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The degeneracy of systems and the exceptional linear combinations of entire functions

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§ 1. Introduction and preliminaries.

Let $f=(f_0, \cdots, f_n)$ $(n\geqq 1)$ be a transcendental system in $|z|<\infty$. That is, $f_0, \cdots, f_n$ are entire functions without common zero and

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r}=\infty,$$

where $T(r, f)$ is the characteristic function of f defined by Cartan ([1]).

Let $X=\{F_i;\, F_i=\sum_{j=0}^{n} a_{ij}f_j\not\equiv 0\}_{i=0}^{N}$ $(n\leqq N\leqq\infty)$ where a_{ij} are constants and matrices $(a_{i_\nu j})_{j=0,\cdots,n}^{\nu=0,\cdots,n}$ are regular for all $n+1$ integers $\{i_\nu\}_{\nu=0}^{n}$ $(0\leqq i_\nu\leqq N)$ and λ be the maximum number of linearly independent linear relations among $f_0, \cdots, f_n$ over $\boldsymbol{C}$. ($\boldsymbol{C}$ means the field of complex numbers.) We know that $0\leqq\lambda\leqq n-1$. When $\lambda>0$, we say that the system f is degenerate.

In this paper, we discuss some relations between the number "λ" and exceptional linear combinations in X.

For $F\in X$, we set

$$\delta(F)=1-\limsup_{r\to\infty} \frac{N(r, 0, F)}{T(r, f)},$$

$$\delta_m(F)=1-\limsup_{r\to\infty} \frac{N_m(r, 0, F)}{T(r, f)} \qquad (m\geqq 1)$$

and $m(F)$=the minimum of multiplicities of all zeros of F ($m(F)=\infty$ when $F(z)\neq 0$), where

$$N_m(r, 0, F)=\sum_{|z_k|<r} \min(m_k, m)\log^+\frac{r}{|z_k|}+\min(m_0, m)\log r,$$

$\{z_k\}=\{z\neq 0;\, F(z)=0\}$ and m_k $(\geqq 1)$ is the multiplicity of zero of F at z_k $(k=1, 2, \cdots)$ and m_0 $(\geqq 0)$ is that of F at the origin.

Cartan ([1]) proved

THEOREM A. *If* $\lambda=0$, *then*

1) $\sum_{F\in X}\delta_n(F)\leqq n+1$,

2) *for any* $n+2$ *combinations* $\{F_i\}_{i=0}^{n+1}$ *in* X,

$$\sum_{i=0}^{n+1} \frac{1}{m(F_i)} \geqq \frac{1}{n}.$$

Further, the second author (Theorem B′, [8]) proved the following

THEOREM B. *If there exist* $n+2$ *combinations* $F, F_0, \cdots, F_n$ *in* X *such that*

i) *arbitrary* $n-1$ *combinations in* $\{F_i\}_{i=0}^{n}$ *are linearly independent;*

ii) $\delta(F)+\sum_{i=0}^{n}\delta(F_i)>n+1$,

then

1) $\lambda=1$,

2) *there exists an* F_{i_0} $(0\leqq i_0\leqq n)$ *such that* F *and* F_{i_0} *are proportional,*

3) *for any* G *in* $X-\{F, F_0, \cdots, F_n\}$, $\delta(G)+\sum_{i=0}^{n}\delta(F_i)\leqq n+1$.

This is a refinement of results of Niino and Ozawa ([4]), Ozawa ([6]), Suzuki ([7]), Noguchi ([5]) and the first author ([2]).

In § 3 we give some relations between the number "λ" and the multiplicities of zeros of combinations in X (Theorems 1, 3) and a generalization of Theorem B (Theorem 2).

We use the symbols

$$T(r, f),\ m(r, a),\ N(r, a),\ N(r, f),\ \delta(a, f),\ S(r, f) \text{ etc.}$$

of the Nevanlinna theory of meromorphic functions (see [3]).

§ 2. Lemmas.

Here we give some lemmas.

LEMMA 1. *For any* $F_1, \cdots, F_k$ $(2\leqq k\leqq n+1)$ *in* X,

$$m(r, \|F_1, \cdots, F_k\|/F_1\cdots F_k)=O(\log r\,T(r, f)) \qquad (r\to\infty\,;\ r\notin E)$$

where E *is a set of finite linear measure and* $\|F_1, \cdots, F_k\|$ *denotes the Wronskian of* $F_1, \cdots, F_k$ (Cartan [1]).

