J. Math. Soc. Japan
Vol. 29, No. 2, 1977

The asymptotic distribution of discrete eigenvalues for Schrödinger operators

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(Received June 17, 1975)
(Revised Aug. 9, 1976)

§ 0. Introduction.

We consider the following eigenvalue problem:

$$-\Delta u-p(x)u=\lambda u^{0)}, \qquad u\in L^2(R^n).$$

If $p(x)$ does not decay too rapidly at infinity, the above problem has an infinite sequence of negative eigenvalues (bound states) approaching to zero. We denote by $n(r;p)$ $(r>0)$ the number of eigenvalues less than $-r$. In this paper we are concerned with the asymptotic behavior of $n(r;p)$ as r tends to zero.

The asymptotic distribution of negative eigenvalues for the Schödinger operators has been studied in Brownell and Clark [**4**] and McLeod [**6**] under the condition that the potential $p(x)$ is non-negative and sufficiently close to a spherically symmetric potential.

The purpose of the present paper is to study the distribution of eigenvalues by a different method from those in [**4**] and [**6**] without assuming the above condition.

we shall briefly explain our approach. Our method is based on the following proposition (see Birman [**1**]).

PROPOSITION 0.1. *Let H_0 be the unique self-adjoint realization of $-\Delta$ with domain $\mathcal{D}(H_0)=H^2(R^n)$ (the Sobolev space of order* 2). *Assume that $|p|^{1/2}$ is a $H_0^{1/2}$-compact operator as a multiplicative operator. Let $\mathcal{D}$ be a core of $H_0^{1/2}$. Then, $n(r;p)$ coincides with the maximal dimension of subspaces lying in $\mathcal{D}$ such that*

$$(H_0^{1/2}u,\ H_0^{1/2}u)-(pu,\ u)<-r(u,\ u),$$

where (,) stands for the usual scalar product in $L^2(R^n)$.

By making use of this proposition, we shall prove the following theorem.

0) It is convenient to write the Schrödinger operator as $-\Delta-p(x)$ instead of as the usual notation $-\Delta+p(x)$ since we are mainly concerned with an eigenvalue problem of the following form: $-\Delta u+u=\lambda p(x)u$, $p(x)>0$.

THEOREM 0.2. *Under the assumption of Proposition* 0.1, $n(r;p)$ *is equal to the number of positive eigenvalues less than one of the following problem*:

$$-\Delta v+rv=\lambda p(x)v\,, \qquad v\in\mathcal{D}(H_0^{1/2})\,, \tag{0.1}$$

where $\mathcal{D}(H_0^{1/2})$ *is the domain of the operator* $H_0^{1/2}$.

PROOF. It follows from Proposition 0.1 that $n(r;p)$ coincides with the maximal dimension of subspaces lying in $\mathcal{D}$ such that

$$((H_0+r)^{1/2}u,\ (H_0+r)^{1/2}u)<(pu,\ u)\,,$$

since $(H_0^{1/2}u,\ H_0^{1/2}u)+r(u,\ u)=((H_0+r)^{1/2}u,\ (H_0+r)^{1/2}u)$ for $u\in\mathcal{D}(H_0^{1/2})$. We put $v=(H_0+r)^{1/2}u$. Note that this transform preserves a linear independence and that $\mathcal{D}'=(H_0+r)^{1/2}\mathcal{D}=\{(H_0+r)^{1/2}w\,|\,w\in\mathcal{D}\}$ is a dense subspace in $L^2(R^n)$. Then we see that $n(r;p)$ is equal to the maximal dimension of subspaces lying in $\mathcal{D}'$ such that

$$(v,\ v)<((H_0+r)^{-1/2}p(H_0+r)^{-1/2}v,\ v)\,. \tag{0.2}$$

Furthermore we note that the operator $(H_0+r)^{-1/2}p(H_0+r)^{-1/2}$ is a compact operator in $L^2(R^n)$ by the assumption that $|p|^{1/2}$ is a $H_0^{1/2}$-compact operator. (0.2) implies that $n(r;p)$ is equal to the number of positive eigenvalues greater than one of the operator $(H_0+r)^{-1/2}p(H_0+r)^{-1/2}$. Therefore, if we put $w=(H_0+r)^{-1/2}v$ in the eigenvalue problem

$$(H_0+r)^{-1/2}p(H_0+r)^{-1/2}v=\mu v\,, \qquad \mu>1\,,$$

then we have

$$-\Delta w+rw=\lambda pw\,, \qquad 0<\lambda=1/\mu<1\,.$$

This completes the proof. q. e. d.

Putting $r=1/h$ in (0.1), we have the following eigenvalue problem with a parameter h:

$$-h\Delta v+v=\lambda p(x)v\,. \tag{0.3}$$

Without loss of generality, we may assume that $h\geqq 1$. Let $N_h(\lambda;p)$ be the number of positive eigenvalues less than λ of problem (0.3). Under some assumptions on $p(x)$, we shall show in § 3 that for any $\delta>0$ small enough, there exists a constant $C(\delta)$ independent of h and λ such that for $\lambda\geqq C(\delta)h^\alpha$,

$$|N_h(\lambda;p)-Ch^{-n/2}\lambda^\beta|\leqq\delta h^{-n/2}\lambda^\beta\,, \tag{0.4}$$

where α and β are some positive constants satisfying $0<\alpha<1$ and $\beta>n/2$ respectively, and C is a constant independent of h, λ and δ. Since $0<\alpha<1$, we see that there exists a constant $h(\delta)$ such that for $h>h(\delta)$,

$$|N_h(h;p)-Ch^{-n/2+\beta}|\leqq\delta h^{-n/2+\beta}\,. \tag{0.5}$$

Noting that $n(r;p)=N_h(h;p)$ with $r=1/h$, we can obtain the asymptotic formula for $n(r;p)$.

We shall give an outline of this paper. In § 1 we introduce some notations and state our main theorem in this paper. In § 2 we prove some lemmas which will be often used later. In § 3 we study the eigenvalue problem (0.3) in the case where $p(x)$ is a smooth positive function satisfying some conditions and obtain the asymptotic formula for $n(r;p)$. In § 4 we investigate an eigenvalue problem with zero boundary conditions in an unbounded domain. Let $p(x)$ be a not necessarily positive function (smooth) and let Ω be the open set given by $\Omega=\{x \mid p(x)>0\}$. Then we consider the following eigenvalue problem:

$$-\Delta u-p(x)u=\lambda u\,, \qquad u\in H_0^1(\Omega)\,,$$

where $H_0^1(\Omega)$ is the usual Sobolev space. We denote by $n(r;p,\Omega)$ the number of negative eigenvalues less than $-r$ ($r>0$) of the above problem. We give the estimate of $n(r;p,\Omega)$ from below for $r\to 0$ by studying the eigenvalue problem

$$-h\Delta v+v=\lambda p(x)v\,, \qquad v\in H_0^1(\Omega)\,.$$

This result is applied to the estimate of $n(r;p)$ from below. In fact, we easily see that $n(r;p)\geqq n(r;p,\Omega)$. In § 5 we give the estimate of $n(r;p)$ from above for the singular potentials $p(x)$, using the result obtained for smooth potentials in § 3 and the theorems by Birman and Borzov [2], Birman and Solomjak [3], and Rozenbljum [8] and we prove the main theorem in § 6, combinig the result in § 5 with the estimate from below obtained in § 4. In § 7 we apply the method developed for the Schrödinger operators to the Dirac operators with scalar potentials. The further development of our method to the Dirac operators with not necessarily scalar potentials will be discussed in a forthcoming paper [11]. Some of results of this paper were reported in [9] without detailed proofs.

Finally we note that throughout this paper we use the same symbol C to denote positive constants which may differ from each other. When we specify the dependence of such a constant on a parameter, say m, we denote it by $C(m)$. When we take integrations over the whole space, we write $\int f(x)dx$ instead of $\int_{R^n} f(x)dx$.

§ 1. Notations and main results.

Let us introduce some classes of functions. Consider a smooth function $p(x)$ (real-valued) satisfying the following condition:

(K-1) $$\lim_{r\to\infty} r^m p(r\omega)=a(\omega;p)$$

uniformly for $\omega \in S^{n-1}$, where $m>0$, $r=|x|$, $x=r\omega$, S^{n-1} is the $n-1$ dimensional unit sphere and $a(\omega;p)$ is a continuous function defined on S^{n-1}.

Let Ω be the open set defined by $\Omega=\{x|p(x)>0\}$ and Σ_γ be the subset in S^{n-1} defined by $\Sigma_\gamma=\{\omega|a(\omega;p)>\gamma\}$ for each fixed $\gamma>0$. Then, by the condition (K-1), we can take a constant $R(\gamma)$ so large that Ω contains $(R(\gamma),\infty)\times\Sigma_\gamma=G_\gamma$ in the polar coordinate system. From now on, we fix G_γ. In addition, we assume that $p(x)$ satisfies the following condition:

(K-2) There exist constants $C_1(\gamma)$ and $C_2(\gamma)$ such that for $x\in G_\gamma$ and

$$|x-y|\leqq C_2(\gamma)(1+|x|^2)^{1/2},$$

$$|p(x)-p(y)|\leqq C_1(\gamma)p(x)(1+|x|^2)^{-1/2}|x-y|,$$

where we note that $C_2(\gamma)$ is taken small so that y belongs to Ω.

From now on, for brevity, we put $\rho(x)=(1+|x|^2)^{1/2}$.

DEFINITION 1.1 (Class $K(m)$). If a smooth function $p(x)$ satisfies the conditions (K-1) and (K-2), we say that $p(x)$ belongs to $K(m)$.

DEFINITION 1.2 (Class $K^+(m)$). We denote by $K^+(m)$ the set of all functions $p(x)$ satisfying the following conditions:

(K^+-1) $p(x)$ belongs to $K(m)$;

(K^+-2) There exist constants C_1 and C_2 such that

$$C_1\rho(x)^{-m}\leqq p(x)\leqq C_2\rho(x)^{-m};$$

(K^+-3) For $|x-y|\leqq 1/2\rho(x)$,

$$|p(x)-p(y)|\leqq Cp(x)\rho(x)^{-1}|x-y|.$$

Here constants C_1, C_2 and C are independent of x and y.

From the condition (K^+-2) it readily follows that for $|x-y|\leqq 1/2\rho(x)$,

$$C_3p(x)\leqq p(y)\leqq C_4p(x). \tag{1.1}$$

DEFINITION 1.3 (Class $S(m)$). We denote by $S(m)$ the set of all real-valued functions satisfying the following conditions:

(S-1) $p(x)$ is decomposed into $p(x)=p_1(x)+p_2(x)$;

(S-2) $p_1(x)$ belongs to $K(m)$ and there exists a sequence $\{q_k(x)\}_{k=1}^\infty$ such that for each k $q_k(x)$ belongs to $K^+(m)$ and satisfies $C\rho(x)^{-m}\geqq q_k(x)\geqq p^+(x)=\max(0,p(x))$ with some constant C independent of k and x and that for each ω $a(\omega;q_k)$ tends to $a^+(\omega;p_1)=\max(0,a(\omega;p_1))$ as $k\to\infty$;

(S-3) $p_2(x)$ is a non-negative function such that if $n\leqq 2$, $p_2(x)$ has compact support and belongs to $L^p(R^n)$, $p>1$, and that if $n\geqq 3$, $p_2(x)$ belongs to $L^p(R^n)\cap L^{n/2}(R^n)$, $p>n/2$.

We denote the usual scalar product in $L^2(R^n)$ by $(\ ,\ )$ and the norm by $\|\ \|$. For a domain G, we denote by $H^j(G)$ the usual Sobolev space of order j with the norm $\|\ \|_j$ on G and by $H_0^j(G)$ the subspace of $H^j(G)$ obtained by the completion of $C_0^\infty(G)$ (the set of all smooth functions with compact support in G) under the norm $\|\ \|_j$.

For a linear operator A, we denote by $\mathcal{D}(A)$ and $\mathcal{R}(A)$ the domain and the range of A respectively.

Now we shall state our main result in the present paper, which will be proved in § 6.

