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A normalization theorem in formal theories of natural numbers

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In this paper we deal with two formal number theories, i.e. a classical and an intuitionistic one. They are obtained from Gentzen's well-known logical systems LK and LJ by adding the principle of mathematical induction as an inference rule. Our aim is to prove that we can transform any derivation in these systems into its so-called normal form. We shall explain later what a normal derivation is. To put it briefly, it means a proof without redundancy. It will be defined not as the derivation which can not be transformed any more but as the derivation satisfying some conditions on variables and inference rules. In the proof of our assertion we apply the transfinite induction up to ε_0 just as Gentzen did in his second consistency proof of number theory.

Our normalization theorem yields some by-products. Examining structures of normal derivations, we shall obtain the following results: the consistency of number theory, and Harrop's result (a disjunctive and an existential property for intuitionistic number theory). These results are obtained as the direct consequences of our theorem.

There are several investigations on normalization theorems in formal number theories. For example, Jervell [**5**], Martin-Löf [**6**], Prawitz [**7**], Troelstra [**13**] and Zucker [**14**] study them for several systems of natural deduction. And Scarpellini [**8**]-[**11**] obtain some related results for systems of sequent calculus. Later we shall refer to them again.

In §1 we introduce our formal systems. We state our main theorem and show its applications there. In §2 we paraphrase the theorem so that we could simplify our way of thinking for the proof. After some preparations the paraphrased theorem is proved. In these sections we treat both the classical and the intuitionistic system simultaneously. And, if necessary, we give a certain notice to distinguish them.

Most of our concepts are due to Gentzen [**2**]. With a few exceptions we use the same terminology in the same senses as those in the English translation of it.

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§ 1. Main theorem and its applications.

1. Formal systems.

1.1. In our systems we use only 0 (zero) as the individual constant and only $'$ (successor function) as the function symbol. We use different letters to represent the free and the bound variables. Predicate symbols are admitted according to the need. The logical symbols which we shall use are $\wedge$, $\vee$, $\neg$, $\supset$, $\forall$ and $\exists$.

The concepts 'term', 'formula' and 'sequent' are defined as usual. A term, a formula, or a sequent is said to be *closed*, if it contains no free variables. In intuitionistic case any sequent shall have at most one formula in the succedent.

1.2. We use all the *inference figures* admitted in Gentzen [2] and the ones concerning the implication $\supset$ introduced in the original *LK* or *LJ* of Gentzen [1]. But the *CJ*-inference figures (*VJ*-Schlußfiguren) are used in the following reformed form:

$$\frac{\Gamma \rightarrow \Theta, F(0) \quad F(a), \Gamma \rightarrow \Theta, F(a') \quad F(t), \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda}$$

where the free variable designated by a—which we call the *eigenvariable* of the inference figure—must not occur in the sequence $F(0)$, $F(t)$, Γ, Δ, Θ, Λ. In each *CJ*-inference figure the formula designated by $F(a)$ and the term designated by t are called the *induction formula* and the *induction term*, respectively.

A free variable in an inference figure is said to be *redundant* [1)], if it appears in one or more upper sequents but does not in the lower sequent, and if it is not the eigenvariable of the inference figure.

1.3. The concept 'derivation (Herleitung)' is defined similarly as in [2]. We use, however, only *basic logical sequents* (logische Grundsequenzen) as the *basic sequents* (Grundsequenzen); consequently, every uppermost sequent of a derivation must have the form $D \rightarrow D$.

If a derivation includes an inference figure with a redundant variable, then we say simply that the derivation contains a redundant variable.

2. Main theorem.

2.1. A *CJ*-inference figure is said to be *irreducible*, if the induction formula contains at least one occurrence of the eigenvariable and if the induction term

1) The same terminology has been used in different meanings: Cf. Jervell [5], Troelstra [13] and Zucker [14].

is a free variable.

We shall often use the following property:

I. *If an irreducible CJ-inference figure is included in a derivation without any redundant variables, then the lower sequent must contain the same free variable as the induction term.*

2.2. Our main theorem holds both in the classical system and in the intuitionistic system.

MAIN THEOREM. *We can transform any derivation into the one with the same end sequent and having the following properties:*

$N1$. *It includes no cuts.*

$N2$. *It includes no CJ-inference figures except irreducible ones.*

$N3$. *It contains no redundant variables.*

The derivation having the properties $N1$-$N3$ is said to be *normal* or *in normal form*. We postpone the proof of this theorem until § 2.