For any $n+1$ combinations $F_0, \cdots, F_n$ in X, there exist $n+1-\lambda$ combinations (say $G_0, \cdots, G_{n-\lambda}$) in $\{F_i\}_{i=0}^{n}$ such that any element in X may be represented by $G_0, \cdots, G_{n-\lambda}$ with constant coefficients. We say that $G_0, \cdots, G_{n-\lambda}$ form a basis of X.

LEMMA 2. *Let* $\{G_i\}_{i=0}^{n-\lambda}$ *be a basis of* X, *then* $G=(G_0, \cdots, G_{n-\lambda})$ *is a system in* $|z|<\infty$ *and*

$$|T(r, f)-T(r, G)|<O(1).$$

This follows at once from the definitions of $T(r, f)$, $T(r, G)$ and G.

LEMMA 3. *Let* $\{G_i\}_{i=0}^{n-\lambda}$ *be a basis of* X *and* H_1, H_2 *in* X *be represented by* $G_0, \cdots, G_{n-\lambda}$ *as follows:*

$$H_1=\alpha_0 G_0+\cdots+\alpha_{n-\lambda}G_{n-\lambda},$$

$$H_2=\beta_0 G_0+\cdots+\beta_{n-\lambda}G_{n-\lambda}.$$

If i) *there exists at least one* i_0 $(0\leqq i_0\leqq n-\lambda)$ *such that* $\alpha_{i_0}\cdot\beta_{i_0}\neq 0$, *and*
ii) *for all* i $(0\leqq i\leqq n-\lambda)$ $|\alpha_i|+|\beta_i|\neq 0$, *then*

$$T(r,f)\leqq N(r,0,H_1)+N_n(r,0,H_2)+\sum_{j=0}^{n-\lambda}N_n(r,0,G_j)+S(r),$$

where $S(r)=o(T(r,f))$ *for* $r\to\infty$ *except for a set of finite linear measure.*

PROOF. As $H_i\not\equiv 0$, there are some non-zero elements in both $\{\alpha_i\}_{i=0}^{n-\lambda}$ and $\{\beta_i\}_{i=0}^{n-\lambda}$. If there exists only one non-zero element in $\{\alpha_i\}_{i=0}^{n-\lambda}$ or $\{\beta_i\}_{i=0}^{n-\lambda}$, this lemma is easily proved by using Theorem A. Therefore we will prove this lemma in the case that there exist at least two non-zero elements in both $\{\alpha_i\}_{i=0}^{n-\lambda}$ and $\{\beta_i\}_{i=0}^{n-\lambda}$. We may assume that the non-zero elements of $\{\alpha_i\}_{i=0}^{n-\lambda}$ are $\alpha_k,\cdots,\alpha_{n-\lambda}$ $(0\leqq k\leqq n-1-\lambda)$ without loss of generality, so that

(1) $$H_1=\alpha_k G_k+\cdots+\alpha_{n-\lambda}G_{n-\lambda}.$$

Next by the hypothesis ii), we know that all coefficients $\{\beta_i\}_{i=0}^{k-1}$ are different from zero and moreover by hypothesis i) at least one of $\{\beta_i\}_{i=k}^{n-\lambda}$ is different from zero. Hence we may assume that the non-zero elements of $\{\beta_i\}_{i=0}^{n-\lambda}$ are $\beta_0,\cdots,\beta_{k+l}$, $0\leqq l\leqq n-k-\lambda$, so that

(2) $$H_2=\beta_0 G_0+\cdots+\beta_k G_k+\cdots+\beta_{k+l}G_{k+l}.$$

From (1), we have

$$\alpha_j G_j=H_1\frac{\Delta'_j}{\Delta'}\qquad(k\leqq j\leqq n-\lambda),$$

where

$$\Delta'_j=\|G_k,\cdots,G_{j-1},H_1,G_{j+1},\cdots,G_{n-\lambda}\|/G_k\cdots G_{j-1}H_1G_{j+1}\cdots G_{n-\lambda},$$

$$\Delta'=\|G_k,\cdots,G_{n-\lambda}\|/G_k\cdots G_{n-\lambda}.$$

Consequently we have

(3) $$\max_{k\leqq j\leqq n-\lambda}\log|G_j|\leqq\log|H_1|+\sum_{j=k}^{n-\lambda}\log^+|\Delta'_j|+\log^+\left|\frac{1}{\Delta'}\right|+O(1).$$

