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Embeddings of infinite-dimensional manifold pairs and remarks on stability and deficiency

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Abstract. In this paper, we treat of an E-manifold pair (M, N) with N a Z-set in M where E is an infinite-dimensional locally convex linear metric space which is homeomorphic to E^{ω} or E_f^{ω}. And we study the condition under which M can be embedded in E such that N is the topological boundary under the embedding (Anderson's Problem in [2]). Moreover we extend the results on topological stability and deficiency, the Homeomorphism Extension Theorem and the results in [18].

§ 0. Introduction.

For each space X, we denote by X^{ω} the countable infinite product of X by itself. And for each space X with a base point 0, $X_f^{\omega}=\{(x_i)\in X^{\omega} \mid x_i=0$ for almost all $i\}$. A closed subset K of a space X is a *Z-set in X* if for each non-empty homotopically trivial open set U, $U\backslash K$ is also non-empty and homotopically trivial ([1]). An *E-manifold* is a paracompact manifold modelled on a space E. As a modelled space, let E be a locally convex linear metric space (LCLMS) homeomorphic ($\cong$) to E^{ω} or E_f^{ω}. In an E-manifold pair (M, N), N is a Z-set in M if and only if N is a collared closed set in M (collared in the sense of M. Brown [7] (see 4-4 in this paper). Then (M, N) may be considered as a manifold-with-boundary, N being the boundary. Thus the study of E-manifolds-with-boundary becomes the study of such E-manifold pairs. However circumstances of infinite-dimensional manifolds are different from finite-dimensional case (e.g., see Examples 1 and 2 of Sect. 7 in this paper).

In this paper, we study the problem for such an E-manifold pair (M, N): *Under what condition can M be embedded in E such that N is a topological boundary under the embedding?* This problem for separable l^2-manifold pairs was raised by R. D. Anderson in [2]. In the previous paper [20], we found a sufficient condition of this problem: *N contains some deformation retract of M.* And we saw that even if N is homeomorphic to E, M cannot always be embedded such a way. But we have an easy example of E-manifold pairs which do not satisfy the above condition but which can be embedded such a way.

By the result of Toruńczyk (Theorems 3.1 and 3.2 in [**25**]), every complete metrizable AR (ANR) X admitting a closed embedding into E is an E-factor (an E-manifold factor), i.e., $X\times E\cong E$ ($X\times E$ is an E-manifold). Let D is a 2-dimensional closed disk with two holes in $\boldsymbol{R}^2$. Then

$$(M, N)=(D\times E, (\mathrm{bd}_{\boldsymbol{R}^2}D)\times E)$$

is an E-manifold pair with N a Z-set in M which does not satisfy the above condition. But M can be embedded in E such that N is the topological boundary of M in E.

In Sect. 6, we shall find a little more mild sufficient condition including this example. Furthermore in Sect. 5, we shall obtain a necessary and sufficient condition under which M can be embedded in E such that N is the topological boundary of M and such that the closure (or each component of the closure) of the complement of M is contractible.

In Sect. 2, we generalize the results on topological stability (Geoghegan-Henderson [**12**] Theorem 1 and Schori [**21**] Theorem 2.2) and the results on E-deficient sets in E-manifolds (Chapman [**9**] Theorem 3.1 and Cutler [**10**] Lemma 3), and in Sect. 3, the Homeomorphism Extension Theorem (HET) which was established by Anderson-McCharen [**4**] and was extended by Chapman [**9**]. In Sect. 4, we generalize the results of [**18**].

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§ 1. Notations.

Let α, β be collections of subsets of a set X. We write

$$\mathrm{st}(\alpha;\beta)=\{\mathrm{st}(A;\beta)\mid A\in\alpha\}$$

where $\mathrm{st}(A;\beta)=\cup\{B\in\beta\mid A\cap B\neq\emptyset\}$, and then $\mathrm{st}(\alpha)=\mathrm{st}^1(\alpha)=\mathrm{st}(\alpha:\alpha)$ and inductively $\mathrm{st}^n(\alpha)=\mathrm{st}(\mathrm{st}^{n-1}(\alpha);\alpha)$. We say that α *refines* β (or α is a *refinement* of β) provided that for each $A\in\alpha$ there is some $B\in\beta$ containing A. Let γ be another collection of subsets of X. It is easy to see that $\mathrm{st}(\mathrm{st}(\alpha;\beta);\gamma)$ refines both $\mathrm{st}(\alpha;\mathrm{st}(\beta;\gamma))$ and $\mathrm{st}(\alpha;\mathrm{st}(\gamma;\beta))$. Maps $f, g: Y\to X$ are said to be α-*near* (or f is α-*near* to g) provided that for each $y\in Y$, $f(y)=g(y)$ or $f(y)$ and $g(y)$ are both contained in some $A\in\alpha$. We write

$$\alpha\times Y=\{A\times Y\mid A\in\alpha\}.$$

Let X, Y be topological space and let $X'\subset X$, $Y'\subset Y$. Continuous map $f, g:(X, X')\to(Y, Y')$ are said to be *homotopic* (or f is *homotopic* ($\sim$) to g) if there is a homotopy $h:(X\times I, X'\times I)\to(Y, Y')$ ($I=[0, 1]$) such that $h_0=f$ and $h_1=g$ where $h_t:(X, X')\to(Y, Y')$ ($t\in I$) is defined by $h_t(x)=h(x, t)$ for each $x\in X$. Let α be an open cover of Y. A homotopy (an isotopy) $h:X\times I\to Y$ is said to be an *α-homotopy* (*α-isotopy*) provided that for each $x \in X$ there is some $U \in \alpha$ containing $h(\{x\} \times I)$. Continuous maps $f, g: X \to Y$ are said to be *α-homotopic* (or f is *α-homotopic* to g) if there is an α-homotopy $h: X \times I \to Y$ such that $h_0 = f$ and $h_1 = g$. An isotopy $h: X \times I \to Y$ is said to be *ambient* if $h_t(X) = Y$ for each $t \in I$, and said to be *invertible* if $\bar{h}: X \times I \to Y \times I$ defined by $\bar{h}(x, t) = (h(x, t), t)$ for each $(x, t) \in X \times I$ is an embedding.

"*AR*" and "*ANR*" mean "absolute retract for metric spaces" and "absolute neighbourhood retract for mertic spaces" respectively. As concerns *AR* and *ANR*, refer to the books of K. Borsuk [**6**] and S.-T. Hu [**15**].

§ 2. Remarks on E-stability and E-deficiency.

In this section, we generalize the Geoghegan-Henderson's result on strong E-stability (Theorem 1 in [**12**] and the Schori's Stability Theorem for open sets in an *LTS* (Theorem 2.2 in [**21**]) and we give an alternative proof of Theorem 4.1 in [**9**] and its extension. T. A. Chapman showed the equivalence of E-deficiency and l^2-deficiency for an *LMS* $E \cong E^\omega$ in [**9**]. We show the equivalence of E-deficiency and l^2_f-deficiency for an *LMS* $E \cong E^\omega_f$ and, as its corollary, we get the extension of Cutler's result on negligibility of E-deficient sets (Lemma 3 in [**10**]).

Let E be an *LTS*. A space X is said to be *E-stable* if $X \times E \cong X$. A subset K of an E-stable space X is said to be *E-deficient* in X if there is a homeomorphism $h: X \to X \times E$ such that $h(K) \subset X \times \{0\}$. An embedding $f: Y \to X$ is said to be *E-deficient* if $f(Y)$ is E-deficient in X.

Although the following lemma is proved by the same way as Theorem 2.2 in [**21**], we give the proof to make sure, because some detailed remarks are required and this is important.