3. Applications of the Main Theorem.

We show two applications as the immediate consequences of our Main Theorem. The concepts 'path (Faden)' and 'a sequent stands *below* (unter) another sequent' will be used in the same senses as those of the same terminology in Gentzen [**2**].

3.1. *Consistency.* We remark in advance that ordinary number-theoretic axioms can be expressed in closed formulas in universal prenex normal form. On the consistency of number theory we can prove the following result.

PROPOSITION 1. *Let Γ be any sequence of closed formulas in universal prenex normal form. If Γ is consistent in LK or LJ, then so is Γ in our classical or intuitionistic number-theoretic system, respectively.*

PROOF. We prove both cases simultaneously. Suppose the contrary. And assume that the sequent $\Gamma \rightarrow$ is derivable in one of our systems. From the normal derivation we take, if any, a lowermost sequent containing a free variable. Clearly it is not the end sequent, nor an upper sequent of an inference figure. This is impossible. Therefore the derivation contains no free variables and hence includes no *CJ*-inference figures: i.e., it is a derivation in *LK* or *LJ.* This contradicts our hypothesis.

In the intuitionistic case the above assertion can be slightly sharpened as follows: If a closed sequent which contains neither a positive occurrence of the symbol $\forall$ nor a negative occurrence of the symbol $\exists$ is derivable in our number-theoretic system, then so is the sequent in *LJ.* We use the terminology 'positive' and 'negative' in the usual senses.

3.2. *Harrop's result.* We give a proof for some results in Harrop [**3**]. We treat here only the intuitionistic system. First we list some conditions on a set of formulas:

Hp 1. *If* $A \wedge B$ *is contained in the set, then so are both* A *and* B.

Hp 2. *If* $A \supset B$ *is contained in the set, then so is* B.

Hp 3. *If* $\forall x F(x)$ *is contained in the set, then so is* $F(\bar{n})$ *for each numeral* $\bar{n}$.

Hp 4. $A \vee B$ *is not contained in the set.*

Hp 5. $\exists x F(x)$ *is not contained in the set.*

All of these conditions are satisfied, for example, by the empty set and by the set of formulas which, in Harrop's terminology, contain neither *relevant* occurrences of the disjunction nor those of the existential quantifiers. We remark that ordinary number-theoretic axioms can be expressed by formulas in some suitably chosen set satisfying all of these conditions.

PROPOSITION 2. *Let* Γ *be a finite (possibly empty) sequence of elements in a set, say* X, *of closed formulas which satisfies Hp* 1–*Hp* 5. *And let* $A \vee B$ *and* $\exists x F(x)$ *be closed formulas.*

1) *If the sequent* $\Gamma \rightarrow A \vee B$ *is derivable, then so is at least one of the sequents* $\Gamma \rightarrow A$ *and* $\Gamma \rightarrow B$.

2) *If the sequent* $\Gamma \rightarrow \exists x F(x)$ *is derivable, then so is the sequent* $\Gamma \rightarrow F(\bar{n})$ *for some numeral* $\bar{n}$.

PROOF. We treat both cases simultaneously. Let D be either $A \vee B$ or $\exists x F(x)$, and suppose that the sequent $\Gamma \rightarrow D$ is derivable.

From the rightmost path in the normal derivation of this sequent we choose the uppermost sequent, say S, satisfying the following: The succedent of the sequent concerned as well as that of any sequent standing below the sequent is D. We denote by $\mathscr{P}$ the (sub)path starting with S and ending with the end sequent.

We take, if any, the lowermost sequent, say S^*, from $\mathscr{P}$ whose antecedent includes a formula not contained in X. Clearly S^* is not the end sequent. By the properties *Hp* 1–*Hp* 5, N 1 and N 3 the sequent must be an upper sequent of a *CJ*-inference figure, since D is in the succedent of every sequent of $\mathscr{P}$. But the properties N 2 and I derive a contradiction from this. Thus the antecedent of each sequent in $\mathscr{P}$ contains at most some elements of X.

Consequently S is not a basic sequent. It must be either the lower sequent of a *thinning* (Verdünnung) introducing D on the succedent or the lower sequent of an *operational inference figure* (Verknüpfungs-Schlußfigur) introducing D as the *principal formula* (Hauptformel) on the succedent.

Since each antecedent formula in any sequent of $\mathscr{P}$ does not contain the disjunction as the outermost logical symbol, we can transform $\mathscr{P}$ so as to prove the assertions.

