOF. (a) Let V be a left G-module which affords R, and $v\in V$ a highest weight vector, i. e.

(1) $\quad xv=v \quad$ for all $\quad x\in\mathfrak{U}$,

and

(2) $\quad hv=(\sum\lambda(\alpha)\omega_\alpha)(h)v \quad$ for all $\quad h\in\mathfrak{H}$.

For a proof of part (a) it suffices to show that

(1′) $x^{\sigma}v=v$ for all $x\in \mathfrak{U}$,

and

(2′) $h^{\sigma}v=(\sum q(\alpha)\lambda(\rho\alpha)\omega_{\alpha})(h)v$ for all $h\in \mathfrak{H}$.

(1′) follows from (1) and the fact that $\mathfrak{U}$ is fixed by σ. For $h\in\mathfrak{H}$ we can write $h=\prod_{\beta\in\Pi}h_{\beta}(t_{\beta})$ by Lemma 3.1 (c). Then, by Lemma 3.1 (a) and Lemma 3.2 (c), we get

$$h^{\sigma}=\prod_{\beta\in\Pi}h_{\rho\beta}(t_{\beta}^{q(\beta)}).$$

Hence, by Lemma 3.1 (d),

$$\omega_{\alpha}(h^{\sigma})=t_{\eta\alpha}^{q(\eta\alpha)}=(q(\eta\alpha)\omega_{\eta\alpha})(h),$$

where $\eta=\rho^{-1}$. Therefore, we see from (2) that

$$h^{\sigma}v=(\sum q(\eta\alpha)\lambda(\alpha)\omega_{\eta\alpha})(h)v=(\sum q(\alpha)\lambda(\rho\alpha)\omega_{\alpha})(h)v,$$

which is (2′). The proof of part (a) is over. Next, we prove part (b). Let $v_i(i=0,1,2,\cdots,m-1)$ be highest weight vectors of $R_i\circ\sigma^i$ respectively. Using part (a) repeatedly we see that $\otimes_{i=0}^{m-1}v_i$ is a highest weight vector of $\otimes_{i=0}^{m-1}R_i\circ\sigma^i$ with the required weight.

Let m be a positive integer. Then $\mathfrak{G}_{\sigma m}$ is also finite by [**15**; 10.6].

LEMMA 3.5. *Let $\mathfrak{R}_{\mathfrak{G}}$ be as in Theorem 3.3. For a positive integer m, let $\mathfrak{R}_{\mathfrak{G},m}$ be the set $\{\otimes_{i=0}^{m-1}R_i\circ\sigma^i\mid R_i\in\mathfrak{R}_{\mathfrak{G}}\}$ of irreducible rational representations of $\mathfrak{G}$.*

(a) *The elements of $\mathfrak{R}_{\mathfrak{G},m}$ remain distinct and irreducible on restriction to $\mathfrak{G}_{\sigma m}$.*

(b) *A complete set of irreducible representations of $\mathfrak{G}_{\sigma m}$ is obtained in this way.*

PROOF. By theorem 3.3 and the definition (Lemma 3.2 (b)) of $q(\alpha)$, it suffices to show that $\mathfrak{R}_{\mathfrak{G},m}$ is the set of irreducible representations of $\mathfrak{G}$ for which the highest weight $\lambda=\sum\lambda(\alpha)\omega_{\alpha}$ satisfies $0\leqq\lambda(\alpha)\leqq Q(\alpha)-1$, where $Q(\alpha)=q(\alpha)q(\rho\alpha)q(\rho^2\alpha)\cdots q(\rho^{m-1}\alpha)$. This, in turn, follows easily from Lemma 3.4 (b).

Put $G=\mathfrak{G}_{\sigma m}$. As in § 2, we denote by A the cyclic group of order m generated by $\sigma|G$. In the following we write σ for $\sigma|G$. Assume that m is not divisible by p. Then it is easy to see that an element $\sigma\chi$ of the semi-direct product AG is p-regular if and only if $N(x)$ is a p-regular (i. e. semi-simple) element of G. The main result of this section is:

THEOREM 3.6. *Assume that m is not divisible by p and that K is the algebraic closure of the finite field with p elements. Let $\tilde{T}$ be an irreducible representation of the semi-direct product AG over K, and T its restriction to G. Let ϕ be an injective homomorphism from K^* into $\boldsymbol{C}^*$.*

(a) *If the representation T of G is reducible, we have*

$$\beta_{\phi}[\tilde{T}](\sigma x)=0 \qquad (x\in G),$$

where $\beta_\phi[\tilde{T}]$ *is defined by* (1.1).

(b) *If T is still irreducible, then there exists an irreducible representation S_T of G_σ over K which depends only on T and satisfies*

$$\beta_\phi[\tilde{T}](\sigma x) = \zeta_m^a \beta_\phi[S_T](n(x))$$

for all $x \in G$ such that $N(x) = x^{\sigma^{m-1}} x^{\sigma^{m-2}} \cdots x^\sigma x$ is semisimple, where $n(x)$ is an arbitrary element of $\mathfrak{C}_G(N(x)) \cap G_\sigma$, $\zeta_m = \exp(2\pi i/m)$, and a is an integer.

(c) *The correspondence $T \to S_T$ induces a bijection between the set of σ-fixed equivalence classes of irreducible representations of G and the set of equivalence classes of irreducible representations of G_σ.*

PROOF. (a) This is a special case of Lemma 1.2.

(b) For each $R \in \mathscr{R}_{\mathfrak{G}}$, we put

$$T_R = \{(R \circ \sigma^{m-1}) \otimes (R \circ \sigma^{m-2}) \otimes \cdots \otimes (R \circ \sigma) \otimes R\} \mid G .$$

By Lemma 3.5, these representations of G are irreducible and pairwise inequivalent. Since the action of σ^m is trivial on G, T_R is equivalent to $T_R \circ \sigma$. Conversely, by Lemma 3.5, irreducible representation T of G over K is equivalent to $T \circ \sigma$ if and only if it is equivalent to some T_R. Let R be an element of $\mathscr{R}_{\mathfrak{G}}$ and V its representation space. Define a linear transformation I_σ of $V \otimes V \otimes \cdots \otimes V$ (m times) by

$$I_\sigma(v_{m-1} \otimes v_{m-2} \otimes \cdots v_1 \otimes v_0) = v_{m-2} \otimes v_{m-3} \otimes \cdots \otimes v_1 \otimes v_0 \otimes v_{m-1} \qquad (v_i \in V) .$$