Let $p(x)$ be a function belonging to $S(m)$ with $0<m<2$. Consider the following eigenvalue problem:

$$Hu=-\Delta u-p(x)u=\lambda u\,, \qquad u\in L^2(R^n)\,. \tag{1.2}$$

Here H is the semi-bounded self-adjoint operator associated with the symmetric bilinear form

$$h(u, v)=\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}u, \frac{\partial}{\partial x_j}v\right)-(pu, v)\,, \quad u, v\in H^1(R^n)\,.$$

Then we have the following theorem.

THEOREM 6.1. *Assume that $p(x)$ belongs to $S(m)$ with $0<m<2$ and that according to* (S-1) *in Definition* 1.3 *$p(x)$ is decomposed into two parts; $p(x)=p_1(x)+p_2(x)$. Let $n(r;p)$ be the number of eigenvalues less than $-r$ of problem* (1.2). *Then, as $r\to 0$,*

$$n(r;p)=C_0r^{n/2-n/m}+o(r^{n/2-n/m})\,,$$

where

$$C_0=(2\pi)^{-n}(2n)^{-1}\sigma_{n-1}\frac{\Gamma(n/m-n/2)\Gamma(n/2)}{\Gamma(n/m)}\int_{S^{n-1}}a^+(\omega;p_1)^{n/m}d\omega\,, \tag{1.3}$$

$a^+(\omega;p_1)=\max(0, a(\omega;p_1))$, σ_{n-1} is the surface measure of S^{n-1} ($\sigma_0=2$ if $n=1$), and $d\omega$ is the Lebesgue measure on S^{n-1}.

As an immediate consequence of Theorem 6.1, we have

COROLLARY 6.2. *Let λ_j be the j-th eigenvalue of problem* (1.2). *Then, λ_j is asymptotically represented as*

$$\lambda_j\sim -C_0^{-\alpha}j^{\alpha}$$

as $j\to\infty$, where $\alpha=(n/2-n/m)^{-1}$.

REMARK. The constant C_0 given by (1.3) does not depend on the way of the decomposition of $p(x)$. In fact, it is not difficult to show that $a(\omega;p)=0$ a.e. for $p(x)\in L^{n/2}(R^n)$.

§ 2. Preliminaries.

In this section we shall state some lemmas which will be often used later.

LEMMA 2.1. *Assume that* $p(x)$ *belongs to* $K^+(m)$. *Let* k *and* j *be positive numbers such that* $km>n$ *and* $n>(k-j)m>0$. *Then, as* $\lambda\to\infty$,

$$\int p(x)^k(1+\lambda p(x))^{-j}dx=C_1\lambda^{n/m-k}+o(\lambda^{n/m-k}),$$

where

$$C_1=m^{-1}\frac{\Gamma(n/m-(k-j))\Gamma(k-n/m)}{\Gamma(j)}\int_{S^{n-1}}a(\omega;p)^{n/m}d\omega.$$

PROOF. By the condition (K^+-1), for any $\varepsilon>0$ small enough, there exists a constant $R(\varepsilon)$ such that for $|x|=r\geqq R(\varepsilon)$,

$$(a(\omega;p)-\varepsilon)r^{-m}\leqq p(x)\leqq(a(\omega;p)+\varepsilon)r^{-m}. \tag{2.1}$$

Since $f(t)=t^k(1+\lambda t)^{-j}$ $(t>0)$ is a monotone increasing function, we have for $|x|\geqq R(\varepsilon)$,

$$f((a(\omega;p)-\varepsilon)r^{-m})\leqq f(p(x))\leqq f((a\omega;p)+\varepsilon)r^{-m}). \tag{2.2}$$

We shall first give the estimate from above. It follows from (2.2) that

$$\int p(x)^k(1+\lambda p(x))^{-j}dx\leqq\int_{|x|\geqq R(\varepsilon)}f((a(\omega;p)+\varepsilon)r^{-m})dx+C(\varepsilon)\lambda^{-j}.\text{[1]} \tag{2.3}$$

Furthermore, a change of variable and the obvious relation $0(\lambda^{-j})=o(\lambda^{n/m-k})$ which follows from the condition $n>(k-j)m$ yield that the right side of (2.3) is equal to

$$\lambda^{n/m-k}\int_{S^{n-1}}(a(\omega;p)+\varepsilon)^{n/m}\int_{R(\lambda,\varepsilon)}^{\infty}t^{(j-k)m+n-1}(t^m+1)^{-j}dt+o(\lambda^{n/m-k}), \tag{2.4}$$

where $R(\lambda,\varepsilon)=\lambda^{-1/m}(a(\omega;p)+\varepsilon)^{-1/m}R(\varepsilon)$. Since $g(t)=t^{(j-k)m+n-1}(t^m+1)^{-j}$ is integrable on $(0,\infty)$ by the conditions $km>n$ and $n>(k-j)m$, we have

$$\int_{R(\lambda,\varepsilon)}^{\infty}g(t)dt=\int_0^{\infty}g(t)dt+o(1)=m^{-1}\frac{\Gamma(n/m-(k-j))\Gamma(k-n/m)}{\Gamma(j)}+o(1)$$

as $\lambda\to\infty$. This gives the desired estimate from above. Similarly we can obtain the estimate from below. Thus the proof is completed since ε is arbitrary. q. e. d.

LEMMA 2.2. *Assume that* $p(x)$ *belongs to* $K^+(m)$ *with* $0<m<k$. *Let* $j>0$. *Then, if we take a constant* q, $q>0$, *large enough, we have*

$$\int \rho(x)^{-q+j}(1+\lambda p(x))^{-q/k}dx=0(\lambda^{-q/k}),\qquad\text{as }\lambda\to\infty. \tag{2.5}$$

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1) This estimate follows from the following estimate:

$$C_1(\varepsilon)\leqq p(t)\leqq C_2(\varepsilon),\qquad\text{for }|t|\leqq R(\varepsilon).$$

PROOF. (2.5) is a consequence of the fact that by the condition $m<k$, we can choose a constant q such that $\rho(x)^{-q+j}p(x)^{-q/k}$ is integrable. In fact, the function $\rho(x)^{-q+j}p(x)^{-q/k}$ behaves like $C|x|^{-(1-m/k)q+j}$ as $|x|\to\infty$. Hence, from this our assertion follows. q. e. d.

Next we shall introduce some operators. Let $p(x)$ belong to $K^+(m)$. For each fixed $t\in R^n$, $h\geqq 1$ and $\lambda>0$, we define the operator $R_{t,h}(\lambda)$ $(=R_{t,h}^{(0)}(\lambda))$ as

$$R_{t,h}(\lambda)=(-h\Delta+1+\lambda p(t))^{-1} \tag{2.6$'$}$$

and $R_{t,h}^{(j)}(\lambda)$ (j; positive integer) as

$$R_{t,h}^{(j)}(\lambda)=\left(\frac{d}{d\lambda}\right)^j R_{t,h}(\lambda)=(-1)^j(j\,!)p(t)^j(-h\Delta+1+\lambda p(t))^{-(j+1)}. \tag{2.6}$$

The operator $R_{t,h}^{(j)}(\lambda)$ (j; non-negative integer) is an integral operator with kernel $F_{t,h}^{(j)}(x-y\,;\lambda)$ (real-valued), which is defined by

$$\begin{aligned} F_{t,h}^{(j)}(y\,;\lambda) &=(-1)^j(j\,!)p(t)^j(2\pi)^{-n}\int e^{iy\xi}(h|\xi|^2+1+\lambda p(t))^{-(j+1)}d\xi \\ &=(-1)^j(j\,!)(2\pi)^{-n}h^{-n/2}p(t)^j(1+\lambda p(t))^{n/2-(j+1)}G^{(j)}(g(t,h,\lambda)y)\,, \end{aligned} \tag{2.7}$$

$$(i=\sqrt{-1})$$

where

$$g(t,h,\lambda)=h^{-1/2}(1+\lambda p(t))^{1/2}\,,$$

$$G^{(j)}(y)=\int e^{iy\xi}(|\xi|^2+1)^{-(j+1)}d\xi\,.$$

We shall state the well-known properties of the fundamental solution $G^{(j)}(y)$ (Mizohata [7]).

LEMMA 2.3. *$G^{(j)}(y)$ satisfies the following estimates:*

$$\begin{aligned} |G^{(j)}(y)| &\leqq H(y)|y|^{2(j+1)-n}\,, && \text{if } n>2(j+1)\,, \\ &\leqq H(y)|y|^{-1} && \text{if } n=2(j+1)\,, \\ &\leqq H(y) && \text{if } n<2(j+1)\,, \\ \left|\frac{\partial}{\partial y_k}G^{(j)}(y)\right| &\leqq H(y)|y|^{2(j+1)-(n+1)}\,, && \text{if } n>2(j+1)-1\,, \\ &\leqq H(y)|y|^{-1}\,, && \text{if } n=2(j+1)-1\,, \\ &\leqq H(y)\,, && \text{if } n<2(j+1)-1\,. \end{aligned}$$

Here $H(y)$ is a rapidly decreasing bounded function.

Finally we shall state the Tauberian theorem of Hardy and Littlewood.

LEMMA 2.4. *Let α and β be positive numbers satisfying $\beta>\alpha>0$. Let $\sigma_h(\lambda)$ be a non-negative non-decreasing function on $[0,\infty)$ with a positive parameter h and let $\sigma_h(+0)=\lim_{\lambda\to 0}\sigma_h(\lambda)=0$. Assume that for any $\delta>0$ small enough, there exists a constant $C(\delta)$ independent of h such that for $t\geqq C(\delta)h^\gamma$, $\gamma>0$,*

$$\left|\int_0^\infty (\lambda+t)^{-\beta} d\sigma_h(\lambda) - t^{\alpha-\beta}\right| \leqq \delta t^{\alpha-\beta}.$$

Then, there exists a constant $C_1(\delta)$ *independent of* h *such that for* $\lambda \geqq C_1(\delta) h^\gamma$,

$$\left|\sigma_h(\lambda) - \frac{\Gamma(\beta)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha)} \lambda^\alpha\right| < \delta \lambda^\alpha.$$

The proof of this lemma is done by examining the proof of the Tauberian theorem in [**5**].

§ 3. Eigenvalue problem on the whole space.

In the first half of this section, we shall consider the following auxiliary problem:

$$H(k)v = h(-\Delta)^k v + v = \lambda q(x) v, \qquad v \in L^2(R^n), \tag{3.1}$$

where $q(x)$ is assumed to belong to $K^+(m)$ with $n<m<2k$. It is clear that $H(k)$ is a positive self-adjoint operator with domain $\mathcal{D}(H(k)) = H^{2k}(R^n)$.

Problem (3.1) is transformed into an equivalent eigenvalue problem of the following form:

$$A(k)u = q^{-1/2}(h(-\Delta)^k + 1) q^{-1/2} u = \lambda u. \tag{3.2}$$

Here the operator $A(k)$ is a positive self-adjoint operator with domain $\mathcal{D}(A(k)) = \mathcal{R}(q^{1/2} H(k)^{-1} q^{1/2})$.

For each fixed $t \in R^n$ we define the operator $E_{t,h}(\lambda)$ $(h \geqq 1, \lambda > 0)$ as

$$E_{t,h}(\lambda) = (h(-\Delta)^k + 1 + \lambda q(t))^{-1}.$$

The operator $E_{t,h}(\lambda)$ is an integral operator with kernel

$$\begin{aligned} K_{t,h}(x-y;\lambda) &= (2\pi)^{-n}\int e^{i(x-y)\xi}(h|\xi|^{2k}+1+\lambda q(t))^{-1} d\xi \\ &= (2\pi)^{-n} h^{-n/2k}(1+\lambda q(t))^{n/2k-1} E(f(t,h,\lambda)(x-y)), \end{aligned} \tag{3.3}$$

where

$$f(t,h,\lambda) = h^{-1/2k}(1+\lambda q(t))^{1/2k},$$

$$E(y) = \int e^{iy\cdot\xi}(|\xi|^{2k}+1)^{-1} d\xi.$$

It is clear that $E(y)$ is a rapidly decreasing bounded function by the condition $n<2k$.

We note that all constants appearing throughout this section are independent of h, λ and t.