4. Others' works.

There are several papers on normalization theorems in formal number the ries.

Applying Gentzen's reduction-method in [**2**] to classical and intuitionistic system of sequent calculus, Scalpellini [8] obtains several results. By one of his theorems we see that any derivation with the closed end sequent can be reduced to a kind of normal form, i. e. to a derivation of it without redundant variables which itself coincides with its *final part*[1] or which includes an operational inference figure[2] with the lower sequent in the final part whose principal formula has an *image* in the end sequent. The consistency of number theory can be obtained as an immediate consequence; he carries out further Gentzen-type reductions to obtain Harrop's result. We can conclude the above assertion from our Main Theorem. In [**9**]-[**11**] he obtains similar results in variously extended systems.

Making use of his correspondence between a natural deduction and a sequent calculus, Zucker [**14**] gets a normalization theorem for intuitionistic system of natural deduction. But, as he remarks it, his theorem works only for the derivations in a restricted class. Let A, Γ be the conclusion and the sequence of assumptions of a derivation, respectively. He uses the following conditions. The sequent $\Gamma \rightarrow A$ is closed; the logical symbol $\forall$ does not occur positively in the sequent, while the logical symbol(s) $\exists$ (and $\vee$) do(es) not occur negatively in it. Under these assumptions he reduces each derivation to a normal form, which includes no *CJ*-inference figures. The result can be obtained, as remarked before, as the immediate consequence of our Main Theorem.

In relation to the investigations of Sanchis's and Tait's on computability of terms of finite type, Jervell [**5**] and Martin-Löf [**6**] obtain their normalization theorems for several systems of natural deduction. Jervell [**5**] exemplifies an important property of redundant variables. The elimination of redundant variables is essential for the normalization theorem to be practical. In the intuitionistic case our Main Theorem is essentially the same as his theorem. In the classical case, however, the expression of his theorem seems not so complete. Martin-Löf [**6**], which treats an intuitionistic system, is likely to intend to take consideration on redundant variables by using the rules of inference for the elimination of an inductively defined predicate. But his contraction-rules for reductions seem to lack those for permutative reductions in Troelstra [**13**], which Troelstra remarks.

Troelstra [**13**] proves several normalization theorems for an intuitionistic system of natural deduction and shows their applications. Our definition of a normal form, however, differs from his. He defines it as the deduction which

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1) Essentially, it corresponds to our 'end place' (cf. 1.2 in § **2**).

2) This inference figure corresponds to our 'boundary' (cf. 1.2 in § **2**) satisfying the condition in 2.251 of § **2**.

can not be reduced any more. He adapts and extends Prawitz's method which he used in Appendix A of [7] to prove a normalization theorem for some logical systems. Their ideas are based on 'strong validity' of deductions. Then he formalizes the normalization theorem (for subsystems of bounded logical complexity) in his intuitionistic system itself. He argues also on a normalization theorem for a second order intuitionistic arithmetic.

§ 2. Proof of the Main Theorem.

1. Preparations for the proof.

1.1. We shall prove the following theorem instead of the Main Theorem. From now on Γ^* means the result obtained from Γ by omitting some formulas.

THEOREM. *Let the sequent* $\Delta \rightarrow \Lambda$ *be derivable. We can transform any derivation of the sequent into the one with the end sequent* $\Delta^* \rightarrow \Lambda^*$ *and having the following properties:*

1) *It includes no cuts.*
2) *It includes no CJ-inference figures except irreducible ones.*
3) *It contains no redundant variables.*

This theorem holds, as later seen, both in the classical system and in the intuitionistic system. Therefore we can see that it implies our Main Theorem.

1.2. We define here some concepts. The *end place* of a derivation consists of the end sequent and the sequents below which only the lower sequents of *contractions* (Zusammenziehungen), or of *interchanges* (Vertauschungen), or of cuts are standing. Gentzen's concept 'ending (Endstück)' and ours 'end place' are similar but different. The uppermost sequents of the end place are the uppermost sequents of the entire derivation, or the lower sequents of thinnings, or those of operational inference figures, or those of *CJ*-inference figures. The inference figure which divides a derivation into the end place and the other part is called the *boundary* of the end place.

The concept 'clustered (verbunden)' is defined similarly as in Gentzen [2]. And the concepts 'cluster of formulas (Formelbund)' and 'the cluster associated with ⋯ (der zu ⋯ gehörige Bund)' can be also defined similarly, since the end place includes at most contractions, interchanges and cuts as the inference figures. But a cut is not always associated with every cluster.

