J. Math. Soc. Japan
Vol. 29, No. 3, 1977

On the growing up problem for semilinear heat equations

By Kusuo KOBAYASHI, Tunekiti SIRAO and Hiroshi TANAKA

(Received Jan. 17, 1976)

§ 1. Introduction.

We will consider the Cauchy problem for the semilinear heat equation

$$\frac{\partial u}{\partial t}=\Delta u+f(u), \qquad t>0,\ x\in R^{d}, \tag{1.1}$$

with the initial condition $u(0, x)=a(x)$. It is assumed that the function f is defined, non-negative and locally Lipschitz continuous in $[0, \infty)$. If the initial value $a(x)$ is a bounded non-negative continuous function in R^{d}, not vanishing identically, then it is well-known that there exists a positive local solution $u(t, x)$ of (1.1); more precisely, there exist positive T $(\leqq\infty)$ and $u(t, x)$ satisfying the following conditions (i), (ii) and (iii).

(i) $u(t, x)$ is defined on $[0, T)\times R^{d}$, strictly positive in $(0, T)\times R^{d}$ and $u(0, x)=a(x)$.

(ii) For any $T'<T$, $u(t, x)$ is bounded and continuous on $[0, T']\times R^{d}$.

(iii) $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial^{2}u}{\partial x_{i}\partial x_{j}}$ $(1\leqq i, j\leqq d)$ exist in $(0, T)\times R^{d}$ and $u(t, x)$ satisfies (1.1) in the classical sense.

If $T_{\infty}=T_{\infty}(a, f)$ denotes the supremum of all T satisfying the above three conditions, then the existence of *global solution* is the case $T_{\infty}=\infty$, and in the general situation $(T_{\infty}\leqq\infty)$ the unique existence assertion amounts to say that there exists a unique solution $u(t, x)$ of (1.1) up to T_{∞} satisfying the above three conditions with $T=T_{\infty}$. In this paper, such a solution is called simply a *positive solution* of (1.1), and is denoted by $u(t, x; a, f)$ when we want to elucidate the initial value $a(x)$ and the nonlinear term $f(u)$. A positive solution of (1.1) is said to *blow up in a finite time* and the corresponding T_{∞} is called *the blowing-up time* of the solution, provided that $T_{\infty}<\infty$. A global positive solution $u(t, x)$ of (1.1) is said to *grow up to infinity*, if for each positive constant M and each compact set K in R^{d} there exists $T<\infty$ such that $t>T$ and $x\in K$ imply $u(t, x)>M$.

The purpose of this paper is to investigate the following problem: How does the nonlinear term f affect the growth of positive solutions of (1.1) as $t\rightarrow\infty$? More precisely, we assume that $f(\lambda)>0$ for $\lambda>0$ and consider the following problems.

(A) Under what condition on f, does any positive solution of (1.1) blow up in a finite time?

(B) Under what condition on f, does any positive *global* solution of (1.1) grow up to infinity?

These problems were investigated by several authors, and our results will sharpen theirs.

When $f(\lambda)=\lambda^{1+\alpha}$, $\alpha>0$, the problem (A) was first considered by H. Fujita [**1**]. His main result, combined with a recent work of K. Hayakawa [**3**] for the case $\alpha d=2$ that was not covered by [**1**], can be stated as follows: in the case $\alpha d\leqq 2$ *all positive* solutions of (1.1) blow up in finite times, and in the contrary case *some positive* solutions of (1.1) converge uniformly to 0 as $t\rightarrow\infty$. The analogous problem for the equation

$$\frac{\partial u}{\partial t}=-(-\Delta)^{\beta}u+u^{1+\alpha},\ t>0,\ x\in R^{d}\qquad (0<\beta\leqq 1)$$

was treated by S. Sugitani [**9**]. Fujita [**2**] also treated the problem (A) for (1.1) when f is of a general type. The conditions on f naturally contain both the local condition of $f(\lambda)$ near $\lambda=0$ and the growth one of $f(\lambda)$ as $\lambda\uparrow\infty$.

Ya. I. Kanel' [**6**] considered different problems. He assumes that $f(0)=f(1)=0$ and $f(\lambda)>0$ for $0<\lambda<1$. Then, one of the results due to Kanel' is that in case of $d=1$ any positive solution ($\leqq 1$) converges to 1 uniformly on each compact set of R^1 as $t\rightarrow\infty$, provided that $f'(0)>0$. This type of problem was also considered by N. Ikeda and K. Kametaka (a part of their results is found in [**5**]), and recently by K. Hayakawa [**4**] in which the condition $f'(0)>0$ was replaced by $f(\lambda)\geqq c\lambda^{1+(2/d)}$ near $\lambda=0$ ($c>0$). The treatment of this problem and (B) are quite similar, and only the local behavior of $f(\lambda)$ near $\lambda=0$ becomes essential.

Our main results are Theorems 2.1, 3.5, 3.7 and 5.1. The first theorem, which gives a sufficient condition on f for the blowing up of any positive solution, plays a fundamental role, since most of the other results in this paper are derived on the basis of this. As we mentioned above, it is expected that the condition on f for the problem (B) is concerned only with the local behavior of $f(\lambda)$ near $\lambda=0$. In § 3 we will examine this situation, and prove that under certain additional conditions on f the divergence of

$$\int_{0+}^{\varepsilon} f(\lambda)/\lambda^{2+(2/d)}d\lambda,\qquad \varepsilon>0$$

implies the growing up of positive global solutions if they exist (Theorem 3.5

and the similar result in Theorem 3.7). Finally using the comparison with the solution of certain ordinary differential equation we prove, under some additional conditions on f, that the convergence of the above integral implies the existence of positive solutions tending to 0 uniformly in x as $t\to\infty$ (Theorem 5.1).

In starting of our present work we owe much to the recent work of K. Hayakawa [**4**] and to conversations with N. Ikeda which were valuable to us. Hayakawa informed us of his result [**4**] yet unpublished at that time. We wish to express our best thanks to both of them.

§ 2. The blowing up problem.

To begin with, we consider a positive solution $u(t, x)=u(t, x; a, f)$ of (1.1) in the sense of § 1. The following properties are well known (cf. [7]).

(2.1) $$u(t, x)=u(t-s, x; u(s, \cdot), f), \qquad 0\leqq s\leqq t<T.$$

(2.2) $a_1\leqq a$ and $f_1\leqq f$ imply that $u(t, x; a_1, f_1)\leqq u(t, x; a, f)$, $0\leqq t<T_\infty$.

Denote by $H(t, x, y)$ the fundamental solution (heat kernel) for the Cauchy problem $\partial u/\partial t=\Delta u$, and by $\{H_t\}$ the corresponding semigroup:

$$H(t, x, y)=(4\pi t)^{-d/2}\exp\left(-\frac{|x-y|^2}{4t}\right),$$

$$H_t a(x)=\int_{R^d} H(t, x, y)a(y)dy.$$

Now assume that $a(x)$ is not identically zero. Since the solution $u(t, x)=u(t, x; a, f)$ satisfies the equation

$$u(t, x)=H_t a(x)+\int_0^t H_{t-s}f(u(s, \cdot))ds, \qquad 0\leqq t<T_\infty$$

we have

(2.3) $$u(t_0, x)\geqq H_{t_0}a(x)>\alpha e^{-\beta|x|^2}, \qquad 0<t_0<T_\infty,$$

where α, β are some positive constants depending upon $t_0>0$. By (2.1), (2.3) and (2.2) we have

$$u(t, x)\geqq u(t-t_0, x; a_1, f), \qquad a_1(x)=\alpha e^{-\beta|x|^2},$$

and this tells us that *when we deal with the blowing up or growing up problem, we may consider only those solutions with initial values* $\alpha e^{-\beta|x|^2}$, α, $\beta>0$.