THEOREM 3.1. *Assume that* $q(x)$ *belongs to* $K^+(m)$ *with* $n<m<2k$. *Let*

$\{\nu_j\}_{j=1}^{\infty}$ *be eigenvalues of problem* (3.2). *Then, there exist constants* C *and* α, $0<\alpha<1$, *such that for* $\lambda\geqq h^{\alpha}$,

$$\sum_{j=1}^{\infty}(\nu_j+\lambda)^{-2}\leqq Ch^{-n/2k}\lambda^{n/m-2}. \tag{3.4}$$

PROOF. Define the operator $E_h(\lambda)$ as $E_h(\lambda)=(h(-\Delta)^k+1+\lambda q)^{-1}$. Then we have

$$q^{1/2}E_h(\lambda)q^{1/2}=q^{1/2}E_{t,h}(\lambda)q^{1/2}+\lambda q^{1/2}E_{t,h}(\lambda)(q(t)-q)E_h(\lambda)q^{1/2}. \tag{3.5}$$

Let $\{v_j\}_{j=1}^{\infty}$ be the normalized eigenfunctions (real-valued) corresponding to the eigenvalues $\{\nu_j\}_{j=1}^{\infty}$. Then, letting (3.5) operate on each v_j, we have

$$(\nu_j+\lambda)^{-1}v_j=q^{1/2}E_{t,h}(\lambda)q^{1/2}v_j+\lambda(\nu_j+\lambda)^{-1}q^{1/2}E_{t,h}(\lambda)(q(t)-q)q^{-1/2}v_j \tag{3.6}$$

since $q^{1/2}E_h(\lambda)q^{1/2}v_j=(\nu_j+\lambda)^{-1}v_j$. Set $\tilde{\theta}_j(t, y)=(q(t)-q(y))q(y)^{-1/2}v_j(y)$. Then, rewriting (3.6) in the form of the integral equation, we have

$$\begin{aligned}(\nu_j+\lambda)^{-1}v_j(x)&=q(x)^{1/2}\int K_{t,h}(x-y;\lambda)q(y)^{1/2}v_j(y)dy\\ &\quad+\lambda(\nu_j+\lambda)^{-1}q(x)^{1/2}\int K_{t,h}(x-y;\lambda)\tilde{\theta}_j(t, y)dy\\ &=a_j(x, t)+b_j(x, t)\qquad(\text{cf. [}\mathbf{10}\text{]}).\end{aligned} \tag{3.7}$$

Since each $v_j(x)$ is a smooth function by the regularity theorem for elliptic operators, (3.7) is well-defined for all x. Hence, putting $x=t$ in (3.7), we have

$$(\nu_j+\lambda)^{-1}v_j(t)=a_j(t)+b_j(t), \tag{3.8}$$

where we have set $a_j(t)=a_j(t, t)$ and $b_j(t)=b_j(t, t)$. By taking the square of both sides of (3.8), summing up with respect to j, and integrating over the whole space, we have

$$\sum_{j=1}^{\infty}(\nu_j+\lambda)^{-2}\leqq 2\Big(\sum_{j=1}^{\infty}\int a_j(t)^2dt+\sum_{j=1}^{\infty}\int b_j(t)^2dt\Big). \tag{3.9}$$

The proof is completed as an immediate consequence of the following lemma.

LEMMA 3.2. *There exists a constant* α, $0<\alpha<1$, *such that for* $\lambda\geqq h^{\alpha}$,

$$\sum_{j=1}^{\infty}\int a_j(t)^2dt\leqq Ch^{-n/2k}\lambda^{n/m-2}, \tag{3.10}$$

$$\sum_{j=1}^{\infty}\int b_j(t)^2dt\leqq Ch^{-n/2k}\lambda^{n/m-2}. \tag{3.11}$$

PROOF. We shall first prove (3.10). Using the Parseval equality, we have

$$\sum_{j=1}^{\infty}a_j(t)^2=q(t)\int K_{t,h}(t-y;\lambda)^2q(y)dy=I(t).$$

We write $I(t)$ as follows:

$$I(t)=q(t)^2\int K_{t,h}(t-y\,;\lambda)^2dy+q(t)\int_{\Omega_{1/2}}K_{t,h}(t-y\,;\lambda)^2(q(y)-q(t))dy$$

$$+q(t)\int_{\Omega_{1/2}^c}K_{t,h}(t-y\,;\lambda)^2(q(y)-q(t))dy$$

$$=I_1(t)+I_2(t)+I_3(t)\,,$$

where we denote by $\Omega_{1/2}$ and $\Omega_{1/2}^c$ the set $\{y\,|\,|t-y|\leqq 1/2\rho(t)\}$ and its complement respectively. We shall first consider the term $I_1(t)$. By the Parseval equality and a change of variable, we obtain

$$\int K_{t,h}(t-y\,;\lambda)^2dy=(2\pi)^{-n}\int(h|\xi|^{2k}+1+\lambda q(t))^{-2}d\xi$$

$$=Ch^{-n/2k}(1+\lambda q(t))^{n/2k-2}\,.$$

Hence, by means of Lemma 2.1, we have

$$\int I_1(t)dt=Ch^{-n/2k}\int q(t)^2(1+\lambda q(t))^{n/2k-2}dt\leqq Ch^{-n/2k}\lambda^{n/m-2}\,. \tag{3.12}$$

Next we investigate the term $I_2(t)$. By the condition (K^+-3) in Definition 1.2, we have for $y\in\Omega_{1/2}$,

$$|q(y)-q(t)|\leqq Cq(t)\,.$$

Using this inequality, we have

$$\int I_2(t)dt\leqq C\int q(t)^2dt\int_{\Omega_{1/2}}K_{t,h}(t-y\,;\lambda)^2dy\leqq C\int I_1(t)dt\leqq Ch^{-n/2k}\lambda^{n/m-2}\,. \tag{3.13}$$

Finally we shall estimate the term $I_3(t)$. We have

$$\int I_3(t)dt\leqq\int q(t)^2dt\int_{\Omega_{1/2}^c}K_{t,h}(t-y\,;\lambda)^2dy+\int q(t)dt\int_{\Omega_{1/2}^c}K_{t,h}(t-y\,;\lambda)^2q(y)dy\,. \tag{3.14}$$

We see that the first term on the right side of (3.14) is dominated by $Ch^{-n/2k}\lambda^{n/m-2}$. On the other hand, recalling the definition of $K_{t,h}(t-y\,;\lambda)$ given by (3.3), we have for any $p>0$ large enough and $y\in\Omega_{1/2}^c$,

$$\begin{aligned}|K_{t,h}(t-y\,;\lambda)|&\leqq Ch^{(p-n)/2k}(1+\lambda q(t))^{(n-p)/2k-1}|t-y|^{-p}\\&\leqq Ch^{(p-n)/2k}(1+\lambda q(t))^{(n-p)/2k-1}\rho(t)^{-p}\,,\end{aligned} \tag{3.15}$$

since $E(y)$ is a rapidly decreasing bounded function. Hence, by means of Lemma 2.2 and (3.15), the second term on the right side of (3.14) is dominated by

$$\begin{aligned}&Ch^{(p-n)/k}\int q(t)(1+\lambda q(t))^{(n-p)/k-2}\rho(t)^{-2p}dt\int q(y)dy\\&\leqq Ch^{-n/2k}\lambda^{n/m-2}h^{\gamma}\lambda^{-\beta}\,,\end{aligned} \tag{3.16}$$

where $\gamma=p/k-n/2k<\beta=p/k-n/k+n/m$ and we have used the fact that $q(y)$ is integrable by the condition $n<m$. Choosing p appropriately and setting $\alpha=\gamma/\beta$, we can obtain (3.10).

Next we shall prove (3.11). Using the Parseval equality, we have

$$\begin{aligned}\sum_{j=1}^{\infty} b_j(t)^2 &\leqq q(t)\int K_{t,h}(t-y\,;\lambda)^2(q(t)-q(y))^2 q(y)^{-1}dy\\ &\leqq 2q(t)\int K_{t,h}(t-y\,;\lambda)^2 q(y)dy+2q(t)^3\int_{\Omega_{1/2}} K_{t,h}(t-y\,;\lambda)^2 q(\lambda)^{-1}dy\\ &\quad +2q(t)^3\int_{\Omega_{1/2}^c} K_{t,h}(t-y\,;\lambda)^2 q(y)^{-1}dy\\ &=I(t)+II(t)+III(t)\,.\end{aligned}$$

The term $\int I(t)dt$ has been already estimated in the proof of (3.10) and satisfies the desired estimate. By (1.1), we easily see that $q(y)^{-1}\leqq Cq(t)^{-1}$ for $y\in\Omega_{1/2}$. Hence, it readily follows that the term $\int II(t)dt$ is dominated by $Ch^{-n/2k}\lambda^{n/m-2}$. It remains to show that $\int III(t)dt$ satisfies the desired estimate. Noting that for any $p>0$ large enough,

$$\int_{\Omega_{1/2}^c}|t-y|^{-2p}q(y)^{-1}dy\leqq C\rho(t)^{-2p+n+m}\,,$$

we have by means of Lemma 2.2 and (3.15) that

$$\begin{aligned}\int III(t)dt &\leqq Ch^{(p-n)/k}\int q(t)^3(1+\lambda q(t))^{(n-p)/k-2}\rho(t)^{-2p+m+n}dt\\ &\leqq Ch^{-n/2k}\lambda^{n/m-2}h^{\gamma}\lambda^{-\beta}\,,\end{aligned}$$

where $\gamma=p/k-n/2k<\beta=p/k+n/m-n/k$. This completes the proof of (3.11).

q. e. d.

We now study the following eigenvalue problem:

$$Au=p^{-1/2}(-h\Delta+1)p^{-1/2}u=\lambda u\,,\qquad u\in L^2(R^n)\,, \tag{3.17}$$

where $p(x)$ is assumed to belong to $K^+(m)$ with $0<m<2$, and A is a positive self-adjoint operator with domain $\mathcal{D}(A)=\mathcal{R}(p^{1/2}(-h\Delta+1)^{-1}p^{1/2})$.

Problem (3.17) is equivalent to problem (0.3). Let $\{\mu_j\}_{j=1}^{\infty}$ be eigenvalues of problem (3.17).

We now fix a positive integer l such that $k_0=2^l>n/m$. Then, we have

LEMMA 3.3. *Let $k_0=2^l$. Assume that $p(x)$ belongs to $K^+(m)$ with $0<m<2$. Let $\{\nu_j(k)\}_{j=1}^{\infty}$ be eigenvalues of the problem*

$$h^k(-\Delta)^k v+v=\lambda p(x)^k v\,, \qquad v\in L^2(R^n)\,.$$

Then, we have for each j

$$\mu_j^{k_0}\geqq \nu_j(k_0)\,.$$

PROOF. We first note that for any $v\in H^2(R^n)$,

$$((-h\Delta+1)v,\ (-h\Delta+1)v)\geqq((h^2(-\Delta)^2+1)v,\ v)\,. \tag{3.18}$$

Let μ_j and w_j be the j-th eigenvalue of problem (0.3) and the eigenfunction corresponding to μ_j respectively. Then, by (3.18), we have

$$((h^2(-\Delta)^2+1)w_j,\ w_j)\leqq \mu_j^2(p^2 w_j,\ w_j)\,, \tag{3.19}$$

from which it follows that $\mu_j^2\geqq\nu_j(2)$. In fact, by (3.19), we see that there exists a subspace of j-dimensions lying in $H^2(R^n)$ such that

$$((h^2(-\Delta)^2+1)u,\ u)\leqq \mu_j^2(p^2 u,\ u)\,.$$

This implies that $\mu_j^2\geqq\nu_j(2)$. By repeating this step, we obtain the conclusion. q. e. d.

From now on, we denote by α some constants satisfying $0<\alpha<1$ and independent of h, λ and δ, which may differ from each other.

LEMMA 3.4. *Let k be an integer satisfying $k\geqq k_0$. Then, we have for $\lambda\geqq h^\alpha$,*

$$\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2k}\leqq Ch^{-n/2}\lambda^{n/m-2k}\,. \tag{3.20}$$

PROOF. (3.20) is an immediate consequence of Theorem 3.1 and Lemma 3.3. In fact, we have for $\lambda\geqq h^\alpha$,

$$\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2k}\leqq\lambda^{-2(k-k_0)}\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2k_0}\leqq\lambda^{-2(k-k_0)}\sum_{j=1}^{\infty}(\nu_j(k_0)+\lambda^{k_0})^{-2}$$
$$\leqq Ch^{-n/2}\lambda^{n/m-2k}\,.$$

This completes the proof.