On the other hand, from (2) we obtain

$$-H_2+\beta_0G_0+\cdots+\beta_{k-1}G_{k-1}=-\beta_kG_k-\cdots-\beta_{k+l}G_{k+l}$$

and we have

$$\beta_m G_m=G\frac{\Delta''_m}{\Delta''}\qquad(0\leqq m\leqq k-1),$$

where

$$G=-\beta_kG_k-\cdots-\beta_{k+l}G_{k+l},$$

$$\Delta''_m = \|H_2, G_0, \cdots, G_{m-1}, G, G_{m+1}, \cdots, G_{k-1}\| / H_2 \cdot G_0 \cdots G_{m-1} G G_{m+1} \cdots G_{k-1},$$

$$\Delta'' = \|H_2, G_0, \cdots, G_{k-1}\| / H_2 \cdot G_0 \cdots G_{k-1}.$$

Consequently we obtain

$$\max_{0 \leqq m \leqq k-1} \log|G_m| \leqq \log|G| + \sum_{m=0}^{k-1} \log^+|\Delta''_m| + \log^+\left|\frac{1}{\Delta''}\right| + O(1). \tag{4}$$

Moreover, by the inequality

$$|G| \leqq \sum_{j=k}^{k+l} |\beta_j| |G_j| \leqq K \max_{k \leqq j \leqq k+l} |G_j|, \qquad K = \sum_{j=k}^{k+l} |\beta_i|,$$

we have

$$\log|G| \leqq \max_{k \leqq j \leqq k+l} \log|G_j| + \log K. \tag{5}$$

By (3), (4) and (5), we have

$$\max_{0 \leqq j \leqq n-\lambda} \log|G_j| \leqq \log|H_1| + \sum_{j=k}^{n-\lambda} \log^+|\Delta'_j| + \sum_{m=0}^{k-1} \log^+|\Delta''_m|$$

$$+ \log^+\left|\frac{1}{\Delta'}\right| + \log^+\left|\frac{1}{\Delta''}\right| + O(1),$$

so that we obtain, using Lemma 2,

$$T(r, f) \leqq N(r, 0, H_1) + \sum_{j=k}^{n-\lambda} m(r, \Delta'_j) + \sum_{m=0}^{k-1} m(r, \Delta''_m) + m\left(r, \frac{1}{\Delta'}\right) + m\left(r, \frac{1}{\Delta''}\right) + O(1)$$

$$\leqq N(r, 0, H_1) + N_n(r, 0, H_2) + \sum_{j=0}^{n-\lambda} N_n(r, 0, G_j) + S(r),$$

where $S(r) = o(T(r, f))$ for $r \to \infty$ except for a set of finite linear measure, because

$$m\left(r, \frac{1}{\Delta'}\right) = N(r, \Delta') + m(r, \Delta') - N\left(r, \frac{1}{\Delta'}\right) + O(1),$$

$$m\left(r, \frac{1}{\Delta''}\right) = N(r, \Delta'') + m(r, \Delta'') - N\left(r, \frac{1}{\Delta''}\right) + O(1),$$

$$N(r, \Delta') \leqq \sum_{j=k}^{n-\lambda} N_n(r, 0, G_j), \qquad N(r, \Delta'') \leqq \sum_{m=0}^{k-1} N_n(r, 0, G_m) + N_n(r, 0, H_2).$$

Thus we have the desired result.

§ 3. Results.

Let f, X and λ be as in § 1.

THEOREM 1. *If there exist $n+2$ combinations $F, F_0, \cdots, F_n$ in X such that*

i) *arbitrary $n-1$ combinations in $\{F_i\}_{i=0}^n$ are linearly independent;*

ii) $\sum_{i=0}^{n} \frac{1}{m(F_i)} < \frac{1}{n}$ *and* $\delta(F) = 1$, *then* $\lambda=1$ *and there exists an* F_{i_0} *in* $\{F_i\}_{i=0}^{n}$ *such that* F *and* F_{i_0} *are proportional.*

PROOF. By Theorem A and the hypotheses i), ii), we see easily $\lambda=1$ or 2.

Suppose $\lambda=2$, then we may assume that $F_0, \cdots, F_{n-2}$ form a basis of X without loss of generality and so F_{n-1} is represented by $\{F_i\}_{i=0}^{n-2}$ as follows:

$$F_{n-1}=\alpha_0 F_0+\cdots+\alpha_{n-2}F_{n-2},$$

where all coefficients α_i $(i=0, \cdots, n-2)$ are different from zero because of the hypothesis i). From this equation, we have, by Lemma 2 and the method used in the proof of the second fundamental theorem by Cartan (see [**1**]),

$$T(r, f)\leqq\sum_{i=0}^{n-1} N_{n-2}(r, 0, F_i)+S(r),$$

where $S(r)=O(\log rT(r, f))$ for $r\to\infty$ except for a set of finite linear measure. Using the inequality $N_{n-2}(r, 0, F_i)\leqq\frac{n-2}{m(F_i)}N(r, 0, F_i)$ $(i=0, \cdots, n-1)$, we have

$$\frac{1}{n-2}\leqq\sum_{i=0}^{n-1}\frac{1}{m(F_i)}$$

which is contradictory to the hypothesis ii). Therefore we have $\lambda=1$.