Our result for the problem (A) is the following

THEOREM 2.1. *Suppose that f satisfies the following three conditions.*

(A.1) *f is a locally Lipschitz continuous and non-decreasing function in $[0, \infty)$ with $f(0)=0$ and $f(\lambda)>0$ for $\lambda>0$.*

(A.2) $\int_{0+}^{\varepsilon} f(\lambda)/\lambda^{2+(2/d)}d\lambda=\infty$ *for some* $\varepsilon>0$.

(A.3) *There exists a positive constant* c ($\leqq 1$) *such that*

(a) $f(\lambda\mu)\geqq c\mu^{1+(2/d)}f(\lambda)$ *for* $0<\lambda\leqq\mu,\ \lambda<c$,

(b) $f(\lambda\mu)\geqq c\mu^{2+(2/d)}f(\lambda)$ *for* $0<\mu\leqq\lambda<c$.

Then each positive solution of (1.1) *blows up in a finite time.*

For any initial value $a(x)=\alpha e^{-\beta|x|^2}$ with $\alpha,\ \beta>0$, we will prove that the solution $u(t, x)=u(t, x; a, f)$ blows up in a finite time. To simplify the notation, we set $\gamma=1+2/d$ and

$$\alpha(s)=\alpha(1+4\beta s)^{-d/2}$$

$$\varphi(t)=\frac{1}{\alpha}\int_0^t(1+4\beta s)^{d/2}f(\alpha(s))ds=\frac{\alpha^{2/d}}{2\beta d}\int_{\alpha(t)}^{\alpha}f(\lambda)\lambda^{-2-2/d}d\lambda .$$

Then $\varphi(t)\uparrow\infty$ as $t\uparrow\infty$ by (A.2). The solution $u(t, x)$ is constructed by iteration as follows. Namely, we set

$$u_0(t, x)=H_t a(x)=\alpha(t)\exp\{-\beta(1+4\beta t)^{-1}|x|^2\},$$

$$u_n(t, x)=H_t a(x)+\int_0^t H_{t-s}f(u_{n-1}(s, \cdot))ds, \qquad n\geqq 1 .$$

Then $u_n(t, x)\uparrow u(t, x)$ as $n\uparrow\infty$ for $0\leqq t<T_\infty$.

LEMMA 2.2. *For any positive integer n, we have*

(2.4) $$u_n(t, x)-u_0(t, x)\geqq B_n(t, |x|)u_0(t, x),$$

where

$$B_n(t, |x|)=C_n\varphi(t)^{1+\gamma+\cdots+\gamma^{n-1}}\cdot\exp\left\{-\frac{\beta(\gamma+\gamma^2+\cdots+\gamma^n)}{1+4\beta t}|x|^2\right\},$$

$$C_n=c^{(\gamma^n-1)/(\gamma-1)}\cdot(1+\gamma)^{-(d/2)\gamma^n\sum_1^n k\gamma^{-k}}/\prod_{k=0}^{n-1}\left(\sum_{p=0}^{k}\gamma^p\right)^{\gamma^{n-1-k}} .$$

PROOF. We prove the lemma by induction. First we consider the case $n=1$. We put

$$\lambda=\alpha(s), \qquad \mu=\exp\{-\beta(1+4\beta s)^{-1}|y|^2\},$$

and then apply (A.3) to $f(u_0(s, y))=f(\lambda\mu)$. Assume that α is so small that $\alpha(s)<c$. In the case $\lambda<\mu$ we have from (a) of (A.3)

$$f(u_0(s, y))\geqq c\mu^\gamma f(\lambda)=c\exp\{-\beta\gamma(1+4\beta s)^{-1}|y|^2\}f(\alpha(s))$$

$$\geqq c\exp\{-\beta(1+\gamma)(1+4\beta s)^{-1}|y|^2\}f(\alpha(s)),$$

while in the case $\lambda\geqq\mu$ we have from (b) of (A.3)

$$f(u_0(s, y)) \geqq c \exp\{-\beta(1+\gamma)(1+4\beta s)^{-1}|y|^2\} f(\alpha(s)).$$

Thus we have the inequality

$$f(u_0(s, y)) \geqq c f(\alpha(s)) \left\{ \frac{\pi(1+4\beta s)}{\beta(1+\gamma)} \right\}^{d/2} H\Big(\frac{1+4\beta s}{4\beta(1+\gamma)}, y, 0\Big)$$

holding for all $y \in R^d$, and therefore from the monotonicity of f we obtain

$$(2.5) \qquad u_1(t, x) - u_0(t, x) = \int_0^t H_{t-s} f(u_0(s, \cdot)) ds$$

$$\geqq c \int_0^t f(\alpha(s)) \left\{ \frac{\pi(1+4\beta s)}{\beta(1+\gamma)} \right\}^{d/2} H_{t-s} H\Big(\frac{1+4\beta s}{4\beta(1+\gamma)}, \cdot, 0\Big) ds$$

$$= c \int_0^t f(\alpha(s)) \left\{ \frac{\pi(1+4\beta s)}{\beta(1+\gamma)} \right\}^{d/2} H\Big(t-s+\frac{1+4\beta s}{4\beta(1+\gamma)}, x, 0\Big) ds$$

$$= c \int_0^t f(\alpha(s)) \left\{ \frac{\delta_0(t, s)}{1+4\beta s} \right\}^{-d/2} \exp\left\{ -\frac{\beta(1+\gamma)|x|^2}{\delta_0(t, s)} \right\} ds,$$

where $\delta_0(t, s) = 1+4\beta s+4\beta(1+\gamma)(t-s)$. On the other hand, we have for $0 \leqq s \leqq t$

$$1+4\beta t \leqq \delta_0(t, s) \leqq 1+4\beta(1+\gamma)t < (1+\gamma)(1+4\beta t),$$

and hence the integrand in the last line of (2.5) is bounded below by

$$f(\alpha(s))(1+\gamma)^{-d/2} \Big(\frac{1+4\beta s}{1+4\beta t}\Big)^{d/2} \exp\left\{ -\frac{\beta(1+\gamma)}{1+4\beta t}|x|^2 \right\}.$$

Therefore we obtain

$$u_1(t, x) - u_0(t, x) \geqq c \int_0^t f(\alpha(s))(1+\gamma)^{-d/2} \Big(\frac{1+4\beta s}{1+4\beta t}\Big)^{d/2} \exp\left\{ -\frac{\beta(1+\gamma)}{1+4\beta t}|x|^2 \right\} ds$$

$$= c(1+\gamma)^{-d/2} \int_0^t f(\alpha(s)) \alpha^{-1} (1+4\beta s)^{d/2} ds \exp\{-\beta\gamma(1+4\beta t)^{-1}|x|^2\} u_0(t, x)$$

$$= B_1(t, |x|) u_0(t, x).$$

Next, assuming that (2.4) holds for n we prove that (2.4) holds also for $n+1$. We now apply (A.3) to $f(u_n(s, y)) \geqq f(\lambda\mu)$ with

$$\lambda = \alpha(s), \qquad \mu = \{1+B_n(s, |y|)\} \exp\{-\beta(1+4\beta s)^{-1}|y|^2\}.$$

