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On indefinite quadratic forms

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Let V be a vector space of dimension n over the real field. A lattice $M\subset V$ is a subgroup of V generated by n linearly independent vectors. Let Λ denote the set of all lattices in V. The general linear group $GL(V)$ acts transitively on Λ and the stabilizer $GL(M)$ of a lattice M is a discrete subgroup of $GL(V)$. We can introduce a topology in Λ so that the natural mapping of $GL(V)/GL(M)$ onto Λ is a homeomorphism. Let dx be the Lebesgue measure on V. We define $D(M)$ by

$$D(M)=\int_{V/M}dx$$

where the integral is over a fundamental domain of M. We have the following:

MAHLER'S CRITERION: If C is a closed subset of Λ, then C is compact if and only if $D(M)$ is bounded on C and there exists a neighborhood U of 0 such that $U\cap M=\{0\}$ for all $M\in C$.

Let q be a non-degenerate quadratic form on V. We denote the bilinear form $q(x+y)-q(x)-q(y)$ by $b(x, y)$. We also fix a Euclidean inner product (x, y) on $V\times V$ such that

$$|q(x)|\leqq\|x\|^2=(x, x)$$

for all $x\in V$. A lattice M is called integral if $q(x)$ is an integer for every $x\in M$. Let G denote the group of all $\rho\in GL(V)$ such that $q(\rho x)=q(x)$, $x\in V$. We have the following:

PROPOSITION 1. *If M is an integral lattice, then the orbit $O(M)=\{\rho M: \rho\in G\}\subset\Lambda$ is closed.*

PROOF. Our assertion is proved in general form in Mostow and Tamagawa [3]. We give a sketch of the proof in this case. Let $x_1, \cdots, x_n$ be a base of M. For every $\sigma\in GL(V)$ put $S(\sigma)=(b(\sigma x_i, \sigma x_j))$. $S(\sigma)$ is a $n\times n$ symmetric matrix and if $\sigma\in G\cdot GL(M)$, $S(\sigma)$ is integral. Hence the set $\{S(\sigma); \sigma\in G\cdot GL(M)\}$ is closed and the set $G\cdot GL(M)$ is also closed.

If M is an integral lattice, then q induces a quadratic form q_Q on the rational vector space $QM\subset V$. We denote the Witt index of q_Q by $\nu(M)$. It is obvious that ρM is integral for all $\rho\in G$ and $\nu(\rho M)=\nu(M)$. If $\nu(M)=0$ then

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by Mahler's criterion, the orbit $O(M)$ is compact (Mostow and Tamagawa [3]).

Let dL denote a Haar measure on the orbit $O(M)=G/G\cap GL(M)$. C. L. Siegel proved that the volume

$$\mathrm{Vol}\,(O(M))=\int_{O(M)} dL$$

is finite except in the case where $n=2$ and $\nu(M)=1$. Usually the proof calls for the reduction theory (cf. C. L. Siegel [2]). The purpose of this note is to give a proof using the reduction theory as little as possible. If $n=3$ and $\nu(M)=1$ or $n=4$ and $\nu(M)=2$, the finiteness of $\mathrm{Vol}\,(O(M))$ follows from the finiteness of $\mathrm{Vol}\,(SL(2, R)/SL(2, Z))$. Hence we consider the case where $n\geqq 5$ or $n=4$ and $\nu(M)=1$, and assume that the finiteness is already proved in the case where the dimension is $n-2$ and the index is $\nu(M)-1$. In the following lines, $c_1, c_2, \cdots$ are positive constants.

For every $M\in\Lambda$, put

$$h(M)=\mathrm{Min}\,\{\|x\|,\ x\in M,\ x\neq 0\}\,.$$

We will prove the following:

THEOREM 1. *If* $0<\varepsilon<1$, *then*

$$\int_{\substack{O(M)\\ h(M)\leqq\varepsilon}} dL=O(\varepsilon^{n-2})\,.$$

Since $D(\rho M)=D(M)$ for $\rho\in G$, by Mahler's criterion, the set $O(M)_\varepsilon=\{L\,;\,L\in O(M),\ h(L)\geqq\varepsilon\}$ is compact. Therefore the finiteness of $\mathrm{Vol}\,(O(M))$ follows immediately.