1.3. We use the following terminology in the same senses as those in [2]:
the 'degrees (Grade)' of a formula, a cut and a *CJ*-inference figure;
the 'level (Höhe)' of a *derivation sequent* (Herleitungssequenz).

We assign an ordinal number to each *line of inference* (Schlußstrich) in a derivation. We do this just as in [2] for each inference figure except the one concerning the implication and the *CJ*-inference figure.

For the former we correspond an ordinal number to the line of inference similarly as to those for the other operational inference figures.

For the latter we do this as follows. Let α, β and γ be the ordinal numbers of the three upper sequents of a *CJ*-inference figure, and especially β be that of the central sequent. If β is expressed in Cantor's normal form as

$$\omega^{\beta_1}+\cdots+\omega^{\beta_n},$$

then the ordinal number of the line of inference is defined as

$$\alpha \# \omega^{\beta_1+1} \# \gamma,$$

where $\#$ means the natural sum of two ordinal numbers.

Thus we can calculate the ordinal number of any derivation in our systems.

2. Proof of the Main Theorem.

2.1. *Course of the proof.* We prove the theorem stated in 1.1 of this section. Our method of the proof is fundamentally based upon those of Gentzen [**2**] and Takeuti [**12**]. And the same idea is also applied to the author's earlier paper [**4**] which Zucker [**14**] refers to.

We take arbitrarily a derivation, say P. If there are no boundaries, then it can be easily transformed into a required derivation. If there are any boundaries, then we need to classify this case according to the types and the kinds of the boundaries. And we reduce the problem concerning P to the one concerning a simpler derivation with less ordinal number.

To carry out this program we reduce the derivation itself. Of the reduction of a derivation, there are two kinds, i. e. the auxiliary and the essential reduction. The ordinal number of a derivation does not always decrease after an auxiliary reduction. Sometimes it is unchanged. But the reduction is necessary for the elimination of redundant variables and for some essential reductions in succeeding steps, which decrease really the ordinal number of the derivation.

When we are concerned with a case in the following, we suppose that P satisfies none of the conditions of the preceding cases. If P does not become a required derivation after the possible reductions in a case, then we go back to a preceding case or ahead to the next case, according as the resulting derivation satisfies the condition of some preceding case or not, respectively. We omit actual calculation of ordinal numbers for derivations, since it can be carried out just as in Gentzen [**2**].

In 2.2 we prove our theorem for the classical system, and in 2.3 we modify the proof into the one for the intuitionistic system.

2.2. *Proof for the classical system.* Suppose that a derivation P is given.

2.21. *The case where P contains redundant variables.* Substituting 0 for suitable occurrences of these free variables, we can eliminate all redundant variables from P. We omit the minor details here. This is an auxiliary reduction. The ordinal number of the new derivation is equal to that of P.

2.22. *The case where the end place of P includes basic logical sequents.* We take arbitrarily such a sequent. If it is P itself, then it is also a required derivation for P. If it is an upper sequent of an inference figure, then the inference figure must be a cut. Removing both upper sequents of the cut, we reduce P to a new derivation, say P_1. This is an essential reduction. The level of each sequent in P_1 is not greater than that of the corresponding sequent in P. Therefore the ordinal number of P_1 is less than that of P. By induction hypothesis P_1 can be transformed into a required derivation, which is also a required derivation for P.

REMARK. Suppose that P includes no boundaries, i. e. that P coincides with its end place. If there are any cuts in P, then we take one of the uppermost ones. Then both upper sequents of this cut must be basic logical sequents. If there are no cuts in P, then P itself is a basic logical sequent. Consequently we shall treat henceforth only the case that P includes some boundaries.

2.23. *The case where P includes thinnings as boundaries.* By a series of auxiliary reductions we can exclude from P all the thinnings used as boundaries. In these reductions some formulas of the end sequent may be canceled. Let $\mathfrak{J}$ be one of such thinnings and D be the thinning formula of $\mathfrak{J}$.

First we cancel D in the lower sequent of $\mathfrak{J}$. Then we cancel in the next lower sequent the formula clustered with D, and then in the subsequent lower sequent the formula clustered with this canceled formula, $\cdots$. We continue this procedure until we cancel a formula in one of the following sequents:

1) The upper sequent of the uppermost contraction that contracts the two formulas belonging to the cluster associated with D.
2) An upper sequent of the cut whose cut formulas belong to the cluster associated with D.
3) The end sequent.