In the case $\lambda < \mu$ we have from (a) of (A.3)

$$f(u_n(s, y)) \geqq c\{1+B_n(s, |y|)\}^{\gamma} \exp\{-\beta\gamma(1+4\beta s)^{-1}|y|^2\} f(\alpha(s))$$

$$\geqq c B_n(s, |y|)^{\gamma} \exp\{-\beta(1+\gamma)(1+4\beta s)^{-1}|y|^2\} f(\alpha(s)),$$

while in the case $\lambda \geqq \mu$

$$f(u_n(s, y)) \geqq c B_n(s, |y|)^{\gamma} \exp\{-\beta(1+\gamma)(1+4\beta s)^{-1}|y|^2\} f(\alpha(s))$$

using (b) of (A.3). Therefore we obtain

$$(2.6)\qquad u_{n+1}(t, x)-u_0(t, x)=\int_0^t ds\int_{R^d} H(t-s, x, y)f(u_n(s, y))dy$$

$$\geqq c\int_0^t f(\alpha(s))ds\int_{R^d} H(t-s, x, y)B_n(s, |y|)^{\gamma}\exp\{-\beta(1+\gamma)(1+4\beta s)^{-1}|y|^2\}dy$$

$$=c\,C_n^{\gamma}\int_0^t f(\alpha(s))\varphi(s)^{\gamma+\cdots+\gamma^n}ds\int_{R^d} H(t-s, x, y)$$

$$\times\exp\{-\beta(1+4\beta s)^{-1}(\gamma^2+\cdots+\gamma^{n+1})|y|^2\}\exp\{-\beta(1+\gamma)(1+4\beta s)^{-1}|y|^2\}dy$$

$$=c\,C_n^{\gamma}\int_0^t f(\alpha(s))\varphi(s)^{\gamma+\cdots+\gamma^n}\left\{\frac{\delta_n(t, s)}{1+4\beta s}\right\}^{-d/2}\exp\left\{-\frac{\beta(1+\gamma+\cdots+\gamma^{n+1})}{\delta_n(t, s)}|x|^2\right\}ds,$$

where

$$\delta_n(t, s)=1+4\beta s+4\beta(1+\gamma+\cdots+\gamma^{n+1})(t-s).$$

On the other hand, we have for $0\leqq s\leqq t$

$$1+4\beta t\leqq\delta_n(t, s)\leqq 1+4\beta(1+\gamma+\cdots+\gamma^{n+1})t$$

$$=(1+4\beta t)\prod_{k=0}^{n}\frac{1+4\beta(1+\gamma+\cdots\gamma^{k+1})t}{1+4\beta(1+\gamma+\cdots\gamma^k)t}$$

$$\leqq(1+4\beta t)\prod_{k=0}^{n}\frac{1+4\beta(1+\gamma)(1+\gamma+\cdots+\gamma^k)t}{1+4\beta(1+\gamma+\cdots+\gamma^k)t}$$

$$<(1+4\beta t)(1+\gamma)^{n+1},$$

and hence the integrand in the last line of (2.6) is bounded below by

$$f(\alpha(s))\varphi(s)^{\gamma+\cdots+\gamma^n}(1+\gamma)^{-(n+1)d/2}\left(\frac{1+4\beta s}{1+4\beta t}\right)^{d/2}\exp\left\{-\frac{\beta(1+\gamma+\cdots\gamma^{n+1})|x|^2}{1+4\beta t}\right\}.$$

Therefore we finally obtain

$$u_{n+1}(t, x)-u_0(t, x)\geqq c\,C_n^{\gamma}(1+\gamma)^{-(n+1)d/2}\int_0^t f(\alpha(s))\varphi(s)^{\gamma+\cdots+\gamma^n}$$

$$\times\left(\frac{1+4\beta s}{1+4\beta t}\right)^{d/2}\exp\left\{-\frac{\beta(1+\gamma+\cdots+\gamma^{n+1})|x|^2}{1+4\beta t}\right\}ds$$

$$=c\,C_n^{\gamma}(1+\gamma)^{-(n+1)d/2}\int_0^t f(\alpha(s))\alpha^{-1}(1+4\beta s)^{d/2}\varphi(s)^{\gamma+\cdots+\gamma^n}ds$$

$$\times\exp\left\{-\frac{\beta(\gamma+\cdots+\gamma^{n+1})|x|^2}{1+4\beta t}\right\}u_0(t, x)$$

$$=c\,C_n^{\gamma}(1+\gamma)^{-(n+1)d/2}\cdot\frac{\varphi(t)^{1+\gamma+\cdots+\gamma^n}}{1+\gamma+\cdots+\gamma^n}\exp\left\{-\frac{\beta(\gamma+\cdots+\gamma^{n+1})|x|^2}{1+4\beta t}\right\}u_0(t, x)$$

$$=B_{n+1}(t, |x|)u_0(t, x).$$

This completes the proof of the lemma.

We can now proceed to the proof of Theorem 2.1. If we set $A=\sum_1^\infty k\gamma^{-k}$, $B=\sum_1^\infty \gamma^{-k}$ and $\gamma_0=\min\{\gamma-1, 1\}$, then we have

$$\prod_{k=0}^{n-1}\Big(\sum_{p=0}^{k}\gamma^{p}\Big)^{\gamma^{n-1-k}}=\prod_{k=1}^{n}\Big(\frac{\gamma^k-1}{\gamma-1}\Big)^{\gamma^{n-k}}<\prod_{k=1}^{n}\Big(\frac{\gamma^k}{\gamma_0}\Big)^{\gamma^{n-k}}$$

$$=\gamma^{\gamma^n\sum_1^n k\gamma^{-k}}\gamma_0^{-\gamma^n\sum_1^n\gamma^{-k}}<\gamma^{A\gamma^n}\gamma_0^{-B\gamma^n},$$

and hence by Lemma 2.2

$$u_n(t, x)\geqq C_n\varphi(t)^{1+\cdots+\gamma^{n-1}}\exp\Big\{-\frac{\beta(\gamma+\cdots+\gamma^n)|x|^2}{1+4\beta t}\Big\}u_0(t, x)$$

$$>c^{\gamma^n/(\gamma-1)}(1+\gamma)^{-(d/2)A\gamma^n}\gamma^{-A\gamma^n}\gamma_0^{B\gamma^n}\Big\{\varphi(t)\exp\Big(-\frac{\beta\gamma|x|^2}{1+4\beta t}\Big)\Big\}^{\gamma^{n-1}}u_0(t, x)$$

$$=\Big\{\delta\varphi(t)\exp\Big(-\frac{\beta\gamma|x|^2}{1+4\beta t}\Big)\Big\}^{\gamma^{n-1}}u_0(t, x),$$

provided that $\varphi(t)\exp\{-\beta\gamma(1+4\beta t)^{-1}|x|^2\}>1$, where

$$\delta=c^{\gamma/(\gamma-1)}(1+\gamma)^{-(d/2)A\gamma}\gamma^{-A\gamma}\gamma_0^{B\gamma}.$$

Since $\varphi(t)\uparrow\infty$ as $t\uparrow\infty$, we can choose $t_0(x)$ so that

$$\delta\varphi(t)\exp\Big(-\frac{\beta\gamma|x|^2}{1+4\beta t}\Big)>2$$

holds for any $t>t_0(x)$. Then we have $u_n(t, x)>2^{\gamma^{n-1}}u_0(t, x)$ for any $n\geqq 1$ and $t>t_0(x)$, and hence $u_n(t, x)\uparrow\infty$ as $n\uparrow\infty$ for any $t>t_0(x)$. Since $u_n(t, x)\uparrow u(t, x)$ $<\infty$ for $t<T_\infty$, we must have $T_\infty<t_0(x)$, and this means that $u(t, x)$ blows up in a finite time.

§ 3. The growing up problem.