Put

$$\tilde{T}_R(\sigma^l x) = I_\sigma^l \circ T_R(x) \tag{3.1}$$

for $x \in G$ and $l = 0, 1, \cdots, m-1$. Then $\tilde{T}_R$ is an irreducible representation of AG and T_R is its restriction to G. Let x be an element of G such that $N(x)$ is semisimple, and $n(x)$ an element of $\mathfrak{C}_G(N(x)) \cap G_\sigma$. By [**12**; II, 1.1], $n(x)$ is contained in a maximal torus $\mathfrak{H}$ of $\mathfrak{G}$ fixed by σ. To calculate $\beta_\phi[\tilde{T}_R](\sigma x)$ we may assume that x is contained in $\mathfrak{G}$ and $N(x) = n(x)$, by Lemma 2.2 (c) and Corollary 2.9 (a). Then x^{σ^i} $(i = 0, 1, 2, \cdots, m-1)$ are semisimple and commute with each other. So we can choose a basis $\{e_1, e_2, \cdots, e_d\}$ ($d = \dim R$) of V for which there exist $\lambda_{ij} \in K$ $(i = 0, 1, 2, \cdots, m-1;\ j = 1, 2, \cdots, d)$ such that

$$R(x^{\sigma^i}) e_j = \lambda_{ij} e_j . \tag{3.2}$$

Put $\mathscr{B} = \{e_{j_{m-1}} \otimes e_{j_{m-2}} \otimes \cdots \otimes e_{j_1} \otimes e_{j_0} \mid 1 \leqq j_i \leqq d\}$; this is a basis of $V \otimes V \otimes \cdots \otimes V$ (m times). The operator I_σ on $V \otimes V \otimes \cdots \otimes V$ permutes the set $\mathscr{B}$. Let o be an I_σ-orbit in $\mathscr{B}$, and W_o a linear subspace of $V \otimes V \otimes \cdots \otimes V$ spanned by elements of $\mathscr{B}$ contained in o. Then $\tilde{T}_R(\sigma x) W_o \subset W_o$ by (3.1) and (3.2). Clearly, the cardinality l of o is a divisor of m. First, assume that $l > 1$. Let b be a fixed element of o, and (a_{st}) $(1 \leqq s, t \leqq l)$ the matrix representation of $\tilde{T}_R(\sigma x) \mid W_o$

with respect to the basis $\{I_\sigma^{l-1}b, I_\sigma^{l-2}b, \cdots, I_\sigma b, b\}$ of W_o. Then, from (3.1) and (3.2), we see that $a_{st}=0$ if $t-s\not\equiv 1 (\mathrm{mod}\, l)$. Hence, if $r_i(\sigma x)$ $(1\leqq i\leqq l)$ are the characteristic roots of $\tilde{T}_R(\sigma x)|W_o$, we have

$$\sum_i \phi(r_i(\sigma x))=0 \tag{3.3}$$

from Lemma 1.3. Next, consider the case that $l=1$, i. e. $o=\{e_j\otimes e_j\otimes \cdots \otimes e_j\}$ for some j. From (3.1) and (3.2) we get

$$\tilde{T}_R(\sigma x)e_j\otimes e_j\otimes \cdots \otimes e_j=\lambda_{m-1,j}\lambda_{m-2,j}\cdots \lambda_{1j}\lambda_{0j}e_j\otimes e_j\otimes \cdots \otimes e_j \quad (j=1, 2, \cdots, d). \tag{3.4}$$

Combining (3.3) with (3.4) we get

$$\beta_\phi[\tilde{T}_R](\sigma x)=\sum_{j=1}^{d}\phi(\lambda_{m-1,j}\lambda_{m-2,j}\cdots \lambda_{1j}\lambda_{0j}). \tag{3.5}$$

On the other hand, from (3.2), we have

$$R(N(x))e_j=\lambda_{m-1,j}\lambda_{m-2,j}\cdots \lambda_{1j}\lambda_{0j}e_j \qquad (j=1, 2, \cdots, d).$$

Hence

$$\beta_\phi[R](N(x))=\sum_{j=1}^{d}\phi(\lambda_{m-1,j}\lambda_{m-2,j}\cdots \lambda_{1j}\lambda_{0j}). \tag{3.6}$$

Put $S_{T_R}=R|G_\sigma$. This is an irreducible representation of G_σ by Theorem 3.3, and depends only on $T_R=\otimes_{i=0}^{m-1} R\circ\sigma^i|G$ by Lemma 3.5 (a). From (3.5) and (3.6) we have

$$\beta_\phi[\tilde{T}_R](\sigma x)=\beta_\phi[S_{T_R}](N(x)).$$

This, combined with Lemma 1.1, implies part (b) of the Theorem.

(c) The defining domain of the correspondence is the set of σ-fixed equivalence classes of irreducible representations of G by Lemma 1.1 (a). The remaining assertions follow from the proof of (b) and Theorem 3.3.

§ 4. Main theorem.

In this section we denote by $\mathfrak{G}$ the general linear group GL_n considered as an algebraic group defined over an algebraically closed field K of characteristic $p>0$. Let k be a fixed finite subfield of K. For a positive integer l, we denote by $k_l(\subset K)$ the extension of k of degree l. Let τ and σ be the surjective endomorphisms of $\mathfrak{G}$ defined by

$$x^\tau=(x_{ij}^q)_{1\leqq i,j\leqq n} \quad \text{for} \quad x=(x_{ij})\in\mathfrak{G} \tag{4.1}$$

and

(4.2) $$x^{\sigma}=(({}^t x)^{\tau})^{-1} \quad \text{for} \quad x\in\mathfrak{G},$$

where $q=|k|$ and ${}^t x$ is the transposed matrix of $x\in\mathfrak{G}$. If m is a positive integer,

(4.3) $$\mathfrak{G}_{\tau m}=GL_n(k_m)$$

and

(4.4) $$\mathfrak{G}_{\sigma m}=\begin{cases} GL_n(k_m) & \text{if } m \text{ is even,} \\ U_n(k_{2m}) & \text{if } m \text{ is odd} \end{cases}$$

where $U_n(k_{2m})$ is the group of unitary matrices over k_{2m}. Put $G=\mathfrak{G}_{\sigma m}$ for a fixed m. Then $\sigma|G$ is an automorphism of the finite group G, and will be denoted simply by σ. Let A be the cyclic group of order m generated by the automorphism σ of G, and AG the semi-direct product of G with A. Now we can state the main result of the paper.

THEOREM 4.1. *Assume that m is not divisible by p. Let $\tilde{\chi}$ be an irreducible character of AG, and χ its restriction to G.*

(a) *If the character χ of G is reducible, then*

$$\tilde{\chi}(\sigma x)=0 \qquad (x\in G).$$

(b) *If χ is still irreducible, then there exists an irreducible character ψ of G_σ $(=U_n(k_2))$ which depends only on χ and satisfies*

$$\tilde{\chi}(\sigma x)=\varepsilon\zeta_m^a\psi_\chi(n(x)) \qquad (x\in G,\ n(x)\in\mathfrak{C}_G(N(x))\cap G_\sigma),$$

where $N(x)=x^{\sigma^{m-1}}x^{\sigma^{m-2}}\cdots x^{\sigma}x$, $\zeta_m=\exp(2\pi i/m)$, $\varepsilon=\pm1$, and a is an integer.

(c) *The correspondence $\chi\to\psi_\chi$ is a bijection between the set of σ-fixed irreducible characters of G and the set of irreducible characters of G_σ.*

REMARK 4.2. Theorem 4.1, and its proof, are valid even if one replaces σ with τ defined by (4.1). Using Green's construction [7] of irreducible characters of finite general linear groups, Shintani [10] proved the τ-case without assuming that m is not divisible by p. Our proof is independent of the Green's construction.