2-1. LEMMA. *Let E be an LTS, $F = E^\omega$ or $= E^\omega_f$ and X a space such that each finite product $X \times E^n$ is perfectly normal. If α is an open collection in $X \times F$ and $W = \cup \alpha$, then there exists an I-preserving continuous map σ^α: $X \times F \times E \times I \to X \times F \times I$ such that*

i) *$\sigma^\alpha(x, 0, 0, t) = (x, 0, t)$ for each $(x, 0, 0, t) \in X \times F \times E \times I$,*

ii) *$\sigma^\alpha_0: X \times F \times E \to X \times F$ is the projection,*

iii) *$\sigma^\alpha_t | (X \times F \setminus W) \times E: (X \times F \setminus W) \times E \to X \times F \setminus W$ is the projection for each $t \in I$,*

iv) *$\sigma^\alpha | W \times E \times (0, 1]: W \times E \times (0, 1] \to W \times (0, 1]$ is a homeomorphism,*

v) *$\sigma^\alpha | W \times \{0\} \times I: W \times \{0\} \times I \to W \times I$ is a closed embedding, and*

vi) *for each $x \in W$, there is some $U \in \alpha$ such that $\sigma^\alpha(\{x\} \times E \times I) \subset U \times I$.*

PROOF. We denote $x = (x_0; x_1, x_2, \cdots) \in X \times F$. For each positive integer n, let $p_n: X \times F \to X \times E^n$ be the natural projection, i.e., $p_n(x) = (x_0; x_1, \cdots, x_n)$ for $x \in X \times F$. Define an I-preserving continuous map $\theta: (X \times F) \times E \times I \to (X \times F) \times I$

by

$$\theta(x, y, t)=(x_0 ; x_1, \cdots, x_n, y\cos(1-2^n t)\pi+x_{n+1}\sin(1-2^n t)\pi,$$
$$y\sin(1-2^n t)\pi-x_{n+1}\cos(1-2^n t)\pi, -x_{n+2}, -x_{n+3}, \cdots ; t)$$
$$\text{for each } (x, y, t)\in(X\times F)\times E\times[2^{-(n+1)}, 2^{-n}]$$

and

$$\theta(x, y, 0)=(x, 0) \qquad \text{for each } (x, y, 0)\in(X\times F)\times E\times\{0\}.$$

Then $\theta|X\times F\times E\times(0, 1]: X\times F\times E\times(0, 1]\to X\times F\times(0, 1]$ is a homeomorphism. And note that if $t\leqq 2^{-n}$, then $p_n\theta_t(x, y)=p_n(x)$ for $(x, y)\in(X\times F)\times E$. Let β be a basic open cover of W which refines α. By Lemma 5.2 in [21], there is a collection $\{K_n|n\in N\}$ of closed sets in $X\times F$ such that $\bigcup_{n=1}^{\infty}K_n=W$ and for each n, $p_n^{-1}p_n(K_n)=K_n\subset\text{int}\,K_{n+1}\cap\bigcup\{B\in\beta|p_n^{-1}p_n(B)=B\}$. For each $x\in W$, put $n(x)=\min\{n\in N|x\in K_n\}$. Then $x\in K_{n(x)}\setminus K_{n(x)-1}$. By perfect normality, there is a sequence $\{k_n|n\in N\}$ of continuous maps $k_n: p_n(K_n\setminus\text{int}\,K_{n-1})\to[2^{-(n+1)}, 2^{-n}]$ such that

$$k_n^{-1}(2^{-(n+1)})=p_n(\text{bd}\,K_n)=\text{bd}\,p_n(K_n)$$

and

$$k_n^{-1}(2^{-n})=p_n(\text{bd}\,K_{n-1})=\text{bd}\,p_n(K_{n-1})$$

where $K_0=\emptyset$. Define a continuous map $k: W\to(0, 1]$ by $k(x)=k_{n(x)}p_{n(x)}(x)$ fo $x\in W$. Then for each $x=(x_0 ; x_1, x_2, \cdots)\in W$,

$$k(x)=k(x_0 ; x_1, \cdots, x_{n(x)}, *, *, \cdots)\leqq 2^{-n(x)}.$$

Now define $\sigma^\alpha: (X\times F)\times E\times I\to(X\times F)\times I$ by

$$\sigma^\alpha(x, y, t)=\begin{cases}(\theta_{tk(x)}(x, y), t) & \text{for each } (x, y, t)\in W\times E\times I\\ (x, t) & \text{for each } (x, y, t)\in(X\times F\setminus W)\times E\times I.\end{cases}$$

Then it is trivial that this map satisfies conditions i), ii) and iii). We must examine the continuity of this map and the other conditions.

vi) Let $x\in W$. Since $x\in K_{n(x)}$, there is some $B\in\beta$ such that

$$p_{n(x)}^{-1}p_{n(x)}(B)=B\ni x.$$

For each $y\in E$ and each $t\in(0, 1]$

$$p_{n(x)}\sigma_t^\alpha(x, y)=p_{n(x)}\theta_{tk(x)}(x, y)=p_{n(x)}(x)\in p_{n(x)}(B)$$

because $tk(x)\leqq 2^{-n(x)}$. Then $\sigma^\alpha(\{x\}\times E\times I)\subset B\times I$. Since β is a refinement of α, there is some $U\in\alpha$ such that $\sigma^\alpha(\{x\}\times E\times I)\subset U\times I$.

iv) Define a continuous map $\sigma': W\times(0, 1]\to W\times E\times(0, 1]$ by

$$\sigma'(x, t)=(\theta_{tk(x)}^{-1}(x), t) \qquad \text{for each } (x, t)\in W\times(0, 1].$$

Let $(x, t)\in W\times(0, 1]$ and $\sigma'(x, t)=(x', y', t)\in W\times E\times(0, 1]$. Since $x=\theta_{tk(x)}(x', y')$

and $tk(x)\leqq 2^{-n(x)}$, $p_{n(x)}(x)=p_{n(x)}\theta_{tk(x)}(x', y')=p_{n(x)}(x')$ therefore $k(x)=k(x')$. Then

$$\sigma^{\alpha}\sigma'(x, t)=\sigma^{\alpha}(x', y', t)=(\theta_{tk(x')}(x', y'), t)=(\theta_{tk(x)}(x', y'), t)=(x, t).$$

Now let $(x, y, t)\in W\times E\times(0, 1]$ and $\sigma^{\alpha}(x, y, t)=(x', t)\in W\times(0, 1]$. Since $x'=\theta_{tk(x)}(x, y)$ and $tk(x)\leqq 2^{-n(x)}$, $p_{n(x)}(x')=p_{n(x)}\theta_{tk(x)}(x, y)=p_{n(x)}(x)$ therefore $k(x')=k(x)$. Then

$$\sigma'\sigma^{\alpha}(x, y, t)=\sigma'(x', t)=(\theta^{-1}_{tk(x')}(x'), t)=(\theta^{-1}_{tk(x)}\theta_{tk(x)}(x, y), t)=(x, y, t).$$

Continuity of σ^{α}: We may examine that σ^{α} is continuous at $(x, y, 0)\in W\times E\times\{0\}$(1) and at $(x, y, t)\in(X\times F\setminus W)\times E\times I$(2).

1) Let V be any neighbourhood of $\sigma^{\alpha}(x, y, 0)=(x, 0)$ in $X\times F\times I$. Then there are a positive integer n and an open set U in $X\times F$ such that $p_n^{-1}p_n(U)=U$ and $(x, 0)\in U\times[0, 2^{-n})\subset V\cap W\times I$. For each $(x', y', t')\in U\times E\times[0, 2^{-n})$,

$$p_n\sigma^{\alpha}_{t'}(x', y')=p_n\theta_{t'k(x')}(x', y')=p_n(x')\in p_n(U),$$

then $\sigma^{\alpha}_{t'}(x', y')\in U$. Therefore $\sigma^{\alpha}(x', y', t')\in U\times[0, 2^{-n})\subset V$.

2) Now let V be any neighbourhood of $\sigma^{\alpha}(x, y, t)=(x, t)$ in $X\times F\times I$. Then there are a positive integer n, an open set U' in $X\times F$ and a neighbourhood J of t in I such that $p_n^{-1}p_n(U')=U'$ and $(x, t)\in U'\times J\subset V$. Then $U=U'\setminus K_n$ is a neighbourhood of x in X. Let $(x', y', t')\in U\times E\times J$. If $x'\notin W$, $\sigma^{\alpha}(x', y', t')=(x', t')\subset U\times J$. If $x'\in W$, $n(x')>n$ because $x'\notin K_n$. Then $t'k(x')\leqq 2^{-n(x')}<2^{-n}$ therefore $p_n\sigma^{\alpha}_{t'}(x', y')=p_n\theta_{t'k(x')}(x', y')=p_n(x')\in p_n(U)\subset p_n(U')$, that is, $\sigma^{\alpha}_{t'}(x', y')\in U'$. Therefore $\sigma^{\alpha}(x', y', t')\in U'\times J\subset V$.

v) It is trivial that $\sigma^{\alpha}|W\times\{0\}\times I$ is a continuous injection. We may observe that this is a closed map. Let A be a closed set in $W\times\{0\}\times I$ and $(x, t)\in \mathrm{cl}\sigma^{\alpha}(A)\subset W\times I$. When $t\neq 0$,

$$(x, t)\in\mathrm{cl}\sigma^{\alpha}(A)\cap W\times(0, 1]=\sigma^{\alpha}(A\cap W\times E\times(0, 1])\subset\sigma^{\alpha}(A)$$

by iv). When $t=0$, we will see that $(x, 0, 0)\in A$ then $(x, 0)=\sigma^{\alpha}(x, 0, 0)\in\sigma^{\alpha}(A)$. For each neighbourhood V of $(x, 0, 0)$ in $W\times\{0\}\times I$, there are a positive integer n and an open set U in W such that $p_n^{-1}p_n(U)=U$ and $(x, 0, 0)\in U\times\{0\}\times[0, 2^{-n})\subset V$. Since $U\times[0, 2^{-n})$ is a neighbourhood of $(x, 0)$ in $W\times I$, there is some $(x', 0, t')\in A$ such that $\sigma^{\alpha}(x', 0, t')\in U\times[0, 2^{-n})$. Because $t'<2^{-n}$, $p_n(x')=p_n\sigma^{\alpha}_{t'}(x', 0)\in p_n(U)$, that is, $x'\in U$. Then $(x', 0, t')\in U\times\{0\}\times[0, 2^{-n})\subset V$. Therefore $A\cap V\neq\emptyset$, so $(x, 0, 0)\in\mathrm{cl}A=A$. □