In this section we seek for a sufficient condition to be imposed on the local behavior of $f(\lambda)$ near $\lambda=0$ in order that any positive *global* solution of (1.1) grows up to infinity. The result in the first paragraph is of a preliminary nature.

3.1. We begin with two simple lemmas, in which it is assumed that f is non-negative and locally Lipschitz continuous in $[0, \infty)$.

LEMMA 3.1. *Let* $\varepsilon>0$. *If any positive solution* $u(t, x)$ *of* (1.1) *either blows up in a finite time or satisfies*

$$\limsup_{t\to\infty}\|u(t,\cdot)\|_\infty=\infty,$$

then the same holds for

$$\frac{\partial u}{\partial t}=\Delta u+\varepsilon f(u)\,. \tag{3.1}$$

PROOF. Let $\tilde{a}(x)$ be the initial value of a positive solution $\tilde{u}(t, x)$ of (3.1), and set $a_\varepsilon(x)=\tilde{a}(\varepsilon^{-1/2}x)$. Then we have $\tilde{u}(t, x)=u(\varepsilon t, \varepsilon^{1/2}x\,;\,a_\varepsilon, f)$, from which the assertion of the lemma follows.

We introduce a class of monotone radial functions as follows:

$$\mathcal{A}=\{a\in C(R^d):\ a(x)\geqq 0,\ \not\equiv 0\quad\text{and}\quad a(x)\geqq a(y)\quad\text{for } |x|\leqq|y|\}.$$

LEMMA 3.2. *Suppose that f is non-decreasing. If $a(\cdot)\in\mathcal{A}$, then $u(t, \cdot\,;\,a, f)\in\mathcal{A}$ for each $t\in[0, T_\infty)$.*

PROOF. The solution $u(t, x\,;\,a, f)$ can be constructed by iteration:

$$u_0(t, x)=H_t a(x)$$

$$u_n(t, x)=H_t a(x)+\int_0^t H_{t-s}f(u_{n-1}(s, \cdot))ds\,,\qquad n\geqq 1\,,$$

$$u(t, x\,;\,a, f)=\lim_{n\to\infty}u_n(t, x)\,,\qquad 0\leqq t<T_\infty\,.$$

If $a(\cdot)\in\mathcal{A}$, then $f(a(\cdot))\in\mathcal{A}$ by the monotonicity of f, and hence we have only to prove that $H_t a\in\mathcal{A}$ for each t. It is obvious that $H_t a$ is a radial function if $a(\cdot)$ is so, and therefore we may assume that

$$x=(x_1, 0, \cdots, 0)\,,\quad y=(y_1, 0, \cdots, 0)\,,\quad 0\leqq x_1\leqq y_1 \tag{3.2}$$

in proving $H_t a(x)\geqq H_t a(y)$ for $|x|\leqq|y|$. We have

$$\begin{aligned}(4\pi t)^{d/2}\{H_t a(x)-H_t a(y)\}&=\int_{|x-z|\leqq|y-z|}\Big\{\exp\Big(-\frac{|x-z|^2}{4t}\Big)-\exp\Big(-\frac{|y-z|^2}{4t}\Big)\Big\}a(z)dz\\&\quad+\int_{|x-z|>|y-z|}\Big\{\exp\Big(-\frac{|x-z|^2}{4t}\Big)-\exp\Big(-\frac{|y-z|^2}{4t}\Big)\Big\}a(z)dz\\&=I+II\,.\end{aligned}$$

Since $|x-z|\leqq|y-z|$ is equivalent to $|z|\leqq|z-x-y|$ from (3.2), we have

$$\begin{aligned}I&=\int_{|z|\leqq|z-x-y|}\Big\{\exp\Big(-\frac{|x-z|^2}{4t}\Big)-\exp\Big(-\frac{|y-z|^2}{4t}\Big)\Big\}a(z)dz\\&\geqq\int_{|z|\leqq|z-x-y|}\Big\{\exp\Big(-\frac{|x-z|^2}{4t}\Big)-\exp\Big(-\frac{|y-z|^2}{4t}\Big)\Big\}a(z-x-y)dz\,.\end{aligned}$$

Making the change of variable $w=x+y-z$ in the last line of the above and then noticing that $|x+y-w|\leqq|w|$ is equivalent to $|x-w|\geqq|y-w|$, we obtain

$$\begin{aligned}I&\geqq\int_{|x+y-w|\leqq|w|}\Big\{\exp\Big(-\frac{|y-w|^2}{4t}\Big)-\exp\Big(-\frac{|x-w|^2}{4t}\Big)\Big\}a(w)dw\\&=\int_{|x-w|\geqq|y-w|}\Big\{\exp\Big(-\frac{|y-w|^2}{4t}\Big)-\exp\Big(-\frac{|x-w|^2}{4t}\Big)\Big\}a(w)dw=-II\,,\end{aligned}$$

and this proves that $H_t a(x) \geqq H_t a(y)$.

THEOREM 3.3. *Let f and $\tilde{f}$ be locally Lipschitz continuous functions on* $[0, \infty)$, *and assume that* (i) $f(\lambda)>0$ *for* $\lambda>0$, (ii) $\tilde{f}$ *is non-decreasing with* $\tilde{f}(0)=0$, *and* (iii) $\liminf_{\lambda \downarrow 0} f(\lambda)/\tilde{f}(\lambda)>0$. *Suppose that any positive solution* $\tilde{u}(t, x)$ *of*

$$\frac{\partial u}{\partial t}=\Delta u+\tilde{f}(u) \tag{3.3}$$

either blows up in a finite time or satisfies

$$\limsup_{t \to \infty} \|\tilde{u}(t, \cdot)\|_\infty=\infty . \tag{3.4}$$

Then any positive global solution of (1.1) *grows up to infinity.*

PROOF. By the remark we made just before Theorem 2.1 we may consider only those initial values belonging to $\mathcal{A}$, and also by virtue of (2.2) we may even consider only small initial values of compact support. So we take $a(x)$ with compact support from the class $\mathcal{A}$, and prove that $u(t, x ; a, f)$ grows up to infinity assuming that it does not blow up. We take an arbitrary positive constant M satisfying $M>\|a\|_\infty$.

In what follows, $\varepsilon>0$ is assumed to be so small that $f(\lambda)>\varepsilon\tilde{f}(\lambda)$ for $0<\lambda \leqq 3M$. Lemma 3.1 together with the assumption of Theorem 3.3 implies that the solution $u(t, x ; a, \varepsilon\tilde{f})$ either blows up or satisfies $\limsup_{t\to\infty} \|u(t, \cdot\, ; a, \varepsilon\tilde{f})\|_\infty=\infty$, and hence if we define T_ε by

$$T_\varepsilon=\inf \{t>0 : \|u(t, \cdot\, ; a, \varepsilon\tilde{f})\|_\infty>3M\} ,$$

then $T_\varepsilon<\infty$. Also it is clear that $\lim_{\varepsilon\downarrow 0} T_\varepsilon=\infty$. Now the rest of the proof is devided into three steps.