REMARK 4.3. It may be possible to extend Theorem 4.1 to a more general case. See Lemma 2.2 and Corollary 2.3.

For the proof of Theorem 4.1 we need some preliminary results. Let $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_s)$ be a partition of n, i. e. an integer sequence such that $n=\sum_{i=1}^{s}\alpha_i$ and $\alpha_1\geqq\alpha_2\geqq\cdots\geqq\alpha_s>0$. Put

$$\mathfrak{G}_\alpha=\{(B_{ij})_{1\le i,j\le s}\in\mathfrak{G}\,|\,B_{ii}\in GL_{\alpha_i}(1\leqq i\leqq s),\ B_{ij}=0(i\neq j)\}.$$

This is a σ-fixed connected algebraic subgroup of $\mathfrak{G}$. A subgroup of $\mathfrak{G}$ is called of type α if it is conjugate to $\mathfrak{G}_\alpha$.

LEMMA 4.4. (a) *Let u be a unipotent element of $\mathfrak{G}$. Then*

$$Z_{\mathfrak{G}}(u)=\mathfrak{X}\cdot\mathfrak{U} \qquad (\textit{semi-direct product}), \tag{4.5}$$

where $\mathfrak{U}$ is the unipotent radical of $Z_{\mathfrak{G}}(u)$ and $\mathfrak{X}$ is a subgroup of $\mathfrak{G}$ which is of type α for some partition α of n. Moreover, if u is fixed by σ, $\mathfrak{U}$ is fixed by σ and $\mathfrak{X}$ can be chosen to be fixed by σ.

(b) *Let s be a (σ-fixed) semisimple element of $\mathfrak{G}$. Then $Z_{\mathfrak{G}}(s)$ is a (resp. σ-fixed) subgroup of $\mathfrak{G}$ of type α for some partition α of n.*

PROOF. (a) The decomposition (4.5) is proved, for example, in [**12**; IV, 1.7]. If u is fixed by σ, we have

$$Z_{\mathfrak{G}}(u)=\mathfrak{X}\cdot\mathfrak{U}=\mathfrak{X}^\sigma\cdot\mathfrak{U}^\sigma .$$

Since $\mathfrak{U}$ and $\mathfrak{U}^\sigma$ are both unipotent radicals of $Z_{\mathfrak{G}}(u)$, we have $\mathfrak{U}=\mathfrak{U}^\sigma$. Let $\mathfrak{S}$ be the center of $\mathfrak{X}$. Then $\mathfrak{S}^\sigma$ is the center of $\mathfrak{X}^\sigma$. Since $\mathfrak{S}\mathfrak{U}$ and $\mathfrak{S}^\sigma\mathfrak{U}$ are both radicals of $Z_{\mathfrak{G}}(u)$, we have $\mathfrak{S}\mathfrak{U}=\mathfrak{S}^\sigma\mathfrak{U}$. Moreover, $\mathfrak{S}$ and $\mathfrak{S}^\sigma$ are maximal tori of the connected algebraic group $\mathfrak{S}\mathfrak{U}=\mathfrak{S}^\sigma\mathfrak{U}$. Hence $\mathfrak{S}^\sigma=x\mathfrak{S}x^{-1}$ for some element x of $\mathfrak{S}\mathfrak{U}$. By Theorem 2.1, there exists $y\in\mathfrak{S}\mathfrak{U}$ such that $x=y^{-1}y$. Put $\mathfrak{S}'=y\mathfrak{S}y^{-1}$. Then $\mathfrak{S}'$ is σ-stable. Therefore, $\mathfrak{X}'=Z_{\mathfrak{G}}(\mathfrak{S}')$ is fixed by σ. We also have $Z_{\mathfrak{G}}(u)=\mathfrak{X}'\mathfrak{U}$, because $\mathfrak{X}'=yZ_{\mathfrak{G}}(\mathfrak{S})y^{-1}=y\mathfrak{X}y^{-1}$. This proves part (a).

(b) This is well-known.

LEMMA 4.5. *Let $\mathfrak{X}=y\mathfrak{G}_\alpha y^{-1}(y\in\mathfrak{G})$ be a σ-fixed algebraic subgroup of $\mathfrak{G}$ which is of type α for some partition $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_s)$ of n. Let $\mathfrak{Y}$ be an algebraic subgroup of $\mathfrak{X}$ defined by*

$$\mathfrak{Y}=\{y(B_{ij})y^{-1}\in\mathfrak{X}\mid B_{ii}\in SL_{\alpha i}(1\leqq i\leqq s)\,,\ B_{ij}=0(i\neq j)\}\,.$$

Then

(a) *$\mathfrak{Y}$ is fixed by σ.*

(b) *For any positive intger l, there exist sequences $\{a_1,a_2,\cdots,a_h\}\{b_1,b_2,\cdots,b_k\}$ of positive integers and sequences $\{D_1,D_2,\cdots,D_h\}\{F_1,F_2,\cdots,F_k\}$ of finite fields such that*

$$\mathfrak{X}_{\sigma l}\cong GL_{a_1}(D_1)\times GL_{a_2}(D_2)\times \cdots\times GL_{a_h}(D_h)\times U_{b_1}(F_1)\times U_{b_2}(F_2)\times\cdots\times U_{b_k}(F_k) \tag{4.6}$$

and

$$\mathfrak{Y}_{\sigma l}\cong SL_{a_1}(D_1)\times SL_{a_2}(D_2)\times \cdots\times SL_{a_h}(D_h)\times SU_{b_1}(F_1)\times SU_{b_2}(F_2)\times\cdots\times SU_{b_k}(F_k)\,, \tag{4.7}$$

where $SU_{b_i}(F_i)=U_{b_i}(F_i)\cap SL_{b_i}(F_i)$.

PROOF. (a) This follows from the fact that $\mathfrak{Y}$ is the commutator subgroup of $\mathfrak{X}$.

(b) Put $\rho=\sigma^l$. Since $\mathfrak{X}=\mathfrak{X}^\rho$, $n=y^{-\rho}y$ normalizes $\mathfrak{G}_\alpha$. For any element $x=ygy^{-1}$ ($g\in\mathfrak{G}_\alpha$) of $\mathfrak{X}$, we have

$$x^\rho=y^\rho g^\rho y^{-\rho}=y(n^{-1}g^\rho n)y^{-1}.$$

Hence

$$\mathfrak{X}_\rho\cong\{g\in\mathfrak{G}_\alpha \mid n^{-1}g^\rho n=g\}. \tag{4.8}$$

For each index $1\leqq i\leqq s$ satisfying $\alpha_i=\alpha_{i+1}$, let $w_i\in\mathfrak{G}$ be the permutation matrix such that

$$w_i g w_i^{-1}=(g_i, g_2, \cdots, g_{i-1}, g_{i+1}, g_i, g_{i+2}, \cdots, g_s)$$

for any element $g=(g_1, g_2, \cdots, g_s)$ ($g\in GL_{\alpha i}$) of $\mathfrak{G}_\alpha$; we denote by $\mathfrak{W}_\mathfrak{X}$ the subgroup of $\mathfrak{G}$ generated by w_i's. The normalizer group of $\mathfrak{G}_\alpha$ in $\mathfrak{G}$ is generated by $\mathfrak{W}_\mathfrak{X}$ and $\mathfrak{G}_\alpha$. Hence $n=aw$ for some $w\in\mathfrak{W}_\mathfrak{X}$ and $a\in\mathfrak{X}$. By Theorem 2.1, there exists $b\in\mathfrak{G}_\alpha$ such that $a=b^{-\rho}b$. Therefore, by (4.9),

$$\begin{aligned}\mathfrak{X}_\rho&\cong\{g\in\mathfrak{G}_\alpha \mid w^{-1}b^{-1}b^\rho g^\rho b^{-\rho}bw=g\}\\&\cong\{g\in\mathfrak{G}_\alpha \mid (bgb^{-1})^\rho=b(wgw^{-1})b^{-1}\}.\end{aligned}$$

Using this and (4.4) we can easily prove (4.6); (4.7) can be proved in a similar way.