Now we shall state the main result of this section.

THEOREM 3.5. *Assume that $p(x)$ belongs to $K^+(m)$ with $0<m<2$. Let $N_h(\lambda\,;p)$ be the number of eigenvalues less than λ of problem (3.17). Then, for any $\delta>0$ small enough, there exists a constant $C(\delta)$ independent of h and λ such that for $\lambda\geqq C(\delta)h^\alpha$, $0<\alpha<1$,*

$$|N_h(\lambda\,;p)-\tilde{C}_0 h^{-n/2}\lambda^{n/m}|\leqq\delta h^{-n/2}\lambda^{n/m}\,,$$

where

$$\tilde{C}_0=(2\pi)^{-n}(2n)^{-1}\sigma_{n-1}\frac{\Gamma(n/m-n/2)\Gamma(n/2)}{\Gamma(n/m)}\int_{S^{n-1}}a(\omega\,;p)^{n/m}d\omega\,.$$

PROOF. For each fixed $t\in R^n$, we begin with the following equation:

$$(\mu_j+\lambda)^{-1}u_j=p^{1/2}R_{t,h}(\lambda)p^{1/2}u_j+\lambda(\mu_j+\lambda)^{-1}R_{t,h}(\lambda)(p(t)-p)p^{-1/2}u_j\,, \tag{3.21}$$

where u_j is the normalized eigenfunction corresponding to μ_j and $R_{t,h}(\lambda)$ is the operator defined by (2.6)'. (3.21) is the equation corresponding to (3.6) in the proof of Theorem 3.1 and is obtained exactly in the same way as (3.6).

We now fix an integer k such that

$$k>\max(n/2-1,\ n/2m-1)+k_0=\max(n/2-1,\ n/2m-1)+2^l>n/m\,.$$

Then, by differentiating (3.21) k-times with respect to λ in the sense of $L^2(R^n)$, we have

$$\begin{aligned}(3.22)\qquad (-1)^k(k\,!)(\mu_j+\lambda)^{-(k+1)}u_j &= p^{1/2}R_{t,h}^{(k)}(\lambda)p^{1/2}u_j\\ &+\sum_{s=0}^{k-1}C(s)(\mu_j+\lambda)^{-(k-s)}p^{1/2}R_{t,h}^{(s)}(\lambda)(p(t)-p)p^{-1/2}u_j\\ &+\lambda\sum_{r=0}^{k}C(r)(\mu_j+\lambda)^{-(k-r+1)}p^{1/2}R_{t,h}^{(r)}(\lambda)(p(t)-p)p^{-1/2}u_j\,.\end{aligned}$$

Set $\theta_j(t,y)=(p(t)-p(y))p(y)^{-1/2}u_j(y)$. Then, rewriting (3.22) in the form of the integral equation, we have

$$\begin{aligned}(3.23)\qquad (-1)^k(k\,!)(\mu_j+\lambda)^{-(k+1)}u_j(x) &= p(x)^{1/2}\int F_{t,h}^{(k)}(x-y\,;\lambda)p(y)^{1/2}u_j(y)dy\\ &+\sum_{s=0}^{k-1}C(s)(\mu_j+\lambda)^{-(k-s)}p(x)^{1/2}\int F_{t,h}^{(s)}(x-y\,;\lambda)\theta_j(t,y)dy\\ &+\lambda\sum_{r=0}^{k}C(r)(\mu_j+\lambda)^{-(k-r+1)}p(x)^{1/2}\int F_{t,h}^{(r)}(x-y\,;\lambda)\theta_j(t,y)dy\\ &=a_j(x,t)+\sum_{s=0}^{k-1}C(s)b_{j,s}(x,t)+\lambda(\mu_j+\lambda)^{-1}\sum_{r=0}^{k}C(r)b_{j,r}(x,t)\,.\end{aligned}$$

Since each $u_j(x)$ is a smooth function, (3.23) is well-defined for all x. Hence, putting $x=t$, in particular, in (3.23), we have

$$\begin{aligned}(3.24)\qquad (-1)^k(k\,!)(\mu_j+\lambda)^{-(k+1)}u_j(t) &= a_j(t)+\sum_{s=0}^{k-1}C(s)b_{j,s}(t)\\ &+\lambda(\mu_j+\lambda)^{-1}\sum_{r=0}^{k}C(r)b_{j,r}(t)\\ &=a_j(t)+b_j(t)\,,\end{aligned}$$

where we have set $a_j(t)=a_j(t,t)$ and $b_{j,s}(t)=b_{j,s}(t,t)$. This is our basic equality in proving this theorem.

As in the proof of Theorem 3.1, taking the square of both sides of (3.24), summing up with respect to j, and integrating over the whole space, we have

$$(k\,!)^2\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2(k+1)}=\sum_{j=1}^{\infty}\int a_j(t)^2dt+2\sum_{j=1}^{\infty}\int a_j(t)b_j(t)dt+\sum_{j=1}^{\infty}\int b_j(t)^2dt\,.$$

Further it follows that for any $\delta>0$ small enough,

$$(3.25)\qquad \left|(k\,!)^2\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2(k+1)}-\sum_{j=1}^{\infty}\int a_j(t)^2dt\right|$$
$$\leqq \delta\sum_{j=1}^{\infty}\int a_j(t)^2dt+C(\delta)\sum_{j=1}^{\infty}\int b_j(t)^2dt\,.$$

In order to complete the proof, we need the following lemmas concerning the estimates for $\sum_{j=1}^{\infty}\int a_j(t)^2dt$ and $\sum_{j=1}^{\infty}\int b_j(t)^2dt$. These lemmas will be proved after the completion of the proof of this theorem.

LEMMA 3.6. *For any $\delta>0$ small enough, there exists a constant $C(\delta)$ independent of h and λ such that for $\lambda\geqq C(\delta)h^{\alpha}$, $0<\alpha<1$,*

$$\left|\sum_{j=1}^{\infty}\int a_j(t)^2dt-C_0(k)h^{-n/2}\lambda^{n/m-2(k+1)}\right|\leqq\delta h^{-n/2}\lambda^{n/m-2(k+1)}\,,$$

where $C_0(k)=(k\,!)^2n/m\dfrac{\Gamma(2(k+1)-n/m)\Gamma(n/m)}{\Gamma(2(k+1))}\tilde{C}_0$ and $\tilde{C}_0$ is the constant defined in Theorem 3.5.

LEMMA 3.7. *For any $\delta>0$ small enough, there exists a constant $C(\delta)$ independent of h and λ such that for $\lambda\geqq C(\delta)h^{\alpha}$, $0<\alpha<1$,*

$$\sum_{j=1}^{\infty}\int b_{j,r}(t)dt\leqq\delta h^{-n/2}\lambda^{n/m-2(k+1)}\,,\qquad (r=0,1,\cdots,k)\,.$$

Completion of the proof on Theorem 3.5. By vitue of Lemmas 3.6 and 3.7, it follows from (3.25) that for $\lambda\geqq C(\delta)h^{\alpha}$,

$$(3.26)\qquad \left|\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2(k+1)}-n/m\frac{\Gamma(2(k+1)-n/m)\Gamma(n/m)}{\Gamma(2(k+1))}\tilde{C}_0h^{-n/2}\lambda^{n/m-2(k+1)}\right|$$
$$\leqq\delta h^{-n/2}\lambda^{n/m-2(k+1)}\,.$$

Now we are in a position to apply the Tauberian theorem of Hardy and Littlewood (Lemma 2.4) to (3.26). Then, we get the conclusion. q. e. d.

Now we shall prove Lemmas 3.6 and 3.7.

PROOF OF LEMMA 3.6. Using the Parseval equality, we calculate as follows:

$$(3.27)\qquad \sum_{j=1}^{\infty}a_j(t)^2=p(t)\int F_{t,h}^{(k)}(t-y\,;\lambda)^2p(y)dy$$
$$=p(t)^2\int F_{t,h}^{(k)}(t-y\,;\lambda)^2dy+p(t)\int_{\Omega_\delta}F_{t,h}^{(k)}(t-y\,;\lambda)^2(p(y)-p(t))dy$$
$$+p(t)\int_{\Omega_\delta^c}F_{t,h}^{(k)}(t-y\,;\lambda)^2(p(y)-p(t))dy$$
$$=I(t)+II(t)+III(t)\,,$$

where we denote by Ω_δ and Ω_δ^c the set $\{y\,|\,|t-y|\leqq\delta\rho(t)\}$ and its complement respectively. We shall first deal with the term $I(t)$. Recalling the definition of $F_{t,h}^{(k)}(t-y;\lambda)$ given by (2.7), and using the Parseval equality and the condition $k>n/2-1$, we have

$$\int F_{t,h}^{(k)}(t-y;\lambda)^2dy=(2\pi)^{-n}(k!)^2p(t)^{2k}\int(h|\xi|^2+1+\lambda p(t))^{-2(k+1)}d\xi \tag{3.28}$$

$$=(2\pi)^{-n}C_1(k)h^{-n/2}p(t)^{2k}(1+\lambda p(t))^{n/2-2(k+1)},$$

where

$$C_1(k)=(k!)^2\int(|\xi|^2+1)^{-2(k+1)}d\xi=(k!)^2\sigma_{n-1}\frac{\Gamma(2(k+1)-n/2)\Gamma(n/2)}{2\Gamma(2(k+1))}.$$

Hence, by means of Lemma 2.1, we have that as $\lambda\to\infty$,

$$\int I(t)dt=(2\pi)^{-n}C_1(k)h^{-n/2}\int p(t)^{2(k+1)}(1+\lambda p(t))^{n/2-2(k+1)}dt$$

$$=C_0(k)h^{-n/2}\lambda^{n/m-2(k+1)}+h^{-n/2}o(\lambda^{n/m-2(k+1)}),$$

where we note that o-estimate is uniform with respect to h, and $C_0(k)$ is the constant defined in Lemma 3.6.

Next we shall show that

$$\int|II(t)|dt\leqq C\delta h^{-n/2}\lambda^{n/m-2(k+1)}.$$

But this is easily done since $|p(t)-p(y)|\leqq C\delta p(t)$ for $y\in\Omega_\delta$ by the condition (K^+-3) in Definition 1.2. Finally we shall consider the term $III(t)$. Since $k>n/2-1$, $F_{t,h}^{(k)}(y;\lambda)$ is a rapidly decreasing function. Hence, by a method similar to that given to the term $I_3(t)$ in the proof of Lemma 3.2,[2)] we can show that there exist positive constants β and γ satisfying $\beta>\gamma$ such that

$$\int|III(t)|dt\leqq C(\delta)h^{-n/2}\lambda^{n/m-2(k+1)}h^\gamma\lambda^{-\beta}. \tag{3.29}$$

Therefore, choosing λ in (3.29) so that $C(\delta)h^\gamma\lambda^{-\beta}\leqq\delta$, we have for $\lambda\geqq C(\delta)h^\alpha$ ($\alpha=\gamma/\beta$),

$$\int|III(t)|dt\leqq\delta h^{-n/2}\lambda^{n/m-2(k+1)}.$$

Combining the above estimates for $\int I(t)dt$, $\int|II(t)|dt$ and $\int|III(t)|dt$, we obtain the conclusion.

Proof of Lemma 3.7. The proof is divided into two cases and two different methods of estimates are employed.