Step 1 is to prove that the inequality

$$u(t, x ; a, f) \geqq u(t, 0 ; a, \varepsilon\tilde{f})-\varepsilon M_0 t_0-M_1 |x| t_0^{-1/2} \tag{3.5}$$

holds for $0<t_0<t \leqq T_\varepsilon$, where $M_0=\tilde{f}(3M)$ and $M_1=3\sqrt{d/2}\,M$. For this purpose first we notice that

$$u(t, x ; a, \varepsilon\tilde{f})=u(t_0, x ; v, \varepsilon\tilde{f}), \qquad 0<t_0<t \leqq T_\varepsilon$$

holds with $v(x)=u(t-t_0, x ; a, \varepsilon\tilde{f})$. An application of (2.2) then yields

$$u(t_0, x ; v, 0) \leqq u(t, x ; a, \varepsilon\tilde{f}) \leqq u(t_0, x ; v, \varepsilon f_0), \qquad f_0(x) \equiv M_0 ,$$

and hence for $0<t_0<t \leqq T_\varepsilon$ we have

$$H_{t_0} v(x) \leqq u(t, x ; a, \varepsilon\tilde{f}) \leqq H_{t_0} v(x)+\varepsilon M_0 t_0 . \tag{3.6}$$

Putting $x=0$ in the second inequality of (3.6), we have

(3.7) $$H_{t_0}v(0) \geqq u(t, 0; a, \varepsilon\tilde{f}) - \varepsilon M_0 t_0, \qquad 0 < t_0 < t \leqq T_\varepsilon.$$

On the other hand, from the inequalities

$$\begin{aligned} |\nabla H_{t_0}v(x)|^2 &= \sum_{j=1}^{d} \left\{ \int -\frac{x_j - y_j}{2t_0} v(y) H(t_0, x, y) dy \right\}^2 \\ &\leqq \|v\|_\infty^2 \int \frac{|x-y|^2}{4t_0^2} H(t_0, x, y) dy \\ &= \|v\|_\infty^2 d(2t_0)^{-1} \leqq M_1^2 t_0^{-1} \end{aligned}$$

it follows that

(3.8) $$\begin{aligned} |H_{t_0}v(x) - H_{t_0}v(0)| &= \left| \int_0^1 (x, \nabla H_{t_0}v(sx)) ds \right| \\ &\leqq |x| \left\{ \int_0^1 |\nabla H_{t_0}v(sx)|^2 ds \right\}^{1/2} \leqq M_1 t_0^{-1/2} |x|. \end{aligned}$$

Combining (3.7) with (3.8) we have

$$H_{t_0}v(x) \geqq u(t, 0; a, \varepsilon\tilde{f}) - \varepsilon M_0 t_0 - M_1 |x| t_0^{-1/2},$$

and this together with the first inequality of (3.6) implies that

(3.9) $$u(t, x; a, \varepsilon\tilde{f}) \geqq u(t, 0; a, \varepsilon\tilde{f}) - \varepsilon M_0 t_0 - M_1 |x| t_0^{-1/2}.$$

However, we have $u(t, x; a, f) \geqq u(t, x; a, \varepsilon f)$ for $0 \leqq t \leqq T_\varepsilon$ by (2.2), and hence this combined with (3.9) gives (3.5).

Step 2 is to prove that for each compact set K containing the support of $a(x)$ there exist constants δ and T such that $0 < \delta < T$ and

(3.10) $$u(t, x; a, f) > M \qquad \text{for } T - \delta \leqq t \leqq T \text{ and } x \in K.$$

For the proof we choose first a large t_0 and then a small $\varepsilon > 0$ so that both of the inequalities $t_0 < T_\varepsilon$ and

(3.11) $$\varepsilon M_0 t_0 + M_1 |x| t_0^{-1/2} < M, \qquad x \in K$$

hold. Since $u(t, x; a, \varepsilon\tilde{f})$ takes the maximum at $x=0$ as a function of x by Lemma 3.2, we have $u(T_\varepsilon, 0; a, \varepsilon\tilde{f}) = 3M$ by the definition of T_ε, and hence there exists $\delta > 0$ such that $t_0 < T_\varepsilon - \delta$ and

(3.12) $$u(t, 0; a, \varepsilon\tilde{f}) > 2M \qquad \text{for } T_\varepsilon - \delta \leqq t \leqq T_\varepsilon.$$

From (3.5), (3.11) and (3.12), we obtain (3.10) with $T = T_\varepsilon$.

Step 3. Here the proof of the theorem will be completed as follows. For each $t_1 \in [T-\delta, T]$ we set $a_1(x) = u(t_1, x; a, f)$ and $u_1(t, x) = u(t, x; a_1, f)$. Since $a_1 \geqq a$ by (3.10), we have

$$u_1(t, x) \geqq u(t, x) > M, \qquad T-\delta \leqq t \leqq T, \qquad x \in K,$$

and hence

$$u(t, x; a, f) > M, \qquad 2T-2\delta \leqq t \leqq 2T, \quad x \in K, \tag{3.13}$$

because $u_1(t, x)=u(t_1+t, x; a, f)$. Repeating this argument, we can now amplify the inequality (3.13) as follows: for each positive integer n, $u(t, x; a, f)>M$ holds for all t and x such that $nT-n\delta \leqq t \leqq nT$ and $x \in K$. However, there exists $t_2>0$ (for example, $t_2=[T\delta^{-1}]T$) such that

$$\bigcup_{n=1}^{\infty} [nT-n\delta, nT] \supset [t_2, \infty).$$

Consequently we have $u(t, x; a, f)>M$ for any $t>t_2$ and $x \in K$, as was to be proved.

THEOREM 3.4. *Let f be a Lipschitz continuous function on $[0, 1]$ such that $f(\lambda)>0$ for $0<\lambda<1$ and $f(1)=0$, and also let $\tilde{f}$ be a non-decreasing locally Lipschitz continuous function on $[0, \infty)$ with $\tilde{f}(0)=0$. We assume that*

$$\liminf_{\lambda \downarrow 0} f(\lambda)/\tilde{f}(\lambda) > 0,$$

and that any positive solution $\tilde{u}(t, x)$ of (3.3) either blows up in a finite time or satisfies (3.4). Then any positive solution of (1.1) which is less than or equal to 1 converges to 1 uniformly on each compact set of R^d as $t \to \infty$.

Proof of this theorem is quite similar to that of Theorem 3.3. Starting with the initial value $a(x)$ such that $\|a\|_\infty<1$, we choose M so that $\|a\|_\infty<M<1$ and define T_ε as before but with replacement of $3M$ by $(1+M)/2$. The argument goes in the same way with slight changes in constants such as in (3.12).

3.2. Combining Theorem 2.1 with Theorem 3.3 we can obtain a result concerning the problem (B).

THEOREM 3.5. *Suppose that f satisfies the following three conditions.*

(B.1) *f is a locally Lipschitz continuous function in $[0, \infty)$ with $f(0)=0$ and $f(\lambda)>0$ for $\lambda>0$.*

(B.2) $\int_{0+}^{\varepsilon} f(\lambda)/\lambda^{2+(2/d)} d\lambda = \infty$ *for some $\varepsilon>0$.*

(B.3) *There exists a positive constant $c_0(\leqq 1)$ such that*

$$f(\lambda\mu) \geqq c_0 \mu^{1+(2/d)} f(\lambda) \qquad \text{for } 0<\lambda \leqq \mu,\ \lambda<c_0 \text{ and } \lambda\mu<c_0.$$

Then any positive global solution of (1.1) grows up to infinity.

This theorem is an immediate consequence of Theorems 2.1, 3.3 and the following lemma in which the relation between (A.1)-(A.3) (of Theorem 2.1) and (B.1)-(B.3) is examined.