LEMMA 4.6. *Let $\mathfrak{X}$ be as in Lemma 4.5. For a fixed positive integer m which is not divisible by p, we put $X=\mathfrak{X}_{\sigma m}$. Assume that K is the algebraic closure of the finite field with p elements. Let ϕ be an injective homomorphism from K^* into $\boldsymbol{C}^*$. Let $\tilde{T}$ be an irreducible representation of the semi-direct product AX over K, and T its restriction to X.*

(a) *If the representation T of X is reducible we have*

$$\beta_\phi[\tilde{T}](\sigma x)=0 \qquad (x\in X),$$

where $\beta_\phi[\tilde{T}]$ is defined by (1.1).

(b) *If T is still irreducible, then there exists an irreducible representation S_T of X_σ whose equivalence class depends only on the equivalence class of T and satisfies*

$$\beta_\phi[\tilde{T}](\sigma x)=\zeta_m^a\beta_\phi[S_T](n(x))$$

for any $x\in X$ such that $N(x)=x^{\sigma m-1}x^{\sigma m-2}\cdots x^\sigma x$ is semisimple, where $n(x)$ is an arbitrary element of $\mathfrak{C}_X(N(x))\cap X_\sigma$, $\zeta_m=\exp(2\pi i/m)$, and a is an integer.

(c) *The correspondence $T\rightarrow S_T$ induces a bijection between the set of σ-fixed equivalence classes of irreducible representations of X and the set of equivalence classes of irreducible representations of X_σ.*

PROOF. (a) is a special case of Lemma 1.2. The proof of (b) and (c) depends on the following two results.

(1) The number of σ-fixed equivalence classes of irreducible representations of X over K is equal to the number of equivalence classes of irreducible representations of X_σ over K.

(2) Let $\mathfrak{Y}$ be as in Lemma 4.5. For an irreducible rational representation R of $\mathfrak{Y}$, there exists an irreducible rational representation R' of $\mathfrak{X}$ such that $R'|\mathfrak{Y}$ is equivalent to R.

Let us deduce (b) and (c) from (1) and (2). Let $\mathfrak{R}_\mathfrak{Y}$ be the set of irreducible rational representations of $\mathfrak{Y}$ defined in Theorem 3.3. For each $R\in\mathfrak{R}_\mathfrak{Y}$, let R' be an irreducible rational representation of $\mathfrak{X}$ such that $R'|\mathfrak{Y}$ is equivalent to R. Then $R'|X_\sigma$ is an irreducible representation of X_σ, since its restriction to Y_σ (where $Y=\mathfrak{Y}_{\sigma m}$) is already irreducible by Theorem 3.3. Hence, by a theorem of Clifford (see [**2**] or [**3**; Theorem (51.7)]) and Theorem 3.3,

$$\{(R'|X_\sigma)\otimes\xi \mid R\in\mathfrak{R}_\mathfrak{Y},\ \xi\in\Xi\}$$

is a complete set of irreducible representations of X_σ over K, each counted exactly once, where Ξ is the set of irreducible representations of X_σ/Y_σ. By Lemma 4.5, each $\xi\in\Xi$ can be extended to a rational 1-dimensional representation ξ' of $\mathfrak{X}$. Then, by Corollary 2.9 (b),

$$(\bigotimes_{i=0}^{m-1}(\xi'\circ\sigma^i))(x)=\xi(n(x)) \qquad (x\in X), \tag{4.9}$$

where $n(x)$ is an arbitrary element of $\mathfrak{C}_\mathfrak{X}(N(x))\cap X_\sigma$. For $R\in\mathfrak{R}_\mathfrak{Y}$ and $\xi\in\Xi$, we put

$$\begin{aligned} T_{R,\xi} &= \{\otimes_{i=0}^{m-1}(R'\otimes\xi')\circ\sigma^i\}\,|\,X \\ &= \{(\otimes_{i=0}^{m-1}R'\circ\sigma^i)\otimes(\otimes_{i=0}^{m-1}\xi'\circ\sigma^i)\}\,|\,X\,. \end{aligned}$$

Then $T_{R,\xi}$ is an irreducible representation of X, since its restriction to Y is already irreducible by Lemma 3.5. Two representations $T_{R,\xi}$ and $T_{S,\eta}$ ($R, S\in\mathfrak{R}_\mathfrak{Y}$; $\xi,\eta\in\Xi$) are equivalent to each other if and only if $R=S$ and $\xi=\eta$. This follows from Lemma 3.5 and (4.9). Clearly, $T_{R,\xi}$ is equivalent to $T_{R,\xi}\circ\sigma$. Conversely, by (1), an irreducible representation T of X over K is equivalent to $T\circ\sigma$ if and only if it is equivalent to some $T_{R,\xi}$. The rest of the proof is similar to the proof of Theorem 3.6, and is omitted. We now prove (1). The table of irreducible Brauer characters of X is a non-singular matrix by orthogonality relations ([**3**; (84.11)]). Hence we may apply a theorem of Brauer ([**5**; § 12.1]). By this theorem the number of σ-fixed irreducible Brauer characters of X equals the number of σ-fixed p-regular conjugacy classes of X. By Corollary 2.8, the latter number equals the number of p-regular conjugacy classes of X_σ, which, in turn, equals the number of irreducible Brauer characters of X_σ. This proves (1). Next, we prove (2). For this purpose we need some results on rational representations of $\mathfrak{X}$. Let $\mathfrak{B}_{\mathfrak{x}}$ be a Borel subgroup of $\mathfrak{X}$, $\mathfrak{U}_{\mathfrak{x}}$ the unipotent radical of $\mathfrak{B}_{\mathfrak{x}}$, and $\mathfrak{H}_{\mathfrak{x}}$ a maximal torus of $\mathfrak{X}$ contained in $\mathfrak{B}_{\mathfrak{x}}$. Then, $\mathfrak{B}_{\mathfrak{y}}=\mathfrak{B}_{\mathfrak{x}}\cap\mathfrak{Y}$ is a Borel subgroup of $\mathfrak{Y}$, $\mathfrak{U}_{\mathfrak{y}}=\mathfrak{U}_{\mathfrak{x}}$ is the unipotent radical of $\mathfrak{B}_{\mathfrak{y}}$, and $\mathfrak{H}_{\mathfrak{y}}=\mathfrak{H}_{\mathfrak{x}}\cap\mathfrak{Y}$ is a maximal torus of $\mathfrak{Y}$ contained in $\mathfrak{B}_{\mathfrak{y}}$. Let $\mathfrak{W}$ be the Weyl group of $\mathfrak{X}$ with respect to $\mathfrak{H}_{\mathfrak{x}}$. This can be identified with the Weyl group of $\mathfrak{Y}$ with respect to $\mathfrak{H}_{\mathfrak{y}}$. We denote by w_0 the element of $\mathfrak{W}$ such that $(w_0\mathfrak{B}_{\mathfrak{x}}w_0^{-1})\cap\mathfrak{B}_{\mathfrak{x}}=\mathfrak{H}_{\mathfrak{x}}$. Let λ be a rational character of $\mathfrak{H}_{\mathfrak{y}}$. Put $\mathfrak{W}_\lambda=\{w\in\mathfrak{W}\,|\,\lambda(w_0hw_0^{-1})=\lambda(w_0whw^{-1}w_0^{-1})$ for all $h\in\mathfrak{H}_{\mathfrak{y}}\}$. We define the K-valued function a_λ on $\mathfrak{Y}$ by