2-2. THEOREM. *Let $E\cong E^{\omega}$ or $\cong E^{\omega}_f$ be a perfectly normal LTS and G an open set in an E-stable perfectly normal space X. Then for each open cover α of G, there exists an I-preserving continuous map $\Delta^{\alpha}: X\times E\times I\to X\times I$ such that*

i) *$\Delta^{\alpha}_0: X\times E\to X$ is the projection,*

ii) *$\Delta^{\alpha}_t|(X\setminus G)\times E: (X\setminus G)\times E\to X\setminus G$ is the projection for each $t\in I$,*

iii) *$\Delta^{\alpha}|G\times E\times(0, 1]: G\times E\times(0, 1]\to G\times(0, 1]$ is a homeomorphism, and*

iv) *for each $x\in G$, there is some $U\in\alpha$ such that $\Delta^{\alpha}(\{x\}\times E\times I)\subset U\times I$.*

PROOF. Let $F=E^{\omega}$ or $=E_f^{\omega}$. Since X is E-stable, there is a homeomorphism $h: X\to X\times F$. Then $\Delta^{\alpha}=(h^{-1}\times \mathrm{id}_I)\sigma^{h(\alpha)}(h\times \mathrm{id}_{E\times I}): X\times E\times I\to X\times I$ is a desired map. □

A space X is said to be *strongly E-stable* if for each open cover α of X, there is an I-preserving continuous map $\Delta^{\alpha}: X\times E\times I\to X\times I$ such that $\Delta_0^{\alpha}: X\times E\to X$ is the projection, $\Delta^{\alpha}|X\times E\times(0, 1]: X\times E\times(0, 1]\to X\times(0, 1]$ is a homeomorphism and for each $x\in X$, there is some $U\in\alpha$ such that $\Delta^{\alpha}(\{x\}\times E\times I)\subset U\times I$. As a corollary, we get the Geoghegan-Henderson's result on strong E-stability (Theorem 1 in [12]) whose original proof holds a technically wrong part.

2-3. COROLLARY. *Let $E\cong E^{\omega}$ or $E\cong E_f^{\omega}$ be a perfectly normal LTS and X a perfectly normal space. Then X is E-stable if and only if each open subset of X is strongly E-stable.*

2-4. COROLLARY. *Each open set in a perfectly normal LTS $E\cong E^{\omega}$ or $\cong E_f^{\omega}$ is strongly E-stable.*

The following theorem is an extension of Theorem 4.1 in [9].

2-5. THEOREM. *Let $E\cong E^{\omega}$ or $\cong E_f^{\omega}$ be a perfectly normal LTS and K an E-deficient subset of an E-stable perfectly normal space X. Then for each open cover α of X, there exists an invertible α-isotopy $g_t: X\to X$ $(t\in I)$ such that*

i) $g_0=\mathrm{id}$,

ii) $g_t|K=\mathrm{id}$ *for each* $t\in I$, *and*

iii) $g_t: X\to X$ *is an E-deficient closed embedding for each* $t\in(0, 1]$.

PROOF. Let $F=E^{\omega}$ or $=E_f^{\omega}$. Since K is E-deficient in X, there is a homeomorphism $h: X\to X\times F$ such that $h(K)\subset X\times\{0\}$. Define a closed embedding $i: X\times F\to X\times F\times E$ by $i(x, y)=(x, y, 0)$ for each $(x, y)\in X\times F$. Then $g=(h^{-1}\times \mathrm{id}_I)\sigma^{h(\alpha)}(ih\times \mathrm{id}_I): X\times I\to X\times I$ is a desired isotopy. □

The following result is a generalization of Theorem 3.1 in [9].

2-6. THEOREM. *Let $E\cong E^{\omega}$ (or $\cong E_f^{\omega}$) be an LMS and K a subset of an E-stable space X. Then K is E-deficient if and only if K is l^2-deficient (or l_f^2-deficient).*

PROOF. If $E\cong E^{\omega}$, this is Theorem 3.1 in [9].

If $E\cong E_f^{\omega}$, the Bartle-Graves-Michael's Theorem [17] induces $E\cong E\times \boldsymbol{R}_f^{\omega}$ by the same argument as in the proof of Theorem 3.1 in [9]. Since $\boldsymbol{R}_f^{\omega}\cong l_f^2$, we have $E\cong E\times l_f^2$. This enables us to see that E-deficiency implies l_f^2-deficiency by the argument of the proof of Theorem 3.1 in [9]. For the opposite implication, the following remarks enable us to apply the arguments of the proof of Theorem 3.1 in [9] in the case $E\cong E_f^{\omega}$. The homeomorphism in Lemma 3.1 in [9] is obtained from the homeomorphism $f: E\times[0, 1)\times E^{\omega}\to C[E]\times E^{\omega}$ defined by

$$f(x_0, t, x_1, x_2, \cdots)=(x_n\cos(1-2^n t)\pi+x_{n+1}\sin(1-2^n t)\pi, t, -x_0, \cdots, -x_{n-1},$$
$$-x_n\sin(1-2^n t)\pi+x_{n+1}\cos(1-2^n t)\pi, x_{n+2}, x_{n+3}, \cdots)$$
$$\text{for each } (x_0, t, x_1, x_2, \cdots)\in E\times[2^{-(n+1)}, 2^{-n}]\times E^{\omega}$$

and

$$f(x_0, 0, x_1, x_2, \cdots) = (0, -x_0, -x_1, -x_2, \cdots)$$
$$\text{for each } (x_0, 0, x_1, x_2, \cdots) \in E \times \{0\} \times E^\omega .$$

Then Lemma 3.1 in [**9**] is valid for $E \cong E_f^\omega$ by restricting this homeomorphism. Since (S_1, S_1') in an (l^2, l_f^2)-manifold pair (Definition (4) in [**14**]) where $S_1 = \{x \in l^2 \mid \|x\| = 1\}$ and $S_1' = \{x \in l_f^2 \mid \|x\| = 1\}$ and since $S_1 \cong l^2$ (Klee's result [**16**] III 1.3), we have $S_1' \cong l_f^2$ by Theorem 2 in [**14**]. □

An *invertible isotopy pushing* K *off* X is an invertible isotopy $h_t : X \to X$ $(t \in I)$ such that $h_0 = \mathrm{id}$, $h_1(X) = X \backslash K$ and that h_t is onto for each $t \in [0, 1)$. A subset K of X is *extractible from* X if for each open cover α of K in X, there is an invertible α-isotopy pushing K off X. The following corollary is a generalization of Lemma 3 in [**10**].

2-7. COROLLARY. *Let* $E \cong E^\omega$ *or* $\cong E_f^\omega$ *be an LMS and* X *an E-stable metric space. Then an E-deficient locally closed subset* K *of* X *is extractible from* X.

PROOF. The invertible continuous family of invertible isotopies pushing the origin off l^2 which is defined on pp. 284-286 of [**3**] can be restricted to l_f^2 and we have the invertible continuous family of invertible isotopies pushing the origin off l_f^2. Then the proof of Lemma 3 in [**10**] holds also true for $E \cong E_f^\omega$. □

§ 3. The homeomorphism extension theorem.

In this section, we generalize the HET in [**9**]. For this purpose, we provide the extension results of Lemma 5.1 and 5.2 in [**9**].

Let X be a metric space with a metric d. For an open cover α of X, define a continuous function $e : X \to \boldsymbol{R}^+$ by

$$e(x) = \sup \{s \in \boldsymbol{R}^+ \mid B_s(x) \subset U \text{ for some } U \in \alpha\}$$

where $\boldsymbol{R}^+$ is the positive real half-line and $B_s(x)$ means the open ball with the center x and the radius s. This function is called a majorant *for* α *with respect to* d (see [**10**] 2). If $d(f(y), g(y)) < ef(y)$ for each $y \in Y$, $g : Y \to X$ is α-near to $f : Y \to X$.

The following theorem is an extension of Lemma 5.1 in [**9**]. We prove this by means of the technique in the proof of 2-1 and consequently, we can omit the condition of local convexity in Lemma 5.1 in [**9**].