2) Instead of the integrability of $q(y)$ in (3.16), we use the following estimate with some constant $C(\delta)$ independent of t:

$$\int_{\Omega_\delta^c}|t-y|^{-(n+1)}p(y)\leqq C(\delta).$$

Case 1, $k \geqq r \geqq k-k_0 > \max(n/2-1, n/2m-1)$: Using the Parseval equality, we calculate as follows:

$$\sum_{j=1}^{\infty} b_{j,r}(t) \leqq \lambda^{-2(k-r)} p(t) \int F_{t,h}^{(r)}(t-y\,;\lambda)^2 (p(t)-p(y))^2 p(y)^{-1} dy$$

$$\leqq \lambda^{-2(k-r)} p(t) \int_{\Omega_\delta} F_{t,h}^{(r)}(t-y\,;\lambda)^2 (p(t)-p(y))^2 p(y)^{-1} dy$$

$$+2\lambda^{-2(k-r)} p(t)^3 \int_{\Omega_\delta^c} F_{t,h}^{(r)}(t-y\,;\lambda)^2 p(y)^{-1} dy$$

$$+2\lambda^{-2(k-r)} p(t) \int_{\Omega_\delta^c} F_{t,h}^{(r)}(t-y\,;\lambda)^2 p(y) dy$$

$$= I(t) + II(t) + III(t)\,,$$

where we should note that since $r > n/2-1$, $F_{t,h}^{(r)}(t-y\,;\lambda)$ is a rapidly decreasing bounded function as a function of y.

We shall first consider the term $I(t)$. By the condition (K^+-3) and (1.1), we have

$$(p(y)-p(t))^2 p(y)^{-1} \leqq C\delta^2 p(t) \tag{3.30}$$

for $y \in \Omega_\delta$. On the other hand, it is easily seen that

$$\int F_{t,h}^{(r)}(t-y\,;\lambda)^2 dy = Ch^{-n/2} p(t)^{2r} (1+\lambda p(t))^{n/2-2(r+1)}\,. \tag{3.31}$$

Hence, in view of (3.30) and (3.31), it follows from Lemma 2.1 and the assumption $2(r+1)m > n$ that

$$\int |I(t)|\,dt \leqq C\delta^2 h^{-n/2} \lambda^{n/m-2(k+1)}\,. \tag{3.32}$$

Next we shall investigate the term $II(t)$; the term $III(t)$ can be dealt with in the same way. Since $F_{t,h}^{(r)}(y\,;\lambda)$ is a rapidly decreasing bounded function, it follows from the definition of $F_{t,h}^{(r)}(t-y\,;\lambda)$ given by (2.7) and Lemma 2.3 that for any $p > 0$

$$|F_{t,h}^{(r)}(t-y\,;\lambda)| \leqq Ch^{(p-n)/2} p(t)^r (1+\lambda p(t))^{(n-p)/2-(r+1)} |t-y|^{-p}\,. \tag{3.33}$$

Furthermore, we have for any $p > 0$ large enough,

$$\int_{\Omega_\delta^c} |t-y|^{-p} p(y)^{-1} dy \leqq C(\delta) \rho(t)^{-p+m+n}\,.$$

By using these estimates and Lemma 2.2, $\int |II(t)|\,dt$ is estimated as follows:

$$\int |II(t)|\,dt \leqq C(\delta) h^{(p-n)} \lambda^{-2(k-r)} \int p(t)^{2r+3} (1+\lambda p(t))^{n-p-2(r+1)} \rho(t)^{-2p+n+m} dt$$

$$\leqq C(\delta) h^{-n/2} \lambda^{n/m-2(k+1)} h^{\gamma} \lambda^{-\beta}\,,$$

where $\gamma=p-n/2<\beta=p+n/m-n$. Therefore, choosing p appropriately and λ so that $C(\delta)h^{\gamma}\lambda^{-\beta}\leqq\delta$, we have

$$\int|II(t)|dt\leqq\delta h^{-n/2}\lambda^{n/m-2(k+1)} \tag{3.34}$$

for $\lambda\geqq\tilde{C}(\delta)h^{\alpha}$, $\alpha=\gamma/\beta$. Combining (3.34) with (3.32), we obtain the conclusion.

Case 2, $0\leqq r<k-k_0$. We calculate as follows:

$$\begin{aligned}\sum_{j=1}^{\infty}b_{j,r}(t)^2&=\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2(k-r)}p(t)\Big(\int F_{t,h}^{(r)}(t-y;\lambda)\theta_j(t,y)dy\Big)^2\\&\leqq2\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2(k-r)}p(t)\Big(\int_{\Omega_\delta}F_{t,h}^{(r)}(t-y;\lambda)\theta_j(t,y)dy\Big)^2\\&\quad+2\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2(k-r)}p(t)\Big(\int_{\Omega_\delta^c}F_{t,h}^{(r)}(t-y;\lambda)\theta_j(t,y)dy\Big)^2\\&=I(t)+II(t)\,.\end{aligned}$$

We note that for any $r\geqq0$ $F_{t,h}^{(r)}(t-y;\lambda)$ is a rapidly decreasing bounded function on Ω_δ^c as a function of y. Hence, by applying the Parseval equality to $II(t)$, the estimate for $\int|II(t)|dt$ is carried out exactly in the same way as in the proof of Case 1 and we obtain

$$\int|II(t)|dt\leqq\delta h^{-n/2}\lambda^{n/m-2(k+1)}$$

for $\lambda\geqq C(\delta)h^{\alpha}$.

Next we consider the term $I(t)$. We shall show that

$$III=\int p(t)dt\Big(\int_{\Omega_\delta}F_{t,h}^{(r)}(t-y;\lambda)\theta_j(t,y)dy\Big)^2\leqq C\delta^2\lambda^{-2(r+1)}, \tag{3.35}$$

where C is a constant independent of j, h, λ and δ. If we are able to prove (3.35), the desired estimate is obtained as follows: Since $k-r>k_0=2^l>n/m$, we have by virtue of Lemma 3.4

$$\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2(k-r)}\leqq Ch^{-n/2}\lambda^{n/m-2(k-r)} \tag{3.36}$$

for $\lambda\geqq h^{\alpha}$. Hence, combining (3.36) with (3.35), we have for $\lambda\geqq h^{\alpha}$,

$$\begin{aligned}\int|I(t)|dt&\leqq C\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2(k-r)}\int p(t)dt\Big(\int_{\Omega_\delta}F_{t,h}^{(r)}(t-y;\lambda)\theta_j(t,y)dy\Big)^2\\&\leqq C\delta^2h^{-n/2}\lambda^{n/m-2(k+1)}\,.\end{aligned}$$

This completes the proof.

Now we shall prove (3.35). To do so, we note the following two facts.

(a) There exists a constant C independent of of y, z, j and δ such that for $|y|\leqq\delta$ and $|z|\leqq\delta$,

$$\int |u_j(t+y\rho(t))|\,|u_j(t+z\rho(t))|\,dt \leqq C.$$

(b) The following estimate holds:

$$|F_{t,h}^{(r)}(t-y\,;\lambda)| \leqq Cp(t)^r(1+\lambda p(t))^{-(r+1)}|t-y|^{-n}.$$

(a) is obtained by the Schwarz inequality and changes of variables. (b) is proved with the aid of Lemma 2.3. In fact, it follows from the definition of $F_{t,h}^{(r)}(y\,;\lambda)$ given by (2.7) and Lemma 2.3 that for $2(r+1)<n$,

$$\begin{aligned}|F_{t,h}^{(r)}(y\,;\lambda)| &\leqq Ch^{-n/2}p(t)^r(1+\lambda p(t))^{n/2-(r+1)}|G^{(r)}(g(t,h,\lambda)y)|\\ &\leqq Ch^{-n/2}p(t)^r(1+\lambda p(t))^{n/2-(r+1)}|g(t,h,\lambda)y|^{2(r+1)-n}H(g(t,h,\lambda)y),\end{aligned}$$

where $g(t,h,\lambda)=h^{-1/2}(1+\lambda p(t))^{1/2}$ and $H(y)$ is a rapidly decreasing bounded function. Hence, using the estimate

$$|g(t,h,\lambda)y|^{2(r+1)}H(g(t,h,\lambda)y) \leqq C,$$

we easily obtain the statement (b) in the case of $2(r+1)<n$. Also in the case of $2(r+1)\geqq n$, we can obtain the desired estimate in a similar manner.

Taking the above facts (a) and (b) into account and using the following estimate obtained from the condition (K^+-3) and (1.1);

$$|\theta_j(t,y)| = |(p(t)-p(y))p(y)^{-1/2}u_j(y)| \leqq Cp(t)^{1/2}\rho(t)^{-1}|t-y|\,|u_j(y)|$$

for $y\in\Omega_\delta$, we have by changes of variables

$$\begin{aligned}III &\leqq C\int p(t)^{2(r+1)}(1+\lambda p(t))^{-2(r+1)}\rho(t)^{-2}dt\int_{\Omega_\delta}|t-y|^{1-n}|u_j(y)|\,dy\int_{\Omega_\delta}|t-z|^{1-n}|u_j(z)|\,dz\\ &\leqq C\lambda^{-2(r+1)}\int_{|\eta|\leqq\delta}|\eta|^{1-n}d\eta\int_{|\zeta|\leqq\delta}|\zeta|^{1-n}d\zeta\int|u_j(t+\eta\rho(t))|\,|u_j(t+\zeta\rho(t))|\,dt\\ &\leqq C\delta^2\lambda^{-2(r+1)},\end{aligned}$$

from which (3.35) follows. Thus the proof is completed. q. e. d.

As a direct application of Theorem 3.5, we have the following theorem.

THEOREM 3.8. *Assume that $p(x)$ belongs to $K^+(m)$ with $0<m<2$. Let $n(r\,;p)$ be the number of eigenvalues less than $-r$ of problem* (1.2). *Then, as $r\to 0$,*

$$n(r\,;p)=\tilde{C}_0 r^{n/2-n/m}+o(r^{n/2-n/m}),$$

where $\tilde{C}_0$ is the constant defined in Theorem 3.5.

PROOF. We first note that $n(r\,;p)=N_h(h\,;p)$ with $r=1/h$. On the other hand, since $0<\alpha<1$, Theorem 3.5 shows that for any $\delta>0$ small enough, there exists a constant $h(\delta)$ such that for $h\geqq h(\delta)$,

$$|N_h(h\,;p)-\tilde{C}_0h^{n/m-n/2}| \leqq \delta h^{n/m-n/2}.$$

From this fact we can easily obtain the conclusion. q. e. d.

The method developed above can be applied to more general elliptic operators.

Let us consider the following eigenvalue problem:

$$A(D)u-p(x)u=\lambda u\,, \qquad u\in L^2(R^n)\,,$$

where $A(D)$ is a homogeneous elliptic operator of order $2l$ with constant coefficients and $p(x)$ is a function belonging to $K^+(m)$ with $0<m<2l$. Let $n(r\,;p, A(D))$ be the number of eigenvalues less than $-r$ of the above problem. Then, we have

$$n(r\,;p,\ A(D))=\tilde{C}_0 r^{n/2l-n/m}+o(r^{n/2l-n/m})\,,$$

where we can give the explicit expression of the constant $\tilde{C}_0$ but we do not refer to it here (see [**9**]).

§ 4. Eigenvalue problem in an unbounded domain.

Let $p(x)$ be a function belonging to $K(m)$ with $0<m<2$ and let Ω be the open set given by $\Omega=\{x\,|\,p(x)>0\}$. Then, consider the following eigenvalue problem:

$$\text{(4.1)} \qquad Au=-h\Delta u+u=\lambda p(x)u\,, \qquad u\in H_0^1(\Omega)\,.$$

Here A is the positive self-adjoint operator associated with the symmetric bilinear form

$$a(u,\ v)=h\sum_{k=1}^{n}\Big(\frac{\partial}{\partial x_k}u,\ \frac{\partial}{\partial x_k}v\Big)_0+(u,\ v)_0\,, \qquad u,\ v\in H_0^1(\Omega)\,,$$

where $(\ ,\ )_0$ stands for the scalar product in $L^2(\Omega)$. As in § 3, we transform problem (4.1) into the following equivalent eigenvalue problem:

$$\text{(4.2)} \qquad Su=p^{-1/2}(-h\Delta+1)p^{-1/2}u=\lambda u\,,$$

where S is a positive self-adjoint operator with domain

$$\mathcal{D}(S)=\mathcal{R}(p^{1/2}(-h\Delta+1)^{-1}p^{1/2})\,.^{3)}$$

Throughout this section, we use the notations G_γ and Σ_γ which were defined in § 1.

Our aim of this section is to prove the following theorem.