In the cases 1) and 3) we stop our procedure. In the case 2) we replace the ex-cut by some thinnings and interchanges so as to obtain the same sequent as the lower sequent of the ex-cut and we stop the procedure. Then we contract the same successive sequents into one to obtain a new derivation P_1.

The level of each sequent in P_1 is not greater than that of the corresponding sequent in P. Therefore the ordinal number of P_1 is not greater than that of P.

Let m and n be the total number of cuts in P and that of thinnings in P, respectively. Our assertion stated at the beginning can be proved by transfinite induction on $\omega \cdot m+n$.

2.24. *The case where P includes CJ-inference figures as boundaries.* Let $\mathfrak{J}$ be one of such *CJ*-inference figures. And let P be of the form:

$$\mathfrak{J}\ \frac{\begin{matrix}\vdots \\ \Gamma_1 \to \Theta_1, F(0)\end{matrix} \qquad \begin{matrix}\vdots\,(a) \\ F(a), \Gamma_1 \to \Theta_1, F(a')\end{matrix} \qquad \begin{matrix}\vdots \\ F(t), \Gamma_2 \to \Theta_2\end{matrix}}{\begin{matrix}\Gamma_1, \Gamma_2 \to \Theta_1, \Theta_2 \\ \vdots \\ \Delta \to \Lambda .\end{matrix}}$$

According to the forms of the induction term t and the induction formula $F(a)$, we need to classify this case furthermore. In each case the reduction of P is essential.

2.241. *The case where the induction term is a numeral, or where the induction formula does not actually contain the eigenvariable.* We reduce P to a derivation P_1 by replacing $\mathfrak{J}$ by some cuts, interchanges and contractions. Then the level of each sequent in P_1 is equal to that of the corresponding sequent in P. And the ordinal number of P_1 is less than that of P, since the number of these consecutive cuts are finite. Therefore, by induction hypothesis, P_1 can be transformed into a required derivation, which is also a required derivation for P. We omit the minor details here.

2.242. *The case where the induction term is not a numeral and the induction formula contains actually the eigenvariable.* First we reduce P to three preparatory derivations P_1, P_2 and P_3. The first two derivations are obtained from P by replacing $\mathfrak{J}$ by some thinnings and interchanges and by substituting a free variable, say b, not contained in P for suitable occurrences of a in P.

$$P_1: \quad \begin{matrix}\vdots \\ \Gamma_1 \to \Theta_1, F(0) \\ \hline\hline \Gamma_1, \Gamma_2 \to F(0), \Theta_1, \Theta_2 \\ \vdots \\ \Delta \to F(0), \Lambda\end{matrix} \qquad\qquad P_2: \quad \begin{matrix}\vdots\,(b) \\ F(b), \Gamma_1 \to \Theta_1, F(b') \\ \hline\hline \Gamma_1, \Gamma_2, F(b) \to F(b'), \Theta_1, \Theta_2 \\ \vdots \\ \Delta, F(b) \to F(b'), \Lambda\end{matrix}$$

Since t is not a numeral, it should have one of the forms $c, c', c'', \cdots$, where c is a free variable. Changing appropriate free variables into new ones, we can suppose that all the eigenvariables in P differ from c. We obtain P_3 from P by substituting $c, c', c'', \cdots$ for suitable occurrences of a and by replacing $\mathfrak{J}$ by some cuts, interchanges and contractions.

P_3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots(c)}{F(c), \Gamma_1 \to \Theta_1, F(c')} \quad \dfrac{\vdots(c')}{F(c'), \Gamma_1 \to \Theta_1, F(c'')}}{F(c), \Gamma_1, \Gamma_1 \to \Theta_1, \Theta_1, F(c'')}}{F(c), \Gamma_1 \to \Theta_1, F(c'')} \quad \dfrac{\vdots(c'')}{F(c''), \Gamma_1 \to \Theta_1, F(c''')}}{F(c), \Gamma_1, \Gamma_1 \to \Theta_1, \Theta_1, F(c''')}}{F(c), \Gamma_1 \to \Theta_1, F(c''')}
$$

$$
\vdots
$$

$$
\mathfrak{J}_1 \dfrac{\dfrac{\dfrac{F(c), \Gamma_1 \to \Theta_1, F(t) \qquad \vdots\; F(t), \Gamma_2 \to \Theta_2}{F(c), \Gamma_1, \Gamma_2 \to \Theta_1, \Theta_2}}{\Gamma_1, \Gamma_2, F(c) \to \Theta_1, \Theta_2}}{}
$$

$$
\vdots
$$

$$
\Delta, F(c) \to \Lambda .
$$

In the case where t is c itself there does not appear in P_3 the derivation ending with the left upper sequent of $\mathfrak{J}_1$. And the lower sequent of $\mathfrak{J}_1$ is directly obtained by adding some thinnings and interchanges to the derivation ending with the rightmost upper sequent of $\mathfrak{J}$ in P.