3-1. THEOREM. *Let* $E \cong E^\omega$ *or* $\cong E_f^\omega$ *be an LMS,* M *an E-stable metric space and let* X *be a space which can be embedded as a closed subset of* E. *If* $f : X \to M$ *is a continuous map such that* $f|A$ *is an E-deficient closed embedding of a closed subset* A *of* X *in* M, *then for each open cover* α *of* M, *there is an* α-*homotopy* $f_t^* : X \to M$ $(t \in I)$ *such that*

i) $f_0^*=f$,

ii) $f_t^*|A=f|A$ *for each* $t\in I$, *and*

iii) $f_1^*: X\to M$ *is an* E*-deficient closed embedding.*

PROOF. Since $E\cong E^\omega$ (or $\cong E_f^\omega$), $E\cong E\times l^2$ (or $\cong E\times l_f^2$) (cf. the proof of 2-6). Since $l^2\times I\cong l^2$ and $l_f^2\times I\cong l_f^2$ by Klee's result [**16**] III 1.3 and Chapman's result [8] 2.12 or Toruńczyk's result [**24**], $E\times I\cong E$. Let $F=E^\omega$ or $=E_f^\omega$, then $F\times I \cong E$.

Let β be a star-refinement of α. By 2-5, there is an invertible β-isotopy $g_t: M\to M$ $(t\in I)$ such that $g_0=\mathrm{id}$, $g_t|f(A)=\mathrm{id}$ for each $t\in I$ and $g_1(M)$ is E-deficient closed in M. Then there is a homeomorphism $h: M\to M\times F\times I$ such that $hg_1(M)\subset M\times\{0\}\times\{0\}$. Let $p: M\times F\times I\to M$ be the projection. Note that $hg_1(x)=(phg_1(x), 0, 0)$ for each $x\in M$. Let $\theta: M\times F\times E\times I\to M\times F\times I$ be the I-preserving continuous map defined in the proof of 2-1, $i: X\to E$ a closed embedding, d_X, d_M and d_E metrics bounded by 1/4 on X, M and E respectively, d the metric on X defined by $d(x, y)=d_X(x, y)+d_M(f(x), f(y))$ and $e: M\times F\times I\to \boldsymbol{R}^+$ a majorant for $h(\beta)$ with respect to the metric d^* on $M\times F\times I$ defined by

$$d^*((x, (y_i), t), (x', (y_i'), t'))=d_M(x, x')+\sum_{i=1}^{\infty}2^{-i}d_E(y_i, y_i')+2^{-1}|t-t'|\,.$$

Define a continuous map $k: X\to\boldsymbol{R}$ by $k(x)=d(x, A)=\inf\{d(x, y)\,|\,y\in A\}$ for each $x\in X$. Then e is bounded by 1 and k is bounded by 1/2 and non-negative. And $k^{-1}(0)=A$.

Define a homotopy $f_t': X\to M$ $(t\in I)$ by

$$f_t'(x)=h^{-1}\theta(phg_1f(x), 0, i(x), tk(x)ehg_1f(x)) \qquad \text{for each } x\in X.$$

It is easy to see that $f_0'=g_1f$ and $f_t'|A=f|A$ for each $t\in I$. Since

$$k(x)ehg_1f(x)\leqq 1/2 \qquad \text{for each } x\in X,$$

it is easy to see that $f_t'(X)$ is E-deficient in M. Although we must show that f_1' is a closed embedding, we may show that f_1' is a closed map as it is obviously a continuous injection. Let $\{x_n\}$ be any sequence in X such that $\{f_1'(x_n)\}$ is convergent in M. Then $\{\theta(phg_1f(x_n), 0, i(x_n), k(x_n)ehg_1f(x_n))\}$ converges some (x, y, t) in $M\times F\times I$. Since θ is I-preserving, $k(x_n)ehg_1f(x_n)$ converges to t.

i) In case of $t\neq 0$: Since $k(x_n)ehg_1f(x_n)\neq 0$ for sufficiently large n, $\{(phg_1f(x_n), 0, i(x_n), k(x_n)ehg_1f(x_n))\}$ convergent to $\theta^{-1}(x, y, t)$. Then $\{i(x_n)\}$ is convergent. Since i is a closed embedding, $\{x_n\}$ is also convergent.

ii) In case of $t=0$: By definitions, it is easily seen that

$$\{d^*(\theta(phg_1f(x_n), 0, i(x_n), k(x_n)ehg_1f(x_n)), hg_1f(x_n))\}$$

converges to 0. Then $\{hg_1f(x_n)\}$ converges to $(x, y, 0)$. Therefore $y=0$. For sufficiently large n, $d^*(hg_1f(x_n), (x, 0, 0))<\frac{1}{3}e(x, 0, 0)$, then there is some $U\in h(\beta)$ such that $B_{e(x,0,0)/3}(hg_1f(x_n))\subset B_{2e(x,0,0)/3}(x, 0, 0)\subset U$. Therefore $ehg_1f(x_n)\geqq \frac{1}{3}e(x, 0, 0)$, so $ehg_1f(x_n)\neq 0$ for sufficiently large n. Therefore $\{k(x_n)\}$ converges to 0. Moreover $\{f(x_n)\}$ converges to $x'=(hg_1)^{-1}(x, 0, 0)$ because hg_1 is a clo ;ed embedding. For each n, there is $x_n'\in A$ such that $d(x_n, x_n')<2k(x_n)$. Since $d_M(f(x_n'), x')\leqq d_M(f(x_n'), f(x_n))+d_M(f(x_n), x')<2k(x_n)+d_M(f(x_n), x')$, $\{f(x_n')\}$ converges to x'. Then $\{x_n'\}$ is convergent in A because $f|A$ is a closed embedding. Since $\{d(x_n, x_n')\}$ converges to 0, $\{x_n\}$ is also convergent.

For each $x\in X$, $d^*(\theta(phg_1f(x), 0, i(x), tk(x)ehg_1f(x)), hg_1f(x))<2tk(x)ehg_1f(x)<ehg_1f(x)$, then there is some $U\in\beta$ such that $\theta(phg_1f(x), 0, i(x), tk(x)ehg_1f(x))$, $hg_1f(x)\in h(U)$, that is, $f_t'(x), g_1f(x)\in U$. Therefore $f_t': X\to M$ $(t\in I)$ is a β-homotopy such that $f_0'=g_1f$, $f_t'|A=f|A$ for each $t\in I$ and $f_1': X\to M$ is an E-deficient closed embedding. The desired α-homotopy $f_t^*: X\to M$ $(t\in I)$ is defined by

$$f_t^*(x)=\begin{cases} g_{2t}f(x) & \text{for } 0\leqq t\leqq\frac{1}{2} \\ f_{2t-1}'(x) & \text{for } \frac{1}{2}\leqq t\leqq 1. \end{cases} \quad \square$$

Next we extend Lemma 5.2 is [**9**]. This can be proved by the same way as [**9**] but we give an alternative proof.

3-2. LEMMA. *Let $E\cong E^\omega$ or $\cong E_f^\omega$ be an LMS, M a connected E-manifold and K an E-deficient closed set in M. Then there exists an open embedding $h: M\to E$ such that $h|K: K\to E$ is an E-deficient closed embedding.*

PROOF. Since $E\cong E^\omega$ or $\cong E_f^\omega$, $E\times\boldsymbol{R}\cong E$. Since K is E-deficient in M, there is a homeomorphism $f: M\to M\times E\times\boldsymbol{R}$ such that $f(K)\in M\times\{0\}\times\{0\}$. By the Henderson's Open Embedding Theorem [**13**], there is an open embedding $g: M\to E$. Define a continuous map $h': M\times E\times\boldsymbol{R}\to E\times E\times\boldsymbol{R}$ by

$$h'(x, y, t)=(g(x), y, t+k(x)) \qquad \text{for each } (x, y, t)\in M\times E\times\boldsymbol{R}$$

where $k: M\to\boldsymbol{R}$ is a continuous map defined by $k(x)=d(g(x), E\backslash g(M))^{-1}$ (d is a metric on E). Then $h'(M\times E\times\boldsymbol{R})=g(M)\times E\times\boldsymbol{R}$ is open in $E\times E\times\boldsymbol{R}$ and $h'f(K)\subset h'(M\times\{0\}\times\{0\})\subset E\times\{0\}\times\boldsymbol{R}$. It is easy to see that h' is an embedding and that $h'(M\times\{0\}\times\{0\})$ is closed in $E\times\{0\}\times\boldsymbol{R}$. Let $f': E\times E\times\boldsymbol{R}\to E$ be a homeomorphism. Then $h=f'h'f$ is a desired open embedding. $\square$

HET: *Let $E\cong E^\omega$ or $\cong E_f^\omega$ be an AR LMS, M an E-manifold and K an E-deficient closed set in M. If α is an open cover of M and if $h_t: K\to M$ $(t\in I)$ is an α-homotopy such that $h_0=\mathrm{id}$ and h_1 is an E-deficient closed embedding, then for each open cover β, there exists an ambient invertible* $\mathrm{st}(\alpha;\beta)$*-isotopy $\bar{h}_t: M\to M$ $(t\in I)$ such that $\bar{h}_0=\mathrm{id}$ and $\bar{h}_1|K=h_1$.*

PROOF. Now, this theorem can be proved by the same argument as the proof of Theorem 2 in [9] since Lemma 5.1 and 5.2 in [9] have been extended. But to avoid using the unpublished result of D. W. Henderson*) which is used in the proof of Theorem 2 in [9] and to improve the limitation by covers, we show that the case $K\cap h_1(K)=\emptyset$ induces the general case. Note that in the case $K\cap h_1(K)=\emptyset$ we obtain our limitation in the proof of Theorem 4.2 in [4] with obvious modifications.