As $\lambda=1$, there exists a linear relation among $F_0, \cdots, F_n$ over $\boldsymbol{C}$:

$$\beta_0 F_0+\cdots+\beta_n F_n=0.$$

If all β_i are different from zero, similarly to the above we have

$$T(r, f)\leqq\sum_{i=0}^{n} N_{n-1}(r, 0, F_i)+S(r),$$

where $S(r)=O(\log rT(r, f))$ for $r\to\infty$ except for a set of finite linear measure. Hence we have

$$\frac{1}{n-1}\leqq\sum_{i=0}^{n}\frac{1}{m(F_i)}$$

as

$$N_{n-1}(r, 0, F_i)\leqq\frac{n-1}{m(F_i)}N(r, 0, F_i) \qquad (i=0, \cdots, n),$$

which is contradictory to the hypothesis ii). Therefore at least one of $\{\beta_i\}_{i=0}^{n}$ is equal to zero. We may assume $\beta_n=0$ without loss of generality:

(6) $$\beta_0 F_0+\cdots+\beta_{n-1}F_{n-1}=0,$$

where all coefficients β_i $(i=0, \cdots, n-1)$ are different from zero because of the hypothesis i). By the definition of X, we may represent F by $F_0, \cdots, F_n$ as follows:

$$F=p_0 F_0+\cdots+p_n F_n \qquad (p_i\neq 0, i=0, \cdots, n).$$

Eliminating F_0 from this relation and (6), we obtain

$$F=p_1'F_1+\cdots+p_{n-1}'F_{n-1}+p_nF_n \tag{7}$$

where $p_i'=p_i-p_0\frac{\beta_i}{\beta_0}$ $(i=1,\cdots,n-1)$.

If there are some coefficients in $\{p_i'\}_{i=0}^{n-1}$ which are not equal to zero, applying Lemma 3 to (6) and (7), we have

$$T(r,f)\leqq N(r,0,F)+\sum_{i=0}^{n}N_n(r,0,F_i)+S(r),$$

where $S(r)=o(T(r,f))$ for $r\to\infty$ except for a set of finite linear measure, as arbitrary $n-1$ combinations in $\{F_i\}_{i=0}^{n-1}$ and F_n form a basis. Hence we have, as $\delta(F)=1$ or $\lim_{r\to\infty}\frac{N(r,0,F)}{T(r,f)}=0$,

$$\frac{1}{n}\leqq\sum_{i=0}^{n}\frac{1}{m(F_i)},$$

which is contradictory to the hypothesis ii). Consequently we have

$$F=p_nF_n \qquad (p_n\neq 0).$$

The proof is complete.

THEOREM 2. *If there exist $n+3$ combinations $F, G, F_0,\cdots,F_n$ $(n\geqq 4)$ in X such that*

i) *arbitrary $n-2$ combinations in $\{F_i\}_{i=0}^{n}$ are linearly independent;*

ii) $\delta(F)+\delta_n(G)+\sum_{i=0}^{n}\delta_n(F_i)>n+2,$

then

1) $\lambda=2$;

2) *there exists a combination F_{i_0} $(0\leqq i_0\leqq n)$ such that F, G and F_{i_0} are proportional;*

3) *for any H in $X-\{F,G,F_0,\cdots,F_n\}$,*

$$\delta(H)+\delta_n(G)+\sum_{i=0}^{n}\delta_n(F_i)\leqq n+2.$$

PROOF. 1). We will prove that $\lambda=2$. By Theorem A and the hypotheses i), ii), we see easily $1\leqq\lambda\leqq 3$.