The level of each sequent in every preparatory derivation is not greater than that of the corresponding sequent in P. Accordingly the ordinal number of each preparatory derivation is less than that of P. Thus, by induction hypothesis, we get a required derivation, say Q_i, for each P_i. Since b is not contained in the sequence $F(0), F(c), \Delta, \Lambda$, we can combine these three derivations by an irreducible *CJ*-inference figure, say $\mathfrak{J}_2$, to obtain the following derivation Q:

$$
\mathfrak{J}_2 \dfrac{\dfrac{\dfrac{\vdots}{\Delta^* \to F(0)^*, \Lambda^*}}{\Delta \to \Lambda, F(0)} \qquad \dfrac{\dfrac{\vdots}{\Delta^*, F(b)^* \to F(b')^*, \Lambda^*}}{F(b), \Delta \to \Lambda, F(b')} \qquad \dfrac{\dfrac{\vdots}{\Delta^*, F(c)^* \to \Lambda^*}}{F(c), \Delta \to \Lambda}}{\dfrac{\Delta, \Delta \to \Lambda, \Lambda}{\Delta \to \Lambda}} .
$$

Now we consider again the original derivation P. Since P is supposed here not to contain redundant variables, the free variables in $F(t)$ must appear in the lower sequent of $\mathfrak{J}$. And, since the lower sequent of $\mathfrak{J}$ is included in the end place, these free variables must appear also in the end sequent. Allowing for the properties of each Q_i, we can conclude from this that Q is a required derivation for P.

2.25. *The case where P includes operational inference figures as boundaries.* According to the types of these boundaries, we calassify this case furthermore. All the reductions are essential.

2.251. *The case where there exists an inference figure among these boundaries* *whose principal formula and a formula in the end sequent belong to the same cluster of formulas.* Let $\mathfrak{J}$ be one of such inference figures. We treat representatively only the case that $\mathfrak{J}$ is an inference figure introducing a universal quantifier. All the other cases can be treated similarly.

2.251.1. *The case where $\mathfrak{J}$ is an inference figure introducing a universal quantifier on the antecedent.* We reduce P to a preparatory derivation P_1 by replacing $\mathfrak{J}$ by some thinnings and interchanges.

$$
P:\quad \begin{array}{c} \vdots \\ \mathfrak{J}\ \dfrac{F(t), \Gamma \rightarrow \Theta}{\forall x F(x), \Gamma \rightarrow \Theta} \\ \vdots \\ \Delta \rightarrow \Lambda \end{array}
\qquad
P_1:\quad \begin{array}{c} \vdots \\ \dfrac{F(t), \Gamma \rightarrow \Theta}{\forall x F(x), \Gamma, F(t) \rightarrow \Theta} \\ \vdots \\ \Delta, F(t) \rightarrow \Lambda \end{array}
\qquad
Q:\quad \begin{array}{c} \vdots \\ \Delta^*, F(t)^* \rightarrow \Lambda^* \\ \hline\hline F(t), \Delta \rightarrow \Lambda \\ \hline \forall x F(x), \Delta \rightarrow \Lambda \\ \hline\hline \Delta \rightarrow \Lambda \end{array}
$$

The level of each sequent in P_1 is equal to that of the corresponding sequent in P. Therefore the ordinal number of P_1 is less than that of P. By induction hypothesis we get a required derivation, say Q_1, for P_1. Adding some inference figures to Q_1 as above, we obtain a new derivation Q. Note that Δ contains $\forall x F(x)$. Since neither P nor Q_1 contains redundant variables, Q does not contain them, too. Thus, allowing for the properties of Q_1, we see that Q is a required derivation for P.