Let γ be a star-refinement of β. By Theorem 2.1 in [11] which is also valid for $E\cong E_f^\omega$ with suitable modifications, $K\cup h_1(K)$ is also E-deficient. Then there is a homeomorphism $k: M\to M\times \boldsymbol{R}$ such that $k(K\cup h_1(K))\subset M\times\{0\}$. Let $e: M\times\boldsymbol{R}\to\boldsymbol{R}^+$ be a majorant for $k(\gamma)$ with respect to the product metric d. Define an ambient invertible isotopy $f_t: M\times\boldsymbol{R}\to M\times\boldsymbol{R}$ $(t\in I)$ by

$$f_t(x, s)=\begin{cases} \left(x, \left(1+\frac{t}{2}\right)s+\frac{t}{4}e(x, 0)\right) & \text{if } -\frac{1}{2}e(x, 0)\leqq s\leqq 0, \\ \left(x, \left(1-\frac{t}{2}\right)s+\frac{t}{4}e(x, 0)\right) & \text{if } 0\leqq s\leqq \frac{1}{2}e(x, 0), \\ (x, s) & \text{otherwise.} \end{cases}$$

If $d((x, s), (x, 0))<\frac{1}{2}e(x, 0)$ then $d(f_t(x, s), (x, 0))<\frac{1}{2}e(x, 0)$ and if $d((x,s), (x, 0))\geqq\frac{1}{2}e(x, 0)$ then $f_t(x, s)=(x, s)$. Thus f is a $k(\gamma)$-isotopy. Then $k^{-1}f_tk: M\to M$ $(t\in I)$ is an ambient invertible γ-isotopy such that $k^{-1}f_0k=\mathrm{id}$ and

$$k^{-1}f_1k(K\cup h_1(K))\cap(K\cup h_1(K))=\emptyset .$$

Since $k^{-1}f_tkh_t: K\to M$ $(t\in I)$ is a $\mathrm{st}(\alpha;\gamma)$-homotopy such that $k^{-1}f_0kh_0=\mathrm{id}$ and $k^{-1}f_1kh_1$ is an E-deficient closed embedding and $K\cap k^{-1}f_1kh_1(K)=\emptyset$, there exists an ambient invertible $\mathrm{st}(\mathrm{st}(\alpha;\gamma);\gamma)$-isotopy $g_t: M\to M$ $(t\in I)$ such that $g_0=\mathrm{id}$ and $g_1|K=k^{-1}f_1kh_1$. Then $k^{-1}f_t^{-1}kg_t$ fulfills our requirements. □

§4. The relative stability theorem and the relative approximation theorem by embeddings.

In this section, we establish the relative ST (Stability Theorem) which will hold a basic part in our Embedding Theorems in Sect. 6 and we generalize the result of [18] which is called the relative ATE (Approximation Theorem by Embeddings).

A characterization of Anderson's Z-sets in Q or s $(\cong\boldsymbol{R}^\omega)$ by H. Toruńczyk

*) Recently, this appeared in Trans. Amer. Math. Soc., 213 (1975), 205-217. But its proof is complicated.

in [**22**] can be generalized to one of Z-sets in any paracompact Hausdorff space X which is locally homotopically trivial, i.e., admitting a fundamental neighbourhood system consisting of homotopically trivial open sets: *K is a Z-set in X if and only if K is a closed set in X such that for each continuous map $f: I^n \to X$ and for each open cover α of X, there exists a continuous map $g: I^n \to X \setminus K$ which is α-near to f. Thus if $E \cong E^\omega$ or $\cong E_f^\omega$ is an LMS, then E-deficient closed sets in an E-stable locally homotopically trivial metric space are Z-sets* by 2-7. Conversely, T. A. Chapman showed in [**9**] that Z-sets in E-manifold are E-deficient in case that $E \cong E^\omega$ is a Fréchet space. The author does not know whether it is true in more general case that $E \cong E^\omega$ or $\cong E_f^\omega$ is an *(LC) LMS.* But H. Toruńczyk showed that *closed submanifolds of an E-manifold which are Z-sets are E-deficient in case that $E \cong E^\omega$ or $\cong E_f^\omega$ is an LCLMS* (Lemma 6.2 in [**24**]).

We require the following lemma.

4-1. LEMMA. *Let $E \cong E^\omega$ or $\cong E_f^\omega$ be an LCLMS, M be an E-manifold and let K be an E-deficient closed set in M. Then for each open cover α of M, there exists a homeomorphism $h: M \times E \to M$ such that $h(x, 0) = x$ for each $x \in K$ and h is α-near to the projection $p: M \times E \to M$.*

PROOF. Note that Theorem 2.2 in [**11**] is also valid for $E \cong E_f^\omega$ with suitable modifications. This lemma is derived from this theorem, the ST (with 2-3) and the HET as all the same as Lemma 2.1 in [**18**]. □

4-2. PROPORITION. *Let $E \cong E^\omega$ or $\cong E_f^\omega$ be an LCLMS, M_1 and M_2 be E-manifolds, M_1 be connected (more generally, M_1 can be embedded as a closed set in E) and let $f: M_1 \to M_2$ be a continuous map. Then for each open cover α of M_2 and for each E-deficient closed set K in M_1, there exist an open embedding $g: M_1 \to M_2$ and a closed embedding $h: M \to M$ such that g, h are α-homotopic to f and $g(K)$, $h(K)$ are E-deficient closed sets in M_2.*

4-3. COROLLARY. *Let E, M_1, M_2 be as above and let $f: M_1 \to M_2$ be a continuous map such that for some E-deficient closed set K in M_1, $f|K$ is an E-deficient closed embedding. Then for each open cover α of M_2 there exist an open embedding $g: M_1 \to M_2$ and a closed embedding $h: M_1 \to M_2$ such that g, h are α-homotopic to f and $f|K = g|K = h|K$.*

The proofs of the above proposition and corollary are all the same as Proposition 2.2 and Corollary 2.5 in [**18**]. And the following ATE is derived from 4-2, the HET and 2-7 by the same way as the proof of Theorem 2.6 (2.6′) in [**18**].

RELATIVE ATE: *Let $E \cong E^\omega$ or $\cong E_f^\omega$ be an LCLMS, (M_i, N_i) be an E-manifold pair with N_i a Z-set in M_i $(i=1, 2)$, M_1 be connected (more generally, M_1 can be embedded as a closed set in E) and let $f: (M_1, N_1) \to (M_2, N_2)$ be continuous. Then for each open cover α of M_2, there exist an open embedding $g: (M_1, N_1) \to (M_2, N_2)$ and a closed embedding $h: (M_1, N_1) \to (M_2, N_2)$ such that g,* *h are α-homotopic to $f:(M_1, N_1)\to(M_2, N_2)$. Furthermore $g(M_1)\cap N_2=g(N_1)$, that is, $g:(M_1, N_1, M_1\backslash N_1)\to(M_2, N_2, M_2\backslash N_2)$.*

4-4. COROLLARY. *Let $E\cong E^{\omega}$ or $\cong E_f^{\omega}$ be an LCLMS and let (M, N) be an E-manifold pair with N a Z-set in M such that M is connected (more generally, M can be embedded as a closed set in E). Then*

i) *there exists an open embedding $i: M\to E\times E$ such that $i(N)=i(M)\cap E\times\{0\}$,*

ii) *there exists an open embedding $j: M\to E\times[0, 1)$ such that $j(N)=j(M)\cap E\times\{0\}$.*

From 4-4, it is derived that *in an E-monifold pair (M, N), N is a Z-set in M if and only if N is a collared closed set in M* (see the foot note (2) in [**20**]). But this result is directly derived from Theorem 1 in [**7**] and Proposition 5.1 in [**11**].