(I). Suppose $\lambda=1$. Then there exists a linear relation among $F_0,\cdots,F_n$ over $\boldsymbol{C}$:

$$\gamma_0F_0+\cdots+\gamma_nF_n=0.$$

If all γ_i are different from zero, applying Theorem A and Lemma 2, we have

$$\sum_{i=0}^{n}\delta_n(F_i)\leqq n,$$

which is contradictory to the hypothesis ii). Therefore at least one of $\{\gamma_i\}_{i=}^{n}$

is equal to zero. We may assume $\gamma_n=0$ without loss of generality, so that

$$\gamma_0 F_0+\cdots+\gamma_{n-1}F_{n-1}=0. \tag{8}$$

In (8), there exists at most one coefficient which is equal to zero because of the hypothesis i).

By the definition of X, we may represent F and G by $F_0, \cdots, F_n$ as follows:

$$F=p_0F_0+\cdots+p_nF_n \qquad (p_i\neq 0,\ i=0, \cdots, n) \tag{9}$$

$$G=q_0F_0+\cdots+q_nF_n \qquad (q_i\neq 0,\ i=0, \cdots, n). \tag{10}$$

(I)-1. The case that $\gamma_0\neq 0, \cdots, \gamma_{n-1}\neq 0$. We have

$$F_0=-\frac{\gamma_1}{\gamma_0}F_1-\cdots-\frac{\gamma_{n-1}}{\gamma_0}F_{n-1}. \tag{8$'$}$$

Eliminating F_0 from (8)$'$ and (9), we obtain

$$F=p_1'F_1+\cdots+p_{n-1}'F_{n-1}+p_nF_n \tag{11}$$

where $p_i'=p_i-p_0\frac{\gamma_i}{\gamma_0}$ $(i=1, \cdots, n-1)$. If there are some coefficients in $\{p_i'\}_{i=1}^{n-1}$ which are not equal to zero, applying Lemma 3 to (8)$'$ and (11), we have

$$\delta(F)+\sum_{i=0}^{n}\delta_n(F_i)\leqq n+1$$

as arbitrary $n-1$ combinations in $\{F_i\}_{i=0}^{n-1}$ and F_n form a basis. This is a contradiction. Hence we have

$$F=p_nF_n \qquad (p_n\neq 0). \tag{12}$$

Next, from (8)$'$, (10) and (12) we obtain

$$F=q_1'F_1+\cdots+q_{n-1}'F_{n-1}+\frac{p_n}{q_n}G \qquad (p_n\cdot q_n\neq 0). \tag{13}$$

Remarking that $F_1, \cdots, F_{n-1}, G$ form a basis of X, we have as in the above

$$G=q_nF_n \qquad (q_n\neq 0). \tag{14}$$

The two relations (12) and (14) mean $\lambda\geqq 2$, which is absurd.

(I)-2. The case that one of $\gamma_0, \cdots, \gamma_{n-1}$ is equal to zero. Without loss of generality, we may assume $\gamma_{n-1}=0$. Then $\gamma_0\neq 0, \cdots, \gamma_{n-2}\neq 0$ and we have

$$F_0=-\frac{\gamma_1}{\gamma_0}F_1-\cdots-\frac{\gamma_{n-2}}{\gamma_0}F_{n-2}. \tag{8$''$}$$

In this case, arbitrary $n-2$ combinations in $\{F_i\}_{i=0}^{n-2}$ and F_{n-1}, F_n form a basis of X and as in (I)-1 we have from (8)″ and (9)

$$F=p_{n-1}F_{n-1}+p_nF_n \tag{15}$$

Next, from (8)″, (10) and (15) we obtain

$$F=q_1'F_1+\cdots+q_{n-2}'F_{n-2}+q_{n-1}'F_{n-1}+\frac{p_n}{q_n}G\,, \tag{16}$$

where $q_{n-1}'=p_{n-1}-p_n\frac{q_{n-1}}{q_n}$ is not equal to zero because if $q_{n-1}'=0$, (8)″ and (16) imply $\lambda\geqq 2$, which is absurd. Remarking that $F_1,\cdots,F_{n-1}$, G form a basis of X, we have similarly

$$G=q_{n-1}F_{n-1}+q_nF_n\,. \tag{17}$$

The two relations (15) and (17) imply $\lambda\geqq 2$, which is absurd. Therefore, we have $\lambda\neq 1$.

(II) Suppose $\lambda=3$. Then there exist $n-2$ combinations in $\{F_i\}_{i=0}^n$ which form a basis of X. We may assume that $F_0,\cdots,F_{n-3}$ form a basis and the combination F_{n-2} is represented by this basis as follows:

$$F_{n-2}=\alpha_0F_0+\cdots+\alpha_{n-3}F_{n-3}\,,$$

where all coefficients α_i $(i=0,\cdots,n-3)$ are different from zero because of the hypothesis i). From this equation we have, by Theorem A and Lemma 2,

$$\sum_{i=0}^{n-2}\delta_n(F_i)\leqq n-2\,,$$

which is contradictory to the hypothesis ii).