$$a_\lambda(y)=\lambda(h^{-1}) \tag{4.10}$$

if $y\in\mathfrak{Y}$ is in $\mathfrak{B}_{\mathfrak{y}}w_0w\mathfrak{B}_{\mathfrak{y}}$ and is written $y=uhw_0wu_1$ with $u, u_1\in\mathfrak{U}_{\mathfrak{y}}$, $h\in\mathfrak{H}_{\mathfrak{y}}$, $w\in\mathfrak{W}_\lambda$, and

$$a_\lambda(y)=0 \tag{4.11}$$

otherwise. For $z\in\mathfrak{Y}$, we also define the function za_λ on $\mathfrak{Y}$ by

$$(za_\lambda)(y)=a_\lambda(z^{-1}y)\,.$$

Let V_λ be the K-linear space spanned by $\{za_\lambda\,|\,z\in\mathfrak{Y}\}$, considered as a $\mathfrak{Y}$-module. Then, by [**14**; pp. 213-217], the function a_λ is rational on $\mathfrak{Y}$, and the $\mathfrak{Y}$-module V_λ affords an irreducible rational representation with the highest weight λ. To prove (2), it is sufficient to show that the action of $\mathfrak{Y}$ on V_λ can be extended to a rational action of $\mathfrak{X}$ on V_λ. Using explicit descriptions of $\mathfrak{H}_{\mathfrak{x}}$, $\mathfrak{H}_{\mathfrak{y}}$, λ and $\mathfrak{W}$, we can see that there exists a rational character λ' of $\mathfrak{H}_{\mathfrak{x}}$ which satisfies

$$\lambda'\,|\,\mathfrak{H}_{\mathfrak{y}}=\lambda$$

and

$$\mathfrak{W}_\lambda=\{w\in\mathfrak{W}\,|\,\lambda'(w_0h'w_0^{-1})=\lambda'(w_0wh'w^{-1}w_0^{-1}) \text{ for all } h'\in\mathfrak{H}_{\mathfrak{x}}\}\,. \tag{4.12}$$

We choose one such λ' and fix it. Since any $f\in V_\lambda$ satisfies

$$f(yh)=f(y)\lambda(w_0h^{-1}w_0^{-1}) \qquad (y\in\mathfrak{Y},\ h\in\mathfrak{H}_{\mathfrak{y}})\,,$$

and since $\mathfrak{X}$ can be written as a semi-direct product of $\mathfrak{Y}$ with a torus $\mathfrak{T}\subset\mathfrak{H}_{\mathfrak{x}}$, any $f\in V_\lambda$ can be uniquely extended to a rational function on $\mathfrak{X}$ satisfying

$$f(xh')=f(x)\lambda'(w_0h'^{-1}w_0^{-1}) \qquad (x\in\mathfrak{X},\ h'\in\mathfrak{H}_{\mathfrak{x}})\,. \tag{4.13}$$

For $v\in X$ and $f\in V_\lambda$, define the function vf on $\mathfrak{X}$ by

$$(vf)(x)=f(v^{-1}x) \qquad (x\in\mathfrak{X})\,.$$

Our purpose is to show that $vf\in V_\lambda$. It is sufficient to prove this in the case

$v=t\in\mathfrak{T}$ and $f=a_\lambda$. If $x\in\mathfrak{X}$ is written $x=uhw_0wu_1t_1$ with $u, u_1\in\mathfrak{U}_{\mathfrak{Y}}$, $h\in\mathfrak{H}_{\mathfrak{Y}}$, $w\in\mathfrak{W}_\lambda$, $t_1\in\mathfrak{T}$, then

$$\begin{aligned}(ta_\lambda)(x) &= a_\lambda(t^{-1}x) = a_\lambda(t^{-1}xw^{-1}w_0^{-1}tw_0w)\lambda'(w_0w^{-1}w_0^{-1}tw_0ww_0^{-1}) \\ &= a_\lambda(x)\lambda'(t)\end{aligned}$$

by (4.10), (4.12) and (4.13). If $x\in\mathfrak{X}$ is not in $\mathfrak{B}_{\mathfrak{X}}w_0\mathfrak{W}_\lambda\mathfrak{B}_{\mathfrak{X}}$,

$$(ta_\lambda)(x) = a_\lambda(t^{-1}x) = 0$$

by (4.11) and the fact that $t^{-1}x\notin B_Xw_0WB_X$. Hence we have

$$ta_\lambda = \lambda'(t)a_\lambda \in V_\lambda\,,$$

as required. This completes the proof of Lemma 4.6.

LEMMA 4.7. *Let $\mathfrak{X}$, m and X be as in Lemma 4.6. Let $\tilde{\chi}$ be an irreducible character of the semi-direct product AX. Define the class function ψ on X_σ by*

$$\tilde{\chi}(\sigma x) = \psi(n(x)) \qquad (x\in X,\ n(x)\in\mathfrak{C}_X(N(x))\cap X_\sigma)\,.$$

(This is possible by Lemma 2.2 (c) and Corollary 2.10 (b).) Let X_σ^0 be the set of semisimple elements of X_σ. Then there exists a $\boldsymbol{Z}[\zeta_m]$-linear combination ψ' of irreducible characters of X_σ such that

$$\psi\,|\,X_\sigma^0 = \psi'\,|\,X_\sigma^0\,.$$

PROOF. Let $\{\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_l\}$ be the set of irreducible Brauer characters of AX. Then

$$\tilde{\chi}(\sigma x) = \sum_{i=1}^{l} d_i\tilde{\beta}_i(\sigma x) \tag{4.14}$$

for all x in $\{x\in X\,|\,\sigma x \text{ is } p\text{-regular}\} = \{x\in X\,|\,N(x) \text{ is semisimple}\}$, where d_i are non-negative integers called decomposition numbers (see, for example, [3; § 83]). From Lemma 4.5 we have

$$\tilde{\beta}_i(\sigma x) = \zeta_m^{a_i}\beta_i(n(x)) \qquad \text{or} \quad 0 \tag{4.15}$$

for all $x\in X$ such that $N(x)$ is semisimple, where a_i are integers and β_i are irreducible Brauer characters of X_σ. By a theorem [7; Theorem 1] of Green, we can write β_i as

$$\beta_i = \sum_j c_{ij}\chi_j\,|\,X_\sigma^0\,, \tag{4.16}$$

where c_{ij} are integers and χ_j are irreducible complex characters of X_σ. Combining (4.14), (4.15) with (4.16), we obtain the required result.