We denote by $G(M)$ the group $G\cap GL(M)$. A vector $x\in M$ is called primitive if M/Zx is torsion free. Let Ω denote the set of all primitive isotropic (with respect to q) vectors in M. The following Proposition is stated without proof.

PROPOSITION 2. *The set Ω decomposes into a finite number of $G(M)$-orbits.*

Proposition 2 follows easily from Witt's theorem. Let $x_1, \cdots, x_t$ be a set of representatives of orbits of $G(M)$ in Ω. For $x, y\in M$, $x\sim y$ means $y\in G(M)x$.

Let $\boldsymbol{C}$ denote the cone of all isotropic vectors in V. The group G operates on $\boldsymbol{C}$ transitively and there exists an invariant measure d_0x on $\boldsymbol{C}$. We will give an explicit form of d_0x. Let $u, v, z_1, \cdots, z_{n-2}$ be a base of V such that $q(u)=q(v)=0$, $b(u, v)=1$, $b(u, z_i)=b(v, z_i)=0$ for $i=1, \cdots, n-2$. If $x\in\boldsymbol{C}$, we have $x=\xi u+\eta v+\zeta_1 z_1+\cdots+\zeta_{n-2}z_{n-2}$ and $q(x)=\xi\eta+f(\zeta_1, \cdots, \zeta_{n-2})=0$. Hence we may regard $\{\eta, \zeta_1, \cdots, \zeta_{n-2}\}$ as a coordinate system on $\boldsymbol{C}$ (except for points where $\eta=0$). We have

$$d\xi\, d\eta\, d\zeta_1 \cdots d\zeta_{n-2}=\frac{1}{\eta}dt\, d\eta\, d\zeta_1 \cdots d\zeta_{n-2}$$

in V where $t=q(x)=\xi\eta+f(\zeta_1, \cdots, \zeta_{n-2})$. Since dt and $d\xi\, d\eta\, d\zeta_1 \cdots d\zeta_{n-2}$ are G-invariant, $d_0x=\frac{d\eta}{\eta}d\zeta_1 \cdots d\zeta_{n-2}$ is invariant on $\boldsymbol{C}$. From this explicit form of d_0x, we have the following:

PROPOSITION 3. *Let d_0x be a G-invariant measure on $\boldsymbol{C}$. We have*

$$d_0(tx)=t^{n-2}d_0(x) \qquad t>0.$$

COROLLARY. *Let ε be a positive number. If $n>2$, we have*

$$\int_{\|x\|\leqq\varepsilon} d_0x=c_1\varepsilon^{n-2}.$$

Let $\varphi(x)$ be a smooth non-negative function on V such that

$$\varphi(x)=\begin{cases} 1 & \|x\|\leqq 1 \\ 0 & \|x\|\geqq 2. \end{cases}$$

Put $\varphi_\varepsilon(x)=\varphi(\varepsilon^{-1}x)$.

PROOF OF THEOREM 1. We consider the integral

$$\int_{O(M)} \sum_{x\in\Omega}\varphi_\varepsilon(\rho x)d(\rho M)=I(\varphi, \varepsilon).$$

If $h(\rho M)\leqq\varepsilon$, then there exists a primitive $x\in M$ such that $\|\rho x\|\leqq\varepsilon$ and $|q(x)| \leqq\|\rho x\|^2<1$, $q(x)=0$. Therefore we have the inequality

$$I(\varphi, \varepsilon)>\int_{\substack{O(M)\\ h(L)\leqq\varepsilon}} dL.$$

By Proposition 2 we have $I(\varphi, \varepsilon)=\sum_{i=1}^{t} I_i(\varphi, \varepsilon)$ where

$$I_i(\varphi, \varepsilon)=\int_{O(M)} (\sum_{x\sim x_i}\varphi_\varepsilon(\rho x))d(\rho M).$$

Let $G(x_1)$ denote the group of all $\rho\in G$ such that $\rho x_1=x_1$. By a simple transformation of the integral (cf. A. Weil [**5**]), we have

$$I_i(\varphi, \varepsilon)=c_2\int_{G/G(x_i)}\varphi_\varepsilon(\rho x_1)d\bar{\rho}\int_{G(x_1)/G(x_1, M)} d\rho_1$$

where $G(x_1, M)$ is the group $G(x_1)\cap G(M)$, $d\bar{\rho}$ is a Haar measure on $G/G(x_1)$, and $d\rho_1$ is a Haar measure on $G(x_1)$. The integral $\int_{G/G(x_1)}\varphi_\varepsilon(\rho x_1)d\bar{\rho}$ is equal to

$$c_3\int_{\boldsymbol{C}}\varphi_\varepsilon(x)d_0x,$$

which is equal to $c_4\varepsilon^{n-2}$ by Proposition 3.