LEMMA 3.6. *For each f satisfying the conditions* (B.1), (B.2) *and* (B.3), *there exists $\tilde{f}$ satisfying the conditions* (A.1), (A.2), (A.3) *and*

$$\liminf_{\lambda\downarrow 0} f(\lambda)/\tilde{f}(\lambda)>0. \tag{3.14}$$

PROOF. Let $f(\lambda)=\lambda^{1+(2/d)}g(\lambda)$ and define $\bar{g}$, $\tilde{g}$ and $\tilde{f}$ by

$$\bar{g}(\lambda)=\begin{cases}\inf_{\lambda\leqq\xi\leqq c_0} g(\xi) & \text{for } \lambda<c_0,\\ g(c_0) & \text{for } \lambda\geqq c_0,\end{cases}$$

$$\tilde{g}(\lambda)=\bar{g}(\lambda)\wedge 1, \qquad \tilde{f}(\lambda)=\lambda^{1+(2/d)}\tilde{g}(\lambda),$$

where $a\wedge b$ denotes $\min(a, b)$. Then it is easy to see that $\tilde{f}$ satisfies (3.14) and (A.1). As regards (A.2) we first notice that the condition (B.3) on f implies that $g(\lambda\mu)\geqq c_0 g(\lambda)$ for any μ such that $\lambda<1\leqq\mu<c_0/\lambda$, and hence putting $\xi=\lambda\mu$ in the definition of $\bar{g}(\lambda)$ we have

$$\bar{g}(\lambda)\geqq c_0 g(\lambda) \qquad \text{for } 0<\lambda<c_0. \tag{3.15}$$

Therefore, $\int_{0+}\bar{g}(\lambda)/\lambda\, d\lambda\geqq c_0\int_{0+}g(\lambda)/\lambda\, d\lambda=\infty$ and hence $\int_{0+}\tilde{g}(\lambda)/\lambda\, d\lambda=\infty$ by the monotonicity of $\bar{g}$, which implies that (A.2) is satisfied for $\tilde{f}$. Next we prove that $\tilde{f}$ satisfies (a) of (A.3). Suppose that $0<\lambda\leqq\mu$ and $\lambda<c_0$. If $\lambda\mu<c_0$, using (3.15) and $c_0\leqq 1$ we have

$$\tilde{g}(\lambda\mu)\geqq(c_0 g(\lambda\mu))\wedge 1\geqq(c_0^2 g(\lambda))\wedge 1\geqq(c_0^2\bar{g}(\lambda))\wedge 1\geqq c_0^2\tilde{g}(\lambda).$$

If $\lambda\mu\geqq c_0$, then $\tilde{g}(\lambda\mu)=\bar{g}(c_0)\wedge 1\geqq\bar{g}(\lambda)\wedge 1=\tilde{g}(\lambda)$. Therefore we have proved that $0<\lambda\leqq\mu$ and $\lambda<c_0$ imply that $\tilde{g}(\lambda\mu)\geqq c_0^2\tilde{g}(\lambda)$, and this shows that (a) of (A.3) is satisfied for $\tilde{f}$ with $c=c_0^2$. Finally, we prove that $\tilde{f}$ satisfies (b) of (A.3). Since the condition (B.3) for f is equivalent to $g(\xi)\geqq c_0 g(\lambda)$ for $0<\lambda^2\leqq\xi<c_0$, $\lambda<c_0$, we have

$$g(\xi)\geqq c_0^n g(c_0) \qquad \text{for } c_0^{2^n}\leqq\xi\leqq c_0^{2^{n-1}},\ n\geqq 1.$$

Take a positive number δ such that $2^{-\delta}<c_0$ and put

$$N=\min\{n\geqq 1: c_0^m g(c_0)>2^{\delta}(\log c_0^{-2^m})^{-\delta} \quad \text{for all } m>n\},$$

$$n(\lambda)=\max\{n: \lambda\leqq c_0^{2^{n-1}}\}.$$

Noting that $c_0^{2^{n(\lambda)}}<\lambda\leqq c_0^{2^{n(\lambda)-1}}$, we have

$$g(\lambda)\geqq c_0^{n(\lambda)}g(c_0)\geqq 2^{\delta}(\log c_0^{-2^{n(\lambda)}})^{-\delta}=(\log c_0^{-2^{n(\lambda)-1}})^{-\delta}\geqq\left(\log\frac{1}{\lambda}\right)^{-\delta},$$

provided $n(\lambda)>N$, and hence $g(\lambda)\geqq(\log 1/\lambda)^{-\delta}$ for all sufficiently small positive λ. Therefore, if λ and μ are sufficiently small and $\mu<\lambda$, we have

$$\frac{1}{\mu}\tilde{g}(\lambda\mu)\geqq\frac{1}{\mu}\{(c_0 g(\lambda\mu)\wedge 1\}\geqq\frac{1}{\mu}\Big\{\Big(c_0\Big(\log\frac{1}{\lambda\mu}\Big)^{-\delta}\Big)\wedge 1\Big\}$$
$$\geqq\frac{1}{\mu}\Big\{\Big(c_0\Big(2\log\frac{1}{\mu}\Big)^{-\delta}\Big)\wedge 1\Big\}\geqq 1\geqq\tilde{g}(\lambda),$$

and hence $\tilde{f}(\lambda\mu)\geqq\mu^{2+(2/d)}\tilde{f}(\lambda)$ for all sufficiently small λ and μ with $0<\mu<\lambda$. The proof of the lemma is thus completed.

In the case when $f(\lambda)>0$ for $0<\lambda<1$ and $f(1)=0$, we obtain the following result similar to Theorem 3.5 as an immediate consequence of Theorems 2.1, 3.4 and Lemma 3.6. This might be some improvement of earlier results due to Kanel' [6], Ikeda and Kametaka (unpublished, partly found in [5]), Hayakawa [4], and Masuda [8].

THEOREM 3.7. *Let f be a Lipschitz continuous function on* $[0, 1]$ *with* $f(\lambda)>0$ *for* $0<\lambda<1$ *and* $f(0)=f(1)=0$, *and assume that* (B.2) *and* (B.3) *of Theorem* 3.5 *are satisfied. Then, any positive solution of* (1.1) *which is less than or equal to* 1 *converges to* 1 *uniformly on each compact set of* R^d *as* $t\to\infty$.

§ 4. A remark to blowing up condition.

Suppose f satisfies (B.1), (B.2) and (B.3) and let $\tilde{f}$ be the function constructed in the proof of Lemma 3.6. If, in addition, f satisfies

(4.1) $$f(\lambda)>\text{const.}\,\lambda^{1+(2/d)}\quad\text{for all sufficiently large }\lambda,$$

then $\tilde{f}$ satisfies $\inf_{\lambda>0} f(\lambda)/\tilde{f}(\lambda)>0$ in addition to (A.1), (A.2) and (A.3). On the other hand, the condition (a) of (A.3) implies (4.1). Therefore the set of conditions (B.1), (B.2), (B.3) and (4.1) is, *in a sense*, equivalent to (A.1), (A.2) and (A.3), and hence gives a sufficient condition for positive solutions of (1.1) to blow up. The following theorem states this with a slight improvement with respect to (4.1).