2.251.2. *The case where $\mathfrak{J}$ is an inference figure introducing a universal quantifier on the succedent.* Let a be the eigenvariable of $\mathfrak{J}$. Substituting a free variable, say b, not contained in P for suitable occurrences of a in P, we change the eigenvariable of $\mathfrak{J}$ into b. Then, just as in 2.251.1, we reduce this new derivation to P_1 and obtain a required derivation Q for P via a required derivation Q_1 for P_1.

$$
P:\quad \begin{array}{c} \vdots\,(a) \\ \mathfrak{J}\ \dfrac{\Gamma \rightarrow \Theta, F(a)}{\Gamma \rightarrow \Theta, \forall x F(x)} \\ \vdots \\ \Delta \rightarrow \Lambda \end{array}
\qquad
P_1:\quad \begin{array}{c} \vdots\,(b) \\ \dfrac{\Gamma \rightarrow \Theta, F(b)}{\Gamma \rightarrow F(b), \Theta, \forall x F(x)} \\ \vdots \\ \Delta \rightarrow F(b), \Lambda \end{array}
\qquad
Q:\quad \begin{array}{c} \vdots \\ \Delta^* \rightarrow F(b)^*, \Lambda^* \\ \hline\hline \Delta \rightarrow \Lambda, F(b) \\ \hline \Delta \rightarrow \Lambda, \forall x F(x) \\ \hline \Delta \rightarrow \Lambda \end{array}
$$

2.252. *The case where there exists an inference figure among these boundaries whose principal formula and a cut formula belong to the same cluster of formulas.* Recall that P is supposed here to satisfy none of the conditions of the preceding cases. First we show that the end place of P includes a cut with special properties. Then we reduce P, using the properties of the cut.

We consider the following condition C to each sequent in the end place of P:

C. *Starting with a formula in the sequent concerned, we trace upward the cluster associated with the formula to meet with the principal formula of a boundary.*

The lower sequent of each boundary satisfies C, since P satisfies none of the conditions of the cases 2.23 and 2.24; on the other hand the end sequent does not, since P does not satisfy the condition of the case 2.251. And, ascending each path from the end sequent, we encounter necessarily the lower sequent of a boundary since P does not satisfy the condition of the case 2.22. Thus there must be an inference figure in each path (one of) whose upper sequent(s) satisfies C but whose lower sequent does not. We take one of the uppermost ones and denote it by $\mathfrak{I}$. $\mathfrak{I}$ should be a cut. And, above each upper sequent of $\mathfrak{I}$, there must be a boundary whose principal formula and the cut fomulas of $\mathfrak{I}$ belong to the same cluster of formulas.

According to the kind of the outermost logical connective of the cut formula of $\mathfrak{I}$, we need to classify this case furthermore. We concern here only the case that the logical connective is a universal quantifier, since all the other cases can be treated similarly. Then P must be of the following form:

$$
\begin{array}{ccc}
\vdots\,(a) & & \vdots \\
\mathfrak{I}_1\;\dfrac{\Gamma_1 \to \Theta_1, F(a)}{\Gamma_1 \to \Theta_1, \forall x F(x)} & & \mathfrak{I}_2\;\dfrac{F(t), \Gamma_2 \to \Theta_2}{\forall x F(x), \Gamma_2 \to \Theta_2} \\
\vdots & & \vdots \\
\mathfrak{I}\;\dfrac{\Gamma_3 \to \Theta_3, \forall x F(x) \qquad\qquad \forall x F(x), \Gamma_4 \to \Theta_4}{\Gamma_3, \Gamma_4 \to \Theta_3, \Theta_4} \; \rho & & \\
\vdots & & \\
\mathfrak{I}_3\;\dfrac{\vdots}{\Delta_1 \to \Lambda_1} \;\begin{matrix}\rho \\ \tau\end{matrix} & & \\
\vdots & & \\
\Delta \to \Lambda .
\end{array}
$$

Both $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are boundaries. $\mathfrak{I}$ is the above-mentioned cut. Let ρ be the level of the upper sequent of $\mathfrak{I}$. Since ρ is positive and the level of the end sequent is zero, there must be an inference figure, say $\mathfrak{I}_3$, whose upper sequent has the level ρ and whose lower sequent has the less level, say τ, than ρ.