THEOREM 4.1. *Assume that $p(x)$ belongs to $K(m)$ with $0<m<2$. Let $N_h(\lambda\,;p,\ \Omega)$ be the number of eigenvalues less than λ of problem (4.2). Let $\gamma>0$ be fixed arbitrarily. Then, for any $\delta>0$ small enough, there exists a constant $C(\delta)$*

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3) The operator $p^{1/2}(-h\Delta+1)^{-1}p^{1/2}$ is a compact operator in $L^2(\Omega)$ since $p^{1/2}$ is a compact operator from $H_0^1(\Omega)$ to $L^2(\Omega)$. Hence, problem (4.2) has only discrete eigenvalues.

independent of h and λ such that for $\lambda \geqq C(\delta)h^{\alpha}$, $0<\alpha<1$,

$$N_h(\lambda\,;p,\Omega) \geqq C_0(\gamma)h^{-n/2}\lambda^{n/m} - \delta h^{-n/2}\lambda^{n/m},$$

where

$$C_0(\gamma)=(2\pi)^{-n}(2n)^{-1}\sigma_{n-1}\frac{\Gamma(n/m-n/2)\Gamma(n/2)}{\Gamma(n/m)}\int_{\Sigma_\gamma} a(\omega\,;p)^{n/m}d\omega\,.$$

Before proving this theorem, we introduce some functions and operators.

Let $\varphi(x)$, $\psi(x)$ and $\chi(x)$ be real-valued C_0^∞-functions defined on R^n such that $\varphi(x)$, $\psi(x)$ and $\chi(x)=1$ if $|x|\leqq 1$, $=0$ if $|x|\geqq 2$ and that $\varphi(x)\psi(x)=\varphi(x)$ and $\psi(x)\chi(x)=\psi(x)$. For each fixed t in R^n and any $\varepsilon>0$, we define $\varphi_{t,\varepsilon}(x)$ as

$$\varphi_{t,\varepsilon}(x)=\varphi\Big(\frac{x-t}{\varepsilon\rho(t)}\Big)\,.$$

Similarly we define $\psi_{t,\varepsilon}(x)$ and $\chi_{t,\varepsilon}(x)$.

Next we define the operators $A_h(\lambda)$ and $R_h(\lambda)$ for $\lambda>0$ as

$$A_h(\lambda)=-h\Delta+1+\lambda p\,,$$

$$R_h(\lambda)=(-h\Delta+1+\lambda p)^{-1}=A_h(\lambda)^{-1}$$

where $A_h(\lambda)$ is a positive self-adjoint operator with domain $\mathcal{D}(A_h(\lambda))=\mathcal{D}(A)$, and $A_h(\lambda)$ and $R_h(\lambda)$ are regarded as operators acting in $L^2(\Omega)$. Furthermore we define the operator $A_{t,h}(\lambda)$ for each fixed $t\in G_\gamma$ as

$$A_{t,h}(\lambda)=-h\Delta+1+\lambda p(t)$$

where $A_{t,h}(\lambda)$ is a positive self-adjoint operator with domain $\mathcal{D}(A_{t,h}(\lambda))=H^2(R^n)$.

PROOF OF THEOREM 4.1. Let $t\in G_\gamma$. Then, we choose a constant $C(\gamma)$ independent of t such that the set $\{y\,|\,|t-y|\leqq C(\gamma)\rho(t)\}$ is included in Ω. Let ε be fixed so that $\varepsilon<C(\gamma)/2$. Then, for each fixed $t\in G_\gamma$, we have the following equality:

$$\varphi_{t,\varepsilon}R_h(\lambda)=\psi_{t,\varepsilon}R_{t,h}(\lambda)\varphi_{t,\varepsilon}+\psi_{t,\varepsilon}R_{t,h}(\lambda)(A_{t,h}(\lambda)\varphi_{t,\varepsilon}-\varphi_{t,\varepsilon}A_h(\lambda))\chi_{t,\varepsilon}R_h(\lambda)\,, \tag{4.3}$$

where $R_{t,h}(\lambda)=A_{t,h}(\lambda)^{-1}$, which is an integral operator with kernel $F_{t,h}(x-y\,;\lambda)$ defined by (2.7). (4.3) is easily obtained by using the equality $\varphi_{t,\varepsilon}A_h(\lambda)\chi_{t,\varepsilon}=\varphi_{t,\varepsilon}A_h(\lambda)$, which follows from the relation $\varphi(x)\chi(x)=\varphi(x)$. From now on, we simply write φ_ε, ψ_ε and χ_ε instead of $\varphi_{t,\varepsilon}$, $\psi_{t,\varepsilon}$ and $\chi_{t,\varepsilon}$ respectively.

Let $\{\zeta_j\}_{j=1}^\infty$ be eigenvalues of problem (4.2) and $\{w_j\}_{j=1}^\infty$ be the normalized eigenfunction corresponding to $\{\zeta_j\}_{j=1}^\infty$. Then, it follows from (4.3) that

$$\begin{aligned}(\zeta_j+\lambda)^{-1}\varphi_\varepsilon w_j &= \psi_\varepsilon p^{1/2}R_{t,h}(\lambda)\varphi_\varepsilon p^{1/2}w_j\\ &\quad+(\zeta_j+\lambda)^{-1}\psi_\varepsilon p^{1/2}R_{t,h}(\lambda)(A_{t,h}(\lambda)\varphi_\varepsilon-\varphi_\varepsilon A_h(\lambda))\chi_\varepsilon p^{-1/2}w_j\,.\end{aligned} \tag{4.4}$$

By the same method as in the proof of Theorem 3.5, we have the following equality corresponding to (3.23):

$$(4.5)\qquad (-1)^k(k\,!)(\zeta_j+\lambda)^{-(k+1)}w_j(t)=p(t)^{1/2}\int F_{t,h}^{(k)}(t-y\,;\lambda)\varphi_\varepsilon(y)p(y)^{1/2}w_j(y)dy$$
$$+\sum_{r=0}^{k-1}C(r)(\zeta_j+\lambda)^{-(k-r)}p(t)^{1/2}\int F_{t,h}^{(r)}(t-y\,;\lambda)\theta_j(t,y,\varepsilon)dy$$
$$+\lambda\sum_{r=0}^{k}C_1(r)(\zeta_j+\lambda)^{-(k-r+1)}p(t)^{1/2}\int F_{t,h}^{(r)}(t-y\,;\lambda)\theta_j(t,y,\varepsilon)dy$$
$$+h\sum_{r=1}^{k}C_2(r)(\zeta_j+\lambda)^{-(k-r+1)}p(t)^{1/2}\int F_{t,h}^{(r)}(t-y\,;\lambda)B(t,D,\varepsilon)p(y)^{-1/2}w_j(y)dy$$
$$=a_j(t,\varepsilon)+\sum_{r=0}^{k-1}C(r)e_{j,r}(t,\varepsilon)+\lambda(\zeta_j+\lambda)^{-1}\sum_{r=0}^{k}C_1(r)e_{j,r}(t,\varepsilon)$$
$$+h\sum_{r=0}^{k}C_2(r)g_{j,r}(t,\varepsilon)\,,$$

where

$$\theta_j(t,y,\varepsilon)=\varphi_\varepsilon(y)(p(t)-p(y))p(y)^{-1/2}w_j(y)\,,$$
$$B(t,D,\varepsilon)=\Delta\varphi_\varepsilon-\varphi_\varepsilon\Delta\,,$$

and k is the integer fixed in the proof of Theorem 3.5. As in the proof of Theorem 3.5, by taking the square of both sides of (4.5), summing up with respect to j, and integrating over G_γ, we have for any $\delta>0$ small enough,

$$(4.6)\qquad \left|\sum_{j=1}^{\infty}(\zeta_j+\lambda)^{-2(k+1)}\int_{G_\gamma}w_j(t)^2dt-(k\,!)^{-2}\sum_{j=1}^{\infty}\int_{G_\gamma}a_j(t,\varepsilon)^2dt\right|$$
$$\leqq\delta\sum_{j=1}^{\infty}\int_{G_\gamma}a_j(t,\varepsilon)^2dt+C(\delta)\sum_{r=0}^{k}\sum_{j=1}^{\infty}\int_{G_\gamma}e_{j,r}(t,\varepsilon)^2dt$$
$$+C(\delta)h^2\sum_{r=0}^{k}\sum_{j=1}^{\infty}\int_{G_\gamma}g_{j,r}(t,\varepsilon)^2dt\,.$$

In order to complete the proof, we have to prepare some lemmas. The proofs of these lemmas are very similar to those of Lemmas 3.6 and 3.7, and so we shall give only outlines. Constants appearing in these lemmas and their proofs may depend on ε and γ.

LEMMA 4.2. *For any $\delta>0$ small enough, there exists a constant $C(\delta)$ independent of h and λ such that for $\lambda\geqq C(\delta)h^\alpha$, $0<\alpha<1$,*

$$\left|\sum_{j=1}^{\infty}\int_{G_\gamma}a_j(t,\varepsilon)^2dt-C_0(\gamma,k)h^{-n/2}\lambda^{n/m-2(k+1)}\right|\leqq\delta h^{-n/2}\lambda^{n/m-2(k+1)}\,,$$

where $C_0(\gamma,k)=(k\,!)^2n/m\dfrac{\Gamma(2(k+1)-n/m)\Gamma(n/m)}{\Gamma(2(k+1))}C_0(\gamma)$ and $C_0(\gamma)$ is the constant given in Theorem 4.1.

PROOF. By means of the Parseval equality, we get

$$\sum_{j=1}^{\infty}a_j(t,\varepsilon)^2=p(t)\int F_{t,h}^{(k)}(t-y\,;\lambda)^2\varphi_\varepsilon(y)^2p(y)dy\,.$$

If we have only to note the following fact (a), the proof is carried out in the same way as in the proof of Lemma 3.6.

(a)[4] For any $\delta>0$ small enough, there exists a constant $C(\delta)$ such that for $\lambda\geqq C(\delta)$,

$$\left|\int_{G_\gamma} p(t)^{2(k+1)}(1+\lambda p(t))^{n/2-2(k+1)}dt-C_1(\gamma)\lambda^{n/m-2(k+1)}\right|\leqq\delta\lambda^{n/m-2(k+1)},$$

where

$$C_1(\gamma)=1/m\frac{\Gamma(n/m-n/2)\Gamma(2(k+1)-n/m)}{\Gamma(2(k+1)-n/2)}\int_{\Sigma_\gamma} a(\omega\,;p)^{n/m}d\omega.$$

q. e. d.

LEMMA 4.3. *For any $\delta>0$ small enough, there exists a constant $C(\delta)$ independent of h and λ such that for $\lambda\geqq C(\delta)h^\alpha$,* $0<\alpha<1$,

$$\sum_{j=1}^{\infty}\int_{G_\gamma} e_{j,r}(t,\,\varepsilon)^2dt\leqq\delta h^{-n/2}\lambda^{n/m-2(k+1)}.$$

PROOF. We note the following three facts (a), (b) and (c). Taking these facts into account, we obtain the proof by an argument similar to that given in the proof of Lemma 3.7.

(a) There exists a constant C independent of y, z, j and δ (small enough) such that for $|y|\leqq\delta$ and $|z|\leqq\delta$,

$$\int_{G_\gamma}|w_j(t+y\rho(t))|\,|w_j(t+z\rho(t))|\,dt\leqq C.$$

(b) There exists a constant C independent of h and λ such that for $r\geqq k_0>n/m$ and $\lambda\geqq h^\alpha$,

$$\sum_{j=1}^{\infty}(\zeta_j+\lambda)^{-2r}\leqq Ch^{-n/2}\lambda^{n/m-2(r+1)}.$$

(b) is proved as follows: Let $p_0(x)$ be a function belonging to $K^+(m)$ such that $p_0(x)\geqq p^+(x)=\max(0,\,p(x))$. Then, consider the following eigenvalue problem:

$$p_0^{-1/2}(-h\Delta+1)p_0^{-1/2}u=\lambda u\,,\qquad u\in L^2(R^n).$$

Let $\{\zeta_{0,j}\}_{j=1}^{\infty}$ be eigenvalues of the above problem. Then, it holds that $\zeta_{0,j}\leqq\zeta_j$ for each j. Hence, by virtue of Lemma 3.4, we obtain the proof of (b).