PROOF OF THEOREM 4.1.

(a) This is a special case of Lemma 1.2.

(b) By Lemma 2.2 (c) and Corollary 2.9 (b), we can define the class function ψ on G_σ by

$$\tilde{\chi}(\sigma x)=\psi(n(x)) \qquad (x\in G),$$

where $n(x)$ is an arbitrary element of $\mathfrak{C}_G(N(x))\cap G_\sigma$. From Lemma 1.4, Lemma 2.2 (b) and Corollary 2.9 (b) we have

$$|G_\sigma|^{-1}\sum_{g\in G_\sigma}|\psi(g)|^2=1.$$

Hence, by Lemma 1.5, for a proof of (b) it suffices to show that: (*) the restriction $\psi|E$ of ψ to an arbitrary elementary subgroup E of G_σ is a $\boldsymbol{Z}[\zeta_m]$-linear combination of irreducible characters of E.
Recall that an elementary subgroup E can be written as a direct product $H\times\langle g\rangle$, where $\langle g\rangle$ is a cyclic group generated by $g\in G_\sigma$, and H is an r-subgroup of $Z_{G_\sigma}(g)$ for some prime number r which does not divide the order of g. We consider the following three cases separately.

(1) g is semisimple and $r=p$.

(2) g is semisimple and $r\neq p$.

(3) g is not semisimple.

First, we prove (*) for the case (1). Let $\mathfrak{S}$ be the center of $Z_{\mathfrak{G}}(g)$. By Lemma 4.4 (b), $\mathfrak{S}$ is a connected abelian algebraic subgroup of $\mathfrak{G}$. Since $g=g^\sigma$, $\mathfrak{S}$ is σ-stable. Put $S=\mathfrak{S}_{\sigma m}$. Then $S_\sigma\times H$ contains E. Consider the subgroup $Q=S\times H$ of G. Since $AQ=AS\times H$, we can write

$$\tilde{\chi}|AQ=\sum_i e_i(\theta_i\times\omega_i), \tag{4.17}$$

where θ_i and ω_i are irreducible characters of AS and H respectively, and e_i are positive integers. From Lemma 2.5 and the assumption that m is not divisible by p, we see that the functions θ_i' on S_σ and the functions ω_i' on H defined by

$$\theta_i'(N(s))=\theta_i(\sigma s) \qquad (s\in S)$$

and

$$\omega_i'(N(h))(=\omega_i'(h^m))=\omega_i(h) \qquad (h\in H)$$

are $\boldsymbol{Z}[\zeta_m]$-linear combinations of irreducible characters of S_σ and H respectively. This facts combined with (4.17) implies (*) for the present case. Next, let us consider the case (2). In this case every element of E is semisimple. Hence (*) follows from Lemma 4.7. There remains to prove (*) for the case (3). Let s and u be semisimple and unipotent elements of G_σ such that $g=su=us$. Since $u\neq 1$, the order of g is divisible by p. Hence $r\neq p$. This means that every element of H is semisimple.

Put $\mathfrak{L}=Z_{\mathfrak{G}}(s)$ and $\mathfrak{M}=Z_{\mathfrak{L}}(u)$. Then $\mathfrak{M}$ is σ-stable and contains E. From

Lemma 4.4, we have a semi-direct product decomposition

$$\mathfrak{M}=\mathfrak{X}\cdot\mathfrak{U}\,,$$

where $\mathfrak{U}$ is the unipotent radical of $\mathfrak{M}$ and $\mathfrak{X}$ is a σ-fixed algebraic subgroup of type α for some partition α of n. Put $X=\mathfrak{X}_{\sigma m}$ and $D=X\times\langle u\rangle$. The order of $\mathfrak{U}_\sigma$ is a power of p. Hence, by Sylow's theorem, we may assume that X_σ contains $H\times\langle s\rangle$. Then D_σ contains E. Since $AD=AX\times\langle u\rangle$, we can write

$$\tilde{\chi}\,|\,AD=\sum_i f_i(\mu_i\times\nu_i)\,, \tag{4.18}$$

where f_i are positive integers, and μ_i and ν_i are irreducible characters of AX and $\langle u\rangle$ respectively. From Lemma 4.7 and the assumption that m is not divisible by p, we see that the functions μ_i' on $H\times\langle s\rangle$ and the functions ν_i' on $\langle u\rangle$ defined by

$$\mu_i'(n(x))=\mu_i(\sigma x) \qquad (x\in X,\ n(x)\in\mathfrak{C}_X(N(x))\cap(H\times\langle s\rangle))$$

and

$$\nu_i'(N(v))(=\nu_i'(v^m))=\nu_i(v) \qquad (v\in\langle u\rangle)$$

are $\boldsymbol{Z}[\zeta_m]$-linear combinations of irreducible characters of $H\times\langle s\rangle$ and $\langle u\rangle$ respectively. This fact combined with (4.18) implies (*) for the case (3).

(c) The defining domain of the correspondence is the set of σ-fixed irreducible characters of G by Lemma 1.1. Let χ_1 and χ_2 be two distinct σ-fixed irreducible characters of G. Then we have

$$|G_\sigma|^{-1}\sum_{g\in G}\phi_{\chi_1}(g)\overline{\phi_{\chi_2}(g)}=0$$

from Lemma 1.4, Lemma 2.2 (b) and Corollary 2.9 (b). This proves the injectivity of the correspondence. By Corollary 2.3, the number of σ-fixed irreducible characters of G is equal to the number of irreducible characters of G_σ. Hence the correspondence must be bijective. The proof of Theorem 4.1 is now complete.

§ 5. Parametrizations.

For a positive integer l, we denote by G_l the general linear group $GL_l(k_2)$ over the quadratic extension k_2 of a finite field k. Let σ be the automorphism of G_l defined by (4.2). Put $F=k_{2n!}$. We consider that σ also acts on $F^*=GL_1(k_{2n!})$ and $\hat{F}^*=\mathrm{Hom}\,(F^*, \boldsymbol{C}^*)$ by

$$t^\sigma=t^{-q},\ u^\sigma(t)=u(t^{-q}) \qquad (t\in F^*,\ u\in\hat{F}^*,\ q=|k|)\,.$$

We denote by $\mathcal{F}_i$ and $\mathcal{G}_i$ respectively, the set of σ^i-orbits in F^* and $\hat{F}^*$ $(i=1, 2)$.