The group $G(x_1)$ is a subgroup of G and the nilpotent radical H_{x_1} of $G(x_1)$ is a vector group of dimension $n-2$ (cf. Tamagawa [**4**]). Let y_1 be an isotropic vector in M such that $b(x_1, y_1)\neq 0$, and G_1 the orthogonal group of the restriction q_U of q to $U=\{Fx_1+Fy_1\}^{\perp}$. Then we have

$$G(x_1)=H_{x_1}G_1=G_1H_{x_1}.$$

Now $H_{x_1}/H_{x_1}\cap G(x_1, M)$ is compact. On the other hand, $M_U=M\cap U$ is a lattice in U, q_U is non-degenerate and M_U is integral. We also have $\nu(M_U)=\nu(M)-1$. Now the group $G_1(M_U)$ and $G_1\cap G(M)$ are commensurable. Using the finiteness assumption, the volume of $G_1/G_1(M_U)$ is finite, hence the volume of $G(x_1)/G(x_1, M)$ is also finite. Now we have

$$I_i(\varphi, \varepsilon)=c_5\varepsilon^{n-2}$$

and

$$I(\varphi, \varepsilon)=\sum_i I_i(\varphi, \varepsilon)=c_6\varepsilon^{n-2}.$$

The estimate given in Theorem 1 is good enough for many purposes. As an example, we will prove the following.

PROPOSITION 4. *We have*

$$\int_{\substack{O(M)\\ h(\rho M)\leqq\varepsilon}} \sum_{x\in M}\varphi(\rho x)d(\rho M)=O(\varepsilon^{n-\nu-2}),$$

if $n\geqq 5$ *or* $n=4$ *and* $\nu=1$, *where* $\nu=\nu(M)$.

PROOF. Put

$$\sum_{x\in M}\varphi(\rho x)=f(\rho M).$$

By the simplest reduction theory, we can find a base $y_1, \cdots, y_n$ of ρM such that

$$c_7(\xi_1^2\|y_1\|^2+\cdots+\xi_n^2\|y_n\|^2)<\|\xi_1y_1+\cdots+\xi_ny_n\|^2$$
$$<c_8(\xi_1^2\|y_1\|^2+\cdots+\xi_n^2\|y_n\|^2).$$

Put $\|y_i\|=\kappa_i$, $\kappa_1\leqq\kappa_2\leqq\cdots\leqq\kappa_n$. We have the following estimate of $f(\rho M)$:

$$f(\rho M)=O(\prod_{i=1}^{n}(\kappa_i^{-1}+1)).$$

There exists $c_9>0$ such that $|b(y_i, y_j)|<c_9\|y_i\|\|y_j\|$. Therefore there exists $c_{10}>0$ such that if $\kappa_\mu\leqq c_{10}$, we have

$$b(y_i, y_j)=0, \qquad q(y_i)=0$$

for $i=1, \cdots, \mu$ because they are all integers. We now have $\kappa_{\nu+1}\geqq c_{10}>0$, and the estimate

$$f(\rho M)=O(\varepsilon^{-\nu})$$

if $h(\rho M)\geqq\frac{1}{2}\varepsilon$. By Theorem 1, the volume of the set

$$\left\{\rho M; \frac{\varepsilon}{2} \leqq h(\rho M) \leqq \varepsilon\right\}$$

is $O(\varepsilon^{n-2})$. Using $n-\nu-2 \geqq n-\frac{n}{2}-2>0$ if $n \geqq 5$ or $4-1-2=1>0$ if $n=4$ and $\nu=1$, we have

$$\int_{h(\rho M)\leqq\varepsilon} f(\rho M) d(\rho M) = \sum_{l=1}^{\infty} \int_{\varepsilon/2^l \leqq h(\rho M) \leqq \varepsilon/2^{l-1}} f(\rho M) d(\rho M)$$
$$= O(\varepsilon^{n-\nu-2}) .$$