Now we establish the relative ST:

RELATIVE ST: *Let $E\cong E^{\omega}$ or $\cong E_f^{\omega}$ be an LCLMS and let (M, N) be an E-manifold pair with N a Z-set in M. Then for each open cover α of M, there exists a homeomorphism $h:(M\times E, N\times E)\to(M, N)$ which is α-homotopic to the projection $p:(M\times E, N\times E)\to(M, N)$.*

PROOF. Let β be an open cover such that $\mathrm{st}^2(\beta)$ refines α and $i: M\to M\times E$ be defined by $i(x)=(x, 0)$ for each $x\in M$. By 4-1, there is a homeomorphism $f: M\times E\to M$ such that $fi|N=\mathrm{id}$ and f is β-near to p. And there is a homeomorphism $g: N\times E\to N$ which is β-near to p by the Schori's Stability Theorem [**21**] and 2-3. Since $ig: N\times E\to M\times E$ is an E-deficient closed embedding which is $(\beta\times E)$-near to id and since $N\times E$ is an E-deficient closed set in $M\times E$, there exists a homeomorphism $h: M\times E\to M\times E$ such that $h|N\times E=ig$ and h is $(\mathrm{st}(\beta)\times E)$-near to id. Thus $fh: M\times E\to M$ is a homeomorphism such that $fh|N\times E=fig=g$. Since ph, $p: M\times E\to M$ are $\mathrm{st}(\beta)$-near and since fh, ph: $M\times E\to M$ are β-near, then fh is $\mathrm{st}^2(\beta)$-near (then, α-near) to $p: M\times E\to M$. As the modification of 2.6 to 2.6′ in [**18**], we obtain the theorem by 4-4. □

§ 5. Embedding theorem with compliment conditions.

In this section, we consider the conditions for an E-manifold pair (M, N) with N a Z-set in M under which M can be embedded in E such that N is the topological boundary of M and the closure (or each component of the closure) of the complement of M is contractible.

By a *cone* over a metric space X, we mean the topological space

$$(C(X), \tau)=(\{0_X\}\cup X\times(0, 1], \tau)$$

where τ is the topology generated by open sets in $X\times(0, 1]$ and sets $\{0_X\}\cup X\times(0, t)$ $(0<t<1)$. Let d be a bounded metric on X. By the Arens-Eelles' Theorem [**5**] (for a shorter proof, see [**23**]) X has an isometric closed copy

X' in some ball B of some normed linear space E. It is easy to see that the natural map of $C(X)$ onto $\{(tx, t)\in E\times\boldsymbol{R}\,|\,t\in[0, 1], x\in X'\}$ is a homeomorphism. Then these may be identified. Moreover, note that $C(X)\cong\{(tx+(1-t)x_0, t)\in E\times\boldsymbol{R}\,|\,t\in[0, 1], x\in X'\}$ where x_0 is any given point of E. If X is an *ANR*, then $C(X)$ is a neighbourhood retract of $C(B)$. Since

$$C(B)\cong\{(tx, t)\in B\times[0, 1]\,|\,x\in B,\ t\in[0, 1]\}$$

is a retract of an *AR* $B\times[0, 1]$, $C(X)$ is an *ANR*. Hence $C(X)$ is an *AR* because it is contractible. (This may be derived from Lemma 4.1 in [**24**] as in the proof of Theorem 4.2 in [**24**].)

By a *mapping cylinder* of a continuous map $f: X\rightarrow Y$ of a metric space X to a metric space Y, we mean the topological space

$$(M(f), \tau)=(Y\times\{0\}\cup X\times(0, 1], \tau)$$

where τ is the topology generated by open sets in $X\times(0, 1]$ and sets

$$V\times\{0\}\cup f^{-1}(V)\times(0, t),$$

V is open in Y and $0<t<1$. By the Arens-Eelles' Theorem, X and Y have homeomorphic bounded closed copies X' and Y' in some normed linear spaces E and F, respectively. Let $f': X'\rightarrow Y'$ be induced from $f: X\rightarrow Y$. Then it is easy to see that the natural map $M(f)$ to

$$\{(tx, (1-t)f'(x), t)\in E\times F\times\boldsymbol{R}\,|\,t\in[0, 1], x\in X'\}\cup\{(0, y, 0)\in E\times F\times\boldsymbol{R}\,|\,y\in Y'\}$$

is a homeomorphism. When Y is a one-point space and $f: X\rightarrow Y$ is constant, $M(f)$ is a cone $C(X)$ over X.

The following lemmas are useful in the proofs of our Embedding Theorems.

5-1. LEMMA. *Let X be an ANR and Y a contractible closed subset of X. Then the quotient map $q:(X, Y)\rightarrow(X/Y, X/Y)$ is a homotopy equivalence.*

PROOF. Since X has the homotopy extension property for (X, Y), there is a homotopy $f_t:(X, Y)\rightarrow(X, Y)$ such that $f_0=\mathrm{id}$ and $f_1|Y$ is a contraction of Y. This homotopy f_t induces a homotopy $\bar{f}_t:(X/Y, Y/Y)\rightarrow(X/Y, Y/Y)$ such that $\bar{f}_tq=qf_t$. And f_1 induces a continuous map $g:(X/Y, Y/Y)\rightarrow(X, Y)$ such that $gq=f_1$ because $f_1(Y)$ is single point. Then $gq=f_1\sim f_0=\mathrm{id}:(X, Y)\rightarrow(X, Y)$ and $qg=\bar{f}_1\sim\bar{f}_0=\mathrm{id}:(X/Y, Y/Y)$. □

5-2. LEMMA. *Let $E\cong E^{\omega}$ or $\cong E_f^{\omega}$ be an LCLMS. If M, X and $M\cap X$ are E-manifolds and if $M\cap X$ is a Z-set in each of M and X, then $M\cup X$ is also an E-manifold.*

PROOF. Since $M\cap X$ is bicollared in $M\cup X$, and since $E\times\boldsymbol{R}\cong E$, the proof is easy. □

5-3. THEOREM. *Let $E\cong E^{\omega}$ or $\cong E_f^{\omega}$ be an LCLMS and let (M, N) be an E-manifold pair with N a Z-set in M such that* $\mathrm{dens}\, M=\mathrm{dens}\, E$. *Then there exists an embedding $h: M\rightarrow E$ such that $\mathrm{bd}_E h(M)=h(N)$ and $\mathrm{cl}_E(E\backslash h(M))$ is contractible if and only if M/N is contractible.*

PROOF. First, assume $M\subset E$, $\mathrm{bd}_E M=N$ and $\mathrm{cl}_E(E\backslash M)$ is contractible. Since $N=\mathrm{bd}_E(E\backslash M)$ is collared in M, $\mathrm{cl}_E(E\backslash M)$ is a neighbourhood retract of E, that is, an ANR. Hence $\mathrm{cl}_E(E\backslash M)$ is an AR. It is straightforward to see that $\mathrm{cl}_E(E\backslash M)$ is a strong deformation retract of E. Therefore $M/N\cong E/\mathrm{cl}_E(E\backslash M)$ is contractible.

Next, assume M/N is contractible. By the Triangulation Theorem (Theorem 3.4 (a) in [**25**]), there is a homeomorphism $h: N\rightarrow|K|\times E$ where K is some locally finite-dimensional simplicial complex. By the Toruńczyk's result (Theorem 3.1 in [**25**]), $C(|K|)\times E\cong E$. Since $M(ph)\cong C(|K|)\times E$ where $p:|K|\times E\rightarrow E$ is the projection, and since $M\times\{1\}\cap M(ph)=N\times\{1\}$ is a Z-set in each of $M\times\{1\}$ and $M(ph)$, $F=M\times\{1\}\cup M(ph)$ is an E-manifold by 5-2. By 5-1, $(F, M(ph))$ is homotopic to $(F/M(ph), M(ph)/M(ph))\cong(M/N, N/N)$, therefore F is contractible. By the Classification Theorem in [**13**], $F\cong E$. And then $\mathrm{bd}_F M\times\{1\}=N\times\{1\}$ and $\mathrm{cl}_F(F\backslash M\times\{1\})=M(ph)\cong E$. This completes the proof. □

The above proof contains the alternative shorter proof of Case II-i) of Theorem in [**19**]. Although the following corollary is directly proved in a general case, to use it in 5-5, we give its proof.

5-4. COROLLARY. *Let (M, N) be as 5-3. If M/N is contractible, the inclusion $i: N\rightarrow M$ induces an isomorphism $i_*: H_*(N)\rightarrow H_*(M)$.*

PROOF. It is well known that $H_*(M, N)\cong H_*(M\times\{1\}\cup C(N), C(N))$. By above proof, $M\times\{1\}\cup C(N)$ and $C(N)$ are contractible. Then $H_*(M, N)=0$. □

Let $\{X_n\}$ be a (finite or infinite) sequence of subsets of M. A sequence $\{L_n\}$ of paths in M is a *chain of paths in M connecting* $\{X_n\}$ provided $\{L_n\}$ is pair-wise disjoint (i.e., $L_n\cap L_{n'}=\emptyset$ if $n\neq n'$) and each L_n intersects only two members X_n and X_{n+1} at its end-points.