(c) If δ is taken sufficiently small, we have for $y\in\Omega_\delta=\{y\,|\,|t-y|\leqq\delta\rho(t)\}$,

$$C_1p(t)\leqq p(y)\leqq C_2p(t).$$

(c) is easily obtained from the condition (K-2). q. e. d.

4) This fact can be verified in the same way as in the proof of Lemma 2.1.

LEMMA 4.4. *For any $\delta>0$ small enough, there exists a constant $C(\delta)$ independent of h and λ such that for $\lambda\geqq C(\delta)h^{\alpha}$,*

$$\sum_{j=1}^{\infty}\int_{G_T}g_{j,r}(t,\varepsilon)^2dt\leqq\delta h^{-n/2-2}\lambda^{n/m-2(k+1)}.$$

PROOF. Integration by parts yields

$$g_{j,r}(t,\varepsilon)=(\zeta_j+\lambda)^{-(k-r+1)}p(t)^{1/2}\int_{{}_{\varepsilon}\Omega_{2\varepsilon}}(B^*(t,D,\varepsilon)F_{t,h}^{(r)}(t,y;\lambda))p(y)^{-1/2}w_j(y)dy,$$

where $B^*(t,D,\varepsilon)$ is the formal adjoint of the operator $B(t,D,\varepsilon)$ and we denote by ${}_{\varepsilon}\Omega_{2\varepsilon}$ the set $\{y\,|\,\varepsilon\rho(t)\leqq|t-y|\leqq2\varepsilon\rho(t)\}$. The coefficients of the operator $B^*(t,D,\varepsilon)$ vanish outside the domain ${}_{\varepsilon}\Omega_{2\varepsilon}$. Hence, $B^*(t,D,\varepsilon)F_{t,h}^{(r)}(t-y;\lambda)$ is a smooth function. Furthermore, by virtue of Lemma 2.3 and the definition of $F_{t,h}^{(r)}(t-y;\lambda)$ given by (2.7), the following estimate holds for any $p>0$ large enough and $y\in{}_{\varepsilon}\Omega_{2\varepsilon}$;

$$|B^*(t,D,\varepsilon)F_{t,h}^{(r)}(t-y;\lambda)|\leqq C(\varepsilon)h^{(p-n)/2}p(t)^r(1+\lambda p(t))^{(n-p)/2-(r+1)}\rho(t)^{-p}. \tag{4.7}$$

If ε is taken sufficiently small, it follows from the fact (c) in the proof of Lemma 4.3 that for $y\in{}_{\varepsilon}\Omega_{2\varepsilon}$,

$$p(y)^{-1/2}\leqq C(\varepsilon)p(t)^{-1/2}. \tag{4.8}$$

Using the Parseval equality and the estimates (4.7) and (4.8), we have

$$\sum_{j=1}^{\infty}g_{j,r}(t,\varepsilon)^2\leqq C(\varepsilon)\lambda^{-2(k-r+1)}h^{p-n}p(t)^{2r}(1+\lambda p(t))^{n-p-2(r+1)}\rho(t)^{-2p+n},$$

from which we obtain by means of Lemma 2.2 that

$$\sum_{j=1}^{\infty}\int_{G_T}g_{j,r}(t,\varepsilon)^2dt\leqq C(\varepsilon)h^{-n/2-2}\lambda^{n/m-2(k+1)}h^{\gamma}\lambda^{-\beta}, \tag{4.9}$$

where $\gamma=p+2-n/2<\beta=p+2+n/m-n$. Hence, we can choose $C(\delta)$ and α so that for $\lambda\geqq C(\delta)h^{\alpha}$, the righthand side of (4.9) is dominated by $\delta h^{-n/2-2}\lambda^{n/m-2(k+1)}$.

q. e. d.

Completion of the proof of Theorem 4.1. By virtue of Lemmas 4.2, 4.3 and 4.4, it follows from (4.6) that for any $\delta>0$ small enough and $\lambda\geqq C(\delta)h^{\alpha}$, $0<\alpha<1$,

$$\Big|\sum_{j=1}^{\infty}(\zeta_j+\lambda)^{-2(k+1)}\sigma_j-(k\,!)^{-2}C_0(\gamma,k)h^{-n/2}\lambda^{n/m-2(k+1)}\Big|\leqq\delta h^{-n/2}\lambda^{n/m-2(k+1)}, \tag{4.10}$$

where $\sigma_j=\int_{G_T}w_j(t)^2dt$, and $C_0(\gamma,k)$ is the constant given in Lemma 4.2.

Now we put $N_{h,T}(\lambda;p,\Omega)=\sum_{\zeta_j<\lambda}\sigma_j$, and apply Lemma 2.4 with $\sigma_h(\lambda)=h^{n/2}N_{h,T}(\lambda;p,\Omega)$ to (4.10). Then, we have

$$|N_{h,\gamma}(\lambda\,;p,\,\Omega)-C_0(\gamma)h^{-n/2}\lambda^{n/m}|\leqq\delta h^{-n/2}\lambda^{n/m}$$

for $\lambda\geqq C(\delta)h^{\alpha}$, where $C_0(\gamma)$ is the constant defined in this theorem. Noting that $N_h(\lambda\,;p,\,\Omega)\geqq N_{h,\gamma}(\lambda\,;p,\,\Omega)$ for each $\gamma>0$, we immediately obtain the conclusion. q. e. d.

As a direct application of Theorem 4.1, we have the following theorem.

THEOREM 4.5. *Assume that $p(x)$ belongs to $K(m)$ with $0<m<2$. Let $n(r\,;p,\,\Omega)$ be the number of eigenvalues less that $-r$ of the problem*

$$-\Delta u-p(x)u=\lambda u\,,\qquad u\in H_0^1(\Omega)\,.$$

Then, we have

$$\liminf_{r\to 0} r^{n/m-n/2}n(r\,;p,\,\Omega)\geqq C_3\,,$$

where

$$C_3=(2\pi)^{-n}(2n)^{-1}\sigma_{n-1}\frac{\Gamma(n/m-n/2)\Gamma(n/2)}{\Gamma(n/m)}\int_{S^{n-1}}a^+(\omega\,;p)^{n/m}d\omega\,.$$

PROOF. We first note that $n(r\,;p,\,\Omega)=N_h(h\,;p,\,\Omega)$ with $r=1/h$. Hence, by virtue of Theorem 4.1, it follows that for each $\gamma>0$,

$$\liminf_{r\to 0} r^{n/m-n/2}n(r\,;p,\,\Omega)=\liminf_{h\to\infty} h^{n/2-n/m}N_h(h\,;p,\,\Omega)\geqq C_0(\gamma)\,,$$

where $C_0(\gamma)$ is the constant given in Theorem 4.1. Since γ is arbitrary, we get the conclusion. q. e. d.

§ 5. Eigenvalue problem with singular potentials.

In this section, using the results in [2], [3] and [8], we shall study the eigenvalue problem with a singular potential.

PROPOSITION 5.1 (H. Weyl). *Let T, T_1 and T_2 be self-adjoint compact operators in a separable Hilbert space $\mathcal{H}$. Assume that T is expressed as $T=T_1+T_2$. We denote by $\lambda_n^+(T)$ and $\lambda_n^+(T_i)$ $(i=1,\,2)$ the n-th positive eigenvalue of T and T_i respectively. Then,*

$$\lambda_{n+m-1}^+(T)\leqq\lambda_n^+(T_1)+\lambda_m^+(T_2)\,.$$

PROPOSITION 5.2 ([2], [3] and [8]). *Let $p_2(x)$ be a non-negative function. Assume that if $n\leqq 2$, $p_2(x)$ has compact support and belongs to $L^p(R^n)$ and that if $n>2$, $p_2(x)$ belongs to $L^{n/2}(R^n)\cap L^p(R^n)$, where $p>\max(1,\,n/2)$. Let $M(\lambda,\,r)$ be the number of eigenvalues less than λ of the following problem for each fixed r $(0\leqq r\leqq 1)$:*

$$-\Delta u+ru=\lambda p_2(x)u\,,\qquad u\in H^1(R^n)\,.$$

Then, as $\lambda\to\infty$,

$$M(\lambda,\,y)=C\int p_2(x)^{n/2}dx\lambda^{n/2}+o(\lambda^{n/2})\,,\tag{5.1}$$

where C is a constant independent of r and $p_2(x)$, *and the remainder estimate is uniform with respect to r.*

Let $p_1(x)$ be a function belonging to $K^+(m)$ $(0<m<2)$ and let $p_2(x)$ be a function satisfying the assumption in Proposition 5.2. For each fixed $0<r<1$ we define the operators $T(r)$, $T_1(r)$ and $T_2(r)$ as follows:

$$T(r)=(-\Delta+r)^{-1}(p_1+p_2)\,,$$

$$T_1(r)=(-\Delta+r)^{-1}p_1,\ T_2(r)=(-\Delta+r)^{-1}p_2\,.$$

We note that the operators $T(r)$, $T_1(r)$ and $T_2(r)$ are self-adjoint compact operators in the Hilbert space $H^1(R^n)$ with the scalar product

$$[u, v]_r=((-\Delta+r)^{1/2}u, (-\Delta+r)^{1/2}v)\,.$$

LEMMA 5.3. *There exists a constant* ε_0 *independent of* $0<r\leqq 1$ *such that for any* $0<\varepsilon<\varepsilon_0$ *and r,* $T_2(r)$ *has at least one eigenvalue in* $(\varepsilon/3, \varepsilon/2)$.

PROOF. The above statement is obvious from the following two facts:

(a) The number of eigenvalues greater than $\varepsilon/3$ of the operator $T_2(r)$ is equal to $M(3/\varepsilon, r)$;

(b) The remainder estimate in (5.1) is uniform with respect to r. q.e.d.

LEMMA 5.4. *Let* $m(\varepsilon, r)$ *be the number of eigenvalues greater than* ε *of the operator* $T_2(r)$. *Then there exists a constant* $C(\varepsilon)$ *independent of r such that*

$$m(\varepsilon, r)\leqq C(\varepsilon)\,. \tag{5.2}$$

PROOF. (5.2) is an immediate consequence of Proposition 5.2. q.e.d.

LEMMA 5.5. *For any* $\varepsilon>0$ *small enough, there exists a constant* $r(\varepsilon)$ *such that for any* $r<r(\varepsilon)$, *the operator* $T_1(r)$ *has at least one eigenvalue in* $(1-2\varepsilon, 1-\varepsilon)$.

PROOF. We note that the number of eigenvalues greater than $(1-2\varepsilon)$ of the operator $T_1(r)$ is equal to $n(r; (1-2\varepsilon)^{-1}p_1)$. In fact, $n(r; (1-2\varepsilon)^{-1}p_1)$ coincides with the maximal dimension of subspaces lying in $H^1(R^n)$ such that

$$(1-2\varepsilon)((-\Delta+r)^{1/2}u, (-\Delta+r)^{1/2}u)<((-\Delta+r)^{1/2}(-\Delta+r)^{-1}p_1u, (-\Delta+r)^{1/2}u)\,.$$

This shows the above fact. On the other hand, by virtue of Theorem 3.5, we have

$$n(r; (1-2\varepsilon)^{-1}p_1)=C(1-2\varepsilon)^{-n/m}r^{n/2-n/m}+o(r^{n/2-n/m})\,,$$

where the constant C is independent of ε and the remainder estimate is uniform for any $\varepsilon>0$ small enough. This completes the proof. q.e.d.