THEOREM 2. *Let f be a smooth function on V with compact support. If $n \geqq 5$ or $n=4$, $\nu(M)=1$, we have*

$$\lim_{\delta\to 0} \delta^n \int_{O(M)} \sum_{x\in M} f(\delta\rho x) d(\rho M) = D(M)^{-1} \operatorname{Vol}(O(M)) \int f(x) dx . \qquad (1)$$

PROOF. By Proposition 4, we have

$$\int_{\substack{O(M) \\ h(\rho M)<\varepsilon}} \sum_{x\in M} f(\delta\rho x) d(\rho M) = O(\delta^{-n} \varepsilon^{n-\nu-2})$$

and

$$\delta^n \int_{h(\rho M)\leqq\varepsilon} = O(\varepsilon^{n-\nu-2}) .$$

Clearly we can exchange the order of $\lim\limits_{\delta\to 0}$ and $\int$ in the following

$$\lim_{\delta\to 0} \delta^n \int_{h(\rho M)\geqq\varepsilon} \sum_{x\in M} f(\delta\rho x) d\rho M$$

because the set $\{\rho M; h(\rho M) \geqq \varepsilon\}$ is compact. Therefore the left side of (1) is equal to

$$\int_{O(M)} \lim_{\delta\to 0} \delta^n \sum_{x\in M} f(\delta\rho x) d(\rho M) = D(M)^{-1} \int f(x) dx \cdot \int_{O(M)} d(\rho M) .$$

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By the same method, we can prove the following:

THEOREM 3. *Under the same assumption as in Theorem 2, the following integral converges for any function f in the Schwartz class $\mathcal{S}(V)$:*

$$\int_{O(M)} \sum_{x\in M} f(\rho x) d(\rho M) .$$

Let C_1 denote the quadric $\{x; q(x)=1\}$. If $\phi(x)$ is a continuous function with compact support on C_1, the invariant integral of ϕ on C_1 is defined by

$$\int_{0\leqq q(x)\leqq 1} \hat{\phi}(x) dx = \int_{C_1} \phi(x) d_1 x$$

where $\hat{\phi}(x)$ is defined by

$$\hat{\phi}(x)=\begin{cases}\phi(q(x)^{-1/2}x) & q(x)>0\\ 0 & q(x)\leqq 0\,.\end{cases}$$

If $x_1\in M$, $q(x_1)=t>0$, we consider the integral

$$\int_{O(M)}\Big(\sum_{x\sim x_1}\hat{\phi}(\rho x)\Big)d(\rho M)=I(\phi, x_1)\,.$$

By the Weil transformation of $I(\phi, x_1)$, we have

$$I(\phi, x_1)=\mu(M, x_1)\int_{C_1}\phi(x)d_1x\,,$$

where $\mu(M, x_1)$ is independent of ϕ. For a given integer $t>0$, put

$$\mu(M, t)=\sum\mu(M, x_i)$$

where x_i runs through a complete set of representatives of $G(M)$-orbits in the set $\Omega_t=\{x\,;\,x\in M,\ q(x)=t\}$. By the reduction theory, the number of x_i is finite. Now let T be a large positive integer, and consider the integral

$$\int_{O(M)}\Big(\sum_{\substack{x\in M\\ q(x)\leqq T}}\hat{\phi}(\rho x)\Big)d(\rho M)$$

$$=\int_{O(M)}\Big(\sum_{\substack{x\in M\\ q(x)\leqq T}}\hat{\phi}(T^{-1/2}\rho x)\Big)d\rho\,.$$

By the definition, the integral is equal to

$$\sum_{t=1}^{T}\mu(M, t)\cdot\int_{C_1}\phi(x)dx\,.$$

By Theorem 2, we have

$$\sum_{t=1}^{T}\mu(M, t)\sim D(M)^{-1}\,\mathrm{Vol}\,(O(M))T^{n/2}\,.$$

It is now easy to see the convergence of the Siegel Z-function

$$\sum_{t=1}^{\infty}\frac{\mu(M, t)}{t^s}=Z^{+}(M, s)$$

if the real part of s is $>\frac{n}{2}$ (cf. C. L. Siegel [**1**]). Siegel did not give a proof of the convergence in his paper. He just wrote "Die Konvergentz der Reihe entnimmt man der Reduktiontheorie".

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