For an element f of $\mathcal{F}_i$ (or $\mathcal{G}_i$), $d(f)$ denotes the cardinality of the orbit f. Let $\mathcal{P}$ be the set of partitions, i.e. integer sequences $\nu=(\nu_1, \nu_2, \cdots, \nu_r)$ satisfying $\nu_1 \geqq \nu_2 \geqq \cdots \geqq \nu_r > 0$. We write $|\nu|=\nu_1+\nu_2+\cdots+\nu_r$. For convention, we suppose that $\mathcal{P}$ contains the empty partition $\emptyset$, and that $|\emptyset|=0$. For a positive integer $l \leqq n$, let $\Lambda_i^{(l)}$ $(i=1, 2)$ be the set of $\mathcal{P}$-valued functions $f \to \lambda(f)$ on $\mathcal{F}_i$, which respectively satisfy

(5.1) $$\sum_{f \in \mathcal{F}_i} |\lambda(f)| d(f) = l,$$

and let $\Theta_i^{(l)}$ $(i=1, 2)$ be the set of $\mathcal{P}$-valued functions $g \to \theta(g)$ on $\mathcal{G}_i$, which respectively satisfy

(5.2) $$\sum_{g \in \mathcal{G}_i} |\theta(g)| d(g) = l.$$

The following two lemmas are easy to verify.

LEMMA 5.1. (a) *Let* $f=\{f_1, f_2, \cdots, f_d\}$ $(f_i \in F^*)$ *be an element of* $\mathcal{F}_2$. *Then* $f=\{f_1^\sigma, f_2^\sigma, \cdots, f_d^\sigma\}$ *is also an element of* $\mathcal{F}_2$, *and the union* $f \cup f^\sigma$ *is an element of* $\mathcal{F}_1$.

(b) *Let* λ *be an element of* $\Lambda_1^{(l)}$. *Define a* $\mathcal{P}$*-valued function* λ' *on* $\mathcal{F}_2$ *by*

(5.3) $$\lambda'(f) = \lambda(f \cup f^\sigma) \qquad (f \in \mathcal{F}_2).$$

Then λ' *is an element of* $\Lambda_2^{(l)}$.

(c) *The mapping* $\lambda \to \lambda'$ *is a bijection between* $\Lambda_1^{(l)}$ *and*

$$\Lambda_{2,\sigma}^{(l)} = \{\lambda \in \Lambda_2^{(l)} \mid \lambda(f) = \lambda(f^\sigma) \quad \textit{for all } f \in \mathcal{F}_2\}.$$

LEMMA 5.2. (a) *Let* $g=\{g_1, g_2, \cdots, g_d\}$ $(g_i \in \hat{F}^*)$ *b ean element of* $\mathcal{G}_2$. *Then* $g=\{g_1^\sigma, g_2^\sigma, \cdots, g_d^\sigma\}$ *is also an element of* $\mathcal{G}_2$, *and the union* $g \cup g^\sigma$ *is an element of* $\mathcal{G}_1$.

(b) *Let* θ *be an element of* $\Theta_1^{(l)}$. *Define a* $\mathcal{P}$*-valued function* θ' *on* $\mathcal{G}_2$ *by*

(5.4) $$\theta'(g) = \theta(g \cup g^\sigma) \qquad (g \in \mathcal{G}_2).$$

Then θ' *is an element of* $\Theta_2^{(l)}$.

(c) *The mapping* $\theta \to \theta'$ *is a bijection between* $\Theta_1^{(l)}$ *and*

$$\Theta_{2,\sigma}^{(l)} = \{\theta \in \Theta_2^{(l)} \mid \theta(g) = \theta(g^\sigma) \quad \textit{for all } g \in \mathcal{G}_2\}.$$

The theory of Jordan normal forms gives a bijection $\lambda \to \mathfrak{C}[\lambda]$ between $\Lambda_2^{(l)}$ and the set of conjugacy classes of G_l. In particular, for each $f=\{f_1, f_2, \cdots, f_d\} \in \mathcal{F}_2$, $|\lambda(f)|$ is the multiplicity of f_i $(1 \leqq i \leqq d)$ as characteristic roots of $x \in \mathfrak{C}[\lambda]$. See [**7**; § 1] or [**11**; § 2] for more details. On the other hand, a theory ([**7**], [**11**]) of J. A. Green gives a bijection $\theta \to \chi[\theta]$ between $\Theta_2^{(l)}$ and the set of irreducible complex characters of G_l. Here, we describe an outline of Green's theory, because we need them later. Let α be an element of $\hat{k}_{2l}^*$ $(l \leqq n)$.

We define the (not necessarily irreducible) character $\chi_l[\alpha]$ of G_l, whose value at $x\in\mathfrak{C}[\lambda]$ $(\lambda\in\Lambda_2^{(l)})$ is given by

$$\chi_l[\alpha](x)=0 \tag{5.5}$$

if $|\lambda(f)|\neq 0$ for at least two elements f of $\mathcal{F}_2$, and

$$\chi_l[\alpha](x)=p_{\lambda(f)}(q^2)\sum_{i=0}^{d(f)-1}\alpha(t^{q^{2i}}) \tag{5.6}$$

if there exists only one $f\in\mathcal{F}_2$ such that $|\lambda(f)|\neq 0$, where p_ν is a polynomial depending on a partition ν, $q=|k|$ and t is an element in the σ^2-orbit f in F^*. (Note that t is an element of k_{2l}^* because of the condition (5.1).) For an element α of k_{2l}^* and a partition $\nu=(\nu_1,\nu_2,\cdots,\nu_r)$ such that $l|\nu|\leqq n$, we can define the character $\chi_{l|\nu|}[\nu;\alpha]$ of $G_{l|\nu|}$ which can be written as

$$\chi_{l|\nu|}[\nu;\alpha]=\sum_{\mu}c_{\mu\nu}\prod_{i=1}^{r}\chi_{l\nu_i}[\alpha\circ N_{k2l\nu_i}/_{k_{2l}}], \tag{5.7}$$

where the sum is over the set of partitions μ such that $|\mu|=|\nu|$, $c_{\mu\nu}$ are rational numbers independent of α, $N_{k_{2l\nu_i}}/_{k_{2l}}$ $(i=1,2,\cdots,r)$ are usual norm mappings from $k_{2l\nu_i}$ to k_{2l}, and $\prod$ is a $\circ$-product ([7]; see the proof of Lemma 5.3 below). We can now describe the irreducible character $\chi[\theta]$ of G_n corresponding to $\theta\in\Theta_2^{(n)}$. Let g be an element of $\mathcal{G}_2$, and u an element of $\hat{F}^*$ contained in g. Since $u=u^{q^{2d}}$ $(d=d(g))$, there exists a unique element α_u of $\hat{k}_{2d}^*$ such that $u=\alpha_u\circ N_{F/k_{2d}}$. For a partition ν, the character $\chi_{d|\nu|}[\nu;\alpha_u]$ does not depend on the choice of u in g. Hence we can define the character $\chi[\theta]$ of G_n by

$$\chi[\theta]=\prod_{g\in\mathcal{G}_2}\chi_{d(g)|\theta(g)|}[\theta(g);\alpha_{u(g)}], \tag{5.8}$$

where $u(g)$ is an element of F^* contained in $g\in\mathcal{G}_2$, and $\prod$ is a $\circ$-product. In [7], it is shown that $\chi[\theta]$ $(\theta\in\Theta_2^{(n)})$ are irreducible and distinct, and any irreducible characters of G_n can be obtained in this way.