From (I) and (II), it must be $\lambda=2$.

2). We will prove the conclusion 2) of this theorem. As $\lambda=2$, we may assume that $F_0,\cdots,F_{n-2}$ form a basis of X and so F_{n-1}, F_n are represented by $\{F_i\}_{i=0}^{n-2}$ as follows:

$$F_{n-1}=\alpha_0F_0+\cdots+\alpha_{n-2}F_{n-2}\,, \tag{18}$$

$$F_n=\beta_0F_0+\cdots+\beta_{n-2}F_{n-2}\,. \tag{19}$$

By the hypotheses i) and ii), applying Theorem A to (18) and (19), only one of $\{\alpha_i\}_{i=0}^{n-2}$ and only one of $\{\beta_i\}_{i=0}^{n-2}$ are equal to zero. We may consider the following two cases A) and B).

A). There exists an i_0 such that $\alpha_{i_0}=0$, $\beta_{i_0}=0$ $(0\leqq i_0\leqq n-2)$. Eliminating F_{n-1}, F_n from (9) using (18) and (19), we have

$$F=p_0''F_0+\cdots+p_{i_0-1}''F_{i_0-1}+p_{i_0}F_{i_0}+p_{i_0+1}''F_{i_0+1}+\cdots+p_{n-2}''F_{n-2}\,. \tag{20}$$

If there are some coefficients in $\{p_i''\}_{i=0}^{n-2}-\{p_{i0}''\}$ which are not equal to zero, applying Lemma 3 to (18) and (20), we have

$$\sum_{i=0}^{n-1}\delta_n(F_i)+\delta(F)\leqq n\,,$$

which is contradictory to the hypothesis ii). This means

$$F=p_{i_0}F_{i_0}\,. \tag{21}$$

Next, from (10), (18), (19) and (21) we have

$$F=q_0'F_0+\cdots+q_{i_0-1}'F_{i_0-1}+\frac{p_{i_0}}{q_{i_0}}G+q_{i_0+1}'F_{i_0+1}+\cdots+q_{n-2}'F_{n-2}\,. \tag{22}$$

Remarking that $F_0,\cdots,F_{i_0-1}, G, F_{i_0+1},\cdots,F_{n-2}$ form a basis of X, we have similarly

$$G=q_{i_0}F_{i_0}\,. \tag{23}$$

B). There exist i_1 and i_2 ($i_1\neq i_2$) such that $\alpha_{i_1}=0$, $\beta_{i_2}=0$. We will prove that this case does not happen. We have

$$F=a_0F_0+\cdots+a_{n-2}F_{n-2}\,, \tag{24}$$

where at least one of $\{a_i\}_{i=0}^{n-2}$ is not equal to zero. In (24), coefficients a_{i_1} and a_{i_2} are equal to zero. Indeed if one of a_{i_1}, a_{i_2} (say a_{i_1}) is not equal to zero, applying Lemma 3 to (18) and (24), we have

$$F=a_{i_1}F_{i_1}\qquad(a_{i_1}\neq 0)\,. \tag{25}$$

Applying Lemma 3 to (18) and the relation obtained by eliminating F_{i_1} from (19) and (25), we have

$$\delta(F)+\sum_{\substack{i=0\\ i\neq i_1}}^{n-1}\delta_n(F_i)+\delta_n(F_n)\leqq n\,,$$

which is contradictory to the hypothesis ii). Consequently F may be represented as follows:

$$\begin{aligned}F=a_0F_0+\cdots+a_{i_1-1}F_{i_1-1}+a_{i_1+1}F_{i_1+1}+\cdots+a_{i_2-1}F_{i_2-1}\\ +a_{i_2+1}F_{i_2+1}+\cdots+a_{n-2}F_{n-2}\,,\end{aligned} \tag{26}$$

where there exists at least one non-zero element in $\{a_i\}_{i=0}^{n-2}-\{a_{i_1}, a_{i_2}\}$.

Next we have

$$G=b_0F_0+\cdots+b_{i_1}F_{i_1}+\cdots+b_{i_2}F_{i_2}+\cdots+b_{n-2}F_{n-2}\,, \tag{27}$$

where at least one of $\{b_i\}_{i=0}^{n-2}$ is not equal to zero. Here $b_{i_1}\cdot b_{i_2}\neq 0$. Indeed, if $b_{i_1}=0$ or $b_{i_2}=0$, then from (18) or (19), (26) and (27) $\lambda\geqq 3$, which is absurd.