If M is a connected E-manifold and $\cup\{X_n\}$ and each X_n are E-deficient closed subsets of M, by 2-7, we can inductively show the existence of a chain of paths in M connecting $\{X_n\}$.

5-5. THEOREM. *Let $E\cong E^\omega$ or $\cong E_f^\omega$ be an LCLMS and let (M, N) be an E-manifold pair with N a Z-set in M such that M is connected and N has at most countable many components. Then following conditions are equivalent:*

i) *There is an embedding $h: M\rightarrow E$ such that $h(N)=\mathrm{bd}_E h(M)$ and each component of $\mathrm{cl}_E(E\backslash h(M))$ is contractible.*

ii) *For any ordering $\{N_n\}$ of all components of N and for any chain of paths $\{L_n\}$ in M connecting $\{N_n\}$, $M/N\cup\bigcup\{L_n\}$ is contractible.*

iii) *There are some ordering $\{N_n\}$ of all components of N and some chain of paths $\{L_n\}$ in M connecting $\{N_n\}$ such that $M/N\cup\bigcup\{L_n\}$ is contractible.*

PROOF. ii)⇒iii) is trivial.

i)⇒ii): Assume $M\subset E$, $N=\mathrm{bd}_E M$ and each component of $\mathrm{cl}_E(E\backslash M)$ is contractible. By the same argument as the proof of 5-3, $\mathrm{cl}_E(E\backslash M)$ is an ANR.

Because M is collectionwise normal, $\mathrm{cl}_E(E\backslash M)\cap\bigcup\{L_n\}=N\cap\bigcup\{L_n\}$ is totally disconnected, then it is an ANR. Hence $\mathrm{cl}_E(E\backslash M)\cup\bigcup\{L_n\}$ is also an ANR.

Let D be a component of $\mathrm{cl}_E(E\backslash M)$. Then D is a contractible ANR, that is, an AR. Since M is connected, so is $\mathrm{cl}_E(E\backslash D)$. Because $\mathrm{bd}_E D=N\cap D$ is open and closed in N, $\mathrm{bd}_E D$ is a collared closed submanifold of M (see 4-4). Since M is a neighbourhood of $\mathrm{bd}_E D$ in $\mathrm{cl}_E(E\backslash D)$, $(\mathrm{cl}_E(E\backslash D), \mathrm{bd}_E D)$ is an E-manifold pair with $\mathrm{bd}_E D$ a Z-set in $\mathrm{cl}_E(E\backslash D)$. By 5-3 and 5-4,

$$H_0(\mathrm{bd}_E D)=H_0(\mathrm{cl}_E(E\backslash D))=0\,.$$

Thus $\mathrm{bd}_E D$ is a connected open and closed subset of N. Hence $\mathrm{bd}_E D$ is a component of N. And so, let $\{D_n\}$ be a sequence of all components of $\mathrm{cl}_E(E\backslash M)$ such that $\mathrm{bd}_E D_n=N_n$. Since each D_n is an AR, it is easily shown that $\mathrm{cl}_E(E\backslash M)\cup\bigcup\{L_n\}=\bigcup\{D_n\}\cup\bigcup\{L_n\}$ deforms to a path or a half open path in inself. Then $\mathrm{cl}_E(E\backslash M)\cup\bigcup\{L_n\}$ is a contractible ANR, that is, an AR. Again by the same argument in the proof of 5-3, $M/N\cup\bigcup\{L_n\}=E/\mathrm{cl}_E(E\backslash D)\cup\bigcup\{L_n\}$ is contractible.

iii)$\Rightarrow$i): By the Triangulation Theorem (3.4 (a) in [**25**]), there are homeomorphisms $h_n: N_n\to|K_n|\times E$ where K_n's are some locally finite-dimensional simplicial complexes. Similarly as the proof of 5-3, $M(p_nh_n)\cong E$ where p_n: $|K_n|\times E\to\{n\}\times E$ is the projection. Since $\bigcup\{M(p_nh_n)\}$ is an E-manifold and since $M\times\{1\}\cap\bigcup\{M(p_nh_n)\}=\bigcup\{N_n\times\{1\}\}=N\times\{1\}$ is a Z-set in each of $M\times\{1\}$ and $\bigcup\{M(p_nh_n)\}$, $F=M\times\{1\}\cup\bigcup\{M(p_nh_n)\}$ is also an E-manifold by 5-2.

Let

$$I_n=\{(n+t,\,(1-t)p_nh_n(a_n)+tp_nh_n(b_n))\,|\,0\leqq t\leqq 1\}\subset \boldsymbol{R}\times E \text{ and}$$

$$J_{n+1}=\{(n+1,\,(1-t)p_{n+1}h_{n+1}(b_n)+tp_{n+1}h_{n+1}(a_{n+1}))\,|\,0\leqq t\leqq 1\}\subset\{n+1\}\times E$$

where $a_n\in N_n$ and $b_n\in N_{n+1}$ are the end-points of L_n. Take a continuous map $f: N\cup\bigcup\{L_n\}\to\bigcup\{\{n\}\times E\}\cup\bigcup\{I_n\}$ such that $f|L_n: L_n\to I_n$ is a homeomorphism and $f|N_n=p_nh_n$ for each n. Since $F\cap\bigcup\{M(f|L_n)\}$ is an ANR because it is homeomorphic to a disjoint union of intervals and since F and $\bigcup\{M(f|L_n)\}$ are ANR's,

$$M\times\{1\}\cup M(f)=M\times\{1\}\cup\bigcup\{M(f|N_n)\}\cup\bigcup\{M(f|L_n)\}=F\cup\bigcup\{M(f|L_n)\}$$

is also an ANR. And it is easy to see that $M\times\{1\}\cup M(f)=F\cup\bigcup\{M(f|L_n)\}$ collapses to F, hence homotopic to F. Note that each J_n is a strong deformation retract of $\{n\}\times E$ because J_n and $\{n\}\times E$ are AR's. Since one can deform $M(f)$ to $\bigcup\{\{n\}\times E\}\cup\bigcup\{I_n\}$, and to $\bigcup\{J_n\}\cup\bigcup\{I_n\}$, $M(f)$ is contractible. By 5-1, $(M\times\{1\}\cup M(f), M(f))$ is homotopic to $(M\times\{1\}\cup M(f)/M(f), M(f)/M(f))\cong(M/N\cup\bigcup\{L_n\}, N\cup\bigcup\{L_n\}/N\cup\bigcup\{L_n\})$. Then $M\times\{1\}\cup M(f)$ is contractible, therefore F is a contractible E-manifold, that is, homeomorphic to E by the Classification Theorem in [**13**]. And we have $\mathrm{bd}_F M\times\{1\}=\bigcup\{N_n\times\{1\}\}=N\times\{1\}$

and $\mathrm{cl}_F(F\backslash M\times\{1\})=\bigcup\{M(p_nh_n)\}\cong\bigcup\{C(|K_n|)\times E\}$. This complete the proof. □

5-6. PROPOSITION. *Let $E\cong E^\omega$ or $\cong E_f^\omega$ be an LCLMS and let (M, N) be an E-manifold pair with N a Z-set in M. If there is an embedding $h: M\to E$ such that $\mathrm{bd}_E h(M)=h(N)$ then there is an embedding $h': M\to E$ such that $\mathrm{bd}_E h'(M)=h'(N)$ is bicollared in E and $\mathrm{cl}_E(E\backslash h'(M))(\cong E\backslash h'(M)$ has the same homotopy type as $\mathrm{cl}_E(E\backslash h(M))$.*

PROOF. Since N is collared in M (see 4-4), there is an open embedding $g: N\times[0, 1)\to M$ such that $g(x, 0)=x$ for each $x\in N$. Let $h': M\to E$ be defined by

$$h'(x)=\begin{cases} h(x) & \text{for } x\in M\backslash g(N\times[0, 2/3)) \\ hg(\mathrm{id}\times k)(x) & \text{for } x\in g(N\times[0, 2/3]) \end{cases}$$

where $k: I\to I$ be defined by $k(s)=(1/2)s+(1/3)$ for $s\in I$.

It is clear that

$$\mathrm{bd}_E h'(M)=h'(N)=hg(N\times\{1/3\})$$

is bicollared in E and that $\mathrm{cl}_E(E\backslash h(M))=E\backslash h(M\backslash g(N\times\{0\}))$ is a deformation retract of E-manifolds $\mathrm{cl}_E(E\backslash h'(M))=E\backslash h(M\backslash g(N\times[0, 1/3]))$ and $E\backslash h'(M)=E\backslash h(M\backslash g(N\times[0, 1/3)))$. By the Classification Theorem [**13**], $\mathrm{cl}_E(E\backslash h'(M))\cong E\backslash h'(M)$. □

REMARK. In 5-3, the condition that $\mathrm{cl}_E(E\backslash h(M))$ is contractible may be changed for the condition $\mathrm{cl}_E(E\backslash h(M))\cong E\backslash h(M)\cong E$ by the above proposition. Similarly, in 5-5, the condition that each component of $\mathrm{cl}_F(E\backslash h(M))$ is contractible may be changed for the condition that each component of $\mathrm{cl}_E(E\backslash h(M))\cong E\backslash h(M)$ is homeomorphic to E.