THEOREM 5.6. *Let* $p(x)$ *be a function satisfying the following assumptions:*

(i) $p(x)$ *is decomposed into* $p(x)=p_1(x)+p_2(x)$;

(ii) $p(x)$ *belongs to* $K^+(m)$ $(0<m<2)$;

(iii) $p_2(x)$ *satisfies the assumption in Proposition* 5.2.
Then, for any $\varepsilon>0$ *small enough, there exist constants* $r(\varepsilon)$ *and* $C(\varepsilon)$ *such that for* $r<r(\varepsilon)$,

$$n(r\,;p)\leqq n(r\,;(1-2\varepsilon)^{-1}p_1)+C(\varepsilon)\,.$$

PROOF. We first note that $n(r\,;p)$ is equal to the number of eigenvalues greater than one of the operator $T(r)$. By virtue of Lemmas 5.3 and 5.4, for any $\varepsilon<\varepsilon_0$, the $m(\varepsilon/3, r)(\leqq C(\varepsilon))$-th eigenvalue of the operator $T_2(r)$ is less than $\varepsilon/2$. On the other hand, by Lemma 5.5, it follows that for any $0<\varepsilon<\varepsilon_0$ there exists $r(\varepsilon)$ such that for any $r<r(\varepsilon)$, the $n(r, (1-2\varepsilon)^{-1}p_1)$-th eigenvalue of the operator $T_1(r)$ is less than $(1-\varepsilon)$. Hence, by proposition 5.1, the $(n(r\,;(1-2\varepsilon)^{-1}p_1)+C(\varepsilon))$-th eigenvalue of the operator $T(r)$ is less than $1-\varepsilon/2$. This completes the proof. q. e. d.

§ 6. Proof of the main theorem.

In this section, we shall prove our main theorem stated in § 1.

THEOREM 6.1. *Assume that* $p(x)$ *belongs to* $S(m)$ *with* $0<m<2$. *Let* $n(r\,;p)$ *be the number of eigenvalues less than* $-r$ *of problem* (1.2). *Then, as* $r\to 0$,

$$n(r\,;p)=C_0 r^{n/2-n/m}+o(r^{n/2-n/m})\,,$$

where C_0 *is the constant given by* (1.3).

PROOF. By the condition (S$-$1), $p(x)$ is decomposed into two parts:

$$p(x)=p_1(x)+p_2(x)\,,$$

where $p_1(x)$ and $p_2(x)$ are functions satisfying the conditions (S-2) and (S-3) respectively. Furthermore, by the condition (S-2), there exists a sequence $\{q_k(x)\}_{k=1}^{\infty}$ such that for each k $q_k(x)$ belongs to $K^+(m)$ and that

$$a(\omega\,;q_k)\leqq C\,, \tag{6.1.1}$$

$$\lim_{k\to\infty} a(\omega\,;q_k)=a^+(\omega\,;p_1)\,. \tag{6.1.2}$$

We shall first give the estimate from above for $n(r\,;p)$. It is clear that for each k $n(r\,;p)\leqq n(r\,;q_k+p_2)$. Furthermore, in virtue of Theorem 5.6, it follows that for each k and any $\varepsilon>0$ small enough,

$$n(r\,;q_k+p_2)\leqq n(r\,;(1-2\varepsilon)^{-1}q_k)+C(k, \varepsilon)\,.$$

Hence, we have by means of Theorem 3.8 that

$$\limsup_{r\to 0} r^{n/m-n/2}n(r\,;q_k+p_2)\leqq(1-2\varepsilon)^{-n/m}C_4\int_{S^{n-1}}a(\omega\,;q_k)^{n/m}d\omega\,,$$

where

$$C_4=(2\pi)^{-n}(2n)^{-1}\sigma_{n-1}\frac{\Gamma(n/m-n/2)\Gamma(n/2)}{\Gamma(n/m)}\,.$$

Since ε is arbitrary, it follows from (6.1.1), (6.1.2) and the Lebesgue convergence theorem that

$$\limsup_{r\to 0} r^{n/m-n/2}n(r;p)\leqq C_0\,, \tag{6.2}$$

where C_0 is the constant given by (1.3). Thus we have established the estimate from above.

Next we shall give the estimate from below. Let Ω be the open set given by $\{x|p_1(x)>0\}$. Then, consider the following eigenvalue problem with zero boundary conditions:

$$-\Delta u-p_1(x)u=\lambda u\,,\qquad u\in H_0^1(\Omega)\,.$$

Let $n(r;p_1,\Omega)$ be the number of eigenvalues less than $-r$ of the above problem. Then, we can easily see that $n(r;p)\geqq n(r;p_1,\Omega)$. Hence, we have by virtue of Theorem 4.5,

$$\liminf_{r\to 0} r^{n/m-n/2}n(r;p)\geqq \liminf_{r\to 0} r^{n/m-n/2}n(r;p_1,\Omega)\geqq C_0\,. \tag{6.3}$$

Combining (6.2) and (6.3), we obtain the conclusion. q. e. d.

§ 7. The asymptotic distribution of discrete eigenvalues for the Dirac operators.

In this section, we shall apply the method developed in the preceding sections to the Dirac operators. Let us consider the following eigenvalue problem:

$$\begin{aligned} S\varphi&=S_0\varphi-p(x)\varphi \\ &=(\sum_{k=1}^{3}\alpha_k\xi_k+\alpha_4)\varphi-p(x)\varphi=\lambda\varphi\,. \end{aligned} \tag{7.1}$$

Here $\xi_k=-i\left(\frac{\partial}{\partial x_k}\right)$ $(i=\sqrt{-1},\ k=1,2,3)$; $\varphi=(\varphi_1,\cdots,\varphi_4)$ is a four-component function belonging to $[L^2(R^3)]^4$; α_k $(k=1,2,3,4)$ are the Dirac numerical 4×4 matrices satisfying the relationship $\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk}$; $p(x)$ is a scalar potential which we suppose, for brevity, to belong to $K^+(m)$ $(0<m<2)$. For later use, we write the explicit form of α_4:

$$\alpha_4=\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}.$$

We denote by $n_0(r;p)$ $(r>0)$ the number of eigenvalues lying in $(-1+r,\ 1-r)$ of problem (7.1). We remark that since $p(x)>0$, discrete eigenvalues cannot admit $\lambda=-1$ as a cluster point.

LEMMA 7.1. *For any $\delta>0$ small enough, there exist operators $A(\pm\delta)$ such that for any $\varphi\in[C_0^\infty(R^3)]^4$,*

$$[A(-\delta)\varphi, \varphi]\leqq[(S^2-1)\varphi, \varphi]\leqq[A(+\delta)\varphi, \varphi],$$

where

$$A(\pm\delta)=\begin{pmatrix} B_1(\pm\delta) & & & 0 \\ & B_1(\pm\delta) & & \\ & & B_2(\pm\delta) & \\ 0 & & & B_2(\pm\delta) \end{pmatrix},$$

$B_1(\pm\delta)=(-1\mp\delta)\Delta-2p\pm C(\delta)p^2$ and $B_2(\pm\delta)=(-1\mp\delta)\Delta+2p\pm C(\delta)p^2$, while [,] stands for the usual scalar product in $[L^2(R^3)]^4$, that is,

$$[\varphi, \psi]=\sum_{j=1}^{4}(\varphi_j, \psi_j) \quad and \quad [\varphi]^2=\sum_{j=1}^{4}(\varphi_j, \varphi_j).$$

PROOF. A simple calculation yields

$$[(S^2-1)\varphi, \varphi]=[(S_0^2-1)\varphi, \varphi]+[p^2\varphi, \varphi] \\ -2\,\mathrm{Re}\,[(S_0-\alpha_4)\varphi, p\varphi]-2[\alpha_4\varphi, p\varphi]. \tag{7.2}$$

Noting that $\alpha_4=\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$, we see that

$$[\alpha_4\varphi, p\varphi]=\sum_{j=1}^{2}(p\varphi_j, \varphi_j)-\sum_{j=3}^{4}(p\varphi_j, \varphi_j). \tag{7.3}$$

On the other hand, we have for any $\delta>0$,

$$2|[(S_0-\alpha_4)\varphi, p\varphi]|\leqq\delta[(S_0-\alpha_4)\varphi]^2+1/\delta[p\varphi]^2 \\ =\delta\sum_{j=1}^{4}(-\Delta\varphi_j, \varphi_j)+1/\delta\sum_{j=1}^{4}(p\varphi_j, \varphi_j). \tag{7.4}$$

Hence, in view of (7.2), (7.3) and (7.4), we obtain the proof since $S_0^2-1=-\Delta$.

q. e. d.

THEOREM 7.2. *Assume that $p(x)$ belongs to $K^+(m)$ with $0<m<2$. Then, as $r\to 0$,*

$$n_0(r\,;p)=C_5r^{3/2-3/m}+o(r^{3/2-3/m}),$$

where

$$C_5=1/12(2\pi^{-1})^{3/2}\frac{\Gamma(3/m-3/2)}{\Gamma(3/m)}\int_{S^2}a(\omega\,;p)^{3/m}d\omega.$$

PROOF. Only an outline of the proof will be presented. We note that $n_0(r; p)$ is equal to the maximal dimension of subspaces lying in $[C_0^\infty(R^3)]^4$ such that

$$[S^2\varphi, \varphi]<(1-r)^2[\varphi, \varphi].$$

Therefore, by virtue of Lemma 7.1, we can apply the same argument as in the proof of Theorem 3.5 to the operators $(-1\mp\delta)\Delta-2p\pm C(\delta)p^2$ in obtaining the estimates from above and from below for $n_0(r; p)$. We should note that since $p(x)^2$ decays at infinity faster than $p(x)$, $C(\delta)p(x)^2$ has no contribution to the leading term of the asymptotic formula for $n_0(r; p)$. q. e. d.

§ 8. Concluding remark.

REMARK 1. The assumption (K-1) is weakened as follows. Consider a smooth function $p(x)$ satisfying

(K′-1) $$\lim_{r\to\infty} r^m p(r\omega)=r(\omega; p),$$

where we do not assume that the convergence is uniform with respect to ω and that $a(\omega; p)$ is a continuous bounded function on S^{n-1}.

Let Σ^+ be the set given by $\Sigma^+=\{\omega \mid a(\omega; p)>0\}$. Then, we assume the following conditions:

(K′-3) Σ^+ is an open set in S^{n-1};

(K′-4) The convergence in (K′-1) is locally uniform in Σ^+ and $a(\omega; p)$ is a continuous function in Σ^+.

We denote by $K'(m)$ the set of all smooth functions satisfying (K′-1), (K-2), (K′-3) and (K′-4) and define $S'(m)$ corresponding to $S(m)$ with $K(m)$ replaced by $K'(m)$ in (S-2). Then, for a function belonging to $S'(m)$ $(0<m<2)$, we have the same conclusion as Theorem 6.1.

References.

[1] M. Š. Birman, On the spectrum of singular boundary problems, Math. Sb., **55** (1961), 125-174 (in Russian); Amer. Math. Soc. Transl., **53**, 23-80.

[2] M. Š. Birman and V. V. Borzov, On the asymptotics of the discrete spectrum of some singular differential operators, Problem of Math. Phys., **5** (1971), 1-24 (in Russian).

[3] M. Š. Birman and M. E. Solomjak, Leading term in the asymptotic spectral formula for nonsmooth elliptic problems, Anal. i Prilozen., **4** (1970), no 4, 1-13 (in Russian) = Functional. Anal. Appl., **4** (1970), 265-275.

[4] F. H. Brownell and C. W. Clark, Asymptotic distribution of the eigenvalues of the lower part of the Schrödinger operator spectrum, J. Math. Mech., **10** (1961), 31-70.

[5] J. Karamata, Neuer Beweis und Verallgemeinerung der Tauberschen Satze, welch die Laplaceshe und Stieltjesshe Transformation betreffen, J. Reine Angew. Math., **164** (1931), 27-39.

[6] J.B. McLeod, The distribution of the eigenvalues for the hydrogen atom and similar cases, Proc. London Math. Soc., **11** (1961), 139-158.

[7] S. Mizohata, Theory of partial differential equations, Iwanami, Tokyo, 1965 (in Japanese). Cambridge Univ. Press, 1973 (English translation).

[8] G.V. Rozenbljum, The distribution of the discrete spectrum for singular differential operators, Dokl. Akad. Nauk SSSR, **202** (1972), 1012-1015 (in Russian); Soviet Math. Dokl., **13** (1972), 245-249.

[9] H. Tamura, The asymptotic distribution of the lower part eigenvalues for elliptic operators, Proc. Japan Acad., **50** (1974), 185-187.

[10] E.C. Titchmarsh, Eigenfunction expansion associated with second order differential Equations, Vol. II. Oxford univ. Press, 1958.

[11] H. Tamura, The asymptotic distribution of discrete eigenvalues for Dirac operators, J. Fac. Sci. Univ. Tokyo, Sec, 1A, **23** (1976), 167-197.

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