Now we consider the case that in (26) there exists only one non-zero element in $\{a_i\}_{i=0}^{n-2}-\{a_{i_1}, a_{i_2}\}$. In this case we have $F=a_lF_l$, $a_l\neq 0$, $l\neq i_1, i_2$. Eliminating F_l from this equation and (19), we have

$$F=\beta_0''F_0+\cdots+\beta_l''F_n+\cdots+\beta_{i_1}''F_{i_1}+\cdots+\beta_{i_2-1}''F_{i_2-1} \tag{28}$$
$$+\beta_{i_2+1}''F_{i_2+1}+\cdots+\beta_{n-2}''F_{n-2},$$

where all coefficients are not equal to zero. Applying Lemma 3 to (27) and (28), we have

$$\delta(F)+\delta_n(G)+\sum_{\substack{i=0\\ i\neq l}}^{n-2}\delta_n(F_i)+\delta_n(F_n)\leqq n,$$

which is contradictory to the hypothesis ii).
Therefore we may assume that there exist at least two non-zero coefficients in (26).

Now we will show that this case ends in a contradiction applying the methods in the proof of Lemma 3 to (19), (26) and (27).

Let $\{a_{i_{h(\nu)}}\}_{\nu=1}^{k}$ $(2\leqq k\leqq n-3)$ be not equal to zero. Then we obtain

$$a_{i_{h(\nu)}}F_{i_{h(\nu)}}=F\frac{\Delta_1^\nu}{\Delta_1} \qquad (a_{i_{h(\nu)}}\neq 0) \tag{29}$$

where $\Delta_1=\|F_{i_{h(1)}},\cdots,F_{i_{h(k)}}\|/F_{i_{h(1)}}\cdots F_{i_{h(k)}}$ and Δ_1^ν is what $F_{i_{h(\nu)}}$ is changed by F in Δ_1. From (29), we have

$$\max_{a_i\neq 0}\log|F_i|\leqq\log|F|+\log^+\left|\frac{1}{\Delta_1}\right|+\sum_{\nu=1}^{k}\log^+|\Delta_1^\nu|+O(1). \tag{30}$$

From (19) we set

$$H_1\equiv-\sum_{\substack{a_i\neq 0\\ \beta_i\neq 0}}\beta_iF_i=-F_n+\sum_{\substack{a_i=0\\ \beta_i\neq 0}}\beta_iF_i. \tag{31}$$

Let $\{\beta_{i_{j(\nu)}}\}_{\nu=1}^{l}$ be the non-zero coefficients which appear in the right side of (31). Then we have

$$\beta_{i_{j(\nu)}}F_{i_{j(\nu)}}=H_1\frac{\Delta_2^\nu}{\Delta_2} \qquad (\beta_{i_{j(\nu)}}\neq 0)$$

and hence

$$\max_{\substack{a_i=0\\ \beta_i\neq 0}}\log|F_i|\leqq\log|H_1|+\log^+\left|\frac{1}{\Delta_2}\right|+\sum_{\nu=1}^{l}\log^+|\Delta_2^\nu|+O(1), \tag{32}$$

where

$$\Delta_2=\|F_n, F_{i_{j(1)}},\cdots,F_{i_{j(l)}}\|/F_nF_{i_{j(1)}}\cdots F_{i_{j(l)}}$$

and Δ_2^ν is what $F_{i_{j(\nu)}}$ is changed by H_1 in Δ_2. By (31) we obtain

$$\log|H_1| \leqq \max_{a_i \neq 0} \log|F_i| + O(1). \tag{33}$$

From (27) we set

$$H_2 \equiv -\sum_{\substack{b_i \neq 0 \\ a_i \neq 0 \text{ or } \beta_i \neq 0}} b_i F_i = -G + \sum_{\substack{a_i = 0 \\ \beta_i = 0, b_i \neq 0}} b_i F_i . \tag{34}$$