§ 6. Another embedding theorem.

In this section, we extend the result of [**20**]. Although we obtain a sufficient condition in the following theorem for our embedding problem, we seem to be away from a necessary and sufficient condition at the observation of Example 4 in the next section.

6-1. THEOREM. *Let $E\cong E^\omega$ or $\cong E_f^\omega$ and let (M, N) be an E-manifold pair with N a Z-set in M such that M is connected. If there exists a chain of paths $\{L_n\}$ in M connecting some (finite or infinite) sequence $\{N_n\}$ consisting of some components of N such that $N_n\neq N_{n'}$ if $n\neq n'$ and $N\cup\bigcup\{L_n\}$ contains a deformation retract of M, then there exists an embedding $h: M\to E$ such that $\mathrm{bd}_E h(M)=h(N)$.*

PROOF. Let $M'\subset N\cup L$ (where $L=\bigcup\{L_n\}$) be a deformation retract of M then a strong deformation retract of M ([**15**] Ch. VII Theorem 2.1). Since M' is connected because so is M, we may assume that $L\subset M'$ and $M'\cap N_n\neq\emptyset$

for each N_n. By the Triangulation Theorem (3.4 (a) in [**25**]), there is a homeomorphism $h: N \to |K| \times E$ where K is some locally finite-dimensional simplicial complex. Let K_n be the subcomplex of K such that $|K_n| \times E = h(N_n)$ and let $a_n \in N_n$ and $b_n \in N_{n+1}$ be the end-points of L_n. Then

$$|K'_n| = |K_n| \times \{(1-t)ph(b_{n-1}) + tph(a_n) \mid 0 \leqq t \leqq 1\} \subset |K_n| \times E$$

is a strong deformation retract of $|K_n| \times E$ where $p: |K| \times E \to E$ is the projection. Then deforming $N \cup L$ to a complete ANR $N' \cup L$ where N' is an E-deficient set (i.e., a Z-set) in N, we may also assume that M' is a complete ANR. Since $M' \times \{1\} \cap \cup \{C(N'_n \cap M')\} = \cup \{(N'_n \cap M') \times \{1\}\}$ is an ANR where $N'_n = N' \cap N_n$, $M' \times \{1\} \cup \cup \{C(N'_n \cap M')\}$ is also an ANR. By the same argument as the proof of iii)$\Rightarrow$i) in 5-5, it can be shown that $M' \times \{1\} \cup \cup \{C(N'_n \cap M')\}$ is contractible, then an AR.

By the Toruńczyk's result (Theorem 3.1 in [**25**]),

$$F' = (M' \times \{1\} \cup \cup \{C(N'_n \cap M)\}) \times E \cong E.$$

Since $N \times \{1\} \times E \cap \cup \{C(N'_n \cap M) \times E\} = \cup \{(N'_n \cap M) \times \{1\} \times E\} \subset N' \times \{1\} \times E$ is a Z-set in each $N \times \{1\} \times E$ and $\cup \{C(N'_n \cap M) \times E\}$,

$$N \times \{1\} \times E \cup F' = N \times \{1\} \times E \cup \cup \{C(N'_n \cap M) \times E\}$$

is an E-manifold by 5-2. Since

$$M \times \{1\} \times E \times \{0\} \cap (N \times \{1\} \times E \cup F') \times [0, 1] = N \times \{1\} \times E \times \{0\}$$

is a Z-set in each $M \times \{1\} \times E \times \{0\}$ and $(N \times \{1\} \times E \cup F') \times [0, 1]$,

$$F = M \times \{1\} \times E \times \{0\} \cup (N \times \{1\} \times E \cup F') \times [0, 1]$$

is also an E-manifold. Since F has the same homotopy type as

$$M \times \{1\} \times E \cup N \times \{1\} \times E \cup F' = M \times \{1\} \times E \cup F'$$

and since $M' \times \{1\} \times E = M \times \{1\} \times E \cap F'$ is a strong deformation retract of $M \times \{1\} \times E$, F has the same homotopy type as $F' \cong E$, hence $F \cong E$ by the Classification Theorem [**13**]. Since $\mathrm{bd}_F M \times \{1\} \times E \times \{0\} = N \times \{1\} \times E \times \{0\}$, it is easy to construct a desired embedding by the relative ST. □

§7. Examples.

First, we give two examples, seeing the relation between the topological boundary of an E-manifold embedded in E as a closed set and its collared submanifold. (First example is suggested by Prof. Y. Kodama.)

EXAMPLE 1. Let $B_0 = S_0 = \{0\}$, $B_n = \{(x, y) \in \boldsymbol{R}^2 \mid ((x-(2/3^n))^2 + y^2 \leqq (1/3^n)^2\}$, $S_n = \mathrm{bd}_{\boldsymbol{R}^2} B_n = \{(x, y) \mid (x-(2/3^n))^2 + y^2 = (1/3^n)^2\}$ for each $n > 0$ and let $M = \cup \{B_n\} \times E \subset \boldsymbol{R}^2 \times E \cong E$. Then M is an E-manifold because $\cup \{B_n\}$ is a finite-dimensional compact AR. But $\mathrm{bd}_{\boldsymbol{R}^2\times E}M=(\bigcup_n S_n)\times E$ is not an E-manifold because $\bigcup_n S_n$ is not an ANR.

EXAMPLE 2. Let $B=\{(x, y, 0)\in\boldsymbol{R}^3 \mid x^2+y^2\leqq 1\}$ and let $I=\{(0, 0, z)\in\boldsymbol{R}^3 \mid 0\leqq z\leqq 1\}$. Then $F=(B\cup I)\times E$, $M=B\times E$ and $\mathrm{bd}_F M=\{0\}\times E$ are homeomorphic to E because $B\cup I$, B and $\{0\}$ are finite-dimensional compact AR's. And $\mathrm{bd}_F M$ is not a Z-set in M, that is, not collared in M.

By Example 1, we see that the topological boundary of an E-manifold embedded in E as a closed set is not generally an E-manifold. By Example 2, we see that the submanifold being the topological boundary of an E-manifold embedded in E as a closed set is not generally a collared submanifold. These show the difference of infinite-dimensional manifolds and finite-dimensional manifolds.

In relation to Theorem 5-3, the following question rises: *In an E-manifold pair (M, N) with N a Z-set in M, the condition that M/N has the homotopy type of S^n is the necessary and sufficient condition under which M can be embedded in E such that N is the topological boundary of M and such that the closure of the complement of M has the homotopy type of S^n, isn't it?* The answer of this problem is "*NO*!" in case of $n=1$. This problem was raised by T. Watanabe when the author had a chat with him. The following example was obtained then.

EXAMPLE 3. Let $(M, N)=(I\times E, \{0, 1\}\times E)$. Then (M, N) is an E-manifold with N a Z-set in M such that M/N is homotopic to S^1. When M is embedded in E with N being the topological boundary, $\mathrm{cl}_E(E\setminus M)$ is homotopic to $S^0=\{0, 1\}$. In fact, assume that (M, N) is embedded in such a way. By 5-6, we may also assume that N is bicollared in E. As same as the example in [**20**], it is easy to see that (E, M) is homotopic to $(I\cup X, I)$ where X is some space such that $I\cap X=\{0, 1\}$. Then $\mathrm{cl}_E(E\setminus M)$ has the same homotopy type as a one point union of two spaces A and B. Since $A\cup B$ contract to $\{p\}=A\cap B$, we can obtain a contraction of A to $\{p\}$ using a contraction of $A\cup B$ to $\{p\}$ and a retraction of B to $\{p\}$. Thus A and B are contractible. Therefore $\mathrm{cl}_E(E\setminus M)$ is homotopic to S^0.

We leave the question: *Under what condition can M be embedded in E such that N is the topological boundary under the embedding and such that the closure of the complement of M in E has the homotopy type of S^n?*

The last example shows that the condition in 6-1 is not necessary.

EXAMPLE 4. Let T be a solid torus in the unit-ball B in $\boldsymbol{R}^3$. Then $\mathrm{cl}_B(B\setminus T)$ and $\mathrm{bd}_B T$ are finite-dimensional compact ANR's. We have an E-manifold pair $(M, N)=(\mathrm{cl}_B(B\setminus T)\times E, (\mathrm{bd}_B T)\times E)$ with N a Z-set in M which does not satisfy the condition in 6-1. But $\mathrm{bd}_{B\times E}M=N$ and $B\times E\cong E$.

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