Changing appropriate free variables into new ones, we can suppose that all the eigenvariables in P differ from the free variable contained in t. We cut off from P the derivation ending with the lower sequent of $\mathfrak{I}_3$. By replacing $\mathfrak{I}_1$ or $\mathfrak{I}_2$ by some thinnings and interchanges, and by substituting t for suitable occurrences of a, we obtain two derivations which are expressed below as those ending with the lower sequents of $\mathfrak{I}_3'$ and $\mathfrak{I}_3''$, respectively. Then, after addition of some interchanges, we combine them by a cut, say $\mathfrak{I}_4$.

to obtain a new derivation P_1:

$$
\begin{array}{cccc}
\vdots\,(t) & \vdots & \vdots\,(a) & \vdots \\
\dfrac{\Gamma_1 \rightarrow \Theta_1, F(t)}{\Gamma_1 \rightarrow F(t), \Theta_1, \forall x F(x)} & \dfrac{F(t), \Gamma_2 \rightarrow \Theta_2}{\forall x F(x), \Gamma_2 \rightarrow \Theta_2} & \dfrac{\Gamma_1 \rightarrow \Theta_1, F(a)}{\Gamma_1 \rightarrow \Theta_1, \forall x F(x)} & \dfrac{F(t), \Gamma_2 \rightarrow \Theta_2}{\forall x F(x), \Gamma_2, F(t) \rightarrow \Theta_2} \\
\vdots & \vdots & \vdots & \vdots \\
\multicolumn{2}{c}{\dfrac{\Gamma_3 \rightarrow F(t), \Theta_3, \forall x F(x) \quad \forall x F(x), \Gamma_4 \rightarrow \Theta_4}{\Gamma_3, \Gamma_4 \rightarrow F(t), \Theta_3, \Theta_4}} & \multicolumn{2}{c}{\dfrac{\Gamma_3 \rightarrow \Theta_3, \forall x F(x) \quad \forall x F(x), \Gamma_4, F(t) \rightarrow \Theta_4}{\Gamma_3, \Gamma_4, F(t) \rightarrow \Theta_3, \Theta_4}} \\
\multicolumn{2}{c}{\vdots} & \multicolumn{2}{c}{\vdots} \\
\multicolumn{2}{c}{\mathfrak{I}_3' \; \dfrac{}{\dfrac{\Delta_1 \rightarrow F(t), \Lambda_1}{\Delta_1 \rightarrow \Lambda_1, F(t)}\; \sigma} \;\rho} & \multicolumn{2}{c}{\mathfrak{I}_3'' \; \dfrac{}{\dfrac{\Delta_1, F(t) \rightarrow \Lambda_1}{F(t), \Delta_1 \rightarrow \Lambda_1}\; \sigma} \;\rho}
\end{array}
$$

$$
\mathfrak{I}_4 \; \dfrac{\Delta_1 \rightarrow \Lambda_1, F(t) \qquad F(t), \Delta_1 \rightarrow \Lambda_1}{\dfrac{\Delta_1, \Delta_1 \rightarrow \Lambda_1, \Lambda_1}{\Delta_1 \rightarrow \Lambda_1}} \; \tau
$$

$$
\begin{array}{c}
\vdots \\
\Delta \rightarrow \Lambda .
\end{array}
$$

The degree of $F(t)$ is less than that of $\forall x F(x)$ by one. Therefore the level σ of the upper sequent of $\mathfrak{I}_4$ is less than ρ. And σ is not less than τ. Consequently the ordinal number of P_1 is less than that of P. By induction hypothesis P_1 can be transformed into a required derivation, which is also a required derivation for P.

2.3. *Proof for the intuitionistic system.* If we attempt to prove our theorem for the intuitionistic system quite similarly as for the classical system, we shall often get stuck since there may exist at most one formula in the succedent of each sequent. If we treat a universal quantifier representatively just as before, we can not obtain the derivations corresponding to the following ones in the classical case: P_1 and P_2 in 2.242, P_1 in 2.251.2 and the derivation in 2.252 which ends with the left upper sequent of $\mathfrak{I}_4$. All the other corresponding derivations can be obtained similarly as in the classical case.

Now we state how to reduce P in those cases. Our method is due to Scarpellini [8]. We can summarize it as follows. Suppose that an intuitionistic derivation P is given. Regarding it as a classical derivation, we first reduce P. The result is, of course, not necessarily intuitionistic. There may appear some sequence of successive sequents each of which has two formulas in the succedent. Then we cancel *proper* one of them in the succedent of each of such sequents, and, if necessary, we replace some ex-cuts by thinnings and interchanges. After contracting the same successive sequents into one, we obtain an intuitionistic derivation to which we want to reduce P. The level of each sequent of this new derivation is not greater than that of the corresponding sequent of P, and the ordinal number of the new derivation is less than that of P.

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