Let $\{b_{i_{s(\nu)}}\}_{\nu=1}^{m}$ be the non-zero coefficients which appear in the right side of (34). Then we have

$$b_{i_{s(\nu)}} F_{i_{s(\nu)}} = H_2 \frac{\Delta_3^\nu}{\Delta_3} \qquad (b_{i_{s(\nu)}} \neq 0)$$

and hence

$$\max_{\substack{a_i = 0 \\ \beta_i = 0, b_i \neq 0}} \log|F_i| \leqq \log|H_2| + \log^+\left|\frac{1}{\Delta_3}\right| + \sum_{\nu=1}^{m} \log^+|\Delta_3^\nu| + O(1), \tag{35}$$

where $\Delta_3 = \|G, F_{i_{s(1)}}, \cdots, F_{i_{s(m)}}\| / G F_{i_{s(1)}} \cdots F_{i_{s(m)}}$ and Δ_3^ν is what $F_{i_{s(\nu)}}$ is changed by H_2 in Δ_3. By (34) we have

$$\log|H_2| \leqq \max_{a_i \neq 0 \text{ or } \beta_i \neq 0} \log|F_i| + O(1). \tag{36}$$

By (30), (32), (33), (35) and (36), we have

$$\begin{aligned} \max_{\substack{a_i \neq 0 \text{ or } \beta_i \neq 0 \\ \text{or } b_i \neq 0}} \log|F_i| \leqq \log|F| &+ \sum_{j=1}^{3} \log^+\left|\frac{1}{\Delta_j}\right| + \sum_{\nu=1}^{k} \log^+|\Delta_1^\nu| \\ &+ \sum_{\nu=1}^{l} \log^+|\Delta_2^\nu| + \sum_{\nu=1}^{m} \log^+|\Delta_3^\nu| + O(1), \end{aligned} \tag{37}$$

so that we obtain, as in the proof of Lemma 3,

$$\begin{aligned} T(r,f) &\leqq N(r,0,F) + \sum_{\nu=1}^{k} m(r,\Delta_1^\nu) + \sum_{\nu=1}^{l} m(r,\Delta_2^\nu) + \sum_{\nu=1}^{m} m(r,\Delta_3^\nu) + \sum_{j=1}^{3} m\left(r, \frac{1}{\Delta_j}\right) + O(1) \\ &\leqq N(r,0,F) + N_n(r,0,G) + \sum_{i=0}^{n-2} N_n(r,0,F_i) + N_n(r,0,F_n) + S(r), \end{aligned}$$

where $S(r) = o(T(r,f))$ for $r \to \infty$ except for a set of finite linear measure. Hence we have

$$\delta(F) + \delta_n(G) + \sum_{i=0}^{n-2} \delta_n(F_i) + \delta_n(F_n) \leqq n+1,$$

which is contradictory to the hypothesis ii).

Thus we have the conclusion 2) of this theorem.

3). If there exists a combination H in $X - \{F, G, F_0, \cdots, F_n\}$ such that

$$\delta(H) + \delta_n(G) + \sum_{i=0}^{n} \delta_n(F_i) > n+2,$$

then by 2), F, G, H and F_{i_0} are proportional. This implies $\lambda=3$, which is absurd.

Hence we have the conclusion 3).

In the same way as we obtained Theorem 2 by generalizing Theorem B, we generalize Theorem 1 as follows.

THEOREM 3. *If there exist* $n+3$ *combinations* $F, G, F_0, \cdots, F_n$ $(n \geqq 4)$ *in* X *such that*

i) *arbitrary* $n-2$ *combinations in* $\{F_i\}_{i=0}^n$ *are linearly independent;*

ii) $\dfrac{1}{m(G)}+\sum_{i=0}^{n}\dfrac{1}{m(F_i)}<\dfrac{1}{n}$ *and* $\delta(F)=1$,

then $\lambda=2$ *and there exists an* F_{i_0} *in* $\{F_i\}_{i=0}^n$ *such that* F, G *and* F_{i_0} *are proportional.*

Applying the method of the proof of Theorem 1 to that of Theorem 2, we obtain this result easily.

COROLLARY. *In Theorem* 2 *or* 3, *if* $F \equiv 1$, *then* G *and at least one combination in* $\{F_i\}_{i=0}^n$ *are lacunary.*

REMARK 1. When $n=3$, Theorems 2 and 3 change into the following:

"If there exist 6 combinations $F, G, F_0, \cdots, F_3$ in X such that

$$\delta(F)+\delta_3(G)+\sum_{i=0}^{3}\delta_3(F_i)>5$$

or

$$\frac{1}{m(G)}+\sum_{i=0}^{3}\frac{1}{m(F_i)}<\frac{1}{3} \quad \text{and} \quad \delta(F)=1,$$

then

1) $\lambda=2$;

2) $\{F, G, F_0, \cdots, F_3\}$ are divided into two classes each of which consists of three elements being proportional".

REMARK 2. Can one replace $\delta(F)$ by $\delta_n(F)$ in Theorem 2 ?

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