LEMMA 5.3. *Let $\nu=(\nu_1,\nu_2,\cdots,\nu_r)$ be a partition such that $|\nu|=l$. Let ϕ_i be a complex valued class function on G_{ν_i} $(i=1,2,\cdots,r)$. Then*

$$\Big(\prod_{i=1}^{r}\phi_i\Big)(x^\sigma)=\Big(\prod_{i=1}^{r}\phi_i^\sigma\Big)(x)\qquad(x\in G_l),$$

where $\prod$ is a $\circ$-product and ϕ_i^σ is the class function on G_{ν_i} defined by

$$\phi_i^\sigma(y)=\phi_i(y^\sigma)\qquad(y\in G_{\nu_i}).$$

PROOF. Let P_ν be the standard parabolic subgroup of G_l corresponding to ν, i.e. the group of matrices

$$b=\begin{pmatrix} B_{11}B_{12}\cdots B_{1r} \\ 0 \quad B_{22}\cdots B_{2r} \\ 0 \quad 0 \qquad \\ \cdots\cdots \qquad \vdots \\ 0 \quad 0\cdots 0 \; B_{rr} \end{pmatrix}\in G_l$$

for which $B_{ii}\in G_{\nu_i}$ $(i=1, 2, \cdots, r)$. Let ϕ be the class function on P_ν defined by

(5.9) $$\phi(b)=\prod_{i=1}^{r}\{\phi_i(B_{ii})\} \qquad (b=(B_{ij})\in P_\nu).$$

Then, by the definition of $\circ$-product,

(5.10) $$\prod_{i=1}^{r}\phi_i=\operatorname{ind}[\phi\,|\,P_\nu\longrightarrow G_l]$$

where the right hand side is the class function on G_l induced from ϕ:

(5.11) $$\operatorname{ind}[\phi\,|\,P_\nu\longrightarrow G_l](x)=|P_\nu|^{-1}|Z_{G_l}(x)|\sum_{y\in\mathfrak{C}G_l}(x)\cap P_\nu\phi(y).$$

From (5.11) we have

(5.12) $$\operatorname{ind}[\phi\,|\,P_\nu\longrightarrow G_l](x^\sigma)=\operatorname{ind}[\phi^\sigma\,|\,P_\nu\longrightarrow G_l](x),$$

where ϕ^σ is defined by

$$\phi^\sigma(y)=\phi(y^\sigma) \qquad (y\in P_\nu).$$

By (5.9), (5.10), (5.12) and the commutativity ([7; Lemma 2.5]) of $\circ$-product, we obtain the required result.

LEMMA 5.4. *Let α be an element of $\hat{k}_{2l}^*$, and ν a partition.*

(a) $\chi_l[\alpha](x^\sigma)=\chi_l[\alpha^{-q}](x) \qquad (x\in G_l).$

(b) $\chi_{l|\nu|}[\nu;\ \alpha](x^\sigma)=\chi_{l|\nu|}[\nu;\ \alpha^{-q}](x) \qquad (x\in G_{l|\nu|}).$

PROOF. (a) Let $t_1, t_2, \cdots, t_l$ be the characteristic roots of $x\in G_l$. Then, clearly, $t_1^\sigma, t_2^\sigma, \cdots, t_l^\sigma$ are the characteristic roots of x^σ. Part (a) follows from this fact and the formulas (5.5) and (5.6).

(b) This follows from (5.7), Lemma 5.3 and part (a).

For each $\lambda\in\Lambda_2^{(n)}$, define the element λ^σ of $\Lambda_2^{(n)}$ by

$$\lambda^\sigma(f)=\lambda(f^\sigma) \qquad (f\in\mathcal{F}_2).$$

Similarly we also define the element θ^σ of $\Theta_2^{(n)}$ for each $\theta\in\Theta_2^{(n)}$.

LEMMA 5.5. (a) *For each $\lambda\in\Lambda_2^{(n)}$, we have $\mathfrak{C}[\lambda]^\sigma=\mathfrak{C}[\lambda^\sigma]$.*

(b) *For each $\theta\in\Theta_2^{(n)}$, we have $\chi[\theta](x^\sigma)=\chi[\theta^\sigma](x) \qquad (x\in G_n).$*

PROOF. (a) This can be easily verified.

(b) This follows from (5.8), Lemma 5.3 and Lemma 5.4 (b).

COROLLARY 5.6. (a) *A conjugacy class* $\mathfrak{C}[\lambda]$ $(\lambda\in\Lambda_2^{(n)})$ *of* G_n *is fixed by* σ *if and only if* λ *is contained in* $\Lambda_{2,\sigma}^{(n)}$ *(see Lemma* 5.1 (c)).

(b) *An irreducible character* $\chi[\theta]$ $(\theta\in\Theta_2^{(n)})$ *of* G_n *is fixed by* σ *if and only if* θ *is contained in* $\Theta_{2,\sigma}^{(n)}$ *(see Lemma* 5.2 (c)).

Let λ be an element of $\Lambda_1^{(n)}$, and λ' an element of $\Lambda_{2,\sigma}^{(n)}$ defined by (5.3). By Corollary 5.6 (a), the conjugacy class $\mathfrak{C}[\lambda']$ of G_n is fixed by σ. Hence, by Corollary 2.8, $\mathfrak{D}[\lambda]=\mathfrak{C}[\lambda']\cap U_n(k_2)$ is a conjugacy class of $U_n(k_2)$. It is easy to see that every conjugacy class of $U_n(k_2)$ can be obtained in this way. Next, let θ be an element of $\Theta_1^{(n)}$, and θ' an element of $\Theta_{2,\sigma}^{(n)}$ defined by (5.4). By Corollary 5.6 (b), the irreducible character $\chi[\theta']$ of G_n is fixed by σ. Hence using Theorem 4.1 with $m=2$, one can define an irreducible character $\psi[\theta]=\psi_{\chi[\theta']}$ of $U_n(k_2)$, if char$(k)\neq 2$. Thus we have proved the following

THEOREM 5.7. *Let the notations be as above.*

(a) *The correspondence* $\lambda\to\mathfrak{D}[\lambda]$ *is a bijection between* $\Theta_1^{(n)}$ *and the set of conjugacy classes of* $U_n(k_2)$.

(b) *The correspondence* $\theta\to\psi[\theta]$ *is a bijection between* $\Theta_1^{(n)}$ *and the set of irreducible characters of* $U_n(k_2)$ *(char*$(k)\neq 2$).

REMARK 5.8. (a) The above parametrization of the conjugacy classes of $U_n(k_2)$ is essentially the same as the one given in Ennola [4].

(b) Ennola constructed a set of class functions $\psi'[\theta]$ $(\theta\in\Theta_1^{(n)})$, and conjectured that these are the irreducible characters of $U_n(k_2)$. It is very probable that our irreducible character $\psi[\theta]$ coincides with Ennola's class function $\psi'[\theta]$ for each $\theta\in\Theta_1^{(n)}$.

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