THEOREM 4.1. *Suppose that f satisfies* (B.1), (B.2), (B.3) *and the following two conditions.*

(4.2) $$\int_{\varepsilon}^{\infty} d\lambda/f(\lambda)<\infty\qquad\text{for some }\varepsilon>0.$$

(4.3) *There exist constants $c>0$ and $\lambda_0>0$ such that $f(\mu)\geqq cf(\lambda)$ for $\lambda_0<\lambda<\mu$.*

Then, any positive solution of (1.1) *blows up in a finite time.*

PROOF. Suppose that a positive solution u of (1.1) does not blow up. Since u grows up to infinity by Theorem 3.5, for any $M>0$ there exists $t_M>0$ such that $u(t_M, x)>M$ for $|x|\leqq 1$. Then u satisfies

(4.4) $$u(t+t_M, x)=H_t u(t_M, \cdot)(x)+\int_0^t H_{t-s} f(u(t_M+s, \cdot))ds$$

for any $0\leqq t<\infty$. We put

$$\rho_M(t)=\min_{|x|\leqq 1} u(t+t_M, x), \qquad t>0,$$

$$\eta=\inf_{0\leqq t\leqq 1}\min_{|x|\leqq 1}\int_{|y|\leqq 1} H(t, x, y)dy>0.$$

From (4.4), we have

$$\rho_M(t)\geqq \eta\rho_M(0)\geqq \eta M>\lambda_0, \qquad 0\leqq t\leqq 1,$$

provided $M>\lambda_0/\eta$. Therefore, again from (4.4) and the assumption (4.3) we have for $M>\lambda_0/\eta$

$$\rho_M(t)\geqq \eta M+c\eta\int_0^t f(\rho_M(s))ds$$

$$\geqq \eta M+c^2\eta\int_0^t g(\rho_M(s))ds, \qquad 0\leqq t\leqq 1,$$

where $g(\mu)=\max_{\lambda_0\leqq\lambda\leqq\mu} f(\lambda)$ $(\mu\geqq\lambda_0)$. Let $\varphi(t)$ be the solution of

$$\varphi(t)=\eta M+c^2\eta\int_0^t g(\varphi(s))ds.$$

Since $g(\mu)$ is non-decreasing, we have $\rho_M(t)\geqq\varphi(t)$ for $0\leqq t\leqq 1$. On the other hand, $\varphi(t)$ satisfies the equation

$$\int_{\eta M}^{\varphi(t)}\frac{d\lambda}{g(\lambda)}=c^2\eta t,$$

which combined with $\int_\varepsilon^\infty \frac{d\mu}{g(\mu)}<\infty$ implies $\varphi(1)=\infty$ provided M is large enough. Therefore, $\rho_M(1)=\infty$ and this contradicts that u does not blow up.

§ 5. Non-growing up condition.

In this section we deal with the situation in which f is so small near the origin that *some* positive solution of (1.1) converges to 0 as $t\to\infty$.

THEOREM 5.1. *Suppose that a locally Lipschitz function f on $[0, \infty)$ satisfies the following conditions.*

(5.1) $$f(\lambda)\geqq 0 \quad \textit{and} \quad f(0)=0.$$

(5.2) $$\int_{0+}^{\varepsilon} f(\lambda)/\lambda^{2+(2/d)}d\lambda<\infty \qquad \textit{for} \quad \varepsilon>0.$$

(5.3) *There exists a positive constant c $(\leqq 1)$ such that $f(\lambda\mu)\geqq c\mu f(\lambda)$ for $\lambda\geqq 0$ and $\mu\geqq 1$.*

Then, some positive solution $u(t, x)$ *of* (1.1) *converges to* 0 *uniformly in* x *as* $t\to\infty$.

REMARK 5.2. The condition (5.3) in the above theorem can be replaced by the following local one.

(5.4) There exists a positive constant c ($\leqq 1$) such that $f(\lambda\mu)\geqq c\mu f(\lambda)$ for $0<\lambda<c$, $\mu\geqq 1$, $\lambda\mu<c$.

In fact, when f satisfies (5.4) instead of (5.3), the function $\tilde{f}$ defined by putting $\tilde{f}(\lambda)=f(c)c^{-1}\lambda$ for $\lambda\geqq c$ and $\tilde{f}(\lambda)=f(\lambda)$ for $0\leqq\lambda<c$ satisfies (5.3), and the conclusion of Theorem 5.1 with respect to $\partial u/\partial t=\Delta u+\tilde{f}(u)$ obviously implies the same one with respect to (1.1). A sufficient condition for (5.4) is

(5.5) $$f(\lambda)/\lambda \text{ is non-decreasing near } \lambda=0\,.$$

We prepare a lemma which gives upper and lower estimates of $u(t, x)$ in terms of solutions of certain ordinary differential equations. For the proof of our present theorem we need only the upper estimate, but we give also the lower one for its own interest.

LEMMA 5.2. (a) *Let* f *be the same as in the preceding theorem except that* f *need not satisfy* (5.2) *this time, and* $a(x)\geqq 0$ *be continuous and bounded. Let* $v(t)$ *be the solution of*

(5.6) $$\frac{dv}{dt}=c^{-1}f(k(t)v(t))/k(t)\,,\qquad v(0)=1\,,$$

where $k(t)=\sup_x H_t a(x)$. *Then we have*

$$u(t, x\,;a, f)\leqq v(t)H_t a(x)\,,\qquad 0\leqq t<t_\infty\,,$$

where $t_\infty(\leqq\infty)$ *is the blowing-up time of* $v(t)$.

(b) *Let* f *be a non-negative convex function with* $f(0)=0$ *and* $a(x)$ *be the same as in* (a). *If* $v(t, \lambda)$ *denotes the solution of* $dv/dt=f(v)$ *with* $v(0)=\lambda$, *then*

$$u(t, x\,;a, f)\geqq v(t, H_t a(x))\,,\qquad 0\leqq t<T_\infty\,.$$

PROOF. (a) For $0\leqq t<t_\infty$ we define $\{v_n(t)\}$ and $\{u_n(t, x)\}$ inductively by

$$\begin{cases} v_0(t)=1, \quad v_n(t)=1+c^{-1}\displaystyle\int_0^t f(k(s)v_{n-1}(s))/k(s)ds\,, \\ u_0(t, x)=H_t a(x), \quad u_n(t, x)=H_t a(x)+\displaystyle\int_0^t H_{t-s}f(u_{n-1}(s, \cdot))ds\,. \end{cases}$$

Since $v_n(t)\to v(t)$ $(0\leqq t<t_\infty)$ and $u_n(t, x)\to u(t, x\,;a, f)$ $(0\leqq t<T_\infty)$ as $n\to\infty$, it is enough to prove that

(5.7) $$u_n(t, x)\leqq v_n(t)H_t a(x)\,,\qquad n\geqq 0\,.$$

We prove this by induction. When $n=0$, (5.7) is trivial and so we assume that it holds for $n-1$. Applying the condition (5.3) with

$$\lambda=u_{n-1}(s, x), \qquad \mu=\frac{v_{n-1}(s)k(s)}{u_{n-1}(s, x)},$$

we have

$$f(v_{n-1}(s)k(s))\geqq c\frac{v_{n-1}(s)k(s)}{u_{n-1}(s, x)}f(u_{n-1}(s, x)),$$

and hence

$$f(u_{n-1}(s, x))\leqq\frac{u_{n-1}(s, x)}{cv_{n-1}(s)k(s)}f(v_{n-1}(s)k(s))$$
$$\leqq c^{-1}H_s a(x)f(v_{n-1}(s)k(s))/k(s);$$

we have used the induction hypothesis $u_{n-1}(s, x)\leqq v_{n-1}(s)k(s)$ in deriving the last inequality of the above. Therefore

$$u_n(t, x)\leqq H_t a(x)+c^{-1}H_t a(x)\int_0^t f(v_{n-1}(s)k(s))/k(s)ds$$
$$=v_n(t)H_t a(x).$$

(b) Let $\{u_n(t, x)\}$ be the same as in the proof of (a) and define $\{v_n(t, \lambda)\}$ as follows:

$$v_0(t, \lambda)=\lambda, \quad v_n(t, \lambda)=\lambda+\int_0^t f(v_{n-1}(s, \lambda))ds, \quad n\geqq 1, \ \lambda\geqq 0.$$

Then, by induction in n it is easy to see that for each $n\geqq 0$ and $t\geqq 0$ $v_n(t, \lambda)$ is a non-negative convex function of $\lambda\geqq 0$ with $v_n(t, 0)=0$. Since $v_n(t, \lambda)\uparrow v(t, \lambda)$ as $n\uparrow\infty$, it is enough to prove that

$$u_n(t, x)\geqq v_n(t, H_t a(x)), \qquad n\geqq 0. \tag{5.8}$$

For $n=0$ (5.8) is trivial, and so assuming that it holds for $n-1$ we have

$$u_n(t, x)\geqq H_t a(x)+\int_0^t H_{t-s}f(v_{n-1}(s, H_s a))(x)ds$$
$$\geqq H_t a(x)+\int_0^t f(H_{t-s}v_{n-1}(s, H_s a)(x))ds$$
$$\geqq H_t a(x)+\int_0^t f(v_{n-1}(s, H_{t-s}H_s a(x))ds$$
$$=H_t a(x)+\int_0^t f(v_{n-1}(s, H_t a(x)))ds$$
$$=v_n(t, H_t a(x));$$

here we have used the monotonicity of f and the induction hypothesis for deriving the first inequality, and Jensen's inequality for the second and the third inequalities. The proof is finished.

PROOF OF THEOREM 5.1. Let $a(x)=\alpha e^{-\beta|x|^2}$, $\alpha>0$, $\beta>0$. Then the function $k(t)$ in the preceding lemma is equal to $\alpha(1+4\beta t)^{-d/2}$. If $v(t)$ denotes the solution of (5.6), then $u(t, x; a, f)\leqq v(t)H_t a(x)$. Since $H_t a(x)$ converges to 0 uniformly in x as $t\to\infty$, it is enough to prove that $v(t)$ is bounded in t for some α, $\beta>0$. We consider

$$\frac{dw(t)}{dt}=\frac{w(t)}{c^2k(t)}f\Big(\frac{k(t)}{\sqrt{\alpha}}\Big), \qquad w(0)=1. \tag{5.9}$$

First applying (5.3) with $\lambda=kw$, $\mu=w^{-1}\alpha^{-1/2}$ and then using $\alpha^{-1/2}\geqq 1$, we have

$$\frac{w}{c^2k}f\Big(\frac{k}{\sqrt{\alpha}}\Big)\geqq\frac{1}{ck}f(kw) \qquad \text{for } 1\leqq w\leqq\frac{1}{\sqrt{\alpha}}.$$

Therefore, a comparison theorem in the theory of differential equations applies to (5.6) and (5.9), yielding

$$v(t)\leqq w(t) \quad \text{for } 0\leqq t<T \text{ whenever } w(t)\leqq\alpha^{-1/2} \quad \text{for } 0\leqq t<T. \tag{5.10}$$

On the other hand, we have

$$w(t)=\exp\Big\{c^{-2}\int_0^t k(s)^{-1}f(\alpha^{-1/2}k(s))ds\Big\},$$

which converges to 1 uniformly in t as $\beta\to\infty$, because

$$\int_0^\infty k(s)^{-1}f(\alpha^{-1/2}k(s))ds$$
$$=(2d)^{-1}\alpha^{1/d-1/2}\beta^{-1}\int_0^{\sqrt{\alpha}} f(\lambda)\lambda^{-2-(2/d)}d\lambda<\infty.$$

Therefore, $w(t)\leqq\alpha^{-1/2}$ for all $t\geqq 0$ if β is sufficiently large, and hence it follows from (5.10) that $v(t)\leqq\alpha^{-1/2}$ for all $t\geqq 0$. This completes the proof.

EXAMPLE. We consider the case when f is given by

$$f(\lambda)=\lambda^{1+(2/d)}\Big\{\log\frac{1}{\lambda}\cdot\log_{(2)}\frac{1}{\lambda}\cdots\log_{(n-1)}\frac{1}{\lambda}\cdot\Big(\log_{(n)}\frac{1}{\lambda}\Big)^{\delta}\Big\}^{-1}$$

near the origin and smooth and positive in the whole of $[0,\infty)$, where $\delta>0$, $n\geqq 1$ and $\log_{(k)}\mu=\log\log\cdots\log\mu$ (k-times).

(a) If $\delta\leqq 1$, then it is easy to see that f satisfies the conditions (B.1) and (B.2) of Theorem 3.5. To check that f satisfies (B.3), we notice that

$$\begin{cases} 0<\lambda\leqq\mu, \quad \log 2\leqq\log_{(k)}\lambda^{-1} \quad \text{and} \quad \log 2\leqq\log_{(k)}(\lambda\mu)^{-1} \\ \text{imply that } \log_{(k)}(\lambda\mu)^{-1}\leqq 2\log_{(k)}\lambda^{-1}, \end{cases} \tag{5.11}$$

which can be proved by induction on k. From (5.11) we obtain

$$\mu^{1+(2/d)}\frac{f(\lambda)}{f(\lambda\mu)}=\prod_{k=1}^{n-1}\Big(\frac{\log_{(k)}(\lambda\mu)^{-1}}{\log_{(k)}\lambda^{-1}}\Big)\Big(\frac{\log_{(n)}(\lambda\mu)^{-1}}{\log_{(n)}\lambda^{-1}}\Big)^{\delta}\leqq 2^{n-1+\delta},$$

provided that λ and $\lambda\mu$ are sufficiently small with $0<\lambda\leqq\mu$, and this implies that f satisfies (B.3). Therefore, any positive solution of (1.1) either blows up in a finite time or grows up to infinity, if $\delta\leqq 1$. (b) If $\delta>1$, then it is easy to see that f satisfies the conditions (5.1), (5.2) and (5.5), and hence some positive solutions of (1.1) converge to 0 uniformly in x as $t\rightarrow\infty$.

References

[1] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t=\Delta u+u^{1+\alpha}$, J. Sci. Univ. Tokyo, Sect. I, **13** (1966), 109-124.

[2] H. Fujita, On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations, Proc. Symp. Pure Math., AMS, **18** (1969), 105-113.

[3] K. Hayakawa, On nonexistence of global solutions of some semilinear parabolic differential equations, Proc. Japan Acad., **49** (1973), 503-505.

[4] K. Hayakawa, On the limit state of solutions of some semilinear diffusion equations, Osaka J. Math., **12** (1975), 767-776.

[5] Y. Kametaka, On nonlinear diffusion equations, Sugaku, **26** (1974), 137-148. (In Japanese.)

[6] Ya. I. Kanel', On the stability of the solutions of the equations of combustion theory with finite initial functions, Mat. Sb., **65** (1964), 398-413. (In Russian.)

[7] A. Kolmogoroff, I. Petrovsky and N. Piscounoff, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bull. Moskov Gos. Univ., Sect. A, **1** (1937), 1-25.

[8] K. Masuda, On the growth of solutions of nonlinear diffusion equation $u_t=\Delta u+F(u)$, Publ. RIMS, Kyoto Univ., **9** (1974), 675-682.

[9] S. Sugitani, On nonexistence of global solutions for some nonlinear integral equations, Osaka J. Math., **12** (1975), 45-51.

Kusuo KOBAYASHI
Department of Mathematics
Hiroshima University

Present address:
Department of Mathematics
Toyama University
Toyama, Japan

Tunekiti SIRAO
Department of Mathematics
Faculty of Science
Tokyo Metropolitan University
Fukazawa, Setagaya-ku, Tokyo
Japan

Hiroshi TANAKA
Department of Mathematics
Faculty of Science
Hiroshima University
Higashisenda-machi, Hiroshima
Japan